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Global well-posedness for the generalized 2D Ginzburg–Landau equation

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ABSTRACT

The local well-posedness for the generalized two-dimensional (2D) Ginzburg–Landau equation is obtained for initial data in $H^s(\mathbb{R}^2)$ $(s > 1/2)$. The global result is also obtained in $H^s(\mathbb{R}^2)$ $(s > 1/2)$ under some conditions. The results on local and global well-posedness are sharp except the endpoint $s = 1/2$. We mainly use the Tao's $[k; Z]$-multiplier method to obtain the trilinear and multilinear estimates.

1. Introduction

The aim in this work is to study the Cauchy problem of the generalized two-dimensional (2D) Ginzburg–Landau equation:

$$u_t - (\alpha + \beta i)\Delta u + |u|^2(\vec{\gamma} \cdot \nabla u) + u^2(\vec{\lambda} \cdot \nabla \bar{u}) + (\alpha_1 + \beta_1 i)|u|^4 u = 0, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,0) = u_0(x) \in H^s(\mathbb{R}^2), \tag{1.2}$$

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where $\bar{u}(x,t)$ is the complex conjugate of $u(x,t)$, β, β_1 are real numbers, $\alpha > 0$, $\alpha_1 > 0$; $\vec{\gamma} = (\gamma_1, \gamma_2)$ and $\vec{\lambda} = (\lambda_1, \lambda_2)$ are complex vectors.

The generalized Ginzburg–Landau (GGL) equation arises as the envelope equation for a weakly sub-critical bifurcation to counter-propagating waves. It is also of importance in the theory of interaction behavior, including complete interpenetration as well as partial annihilation, for collision between localized solutions corresponding to a single particle and to a two particle state. For details of the physical backgrounds of the GGL equation, one can refer to [1,5,6].

If $\vec{\gamma} = \vec{\lambda} = 0$, then Eq. (1.1) becomes the Ginzburg–Landau (GL) equation

$$u_t - (\alpha + \beta i)\Delta u + (\alpha_1 + \beta_1 i)|u|^4 u = 0, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+. \tag{1.3}$$

It is an important model in the description of spatial pattern formation and of the onset of instabilities in non-equilibrium fluid dynamical systems, as well as in the theory of phase transitions and superconductivity [4].

For the well-posedness of 2D GL equation (1.3), Bu [3] showed that the Cauchy problem (1.3) and (1.2) is locally well-posed in H^3 if $\alpha > 0$, $\alpha_1 \geqslant 0$, and globally well-posed in H^3 if $|\beta_1| \leqslant \frac{\sqrt{5}}{2}\alpha_1$ or $\beta\beta_1 > 0$. In fact, the condition $\alpha_1 \geqslant 0$ is redundant for local result, which is killed in this paper without any penalty. One can find it in this paper.

For 1D and 2D GGL equation (1.1), there are several papers [7,8,10,13,14] related to the well-posedness of the Cauchy problem (1.1) and (1.2). Notice that these papers mainly consider the global well-posedness in energy space H^1 or H^2. Moreover, these authors treated Eqs. (1.1) and (1.3) as parabolic equations, used the time–space L^p–L^r estimates method [13,14] or semigroup method [12] to obtain the local results.

Recently, Molinet and Ribaud [11] used the Bourgain's space with dissipation to consider the KdV–Burgers equation

$$u_t + u_{xxx} - u_{xx} + uu_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{1.4}$$

They showed that it is globally well-posed in H^s with $s > -1$. Enlightened by some ideas in [11], we will use this method to consider Eq. (1.1) in both 1D and 2D cases. In fact, for 1D GGL equation, we showed that the Cauchy problem (1.1) and (1.2) is locally well-posed in H^s with $s > 0$, and globally well-posed in H^s with $s > 0$ under some conditions in [9].

In this paper, we consider the Cauchy problem (1.1) and (1.2) in two-dimensional case. We will prove that if $\alpha > 0$, then it is locally well-posed in H^s with $s > 1/2$. Furthermore, it is globally well-posed in H^s with $s > 1/2$ under some conditions. The space $H^{1/2}$ is critical one for Eq. (1.1). Therefore, our results on local and global well-posedness are sharp except the endpoint $s = 1/2$.

1.1. Definitions and notations

The Cauchy problem (1.1) and (1.2) is rewritten as the integral equivalent formulation

$$u(x,t) = S_\alpha(t)u_0 - \int_0^t S_\alpha(t-t')\big(|u|^2(\vec{\gamma}\cdot\nabla u) + u^2(\vec{\lambda}\cdot\nabla\bar{u}) + (\alpha_1 + \beta_1 i)|u|^4 u\big)(t')\,dt', \tag{1.5}$$

where $S_\alpha(t) = \mathcal{F}_x^{-1} e^{-it\beta|\xi|^2} e^{-|t|\alpha|\xi|^2}\mathcal{F}_x$ is the semigroup associated to the linear GGL equation.

For $s, b \in \mathbb{R}$, we define the Bourgain's spaces with dissipation for (1.1) endowed with the norms

$$\|u\|_{Y_{s,b}} = \big\|\langle\xi\rangle^s \big\langle i\big(\tau - |\xi|^2\big) + |\xi|^2\big\rangle^b \hat{u}(\xi,\tau)\big\|_{L^2_{\xi\in\mathbb{R}^2} L^2_{\tau\in\mathbb{R}}}, \tag{1.6}$$

$$\|u\|_{\overline{Y}_{s,b}} = \big\|\langle\xi\rangle^s \big\langle i\big(\tau + |\xi|^2\big) + |\xi|^2\big\rangle^b \hat{u}(\xi,\tau)\big\|_{L^2_{\xi\in\mathbb{R}^2} L^2_{\tau\in\mathbb{R}}}. \tag{1.7}$$

The norm standard spaces $X_{s,b}$ and $\overline{X}_{s,b}$ for the Schrödinger equation are defined [2]

$$\|u\|_{X_{s,b}} = \left\| \langle\xi\rangle^s \langle\tau - |\xi|^2\rangle^b \hat{u}(\xi,\tau)\right\|_{L^2_{\xi\in\mathbb{R}^2} L^2_{\tau\in\mathbb{R}}}, \tag{1.8}$$

$$\|u\|_{\overline{X}_{s,b}} = \left\| \langle\xi\rangle^s \langle\tau + |\xi|^2\rangle^b \hat{u}(\xi,\tau)\right\|_{L^2_{\xi\in\mathbb{R}^2} L^2_{\tau\in\mathbb{R}}}. \tag{1.9}$$

Denote $\hat{u}(\xi,\tau) = \mathcal{F}u(x,t)$ by the Fourier transform in t and x of u and $\mathcal{F}_{(\cdot)}u$ by the Fourier transform in the $(\cdot)$ variable. Notice that $\|\bar{u}\|_{\overline{Y}_{s,b}} = \|u\|_{Y_{s,b}}$. The spaces $Y_{s,b}$ and $\overline{Y}_{s,b}$ turn out to be very useful to consider the well-posedness of the dispersive equation with dissipative term, such as Eqs. (1.1), (1.4), etc.

Define $A \sim B$ by using the statement: $A \leqslant C_1 B$ and $B \leqslant C_1 A$ for some constant $C_1 > 0$, and define $A \ll B$ through the statement: $A \leqslant \frac{1}{C_2} B$ for some large enough constant $C_2 > 0$. We use $A \lesssim B$ to denote the statement that $A \leqslant CB$ for some large constant C.

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp}\psi \subset [-1,1]$, ψ is positive and even. Define $\psi_\delta(\cdot) = \psi(\delta^{-1}(\cdot))$ for some non-zero $\delta \in \mathbb{R}$.

1.2. Main method and results

Considering the local well-posedness of the Cauchy problem (1.1) and (1.2), we would apply a fixed point argument to the following truncation version of (1.5)

$$\begin{aligned} u(x,t) = \psi(t) S_\alpha(t) u_0 - \psi(t) \int_0^t S_\alpha(t-t')\big(|u|^2(\vec{\gamma}\cdot\nabla u) \\ + u^2(\vec{\lambda}\cdot\nabla\bar{u}) + (\alpha_1 + \beta_1 i)|u|^4 u\big)(t')\,dt', \end{aligned} \tag{1.10}$$

for any $u, \bar{u}$ with compact support in $[-T, T]$ in the integral of right side.

Indeed, if u solves (1.10) then u is a solution of (1.5) on $[0, T]$ with $T < 1$. Therefore, following some ideas in [11], we mainly prove the trilinear, multilinear estimates as follows, which will be obtained in Section 3,

$$\left\| |u|^2(\vec{\gamma}\cdot\nabla u)\right\|_{Y_{s,-1/2+\delta}} \leqslant C\|u\|^3_{Y_{s,1/2}}, \tag{1.11}$$

$$\left\| u^2(\vec{\lambda}\cdot\nabla\bar{u})\right\|_{Y_{s,-1/2+\delta}} \leqslant C\|u\|^3_{Y_{s,1/2}}, \tag{1.12}$$

$$\left\| |u|^4 u\right\|_{Y_{s,-1/2+\delta}} \leqslant C\|u\|^5_{Y_{s,1/2}}. \tag{1.13}$$

And from linear estimates obtained in Section 2, we can obtain the local result. Then the global well-posedness will be obtained by some a priori estimates obtained in Section 4 and regularity of solution given in Lemma 2.3.

Denote $Z_T = C([0,T]; H^s) \cap Y^T_{s,1/2}$, the main results of the paper are listed as below.

Theorem 1.1. *Let $u_0 \in H^s(\mathbb{R}^2)$ with $s > 1/2$. Then there exists a constant $T > 0$, such that the Cauchy problem (1.1) and (1.2) admits a unique local solution $u(x,t) \in Z_T$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is smooth from H^s to Z_T and u belongs to $C((0,T); H^{+\infty})$.*

Theorem 1.2. *Let $u_0 \in H^s(\mathbb{R}^2)$ $(s > 1/2)$. Assume $A := 1 - \frac{(|\vec{\gamma}|+|\vec{\lambda}|)^2}{4\alpha\alpha_1} > 0$. Moreover, if one of the following conditions holds*

$$(1) \qquad \beta\beta_1 > 0; \tag{1.14}$$

(2) $$\beta\beta_1 \leqslant 0, \qquad \min\left\{\frac{|\beta|}{\alpha A}, \frac{|\beta_1|}{\alpha_1 A}\right\} < \frac{\sqrt{5}}{2}; \tag{1.15}$$

(3) $$\beta\beta_1 \leqslant 0, \qquad |\beta\beta_1| - \left(\sqrt{\alpha\alpha_1} - \frac{(|\vec{\gamma}| + |\vec{\lambda}|)}{2}\right)^2 < \frac{\sqrt{5}}{2}\big(|\beta|\alpha_1 + |\beta_1|\alpha\big)\left(1 - \frac{(|\vec{\gamma}| + |\vec{\lambda}|)}{2\sqrt{\alpha\alpha_1}}\right), \tag{1.16}$$

then for any $T > 0$, Cauchy problem (1.1) *and* (1.2) *admits a unique solution $u(x,t) \in Z_T$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is smooth from H^s to Z_T and u belongs to $C((0,+\infty); H^{+\infty})$.*

Remark. In fact, similarly with the proofs of Theorems 1.1 and 1.2, we can also prove that the Cauchy problem of 3D GGL equation (1.1) is locally well-posed in H^s $(s > 1)$, and globally well-posed in H^s $(s > 1)$ under some conditions.

2. Linear estimates

In this section, we give some linear estimates for Eqs. (1.1) similarly with the dissipative KdV equation (1.4). In fact, in the proofs of the following lemmas, we only make computations with respect to the time variable t or the Fourier transform in t, which are similar with those of Propositions 2.1–2.4 in [11]. Here, we omit the details.

Lemma 2.1. *Let $s \in \mathbb{R}$ and $\alpha \geqslant 0$. Then*

$$\big\|\psi(t)S_\alpha(t)u_0\big\|_{Y_{s,1/2}} \leqslant C\|u_0\|_{H^s}. \tag{2.1}$$

Lemma 2.2. *Let $s \in \mathbb{R}$, $0 < \delta \ll \frac{1}{2}$ and $\alpha \geqslant 0$. Then*

$$\left\|\psi(t)\int_0^t S_\alpha(t-t')f(t')\,dt'\right\|_{Y_{s,1/2}} \leqslant C\|f\|_{Y_{s,-1/2+\delta}}. \tag{2.2}$$

Lemma 2.3. *Let $s \in \mathbb{R}$, $\alpha > 0$, and $0 < \delta \ll \frac{1}{2}$. Then for $f \in Y_{s,-1/2+\delta}$,*

$$\int_0^t S_\alpha(t-t')f(t')\,dt' \in C\big(\mathbb{R}^+, H^{s+2\delta}\big). \tag{2.3}$$

Moreover, if $\{f_n\}$ is a sequence and $f_n \to 0$ in $Y_{s,-1/2+\delta}$ as $n \to \infty$, then

$$\left\|\int_0^t S_\alpha(t-t')f_n(t')\,dt'\right\|_{L^\infty(\mathbb{R}^+, H^{s+2\delta})} \to 0, \quad \text{as } n \to \infty. \tag{2.4}$$

3. Trilinear, multilinear estimates and local well-posedness

In this section, the trilinear and multilinear estimates are obtained by using Tao's $[k; Z]$-multiplier method. Then, we can obtain the local well-posedness for the Cauchy problem (1.1) and (1.2) by the linear estimates in Section 2 and the trilinear and multilinear estimates. In fact, Theorem 1.1 can be proved by Lemmas 2.1–2.3 and Corollary 3.6.

We firstly list some useful notations and properties for multilinear expressions [15]. Let Z be any abelian additive group with an invariant measure $d\xi$. For any integer $k \geqslant 2$, we denote $\Gamma_k(Z)$ by the "hyperplane"

$$\Gamma_k(Z) = \big\{(\xi_1, \ldots, \xi_k) \in Z^k\colon\ \xi_1 + \cdots + \xi_k = 0\big\},$$

which is endowed with the measure

$$\int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\xi_1, \dots, \xi_{k-1}, -\xi_1 - \dots - \xi_{k-1}) d\xi_1 \dots d\xi_{k-1},$$

and define a $[k; Z]$-multiplier to be any function $m : \Gamma_k(Z) \to \mathbb{C}$. If m is a $[k; Z]$-multiplier, we define $\|m\|_{[k;Z]}$ to be the best constant, such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j) \right| \leqslant \|m\|_{[k;Z]} \prod_{j=1}^{k} \|f_j\|_{L_2(Z)}$$

holds for all test functions f_j defined on Z. It is clear that $\|m\|_{[k;Z]}$ determines a norm on m, for test functions at least. We are interested in obtaining the good boundedness on the norm. We will also define $\|m\|_{[k;Z]}$ in situations when m is defined on all of Z^k by restricting to $\Gamma_k(Z)$.

We give some properties of $\|m\|_{[k;Z]}$, especially for the case $k = 3$. This corresponds to the bilinear $X_{s,b}$ estimates of Schrödinger equation ($Y_{s,b}$ estimates of GGL equation) since multilinear estimates can be reduced to some bilinear estimates (we can find it later).

Let

$$\xi_1 + \xi_2 + \xi_3 = 0, \qquad \tau_1 + \tau_2 + \tau_3 = 0, \tag{3.1}$$

$$\tilde{\sigma}_j = \tau_j + h_j(\xi_j), \qquad h_j(\xi_j) = \pm|\xi_j|^2, \quad j = 1, 2, 3. \tag{3.2}$$

Then we will study the problem of obtaining

$$\big\| m\big((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3)\big) \big\|_{[3, \mathbb{R}^2 \times \mathbb{R}]} \lesssim 1, \tag{3.3}$$

where $m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3))$ is some $[k; Z]$-multiplier in $\Gamma_3(\mathbb{R}^2 \times \mathbb{R})$.

From (3.1) and (3.2), it follows that

$$\tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3 = h(\xi_1, \xi_2, \xi_3). \tag{3.4}$$

By symmetry, there are only two possibilities for the h_j: the $(+++)$ case

$$h_1(\xi) = h_2(\xi) = h_3(\xi) = |\xi|^2; \tag{3.5}$$

and the $(++-)$ case

$$h_1(\xi) = h_2(\xi) = |\xi|^2, \qquad h_3(\xi) = -|\xi|^2. \tag{3.6}$$

Of the two cases, the $(+++)$ case is substantially easier, because the resonance function

$$h(\xi_1, \xi_2, \xi_3) := |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 \tag{3.7}$$

does not vanish except at the origin. The $(++-)$ case is more delicate, because the resonance function

$$h(\xi_1, \xi_2, \xi_3) := |\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 \tag{3.8}$$

vanishes when ξ_1 and ξ_2 are orthogonal. Notice that for $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^2$, the resonance identity is given by

$$\big|h(\xi_1,\xi_2,\xi_3)\big| = \big||\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2\big| = 2|\xi_1 \cdot \xi_2| \sim |\xi_1||\xi_2|\big|\pi/2 - \angle(\xi_1,\xi_2)\big|.$$

In particular, we may assume

$$\big|h(\xi_1,\xi_2,\xi_3)\big| \lesssim |\xi_1||\xi_2|, \tag{3.9}$$

and that

$$\angle(\xi_1,\xi_2) = \frac{\pi}{2} + O\left(\frac{|h(\xi_1,\xi_2,\xi_3)|}{|\xi_1||\xi_2|}\right). \tag{3.10}$$

By dyadic decomposition of ξ_j, $\tilde{\sigma}_j$ and $h(\xi_1,\xi_2,\xi_3)$, we assume that $|\xi_j| \sim N_j$, $|\tilde{\sigma}_j| \sim L_j$ and $|h(\xi_1,\xi_2,\xi_3)| \sim H$. Where N_j, L_j and H are presumed to be dyadic, i.e. these variables range over numbers of form 2^k $(k \in \mathbb{Z})$.

It is convenient to define $N_{max} \geqslant N_{med} \geqslant N_{min}$ to be the maximum, median, and minimum of N_1, N_2, N_3. Similarly, define $L_{max} \geqslant L_{med} \geqslant L_{min}$ whenever $L_1, L_2, L_3 > 0$. Without loss of generality, we can assume

$$N_{max} \gtrsim 1, \qquad L_{min} \gtrsim 1. \tag{3.11}$$

We adopt the following summation conventions. Any summation of the form $L_{max} \sim \cdots$ is a sum over the three dyadic variables $L_1, L_2, L_3 \geqslant 1$. Therefore, denote for abbreviation, for instance,

$$\sum_{L_{max}\sim H} := \sum_{L_1,L_2,L_3 \geqslant 1: L_{max}\sim H}.$$

Similarly, any summation of form $N_{max} \sim \cdots$ sum over three dyadic variables $N_1, N_2, N_3 > 0$:

$$\sum_{N_{max}\sim N_{med}\sim N} := \sum_{N_1,N_2,N_3>0: N_{max}\sim N_{med}\sim N}.$$

By dyadic decomposition of ξ_j, $\tilde{\sigma}_j$, as well as $h(\xi_1,\xi_2,\xi_3)$, we estimate the following expression to replace (3.3)

$$\bigg\| \sum_{N_{max}\geqslant 1} \sum_{H} \sum_{L_1,L_2,L_3\geqslant 1} m\big((N_1,L_1),(N_2,L_2),(N_3,L_3)\big) X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \bigg\|_{[3,\mathbb{R}^2\times\mathbb{R}]} \lesssim 1, \tag{3.12}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3}(\xi,\tau) := \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j|\sim N_j} \chi_{|\tilde{\sigma}_j|\sim L_j}, \tag{3.13}$$

$$m\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3)\big) = m(N_1 \times L_1, N_2 \times L_2, N_3 \times L_3) \quad \text{if } |\xi_j| \sim N_j \text{ and } |\sigma_j| \sim L_j. \tag{3.14}$$

From the identities (3.1) and (3.4), $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{max} \sim N_{med}; \tag{3.15}$$

and

$$L_{max} \sim \max(H, L_{med}). \tag{3.16}$$

By the comparison principle and Schur's test [15], it suffices to prove, for $N_{max} \gtrsim 1$, that

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_1,L_2,L_3\gtrsim 1} m\big((N_1,L_1),(N_2,L_2),(N_3,L_3)\big)\|X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3}\|_{[3,\mathbb{R}^2\times\mathbb{R}]} \lesssim 1, \tag{3.17}$$

or

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{H\ll L_{max}} m\big((N_1,L_1),(N_2,L_2),(N_3,L_3)\big)\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^2\times\mathbb{R}]} \lesssim 1. \tag{3.18}$$

Therefore, we only need to estimate

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^2\times\mathbb{R}]}. \tag{3.19}$$

Then we have the following lemma about the boundedness of (3.19).

Lemma 3.1. *(See [15].) Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ obey* (3.15), (3.16).

- *For the $(+++)$ case, let the dispersion relations be given by* (3.5), *then $H \sim N_{max}^2$. It follows that*

$$(3.19) \lesssim L_{min}^{1/2} N_{max}^{-1/2} N_{min}^{1/2} \min(N_{max}N_{min}, L_{med})^{1/2}. \tag{3.20}$$

- *For the $(++-)$ case, let the dispersion relations be given by* (3.6), *then $H \lesssim N_1N_2$. It follows that:*
  - *The $((++)$ case). If $N_1 \sim N_2 \gg N_3$, then* (3.19) *vanishes unless $H \sim N_1^2$, in which case one has*

$$(3.19) \lesssim L_{min}^{1/2} N_{max}^{-1/2} N_{min}^{1/2} \min(N_{max}N_{min}, L_{med})^{1/2}. \tag{3.21}$$

  - *The $((+-)$ coherence). If we have*

$$N_1 \sim N_3 \gg N_2, \qquad H \sim L_2 \gg L_1, L_3, N_2^2, \tag{3.22}$$

  then we have

$$(3.19) \lesssim L_{min}^{1/2} N_{max}^{-1/2} N_{min}^{1/2} \min\left(H, \frac{H}{N_{min}^2} L_{med}\right)^{1/2}. \tag{3.23}$$

  Similarly with the roles of 1 *and* 2 *reversed.*
  - *In other cases, we have*

$$(3.19) \lesssim L_{min}^{1/2} N_{max}^{-1/2} N_{min}^{1/2} \min(H, L_{med})^{1/2} \min\left(1, \frac{H}{N_{min}^2}\right)^{1/2}. \tag{3.24}$$

Lemma 3.2 *(Comparison principle). (See [15].) If m and M are $[k; Z]$-multipliers and satisfy $|m(\xi)| \leqslant |M(\xi)|$ for all $\xi \in \Gamma_k(Z)$, then $\|m\|_{[k;Z]} \leqslant \|M\|_{[k;Z]}$. Also, if m is a $[k; Z]$-multiplier, and $a_1, \dots, a_k$ are functions from Z to $\mathbb{R}$, then*

$$\left\| m(\xi) \prod_{j=1}^{k} a_j(\xi_j) \right\|_{[k;Z]} \leqslant \|m\|_{[k;Z]} \prod_{j=1}^{k} \|a_j\|_{L_\infty}. \tag{3.25}$$

Lemma 3.3 *(Composition and TT^*). (See [15].) If $k_1, k_2 \geqslant 1$ and m_1, m_2 are functions on Z^{k_1} and Z^{k_2} respectively, then*

$$\begin{aligned}&\big\|m_1(\xi_1,\dots,\xi_{k_1})m_2(\xi_{k_1+1},\dots,\xi_{k_1+k_2})\big\|_{[k_1+k_2;Z]}\\&\quad\leqslant\big\|m_1(\xi_1,\dots,\xi_{k_1})\big\|_{[k_1+1;Z]}\big\|m_2(\xi_1,\dots,\xi_{k_2})\big\|_{[k_2+1;Z]}.\end{aligned}\tag{3.26}$$

As a special case, for all functions $m: Z^k\to\mathbb{R}$, we have the TT^* identity

$$\big\|m(\xi_1,\dots,\xi_k)\overline{m(-\xi_{k+1},\dots,-\xi_{2k})}\big\|_{[2k;Z]}=\big\|m(\xi_1,\dots,\xi_k)\big\|^2_{[k+1;Z]}.\tag{3.27}$$

Using these lemmas above, we will prove the main theorems in this section. We firstly give some notations about the following multilinear estimates. Define

$$\sigma_j=\tau_j-|\xi_j|^2,\qquad\bar\sigma_j=\tau_j+|\xi_j|^2,\quad j=1,2,\dots,k,\tag{3.28}$$

$$\xi_1+\xi_2+\dots+\xi_k=0,\qquad\tau_1+\tau_2+\dots+\tau_k=0.\tag{3.29}$$

Denote $\tilde\xi_j$, $\tilde\tau_j$ by variables different from $\xi_1,\xi_2,\dots,\xi_k$; $\tau_1,\tau_2,\dots,\tau_k$ respectively. Also define $\tilde\sigma_j=\tilde\tau_j-|\tilde\xi_j|^2$ or $\tilde\tau_j+|\tilde\xi_j|^2$.

$$|\sigma|_{max}=\max\big\{|\sigma_{j_1}|,\dots,|\sigma_{j_{k_1}}|;|\bar\sigma_{l_1}|,\dots,|\bar\sigma_{l_{k_2}}|;|\tilde\sigma_{n_1}|,\dots,|\tilde\sigma_{n_{k_3}}|\big\},\tag{3.30}$$

$$|\sigma|_{med}=\operatorname{med}\big\{|\sigma_{j_1}|,\dots,|\sigma_{j_{k_1}}|;|\bar\sigma_{l_1}|,\dots,|\bar\sigma_{l_{k_2}}|;|\tilde\sigma_{n_1}|,\dots,|\tilde\sigma_{n_{k_3}}|\big\},\tag{3.31}$$

$$|\xi|_{max}=\max\big\{|\xi_{j_1}|,\dots,|\xi_{j_{k_1}}|;|\tilde\xi_{l_1}|,\dots,|\tilde\xi_{l_{k_2}}|\big\},\tag{3.32}$$

$$|\xi|_{med}=\operatorname{med}\big\{|\xi_{j_1}|,\dots,|\xi_{j_{k_1}}|;|\tilde\xi_{l_1}|,\dots,|\tilde\xi_{l_{k_2}}|\big\}.\tag{3.33}$$

For convenience, by the dyadic decomposition of ξ_j, σ_j, $\bar\sigma_j$, $\tilde\xi_j$ and $\tilde\sigma_j$, we assume that $|\xi_j|\sim N_j$, $|\sigma_j|\sim L_j$, $|\bar\sigma_j|\sim L_j$; $|\tilde\xi_j|\sim\widetilde N_j$ and $|\tilde\sigma_j|\sim\widetilde L_j$. Define $N_{max}\geqslant N_{med}\geqslant N_{min}$ to be the maximum, median, and minimum of $\{N_{j_1},N_{j_2},\dots,N_{j_{k_1}};\widetilde N_{l_1},\widetilde N_{l_2},\dots,\widetilde N_{l_{k_2}}\}$.

Similarly, define $L_{max}\geqslant L_{med}\geqslant L_{min}$ to be the maximum, median, and minimum of $\{L_{j_1},L_{j_2},\dots,L_{j_{k_1}};\widetilde L_{l_1},\widetilde L_{l_2},\dots,\widetilde L_{l_{k_2}}\}$. Notice that indices above $j_1,\dots,j_{k_1}$; $l_1,\dots,l_{k_2}$ and $n_1,\dots,n_{k_3}$ are different in the following different cases.

Theorem 3.4 *(Trilinear estimates). Let $s>1/2$ and $0<\delta\ll\frac12$. Then*

$$\big\|\nabla(u_1)u_2\bar u_3\big\|_{Y_{s,-1/2+\delta}}\leqslant C\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}},\tag{3.34}$$

$$\|u_1u_2\nabla\bar u_3\|_{Y_{s,-1/2+\delta}}\leqslant C\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}}.\tag{3.35}$$

Remark. In the following proof and that of Theorem 3.5, the claims (3.17) and (3.18) can be obtained similarly. For simplicity, we sometimes prove them without distinguishing. That is, we sometimes define

$$\sum_{L_1,L_2,L_3\geqslant1}:=\sum_{L_1,L_2,L_3\geqslant1:L_{max}\sim H}\quad\text{or}\quad\sum_{L_{max}\sim L_{med}}\sum_{H\ll L_{max}}.\tag{3.36}$$

Proof of Theorem 3.4. First, we prove (3.34). (3.35) can be obtained similarly. By duality and the Plancherel identity, it suffices to show

$$\begin{aligned}&\big\|m\big((\xi_1,\tau_1),\ldots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\\&:=\left\|\frac{K(\xi_1,\xi_2,\xi_3,\xi_4)}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_3+|\xi_3|^2\rangle^{1/2}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2-\delta}}\right\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\\&\lesssim 1,\end{aligned}\tag{3.37}$$

where

$$K(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{|\xi_1|\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s},\tag{3.38}$$

$$\xi_1+\xi_2+\xi_3+\xi_4=0,\qquad \tau_1+\tau_2+\tau_3+\tau_4=0.\tag{3.39}$$

Without loss of generality, we can assume $|\xi_1|\sim|\xi|_{max}\sim|\xi|_{med}$, where $|\xi|_{max}=\max\{|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|\}$; otherwise, we can obtain the result similarly. We separately consider three cases

$$\text{(A)}\quad |\xi_1|\sim|\xi_2|,\qquad \text{(B)}\quad |\xi_1|\sim|\xi_3|,\qquad \text{(C)}\quad |\xi_1|\sim|\xi_4|.\tag{3.40}$$

First, we consider Case (A). It follows that

$$\begin{aligned}&m\big((\xi_1,\tau_1),\ldots,(\xi_4,\tau_4)\big)\\&\quad\lesssim\frac{|\xi_1|}{\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}}\frac{\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2-\delta}\langle i\bar{\sigma}_3+|\xi_3|^2\rangle^{1/2}}\\&\quad\lesssim\frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2+|\xi_2|^2\rangle^{1/4}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2-\delta}}\frac{\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_3+|\xi_3|^2\rangle^{1/2}\langle i\sigma_1+|\xi_1|^2\rangle^{1/4}}\\&\quad:=m_{a-1}\big((\xi_2,\tau_2),(\xi_4,\tau_4)\big)m_{a-2}\big((\xi_1,\tau_1),(\xi_3,\tau_3)\big).\end{aligned}\tag{3.41}$$

By Lemmas 3.2 and 3.3, it suffices to show

$$\begin{aligned}&\big\|m\big((\xi_1,\tau_1),\ldots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\\&\quad\lesssim\big\|m_{a-1}\big((\xi_2,\tau_2),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}^2\times\mathbb{R}]}\big\|m_{a-2}\big((\xi_1,\tau_1),(\xi_3,\tau_3)\big)\big\|_{[3,\mathbb{R}^2\times\mathbb{R}]}\\&\quad\lesssim 1.\end{aligned}\tag{3.42}$$

Then we will prove the two following inequalities separately

$$\big\|m_{a-1}\big((\xi_2,\tau_2),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}^2\times\mathbb{R}]}\lesssim 1,\tag{3.43}$$

$$\big\|m_{a-2}\big((\xi_1,\tau_1),(\xi_3,\tau_3)\big)\big\|_{[3,\mathbb{R}^2\times\mathbb{R}]}\lesssim 1.\tag{3.44}$$

Situation A-I. For $m_{a-1}((\xi_2,\tau_2),(\xi_4,\tau_4))$, we choose two variables $\tilde{\xi}_3$ and $\tilde{\tau}_3$ such that $\xi_2+\xi_4+\tilde{\xi}_3=0$ and $\tau_2+\tau_4+\tilde{\tau}_3=0$. Let $\tilde{\sigma}_3=\tilde{\tau}_3-|\tilde{\xi}_3|^2$ or $\tilde{\tau}_3+|\tilde{\xi}_3|^2$ in different cases. It is the $(++-)$ case. Then $|\sigma_2+\bar{\sigma}_4+\tilde{\sigma}_3|=|h(\xi_2,\xi_4,\tilde{\xi}_3)|\lesssim|\xi|^2_{max}$, where $|\xi|_{max}=\max\{|\xi_2|,|\xi_4|,|\tilde{\xi}_3|\}$.

We can separately consider the following four cases:

$$\text{Case (1): } |\xi_2| \sim |\xi_4| \sim |\tilde{\xi}_3|, \qquad \text{Case (2): } |\xi_2| \sim |\xi_4| \gg |\tilde{\xi}_3|,$$

$$\text{Case (3): } |\xi_2| \sim |\tilde{\xi}_3| \gg |\xi_4|, \qquad \text{Case (4): } |\xi_4| \sim |\tilde{\xi}_3| \gg |\xi_2|.$$

Case A-I-1. Assume that $N_2 \sim N_4 \sim \widetilde{N}_3 \sim N_{max} \sim N_{min} \sim N$. We apply (3.24) to bound the left side of (3.43) by

$$\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{1/2}N^{-1/2}N^{1/2}\min\{H,L_{med}\}^{1/2}\min\{1,H/N^2\}^{1/2}}{\langle L_2+N^2\rangle^{1/4}\langle L_4+N^2\rangle^{1/2-\delta}}$$
$$\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{\delta}\min\{H,L_{med}\}^{1/2}\min\{1,H/N^2\}^{1/2}}{\langle L_{med}+N^2\rangle^{1/4-2\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}. \tag{3.45}$$

If $L_{med} \leqslant H$, then for $s \geqslant 2\delta + 1/2 + 5\varepsilon$ with any small $\varepsilon > 0$, it follows that

$$\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{\delta}L_{med}^{1/2}\min\{1,H/N^2\}^{1/2}}{\langle L_{med}+N^2\rangle^{1/4-2\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$
$$\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{L_{min}^{\delta}L_{med}^{1/4+2\varepsilon}}{\langle N\rangle^{s}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$
$$\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{1}{\langle N\rangle^{s-2\delta-1/2-5\varepsilon}N^{\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$
$$\lesssim 1. \tag{3.46}$$

If $L_{med} \geqslant H$, then for $s \geqslant 2\delta + 1/2 + 5\varepsilon$, it follows that

$$\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{\delta}H^{1/2}\min\{1,H/N^2\}^{1/2}}{\langle L_{med}+N^2\rangle^{1/4-2\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$
$$\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{L_{min}^{\delta}H^{1/4+2\varepsilon}}{\langle N\rangle^{s}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$
$$\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{1}{\langle N\rangle^{s-2\delta-1/2-5\varepsilon}N^{\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$
$$\lesssim 1. \tag{3.47}$$

Case A-I-2. Assume $N \sim N_{max} \sim N_2 \sim N_4 \gg \widetilde{N}_3 \sim N_{min}$.

Subcase A-I-2-1. If $H \sim \widetilde{L}_3 \gg L_2, L_4, N_{min}^2$, then for $s \geqslant 2\delta + 1/2 + 5\varepsilon$, we use (3.23) to bound the left side of (3.43) by

$$\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{1/2}\min\{H,\frac{H}{N_{min}^2}L_{med}\}^{1/2}}{\langle L_2+N^2\rangle^{1/4}\langle L_4+N^2\rangle^{1/2-\delta}}$$
$$\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{\delta}\min\{H,\frac{H}{N_{min}^2}L_{med}\}^{1/2}}{\langle L_{med}+N^2\rangle^{1/4-2\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}$$

$$
\begin{aligned}
&\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{\delta}H^{1/2}}{\langle L_{med}+N^2\rangle^{1/4-2\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}\\
&\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{L_{min}^{\delta}H^{1/2}}{\langle N\rangle^{s+1/2-4\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}\\
&\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_2,L_4,\widetilde{L}_3\gtrsim 1}\frac{L_{min}^{\delta}H^{1/2}}{\langle N\rangle^{s-1/2-5\varepsilon-2\delta}N^{2\delta+1}N^{\varepsilon}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}\\
&\lesssim 1.
\end{aligned}
\tag{3.48}
$$

Subcase A-I-2-2. For other cases, we can obtain the result similarly with Case A-I-1.

Case A-I-3. Assume that $N\sim N_{max}\sim N_2\sim \widetilde{N}_3\gg N_4\sim N_{min}$. Let $\tilde{\sigma}_3=\tilde{\tau}_3-|\tilde{\xi}_3|^2$. Then it holds that $|h(\xi_2,\xi_4,\tilde{\xi}_3)|\sim|\xi|^2_{max}$.

If $L_{max}\sim L_{med}\gg H\sim N^2$, then for $s\geqslant 2\delta+1/2+5\varepsilon$, we use (3.21) to bound the left side of (3.43) by

$$
\begin{aligned}
&\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_{max}\sim L_{med}\gg H}\frac{\langle N\rangle^{-s}L_{min}^{1/2}N^{-1/2}N_{min}^{1/2}\min\{NN_{min},L_{med}\}^{1/2}}{\langle L_2+N^2\rangle^{1/4}\langle L_4+N_{min}^2\rangle^{1/2-\delta}}\\
&\quad\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_{max}\sim L_{med}\gg H}\frac{\langle N\rangle^{-s}N_{min}}{\langle L_{med}+N^2\rangle^{1/4-2\varepsilon-\delta}L_{min}^{\varepsilon}L_{med}^{\varepsilon}}\\
&\quad\lesssim 1.
\end{aligned}
\tag{3.49}
$$

If $L_{max}\sim H\sim N^2$, we can obtain the result similarly as above.

Case A-I-4. Assume that $N\sim N_{max}\sim N_4\sim \widetilde{N}_3\gg N_2\sim N_{min}$. Let $\tilde{\sigma}_3=\tilde{\tau}_3+|\tilde{\xi}_3|^2$. Then one knows that $|h(\xi_2,\xi_4,\tilde{\xi}_3)|\sim|\xi|^2_{max}$. We can obtain the result similarly with Case A-I-3.

Situation A-II. For $m_{a-2}((\xi_1,\tau_1),(\xi_3,\tau_3))$, we choose two variables $\tilde{\xi}_2$ and $\tilde{\tau}_2$ such that $\xi_1+\xi_3+\tilde{\xi}_2=0$ and $\tau_1+\tau_3+\tilde{\tau}_2=0$. Let $\tilde{\sigma}_2=\tilde{\tau}_2-|\tilde{\xi}_2|^2$ or $\tilde{\tau}_2+|\tilde{\xi}_2|^2$ in different cases. It is the $(++-)$ case. Then $|\sigma_1+\bar{\sigma}_3+\tilde{\sigma}_2|=|h(\xi_1,\xi_3,\tilde{\xi}_2)|\lesssim|\xi|^2_{max}$, where $|\xi|_{max}=\max\{|\xi_1|,|\xi_3|,|\tilde{\xi}_2|\}$.

We can separately consider four cases:

$$
\begin{aligned}
&\text{Case (1): } |\xi_1|\sim|\xi_3|\sim|\tilde{\xi}_2|, &&\text{Case (2): } |\xi_1|\sim|\xi_3|\gg|\tilde{\xi}_2|,\\
&\text{Case (3): } |\xi_3|\sim|\tilde{\xi}_2|\gg|\xi_1|, &&\text{Case (4): } |\xi_1|\sim|\tilde{\xi}_2|\gg|\xi_3|.
\end{aligned}
$$

Case A-II-1. If $N_1\sim N_3\sim \widetilde{N}_2\sim N_{max}\sim N_{min}\sim N$, similarly with Case A-I-1, we prove that (3.44) holds for $s\geqslant 1/2+5\varepsilon$.

Case A-II-2. If $N\sim N_{max}\sim N_1\sim N_3\gg \widetilde{N}_2\sim N_{min}$, similarly with Case A-I-2, we prove that (3.44) holds for $s\geqslant 1/2+5\varepsilon$.

Case A-II-3. Assume that $N\sim N_{max}\sim N_3\sim \widetilde{N}_2\gg N_1\sim N_{min}$. Let $\tilde{\sigma}_2=\tilde{\tau}_2+|\tilde{\xi}_2|^2$, it holds that $|h(\xi_1,\xi_3,\tilde{\xi}_2)|\sim|\xi|^2_{max}$.

If $L_{max}\sim H\sim N^2$, for $s\geqslant 1/2+5\varepsilon$, we use (3.21) to bound the left side of (3.44) by

$$
\begin{aligned}
&\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_1,L_3,\widetilde{L}_2\gtrsim 1}\frac{\langle N\rangle^{-s}L_{min}^{1/2}N^{-1/2}N_{min}^{1/2}\min\{NN_{min},L_{med}\}^{1/2}}{\langle L_1+N_{min}^2\rangle^{1/4}\langle L_3+N^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{max}\sim N_{med}\sim N}\sum_{L_1,L_3,\widetilde{L}_2\gtrsim 1}\frac{L_{min}^{1/4}N_{min}}{\langle N\rangle^{s}\langle L_3+N^2\rangle^{1/2-2\varepsilon}L_{med}^{\varepsilon}L_{min}^{\varepsilon}}
\end{aligned}
$$

$$\lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_1,L_3,\widetilde{L}_2\gtrsim 1} \frac{1}{N^{\varepsilon} L_{med}^{\varepsilon} L_{min}^{\varepsilon}}$$
$$\lesssim 1. \tag{3.50}$$

If $L_{max} \sim L_{med} \gg H$, then similarly as above, we can obtain the result for $s \geqslant 1/2 + 5\varepsilon$.

Case A-II-4. Assume that $N \sim N_{max} \sim N_1 \sim \widetilde{N}_2 \gg N_3 \sim N_{min}$. Let $\tilde{\sigma}_2 = \tilde{\tau}_2 - |\tilde{\xi}_2|^2$, it holds that $|h(\xi_1, \xi_3, \tilde{\xi}_2)| \sim |\xi|^2_{max}$. Then for $s \geqslant 1/2 + 5\varepsilon$, we can obtain the result similarly as above.

Next, we consider Case (B). It follows that

$$\begin{aligned} m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big) &\lesssim \frac{|\xi_1|}{\langle i\bar{\sigma}_3 + |\xi_3|^2\rangle^{1/2}\langle i\sigma_1 + |\xi_1|^2\rangle^{1/2}} \frac{\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_4 + |\xi_4|^2\rangle^{1/2-\delta}\langle i\sigma_2 + |\xi_2|^2\rangle^{1/2}} \\ &\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2 + |\xi_2|^2\rangle^{1/2}\langle i\sigma_1 + |\xi_1|^2\rangle^{1/4}} \frac{\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_3 + |\xi_3|^2\rangle^{1/4}\langle i\bar{\sigma}_4 + |\xi_4|^2\rangle^{1/2-\delta}} \\ &:= m_{b-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) m_{b-2}\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big). \end{aligned} \tag{3.51}$$

Situation B-I. For $m_{b-1}((\xi_1,\tau_1),(\xi_2,\tau_2))$, we choose two variables $\tilde{\xi}_0$ and $\tilde{\tau}_0$ such that $\xi_1+\xi_2+\tilde{\xi}_0 = 0$ and $\tau_1+\tau_2+\tilde{\tau}_0 = 0$. Let $\tilde{\sigma}_0 = \tilde{\tau}_3 - |\tilde{\xi}_0|^2$, it is the $(+++)$ case. Then $|\sigma_1+\sigma_2+\tilde{\sigma}_0| = |h(\xi_1,\xi_2,\tilde{\xi}_0)| \sim |\xi|^2_{max}$, where $|\xi|_{max} = \max\{|\xi_1|, |\xi_2|, |\tilde{\xi}_0|\}$. Similarly with Case A-I-3, we can obtain the result for $s \geqslant 1/2 + 5\varepsilon$.

Situation B-II. For $m_{b-2}((\xi_3,\tau_3),(\xi_4,\tau_4))$, we take two variables $\tilde{\xi}_5$ and $\tilde{\tau}_5$ such that $\xi_3+\xi_4+\tilde{\xi}_5 = 0$ and $\tau_3+\tau_4+\tilde{\tau}_5 = 0$. Let $\tilde{\sigma}_5 = \tilde{\tau}_5 + |\tilde{\xi}_5|^2$, it is the $(+++)$ case. Then $|\sigma_3+\sigma_4+\tilde{\sigma}_5| = |h(\xi_3,\xi_4,\tilde{\xi}_5)| \sim |\xi|^2_{max}$, where $|\xi|_{max} = \max\{|\xi_3|, |\xi_4|, |\tilde{\xi}_5|\}$. Similarly with Case A-I-3, we can obtain the result for $s \geqslant 2\delta + 1/2 + 5\varepsilon$.

Finally, we consider Case (C). It follows that

$$\begin{aligned} m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big) &\lesssim \frac{|\xi_1|}{\langle i\bar{\sigma}_4 + |\xi_4|^2\rangle^{1/2-\delta}\langle i\sigma_1 + |\xi_1|^2\rangle^{1/2}} \frac{\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_3 + |\xi_3|^2\rangle^{1/2}\langle i\sigma_2 + |\xi_2|^2\rangle^{1/2}} \\ &\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2 + |\xi_2|^2\rangle^{1/2}\langle i\sigma_1 + |\xi_1|^2\rangle^{1/4}} \frac{\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_3 + |\xi_3|^2\rangle^{1/3}\langle i\bar{\sigma}_4 + |\xi_4|^2\rangle^{1/4-\delta}} \\ &:= m_{b-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) m_{b-2}\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big). \end{aligned} \tag{3.52}$$

Similarly with Case (B), we can obtain the results for $s \geqslant 2\delta + 1/2 + 5\varepsilon$. This completes the proof of Theorem 3.4. □

Theorem 3.5 *(Multilinear estimate). Let $s > 1/2$ and $0 < \delta \ll \frac{1}{2}$. Then*

$$\|u_1u_2u_3\bar{u}_4\bar{u}_5\|_{Y_{s,-1/2+\delta}} \leqslant C\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}}\|u_4\|_{Y_{s,1/2}}\|u_5\|_{Y_{s,1/2}}. \tag{3.53}$$

Proof. Similarly with the proof of Theorem 3.4, by duality and the Plancherel identity, it suffices to prove

$$\begin{aligned} &\big\|m\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}^2\times\mathbb{R}]} \\ &\quad= \left\| \frac{\langle i\bar{\sigma}_4 + |\xi_4|^2\rangle^{-1/2}\langle i\bar{\sigma}_5 + |\xi_5|^2\rangle^{-1/2}\langle i\bar{\sigma}_6 + |\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_1 + |\xi_1|^2\rangle^{1/2}\langle i\sigma_2 + |\xi_2|^2\rangle^{1/2}\langle i\sigma_3 + |\xi_3|^2\rangle^{1/2}} K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) \right\|_{[6,\mathbb{R}^2\times\mathbb{R}]} \\ &\quad\lesssim 1, \end{aligned} \tag{3.54}$$

where

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\frac{\langle\xi_6\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s}, \tag{3.55}$$

$$\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0, \qquad \tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0. \tag{3.56}$$

By symmetry, we separately consider two cases

$$\text{(D)}\quad |\xi_6|\lesssim|\xi_1|=\max\{|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|,|\xi_5|\},$$
$$\text{(E)}\quad |\xi_6|\lesssim|\xi_4|=\max\{|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|,|\xi_5|\}.$$

In fact, the proofs of the cases $|\xi_6|\lesssim|\xi_2|$ and $|\xi_6|\lesssim|\xi_3|$ are similar with that of Case (D). The proof of the case $|\xi_6|\lesssim|\xi_5|$ is similar with that of Case (E).

First, we consider Case (D). It follows that

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\leqslant\frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s}, \tag{3.57}$$

and

$$\begin{aligned}
m\big((\xi_1,\tau_1),\ldots,(\xi_6,\tau_6)\big)&\leqslant\frac{\langle i\bar\sigma_4+|\xi_4|^2\rangle^{-1/2}\langle i\bar\sigma_5+|\xi_5|^2\rangle^{-1/2}\langle i\bar\sigma_6+|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\sigma_3+|\xi_3|^2\rangle^{1/2}}\frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s}\\
&\leqslant\frac{\langle i\bar\sigma_4+|\xi_4|^2\rangle^{-1/2}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}}\frac{\langle i\bar\sigma_5+|\xi_5|^2\rangle^{-1/2}\langle i\bar\sigma_6+|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_3+|\xi_3|^2\rangle^{1/2}\langle\xi_3\rangle^s\langle\xi_5\rangle^s}\\
&:=m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)m_{d-2}\big((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\big).
\end{aligned} \tag{3.58}$$

By Lemmas 3.2 and 3.3, it suffices to prove

$$\begin{aligned}
&\big\|m\big((\xi_1,\tau_1),\ldots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}^2\times\mathbb{R}]}\\
&\quad\lesssim\big\|m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\big\|m_{d-2}\big((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\big)\big\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\\
&\quad\lesssim 1.
\end{aligned} \tag{3.59}$$

Situation D-I. We first prove

$$\big\|m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\lesssim 1. \tag{3.60}$$

We choose two variables $\tilde\xi_3$ and $\tilde\tau_3$ such that $\xi_1+\xi_2+\tilde\xi_3+\xi_4=0$ and $\tau_1+\tau_2+\tilde\tau_3+\tau_4=0$. Let $\tilde\sigma_3=\tilde\tau_3+|\tilde\xi_3|^2$, it follows that $|\sigma_1+\sigma_2+\tilde\sigma_3+\sigma_4|=|h(\xi_1,\xi_2,\tilde\xi_3,\xi_4)|\leqslant|\xi|^2_{max}$, where $|\xi|_{max}=\max\{|\xi_1|,|\xi_2|,|\tilde\xi_3|,|\xi_4|\}$. Moreover, we have

$$|\sigma|_{max}\sim|\sigma|_{med}\gg\big|h(\xi_1,\xi_2,\tilde\xi_3,\xi_4)\big|, \tag{3.61}$$

or

$$|\sigma|_{max}\sim\big|h(\xi_1,\xi_2,\tilde\xi_3,\xi_4)\big|, \tag{3.62}$$

where $|\sigma|_{max}=\max\{|\sigma_1|,|\sigma_2|,|\tilde\sigma_3|,|\sigma_4|\}$. Without loss of generality, we can assume

$$|\sigma|_{max}\sim|\sigma|_{med}\gtrsim|\xi|^2_{max}\sim|\xi|^2_{med}, \tag{3.63}$$

or

$$|\sigma|_{max} \lesssim |\xi|_{max}^2 \sim |\xi|_{med}^2. \tag{3.64}$$

First, we consider the case: $|\sigma|_{max} \sim |\sigma|_{med} \gtrsim |\xi|_{max}^2$.

Case D-I-1. If $|\sigma_1| = |\sigma|_{max}$ or $|\sigma|_{med}$, then it follows that

$$\begin{aligned}
m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big) &\lesssim \frac{\langle i\tilde{\sigma}_3 + |\tilde{\xi}_3|^2\rangle^{1/4}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&\lesssim \frac{\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/4}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/4}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}} \cdot \frac{\langle\xi_4\rangle^{-s}}{\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&:= m_{d-11}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) m_{d-12}\big((\tilde{\xi}_3,\tilde{\tau}_3),(\xi_4,\tau_4)\big).
\end{aligned} \tag{3.65}$$

Then we can obtain (3.60) similarly with Case (B) in the proof of Theorem 3.4 for $s \geqslant 1/2 + 5\varepsilon$.

Case D-I-2. If $|\sigma_2| = |\sigma|_{max}$ or $|\sigma|_{med}$, then it follows that

$$\begin{aligned}
m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big) &\lesssim \frac{\langle i\tilde{\sigma}_3 + |\tilde{\xi}_3|^2\rangle^{1/4}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&\lesssim \frac{\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/4}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/4}} \cdot \frac{\langle\xi_4\rangle^{-s}}{\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&:= m_{d-11}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) \cdot m_{d-12}\big((\tilde{\xi}_3,\tilde{\tau}_3),(\xi_4,\tau_4)\big).
\end{aligned} \tag{3.66}$$

Similarly with the above, we can obtain the result for $s \geqslant 1/2 + 5\varepsilon$.

Case D-I-3. If $|\sigma_4| = |\sigma|_{max}$ or $|\sigma|_{med}$, then

$$\begin{aligned}
m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big) &\lesssim \frac{\langle i\tilde{\sigma}_3 + |\tilde{\xi}_3|^2\rangle^{1/4}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&\lesssim \frac{\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/4}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \\
&\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+|\tilde{\xi}_3|^2\rangle^{1/4}} \cdot \frac{\langle\xi_4\rangle^{-s}}{\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{1/4}\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}} \\
&:= m_{d-11}\big((\xi_2,\tau_2),(\tilde{\xi}_3,\tilde{\tau}_3)\big) \cdot m_{d-12}\big((\xi_1,\tau_1),(\xi_4,\tau_4)\big).
\end{aligned} \tag{3.67}$$

Similarly with Case (A) in the proof of Theorem 3.4, we can obtain the result for $s \geqslant 1/2 + 5\varepsilon$.

Next, we consider the case: $|\sigma|_{max} \lesssim |\xi|_{max}^2 \sim |\xi|_{med}^2$. In fact, we can obtain (3.60) for $s \geqslant 1/2 + 5\varepsilon$ to consider the following cases similarly as above

$$|\xi_1| \sim |\xi|_{max} \sim |\xi|_{med} \quad \text{corresponding to Case D-I-1},$$

$$|\xi_2| \sim |\xi|_{max} \sim |\xi|_{med} \quad \text{corresponding to Case D-I-2},$$

$$|\xi_4| \sim |\xi|_{max} \sim |\xi|_{med} \quad \text{corresponding to Case D-I-3}.$$

Situation D-II. In this situation, we will prove

$$\big\|m_{d-2}\big((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\big)\big\|_{[4,\mathbb{R}^2\times\mathbb{R}]}\lesssim 1. \tag{3.68}$$

We choose two variables $\tilde{\xi}_4$ and $\tilde{\tau}_4$ such that $\xi_3+\tilde{\xi}_4+\xi_5+\xi_6=0$ and $\tau_3+\tilde{\tau}_4+\tau_5+\tau_6=0$. Let $\tilde{\sigma}_4=\tilde{\tau}_4-|\tilde{\xi}_4|^2$. Similarly with Situation D-I, we can obtain (3.68) for $s>2\delta+1/2+5\varepsilon$.

Gathering (3.60) and (3.68), we have (3.59).

Next, we consider Case (E). It follows that

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\leqslant\frac{1}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_5\rangle^s}, \tag{3.69}$$

and

$$\begin{aligned} m_e\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big) &\leqslant \frac{\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{-1/2}\langle i\bar{\sigma}_5+|\xi_5|^2\rangle^{-1/2}\langle i\bar{\sigma}_6+|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}\langle i\sigma_3+|\xi_3|^2\rangle^{1/2}}\frac{1}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_5\rangle^s} \\ &\leqslant \frac{\langle i\bar{\sigma}_4+|\xi_4|^2\rangle^{-1/2}\langle\xi_2\rangle^{-s}\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+|\xi_2|^2\rangle^{1/2}}\cdot\frac{\langle i\bar{\sigma}_5+|\xi_5|^2\rangle^{-1/2}\langle i\bar{\sigma}_6+|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}\langle\xi_3\rangle^s\langle\xi_5\rangle^s} \\ &:= m_{e-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)m_{e-2}\big((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\big). \end{aligned} \tag{3.70}$$

In fact, by symmetry about σ_j and $\bar{\sigma}_j$, similarly with Case (D), we can obtain

$$\big\|m_e\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}^2\times\mathbb{R}]}\lesssim 1. \tag{3.71}$$

Gathering (3.59) and (3.71), we have (3.54). This completes the proof of Theorem 3.5. □

Corollary 3.6. *Let $0<\delta\ll\frac{1}{2}$. Then there exist $\mu, C_\delta>0$ such that for $u_1,u_2,u_3\in Y_{s,1/2}$, $\bar{u}_3,\bar{u}_4,\bar{u}_5\in\overline{Y}_{s,1/2}$ with compact support in $[-T,T]$,*

$$\big\|\nabla(u_1)u_2\bar{u}_3\big\|_{Y_{s,-1/2+\delta}}\leqslant C_\delta T^\mu\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}}, \tag{3.72}$$

$$\|u_1u_2\nabla\bar{u}_3\|_{Y_{s,-1/2+\delta}}\leqslant C_\delta T^\mu\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}}, \tag{3.73}$$

$$\|u_1u_2u_3\bar{u}_4\bar{u}_5\|_{Y_{s,-1/2+\delta}}\leqslant C_\delta T^\mu\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}}\|u_4\|_{Y_{s,1/2}}\|u_5\|_{Y_{s,1/2}}. \tag{3.74}$$

Proof. In fact, from Theorem 3.5, we can complete the proof of Corollary 3.6 by the inequality for $f(t)$ with compact support in $[-T,T]$

$$\left\|\mathcal{F}^{-1}\frac{\hat{f}(\tau,\xi)}{\langle\tau-|\xi|^2\rangle^\delta}\right\|_{L^2}\leqslant C_\delta T^\mu\|f\|_{L^2},\quad\text{for any }\delta>0.\qquad\square \tag{3.75}$$

4. Some a priori estimates and global well-posedness

In this section, we first give some a priori estimates for Eqs. (1.1). Then we can obtain that the local solution obtained in Section 3 can be extended to the global one by using Lemma 2.3 and the a priori estimates. Therefore, Theorem 1.2 can be obtained.

Lemma 4.1. *(See [14].) Assume that* $1-\frac{(|\vec{\gamma}|+|\vec{\lambda}|)^2}{4\alpha\alpha_1}>0$, *$u(t)$ is a smooth solution of the Cauchy problem* (1.1) *and* (1.2)*. Then there exists $\varepsilon>0$ such that $(|\vec{\gamma}|+|\vec{\lambda}|)^2<\alpha\alpha_1(2-\varepsilon)^2$, it follows that*

$$\frac{1}{2}\left\|u(t)\right\|_{L^2}^2+\frac{\alpha\varepsilon}{2}\int_0^t\left\|\nabla u(t')\right\|_{L^2}^2dt'+\frac{\alpha_1\varepsilon}{2}\int_0^t\left\|u(t')\right\|_{L^6}^6dt'\leqslant\frac{1}{2}\|u_0\|_{L^2}^2. \tag{4.1}$$

Moreover, if $\vec{\gamma}$ and $\vec{\lambda}$ are real vectors, then

$$\frac{1}{2}\left\|u(t)\right\|_{L^2}^2+\alpha\int_0^t\left\|\nabla u(t')\right\|_{L^2}^2dt'+\alpha_1\int_0^t\left\|u(t')\right\|_{L^6}^6dt'=\frac{1}{2}\|u_0\|_{L^2}^2. \tag{4.2}$$

Lemma 4.2. *(See [14].) Assume* $1-\frac{(|\vec{\gamma}|+|\vec{\lambda}|)^2}{4\alpha\alpha_1}>0$ *and $\beta\beta_1>0$. If $u(t)$ is a smooth solution of the Cauchy problem* (1.1) *and* (1.2)*, then for some $\eta>0$, $c>0$, $c_1>0$, it holds that*

$$\begin{aligned}&\frac{1}{2}\left\|\nabla u(t)\right\|_{L^2}^2+\frac{\eta}{6}\left\|u(t)\right\|_{L^6}^6+c\int_0^t\left\|\Delta u(t')\right\|_{L^2}^2dt'+c_1\eta\int_0^t\left\|u(t')\right\|_{L^6}^6dt'\\&\quad\leqslant\frac{1}{2}\|\nabla u_0\|_{L^2}^2+\frac{\eta}{6}\|u_0\|_{L^6}^6.\end{aligned} \tag{4.3}$$

Lemma 4.3. *(See [14].) Assume* $A:=1-\frac{(|\vec{\gamma}|+|\vec{\lambda}|)^2}{4\alpha\alpha_1}>0$ *and $\beta\beta_1\leqslant 0$,*

$$\min\left\{\frac{|\beta|}{\alpha A},\frac{|\beta_1|}{\alpha_1 A}\right\}<\frac{\sqrt{5}}{2}, \tag{4.4}$$

or

$$|\beta\beta_1|-\left(\sqrt{\alpha\alpha_1}-\frac{(|\vec{\gamma}|+|\vec{\lambda}|)}{2}\right)^2<\frac{\sqrt{5}}{2}\big(|\beta|\alpha_1+|\beta_1|\alpha\big)\left(1-\frac{(|\vec{\gamma}|+|\vec{\lambda}|)}{2\sqrt{\alpha\alpha_1}}\right). \tag{4.5}$$

Then (4.3) *holds also for some $\eta>0$, $c>0$, $c_1>0$.*

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