NORTH-HOLLAND

Structured Dispersion and Validity in Linear Inference

D. R. Jensen
Department of Statistics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

Submitted by Ingram Olkin

ABSTRACT

Structured matrices $\boldsymbol{\Sigma}(\boldsymbol{\gamma}) = [\mathbf{I}_n + \mathbf{e}\boldsymbol{\gamma}' + \boldsymbol{\gamma}\mathbf{e}' - \bar{\gamma}\mathbf{e}\mathbf{e}']$ arise in nonstandard linear models, where $\mathbf{e}' = [1, \dots, 1]$, $\boldsymbol{\gamma}' = [\gamma_1, \dots, \gamma_n]$, and $\bar{\gamma} = (\gamma_1 + \cdots + \gamma_n)/n$. Their properties are studied, including expressions for eigenvalues, conditions for positive definiteness, and conditioning of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ as $\boldsymbol{\gamma}$ varies. It is shown that if $\boldsymbol{\gamma}$ majorizes $\boldsymbol{\gamma}_0$, then the condition numbers are ordered as $c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma})) \geqslant c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_0))$ for every condition number $\{c_\phi(\cdot);\ \phi \in \Phi\}$ generated by the unitarily invariant matrix norms. Applications are noted in linear inference and in outlier detection. © *Elsevier Science Inc., 1996*

1. INTRODUCTION

Let $\mathbf{Y} \in \mathbb{R}^n$ be Gaussian with mean $\boldsymbol{\mu}$ and dispersion matrix $\boldsymbol{\Sigma}$, i.e., $\mathscr{L}(\mathbf{Y}) = N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\mathscr{L}(\mathbf{Y})$ as the law of distribution of $\mathbf{Y}$. The following issues arise regarding validity of the normal-theory analysis of variance in linear statistical inference. With $\mathbf{B}_n = \mathbf{I}_n - n^{-1}\mathbf{e}\mathbf{e}'$ in standard notation, the typical analysis of variance partitions $\mathbf{Y}'\mathbf{B}_n\mathbf{Y}$ into orthogonal components under conditions assuring the chi-squared (χ^2) character of $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y})$. Standard model assumptions, not necessarily realized in practice, are that $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_n$, and it remains to consider the validity of the analysis under dependence. That $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_n$ is sufficient but not necessary for validity is seen on noting that $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y})$ is a scaled χ^2 variate, even under equicorrelation with $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\rho) = \sigma^2[(1-\rho)\mathbf{I}_n + \rho\mathbf{e}\mathbf{e}']$, where ρ is restricted to $-(n-1)^{-1} < \rho < 1$.

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Here we consider the collection $\Xi(n) = \{\boldsymbol{\Sigma}(\boldsymbol{\gamma});\ \boldsymbol{\gamma} \in \Gamma\}$ of all $n \times n$ structured matrices of the type $\boldsymbol{\Sigma}(\boldsymbol{\gamma}) = \mathbf{I}_n + \mathbf{e}\boldsymbol{\gamma}' + \boldsymbol{\gamma}\mathbf{e}' - \bar{\gamma}\mathbf{e}\mathbf{e}'$ such that $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ is positive definite, where $\mathbf{e}' = [1, \dots, 1]$, $\boldsymbol{\gamma}' = [\gamma_1, \dots, \gamma_n]$, and $\bar{\gamma} = (\gamma_1 + \cdots + \gamma_n)/n$. This class is seen to exhaust the Gaussian dispersion models preserving validity in the analysis of variance, in the sense that $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y})$ has its requisite form.

Structured dispersion matrices have been considered elsewhere in the statistical literature. Baldessari (1966) studied a class of matrices of the form $\boldsymbol{\Sigma}(\lambda, \boldsymbol{\alpha}) = \lambda\mathbf{I}_n + \mathbf{e}\boldsymbol{\alpha}' + \boldsymbol{\alpha}\mathbf{e}'$ arising in certain analysis-of-variance problems. Huynh and Feldt (1970) and Rouanet and Lépine (1970) characterized the class of all within-subject $k \times k$ dispersion matrices preserving validity of the usual F-tests regarding k repeated measurements on each of n experimental subjects. These matrices have the structure $\boldsymbol{\Sigma}(\boldsymbol{\lambda}) = [\lambda_i + \lambda_j + \delta_{ij}\lambda]$ with δ_{ij} as Kronecker's symbol. It is clear that the structures $\boldsymbol{\Sigma}(\lambda, \boldsymbol{\alpha})$, $\boldsymbol{\Sigma}(\boldsymbol{\lambda})$, and $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ are all equivalent, and therefore that $\Xi(n)$ comprises the class of all such matrices in S_n^+. More recently, Srivastava (1980), Young et al. (1989), and Baksalary and Puntanen (1990) have shown that a normal-theory test for a single shifted outlier, due to Grubbs (1950), remains exact in level and power under dispersion matrices belonging to $\Xi(n)$. Matrices of the type $\mathbf{D} + \mathbf{e}\boldsymbol{\gamma}' + \boldsymbol{\gamma}\mathbf{e}'$ arise in the study of Euclidean distance matrices (Gower, 1982) and their applications in linear inference (Farebrother, 1985). An outline of the paper follows.

Preliminary developments occupy Section 2. Section 3 characterizes $\Xi(n)$ in terms of the χ^2 character of $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y})$ through properties of generalized matrix inverses. Section 4 undertakes the spectral analysis of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$, including expressions for its eigenvalues and conditions for positive definiteness. The conditioning of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ as $\boldsymbol{\gamma}$ varies is studied in Section 5 for condition numbers $\{c_\phi(\cdot);\ \phi \in \Phi\}$ generated by the unitarily invariant matrix norms. Section 6 gives some applications and concluding remarks.

2. PRELIMINARIES

To fix notation, $\mathbb{R}^n$, $\mathbb{R}_+^n$, S_n, and S_n^+ designate Euclidean n-space, its positive orthant, the $n \times n$ real symmetric matrices, and their positive definite varieties. The set $C(n) = \{\mathbf{x} \in \mathbb{R}^n : x_1 \geqslant \cdots \geqslant x_n \geqslant 0\}$ is a standard simplex in $\mathbb{R}_+^n$, with $\mathbf{x}' = [x_1, \dots, x_n]$ as the transpose of $\mathbf{x} \in \mathbb{R}^n$. Special arrays include the $n \times n$ identity $\mathbf{I}_n$, the unit vector $\mathbf{e}' = [1, \dots, 1]$ in $\mathbb{R}^n$, and the diagonal matrix $\mathrm{Diag}(a_1, \dots, a_n)$. Special distributions on $\mathbb{R}_+^1$ include the chi-squared distribution $\chi^2(\nu, \lambda)$ having ν degrees of freedom and noncentrality λ, and $O(n)$ designates the group of real orthogonal $n \times n$ matrices.

Let $\mathbf{a} = [a_1, \ldots, a_n]'$ and $\mathbf{b} = [b_1, \ldots, b_n]'$ be two vectors in $\mathbb{R}^n$ such that $\{\alpha_1 \geqslant \cdots \geqslant a_n\}$ and $\{b_1 \geqslant \cdots \geqslant b_n\}$. Then $\mathbf{b}$ is said to be *majorized* by $\mathbf{a}$ if (1) $a_1 + \cdots + a_r \geqslant b_1 + \cdots + b_r$ for $r = 1, \ldots, n-1$, and (2) $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. The ordering is designated by $\mathbf{a} \succcurlyeq \mathbf{b}$. A standard reference is Marshall and Olkin (1979).

A function $\phi(\cdot)$ on $\mathbb{R}^n$ is called a *symmetric gauge* if it satisfies the following conditions:

C_1. $\phi(u_1, \ldots, u_n) = \phi(\varepsilon_1 u_{i1}, \ldots, \varepsilon_n u_{i_n})$ for each $\{\varepsilon_i = \pm 1;\ 1 \leqslant i \leqslant n\}$ and any permutation $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$;

C_2. $\phi(\mathbf{u}) > 0$ when $\mathbf{u} \neq \mathbf{0}$;

C_3. $\phi(c\mathbf{u}) = |c|\phi(\mathbf{u})$ for complex c; and

C_4. $\phi(\mathbf{u} + \mathbf{v}) \leqslant \phi(\mathbf{u}) + \phi(\mathbf{v})$.

Let Φ be the class of all symmetric gauges on $\mathbb{R}^n$. A function $\psi(\cdot)$ on S_n^+ is said to be *unitarily invariant* if $\psi(\mathbf{A}) = \psi(\mathbf{PAP}')$ for each $\mathbf{A} \in S_n^+$ and $\mathbf{P} \in O(n)$, so that $\psi(\mathbf{A})$ depends on $\mathbf{A}$ only through its ordered eigenvalues $\{\alpha_1 \geqslant \cdots \geqslant \alpha_n > 0\}$. If $\psi(\cdot)$ is also a norm on $(S_n^+, \|\cdot\|)$, then the class of all unitarily invariant matrix norms is generated as $\{\|\cdot\|_\phi;\ \phi \in \Phi\}$ with $\|\mathbf{A}\|_\phi = \phi(\alpha_1, \ldots, \alpha_n)$ as shown in von Neumann (1937).

3. CHARACTERIZING $\Xi(n)$

We next characterize those Gaussian models for which $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y}) = \chi^2(n-1, \lambda)$. Throughout we consider only nonsingular distributions on $\mathbb{R}^n$ having dispersion matrices of full rank n. A principal result is the following based on properties of generalized matrix inverses.

THEOREM 1. *Suppose that* $\mathscr{L}(\mathbf{Y}) = N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Then a necessary and sufficient condition for* $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y}) = \chi^2(n-1, \lambda)$ *is that* $\boldsymbol{\Sigma} \in \Xi(n)$, *in which case* $\lambda = \boldsymbol{\mu}'\boldsymbol{\Sigma}(\boldsymbol{\gamma})\boldsymbol{\mu}$ *for some* $\boldsymbol{\gamma} \in \Gamma$.

Proof. First invoke the necessity and sufficiency of the condition $\mathbf{B}_n\boldsymbol{\Sigma}\mathbf{B}_n = \mathbf{B}_n$ in order that $\mathscr{L}(\mathbf{Y}'\mathbf{B}_n\mathbf{Y}) = \chi^2(n-1, \lambda)$; see Johnson and Kotz (1970), for example. Next generate all full-rank g-inverses of $\mathbf{B}_n$, starting with $\mathbf{I}_n$, using results from Rao and Mitra (1971), to infer that $\boldsymbol{\Sigma} \in \Xi(n)$. Further details are given in Jensen (1989). ∎

In summary, Theorem 1 characterizes $\{N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}); \boldsymbol{\Sigma} \in \Xi(n)\}$ as the class of all Gaussian models for which the analysis of variance effects a valid decomposition of $\mathbf{Y}'\mathbf{B}_n\mathbf{Y}$ as a $\chi^2(n-1, \lambda)$ random variate. Further properties of these models are developed next.

4. SPECTRAL PROPERTIES OF $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$

We seek expressions for the eigenvalues of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ and conditions for its positive definiteness, all in terms of $\boldsymbol{\gamma}$. To these ends write $\boldsymbol{\Sigma}(\boldsymbol{\gamma}) = \mathbf{I}_n + \mathbf{A}(\boldsymbol{\gamma})$ with $\mathbf{A}(\boldsymbol{\gamma}) = \mathbf{e}\boldsymbol{\gamma}' + \boldsymbol{\gamma}\mathbf{e}' - \bar{\gamma}\mathbf{e}\mathbf{e}'$, and recall that the eigenvalues of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ are given in terms of $\mathbf{A}(\boldsymbol{\gamma})$ by $\{ch_i(\boldsymbol{\Sigma}(\boldsymbol{\gamma})) = 1 + ch_i(\mathbf{A}(\boldsymbol{\gamma})); 1 \leqslant i \leqslant n\}$. Further let $\tau_1 = \gamma_1 + \cdots + \gamma_n = n\bar{\gamma}$ and $\tau_2 = (\gamma_1 - \bar{\gamma})^2 + \cdots + (\gamma_n - \bar{\gamma})^2$. We now consider two cases, first supposing that $\boldsymbol{\gamma} = c\mathbf{e}$ for some $c \neq 0$. For this case $\mathbf{A}(c\mathbf{e}) = c\mathbf{e}\mathbf{e}'$ has unit rank and $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ takes the familiar form $\boldsymbol{\Sigma}(c\mathbf{e}) = \mathbf{I}_n + c\mathbf{e}\mathbf{e}'$ having eigenvalues 1.0, with multiplicity $n-1$, and $1 + nc$, so that $\boldsymbol{\Sigma}(c\mathbf{e})$ is positive definite if and only if $c > -n^{-1}$. Results for the case $\gamma \neq c\mathbf{e}$ are given in the following.

THEOREM 2. *Suppose that* $\boldsymbol{\Sigma}(\boldsymbol{\gamma}) = \mathbf{I}_n + \mathbf{A}(\boldsymbol{\gamma})$ *with* $\mathbf{A}(\boldsymbol{\gamma}) = \mathbf{e}\boldsymbol{\gamma}' + \gamma\mathbf{e}' - \bar{\gamma}\mathbf{e}\mathbf{e}'$ *and* $\boldsymbol{\gamma} \neq c\mathbf{e}$, *and let* $\Xi(n) = \{\boldsymbol{\Sigma}(\boldsymbol{\gamma}); \boldsymbol{\gamma} \in \Gamma\}$ *be the class of all such matrices for which* $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ *is positive definite. Then*

(i) $\mathbf{A}(\gamma)$ *has rank* $r[\mathbf{A}(\gamma)] = 2$.

(ii) $\mathbf{A}(\gamma)$ *is an indefinite matrix, its positive and negative eigenvalues given by* $\alpha_1 = [\tau_1 + (\tau_1^2 + 4n\tau_2)^{1/2}]/2$ *and* $\alpha_n = [\tau_1 - (\tau_1^2 + 4n\tau_2)^{1/2}]/2$, *respectively.*

(iii) *The ordered eigenvalues* $\{\xi_1 \geqslant \cdots \geqslant \xi_n\}$ *of* $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ *are given by* $\{\xi_1 = 1 + \alpha_1, \xi_2 = \cdots = \xi_{n-1} = 1, \xi_n = 1 + \alpha_n\}$.

(iv) $\boldsymbol{\Sigma}(\boldsymbol{\gamma}) \in \Xi(n)$, *or equivalently,* $\boldsymbol{\gamma} \in \Gamma$, *if and only if* $\tau_1 > n\tau_2 - 1$.

Proof. (i): Rewrite $\mathbf{A}(\boldsymbol{\gamma}) = \mathbf{e}(\boldsymbol{\gamma} - \eta\mathbf{e})' + (\boldsymbol{\gamma} - \eta\mathbf{e})\mathbf{e}' = \mathbf{e}\boldsymbol{\theta}' + \boldsymbol{\theta}\mathbf{e}'$ with $\eta = \bar{\gamma}/2$ and $\boldsymbol{\theta} = \boldsymbol{\gamma} - \eta\mathbf{e}$. This clearly has rank 2 unless $\boldsymbol{\theta} = d\mathbf{e}$ with $d \neq 0$, which is excluded by hypothesis.

To see conclusions (ii), recall that the leading terms of the characteristic polynomial for $\mathbf{A}(\boldsymbol{\gamma})$ are $P_n(\mathbf{A}) = \alpha^n - c_1\alpha^{n-1} + c_2\alpha^{n-2}$, where $c_1 = \operatorname{tr}\mathbf{A}(\boldsymbol{\gamma}) = \tau_1$ and c_2 is the sum of all 2×2 principal minors. Further terms vanish, since $\mathbf{A}(\boldsymbol{\gamma})$ has rank 2. A typical principal 2×2 submatrix, namely,

$$\mathbf{A}(i,j) = \begin{bmatrix} 2\gamma_i - \bar{\gamma} & \gamma_i + \gamma_j - \bar{\gamma} \\ \gamma_i + \gamma_j - \bar{\gamma} & 2\gamma_j - \bar{\gamma} \end{bmatrix}, \tag{4.1}$$

gives the minor $|\mathbf{A}(i,j)| = -(\gamma_i - \gamma_j)^2$, so that $c_2 = -\sum_{i<j}(\gamma_i - \gamma_j)^2$. From a standard formula it follows that $c_2 = -n\sum_{i=1}^{n}(\gamma_i - \bar{\gamma})^2 = -n\tau_2$. Substituting into $P_n(\mathbf{A})$ gives $P_n(\mathbf{A}) = \alpha^{n-2}(\alpha^2 - \tau_1\alpha - n\tau_2)$, the roots of which give conclusions (ii). Conclusion (iii) follows immediately, and conclusion (iv) from the requirement that $1 + \alpha_n = 1 + [\tau_1 - (\tau_1^2 + 4n\tau_2)^{1/2}]/2 > 0$ in order that $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ should be positive definite. ■

For later reference note that conclusion (iv) characterizes Γ as $\Gamma = \{\boldsymbol{\gamma} \in \mathbb{R}^n : \bar{\gamma} > \sum_{i=1}^{n}(\gamma_i - \bar{\gamma})^2 - n^{-1}\}$. We return to this subsequently in constructing mixtures of distributions.

5. CONDITIONING OF $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$

The conditioning of a linear system $\mathbf{Ax} = \mathbf{b}$, and the propagation of errors in its solution, are gauged by the *condition number* $c_g(\mathbf{A}) = g(\mathbf{A})g(\mathbf{A}^{-1})$, where $g(\cdot)$ ordinarily is a norm. The system is well conditioned at $\mathbf{A} = \mathbf{I}_n$, larger values of $c_g(\mathbf{A})$ reflecting more ill-conditioned systems. For further details see Horn and Johnson (1985). In a Gaussian model $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the condition number $c_g(\boldsymbol{\Sigma})$ also may be taken to gauge shapes of elliptical probability contours in comparison with the spherical contours occurring at $\boldsymbol{\Sigma} = \mathbf{I}_n$. If $g(\cdot)$ is unitarily invariant, then $\{c_\phi(\cdot);\ \phi \in \Phi\}$ is the class of condition numbers generated by the unitarily invariant matrix norms as studied in Marshall and Olkin (1965), where $c_\phi(\mathbf{A}) = \|\mathbf{A}\|_\phi \|\mathbf{A}^{-1}\|_\phi$.

We next investigate the manner in which the conditioning of $\boldsymbol{\Sigma}(\boldsymbol{\gamma}) \in \Xi(n)$ depends on $\boldsymbol{\gamma} \in \Gamma$. A basic result is the following.

THEOREM 3. *Let $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ and $\boldsymbol{\Sigma}(\boldsymbol{\gamma}_0)$ belong to $\Xi(n)$. If $\boldsymbol{\gamma}$ majorizes $\boldsymbol{\gamma}_0$, i.e., $\boldsymbol{\gamma} \succcurlyeq \boldsymbol{\gamma}_0$, then the condition numbers are ordered as $c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma})) \geqslant c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_0))$ for every condition number in the class $\{c_\phi(\cdot);\ \phi \in \Phi\}$ generated by the unitarily invariant matrix norms on S_n^+.*

Proof. Let $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_n]'$ and $\boldsymbol{\xi}_0 = [\xi_{01}, \ldots, \xi_{0n}]'$ be the ordered eigenvalues of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ and $\boldsymbol{\Sigma}(\boldsymbol{\gamma}_0)$, respectively. Then $\boldsymbol{\gamma} \succcurlyeq \boldsymbol{\gamma}_0$ implies that $\boldsymbol{\xi} \succcurlyeq \boldsymbol{\xi}_0$ on applying Theorem 2, since $\tau_2 = \sum_{i=1}^{n}(\gamma_i - \bar{\gamma})^2$ is order-preserving under majorization. But since $\|\cdot\|_\phi$ is unitarily invariant, we have that $c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma})) = \phi(\xi_1, \ldots, \xi_n)\phi(\xi_1^{-1}, \ldots, \xi_n^{-1})$. Lemma 3.3 of Marshall and Olkin (1965) shows that if $\mathbf{b} \succcurlyeq \mathbf{a}$, then (1) $\phi(a_1, \ldots, a_n) \leqslant \phi(b_1, \ldots, b_n)$ and (2) $\phi(a_1^{-1}, \ldots, a_n^{-1}) \leqslant \phi(b_1^{-1}, \ldots, b_n^{-1})$. On combining these and identifying $\mathbf{a}$

with $\boldsymbol{\xi}_0$ and $\mathbf{b}$ with $\boldsymbol{\xi}$, we conclude that $c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma})) \geqslant c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_0))$ as claimed, to complete our proof. ■

Now consider models $\boldsymbol{\Sigma}(\boldsymbol{\gamma}_1)$ and $\boldsymbol{\Sigma}(\boldsymbol{\gamma}_2)$ in $\Xi(n)$ with $\boldsymbol{\gamma}_1 = [\gamma_{11}, \dots, \gamma_{1n}]'$ and $\boldsymbol{\gamma}_2 = [\gamma_{21}, \dots, \gamma_{2n}]'$, such that neither $\boldsymbol{\gamma}_1 \succcurlyeq \boldsymbol{\gamma}_2$ nor $\boldsymbol{\gamma}_2 \succcurlyeq \boldsymbol{\gamma}_1$. To make further progress observe that the simplex $(C(n), \succcurlyeq)$ from Section 2 is a lattice with greatest lower bound and least upper bound given respectively by the vectors $\boldsymbol{\gamma}_m = \boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_M = \boldsymbol{\gamma}_1 \vee \boldsymbol{\gamma}_2$. Here the elements of $\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \in \mathbb{R}^n$ are found by working backwards from the scalar quantities $\{\gamma_{11} \wedge \gamma_{21}, (\gamma_{11} + \gamma_{12}) \wedge (\gamma_{21} + \gamma_{22}), \dots, (\gamma_{11} + \cdots + \gamma_{1,n-1}) \wedge (\gamma_{21} + \cdots + \gamma_{2,n-1})\}$, with $\gamma_i \wedge \gamma_j = \min(\gamma_i, \gamma_j)$. Similar developments apply for the vector $\boldsymbol{\gamma}_1 \vee \boldsymbol{\gamma}_2$ with $\gamma_i \vee \gamma_j = \max(\gamma_i, \gamma_j)$. Theorem 3 applies directly to give the bounds

$$c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_m)) \leqslant \{c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_1)), c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_2))\} \leqslant c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_M)) \qquad (5.1)$$

for each condition number in the class $\{c_\phi(\cdot);\ \phi \in \Phi\}$. With apparent modifications we have proved the following.

THEOREM 4. *Let* $\Xi_B(n) = \{\boldsymbol{\Sigma}(\boldsymbol{\gamma});\ \boldsymbol{\gamma} \in \Gamma(\boldsymbol{\gamma}_m, \boldsymbol{\gamma}_M)\}$ *be bounded in the sense that* $\boldsymbol{\gamma}_m \preccurlyeq \boldsymbol{\gamma} \preccurlyeq \boldsymbol{\gamma}_M$ *for each* $\gamma \in \Gamma(\boldsymbol{\gamma}_m, \boldsymbol{\gamma}_M)$. *Then bounds on condition numbers for the ensemble* $\Xi_B(n)$ *are given by*

$$c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_m)) \leqslant \{c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}));\ \boldsymbol{\Sigma}(\boldsymbol{\gamma}) \in \Xi_B(n)\} \leqslant c_\phi(\boldsymbol{\Sigma}(\boldsymbol{\gamma}_M)) \quad (5.2)$$

for each condition number in the class $\{c_\phi(\cdot);\ \phi \in \Phi\}$ *generated by the unitarily invariant matrix norms on* S_n^+.

6. CONCLUSIONS

The one-way analysis of variance partitions $\mathbf{Y}'\mathbf{B}_n\mathbf{Y}$ so that $\mathbf{Y}' = [\mathbf{Y}_1', \dots, \mathbf{Y}_k']$ and $\{\mathbf{Y}_i' = [Y_{i1}, \dots, Y_{in_i}];\ 1 \leqslant i \leqslant k\}$. Standard assumptions are that $\mathbf{Y}_i$ represents an iid sample from $N_1(\mu_i, \sigma^2)$ independently for $1 \leqslant i \leqslant k$, so that $\mathscr{L}(\mathbf{Y}) = N_n(\boldsymbol{\mu}, \sigma^2\mathbf{I}_n)$ with $n = n_1 + \cdots + n_k$. Theorem 1 assures validity of the analysis even when $\mathscr{L}(\mathbf{Y}) \in \{N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma});\ \boldsymbol{\Sigma} \in \Xi(n)\}$. This holds despite highly ill-conditioned models in the class $\{N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma});\ \boldsymbol{\Sigma} \in \Xi(n)\}$ as studied in Section 5. A similar validation applies in every normal-theory analysis of variance partitioning the sum of squared deviations from a grand mean.

Implications of the present study go considerably farther. For fixed $\boldsymbol{\gamma} \in \Gamma$ suppose that the conditional distribution of $\mathbf{Y}$ is given by $\mathscr{L}(\mathbf{Y}|\boldsymbol{\gamma}) = N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}(\boldsymbol{\gamma}))$, and consider testing linear contrasts of the type $H : \mathbf{A}'\boldsymbol{\mu} = \mathbf{0}$ against general alternatives $K : \mathbf{A}'\boldsymbol{\mu} \neq \mathbf{0}$. It follows that normal-theory tests for H are independent of $\boldsymbol{\gamma}$ in level and power. We now mix over $\boldsymbol{\gamma}$ so as to preserve the definiteness of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$, to obtain the typical mixture

$$f(\mathbf{y}; \boldsymbol{\mu}, G) = \int_{\Gamma} g(\mathbf{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}(\boldsymbol{\gamma}))\, dG(\boldsymbol{\gamma}). \tag{6.1}$$

Here $g(\mathbf{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}(\boldsymbol{\gamma}))$ designates the Gaussian density corresponding to $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}(\boldsymbol{\gamma}))$, and $G(\cdot)$ is any cumulative distribution function supported on the set Γ as characterized in Theorem 2 and the comments following. We conclude that normal-theory tests for H remain exact in level and power for all such mixtures. Examples include testing main effects and interactions in factorial experiments. Similar conclusions apply to Grubbs's (1950) test for outliers, as well as to all other tests based on the Studentized residuals.

It may be noted that mixtures of the type (6.1) belong to the class of symmetric star unimodal distributions as studied in Dharmadhikari and Joag-Dev (1988). It is remarkable that some normal-theory tests remain exact in level and power even for joint error distributions having star-shaped contours. Further details and extensions of these properties will be reported elsewhere.

In Section 4 we found explicit expressions for the eigenvalues of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$, and conditions for positive definiteness, all in terms of $\boldsymbol{\gamma}$. These findings in turn support a detailed assessment of the conditioning of $\boldsymbol{\Sigma}(\boldsymbol{\gamma})$ as $\boldsymbol{\gamma}$ varies, as reported in Section 5.

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