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Triple I method based on pointwise sustaining degrees☆

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Abstract

The concept of the pointwise sustaining degrees is first introduced, and based on it the triple I principles for fuzzy modus ponens (FMP for short) and fuzzy modus tollens (FMT for short) are improved. And then, for the sake of making more implications can be used under the same way, general computing formulas of the triple I method for FMP and FMT are established under weaker conditions. Thus, the existing unified forms of triple I method are generalized to new forms.

Keywords: Fuzzy reasoning; Triple I method; Triple I principle; Pointwise sustaining degree; Implication operator

1. Introduction

The method of fuzzy reasoning plays an important role in the design and analysis of fuzzy control and fuzzy expert systems. The basic problems of fuzzy reasoning are the following fuzzy modus ponens (FMP for short) and fuzzy modus tollens (FMT for short):

$$\text{FMP: for a given rule } A \to B \text{ and input } A^*, \text{ to compute } B^* \text{ (output)} \tag{1}$$

$$\text{FMT: for a given rule } A \to B \text{ and input } B^*, \text{ to compute } A^* \text{ (output)} \tag{2}$$

where $A, A^* \in F(U)$ (the set of all fuzzy subsets of universe U) and $B, B^* \in F(V)$ (the set of all fuzzy subsets of universe V). The second author of the present paper pointed out in [1] that the conclusions of the above problems should be deduced from the major premise $A \to B$ and the minor premise jointly, and hence proposed that the fact that $A^*(u)$ implies $B^*(v)$ should be considered and should also be fully sustained by the major premise $A(u) \to B(v)$, i.e.,

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \tag{3}$$

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should take its maximum whenever $u \in U$ and $v \in V$. Evidently, if the employed implication $\to$ is nonincreasing in its first and nondecreasing in its second component, then (3) always takes its maximum in the case $B^*(v) \equiv 1$ $(v \in V)$ (respectively, $A^*(u) \equiv 0$ $(u \in U)$) no matter what the major premise $A(u) \to B(v)$ and the minor premise $A^*(u)$ (respectively, $B^*(v)$) are, hence the conclusion B^* of FMP (respectively, conclusion A^* of FMT) was required to be the smallest fuzzy subset of V (respectively, the largest fuzzy subset of U) making (3) take the maximum. According to above, the so-called triple I (the abbreviation of triple implications) method for solving the above problems was proposed in [1] (see, also [2–4]) and the suggested triple I principle is that the conclusion B^* of FMP (respectively, the conclusion A^* of FMT) is the smallest fuzzy subset of V (respectively, the largest fuzzy subset of U) making (3) take its maximum. Furthermore, the triple I method was generalized to the α-triple I method (see [1–4]). Its basic idea can be summarized as follows: for $\alpha \in [0, 1]$, and A, B and A^* (or B^*), the solution to (1) (respectively, to (2)) is the smallest fuzzy subset B^* of V (respectively, the largest fuzzy subset A^* of U) satisfying

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geq \alpha \tag{4}$$

for all $u \in U$ and $v \in V$. Inequality (4) indicates that the sustaining degree [1] of $A \to B$ to $A^* \to B^*$ should be greater than or equal to α, i.e.,

$$\mathrm{sust}(A \to B, A^* \to B^*) \geq \alpha$$

where $\mathrm{sust}(A \to B, A^* \to B^*) = \inf\{(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) | u \in U, v \in V\}$. In order to control each step in the process of reasoning more flexibly, in the present paper, we replace the constant α in formula (4) by a variable $\alpha(u, v)$, where $(u, v) \in U \times V$, i.e., we request the sustaining degree of $A \to B$ to $A^* \to B^*$ at any point (u, v) to be not less than a corresponding level $\alpha(u, v)$. Thus, the concept of pointwise sustaining degrees is first introduced and therefore the original reasoning principles are improved from the viewpoint of sustaining degrees.

Wang and Fu provided in [3] the unified forms of triple I method for FMP and FMT. Later, Liu and Wang [5] established another unified form of triple I method for FMT. The employed implications in the above unified forms are required to be regular or normal (Definitions 3 and 4 in [3] or Section 2 of this paper). But we find that this request is very strong, and as a result, some common implications, such as Kleene–Dienes implication, Reichenbach implication, Yager implication and so on, cannot be used to the unified forms. In the present paper, we firstly generalize the triple I principles to new forms based on the pointwise sustaining degrees introduced in this paper, and then establish more general expressions for triple I method under weaker conditions make more implications which can be used.

The rest of this paper is organized as follows. Section 2 is the preliminaries. In Section 3, we introduce the concept of the pointwise sustaining degrees, improve the triple I principles and establish new unified forms, i.e. the general expressions, for triple I method. The final section is the conclusion.

2. Preliminaries

In this section, we briefly recall the concepts of some logical operators, and introduce and discuss the concept of residual pair different from the one in [3].

Definition 2.1. A negation on [0, 1] is a decreasing mapping $n : [0, 1] \to [0, 1]$ satisfying $n(0) = 1$ and $n(1) = 0$. If $n(n(x)) = x$ for all $x \in [0, 1]$, then n is called involutive.

The negation n_s defined by $n_s(x) = 1 - x$ for all $x \in [0, 1]$ is called the standard negation on [0, 1].

Definition 2.2. A triangular norm (briefly t-norm) on [0, 1] is a commutative, associative and nondecreasing mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(1, x) = x$ for all $x \in [0, 1]$. A t-norm T is said to be left-continuous if it is left-continuous as a two-place mapping.

A triangular conorm (briefly t-conorm) on $[0, 1]$ is a commutative, associative and nondecreasing mapping $S : [0, 1]^2 \to [0, 1]$ satisfying $S(0, x) = x$ for all $x \in [0, 1]$.

Definition 2.3. The dual of a t-norm T (respectively, t-conorm S) on [0, 1] with respect to (w.r.t. for short) the negation n is the mapping T^* (respectively, S^*) defined by $T^*(x, y) = n(T(n(x), n(y)))$ (respectively, $S^*(x, y) = n(S(n(x), n(y))))$ for $x, y \in [0, 1]$.

It is noted that T^* is a t-conorm and S^* is a t-norm on [0, 1].

Definition 2.4. An implication on [0, 1] is any $[0,1]^2 \to [0,1]$ mapping I satisfying $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. The residual implication generated by a t-norm T is defined as $I_T(x, y) = \sup\{\gamma \in [0,1] | T(x,\gamma) \le y\}$ for all $x, y \in [0,1]$. We also write $I(x, y)$ as $x \to y$ for $x, y \in [0,1]$ in the following discussion.

Definition 2.5. Let T and I be two $[0,1]^2 \to [0,1]$ mappings, (T, I) is said to be a residual pair or, T and I are residual to each other, if the following residuation condition holds, for all $x, y, z \in [0,1]$,

$$T(x, y) \le z \quad \text{if and only if} \quad x \le I(y, z). \tag{5}$$

Remark 2.1. The above definition about residual pair is different from the one in [3], where mapping T was assumed to be a t-norm on [0, 1].

Since the left-continuity of a t-norm T on [0, 1] is equivalent to the residuation condition (5) where $I = I_T$ (see, e.g. [6]), (T, I_T) always forms a residual pair for any left-continuous t-norm T on [0, 1]. The residual implication I generated by a left-continuous t-norm on [0, 1] is called a regular implication [3]. We now list its some properties (see [7–10]):

(P1) $I(y, z) = 1 \Longleftrightarrow y \le z$;
(P2) $x \le I(y, z) \Longleftrightarrow y \le I(x, z)$;
(P3) $I(x, I(y, z)) = I(y, I(x, z))$;
(P4) $I(1, z) = z$;
(P5) $I(x, \inf_{z\in Z} z) = \inf_{z\in Z} I(x, z)$;
(P6) $I(\sup_{z\in Z} z, y) = \inf_{z\in Z} I(z, y)$;
(P7) I is nonincreasing in its first component;
(P8) I is nondecreasing in its second component

where $x, y, z \in [0,1]$ and Z is any nonempty subset of [0, 1].

Definition 2.6. We say that an implication I on [0, 1] has contrapositive symmetry w.r.t. an involutive negation n if $I(x, y) = I(n(y), n(x))$ holds for all $x, y \in [0,1]$. An implication I on [0, 1] is said to be normal if it is regular and has contrapositive symmetry w.r.t. an involutive negation n.

Remark 2.2. The above definition about normal is different from the one in [3], where the involutive negation n was assumed to be the standard negation n_s on [0, 1].

Theorem 2.1. *If a mapping $I : [0,1]^2 \to [0,1]$ satisfies* (P5) *and*

(P) $\{\gamma \in [0,1] | x \le I(y,\gamma)\} \ne \phi$ *for all* $x, y \in [0,1]$,

then the mapping $T : [0,1]^2 \to [0,1]$ defined by

$$T(x, y) = \inf\{\gamma \in [0,1] | x \le I(y,\gamma)\}, \quad x, y \in [0,1] \tag{6}$$

is residual to I, i.e., T and I satisfy the residuation condition (5).

Proof. For any $z_1, z_2 \in [0,1]$ and $z_1 \le z_2$, we have from (P5) that $I(x, z_1) = I(x, z_1 \wedge z_2) = I(x, z_1) \wedge I(x, z_2)$, i.e., $I(x, z_1) \le I(x, z_2)$. This means that (P5) implies (P8). It is obvious that formula (6) follows that T is nondecreasing in its first component. For any $x, y, z \in [0,1]$, if $T(x, y) \le z$, then, it follows from (P5) and (P8) that

$$\begin{aligned} I(y, z) &\ge I(y, T(x, y)) = I(y, \inf\{\gamma \in [0,1] | x \le I(y,\gamma)\}) \\ &= \inf\{I(y,\gamma) | \gamma \in [0,1] \quad \text{and} \quad x \le I(y,\gamma)\} \ge x. \end{aligned}$$

If $x \le I(y, z)$, then, since T is nondecreasing in its first component and (P8) holds, it follows that

$$T(x, y) \le T(I(y, z), y) = \inf\{\gamma \in [0,1] | I(y, z) \le I(y,\gamma)\} \le z.$$

Summarizing above, the residuation condition (5) is proved. $\square$

Remark 2.3. The condition (P) in Theorem 2.1 cannot be omitted.

For counterexample, it is easy to verify that Zadeh implication I_Z, defined by $I_Z(x, y) = (1 - x) \vee (x \wedge y)$ for all $x, y \in [0, 1]$, satisfies (P5), but not (P) since $\{\gamma \in [0, 1] | x \le I_Z(y, \gamma)\} = \phi$ when $x + y > 1$ and $x > y$. The mapping T_Z determined by formula (6) and I_Z is as follows

$$T_Z(x, y) = \begin{cases} 0, & x + y \le 1 \\ x, & x + y > 1 \text{ and } x \le y \\ 1, & x + y > 1 \text{ and } x > y. \end{cases}$$

Since inequality $x \le I_Z(y, z)$ cannot be followed from $T_Z(x, y) \le z$ whenever $x + y > 1$ and $x > y$, residuation condition (5) does not hold for T_Z and I_Z.

Theorem 2.2. *If a mapping* $I : [0, 1]^2 \to [0, 1]$ *satisfies*

(P5)′ $I(x, \inf_{z \in Z} z) = \inf_{z \in Z} I(x, z)$ *for any* $x \in [0, 1]$ *and any subset* Z *of* $[0, 1]$,

then it satisfies (P5) *and* (P), *and hence the mapping* T *determined by* (6) *is residual to* I.

Proof. It is obvious that (P5)′ implies (P5). Taking $Z = \phi$ in (P5)′, we obtain $I(x, 1) = 1$ for all $x \in [0, 1]$ since $\inf \phi = 1$. This follows that $1 \in \{\gamma \in [0, 1] | x \le I(y, \gamma)\}$, i.e., (P) holds. According to Theorem 2.1, the mapping T determined by (6) is residual to I. □

From Theorem 2.2 and its proof, we have the following result.

Corollary 2.1. *If a mapping* $I : [0, 1]^2 \to [0, 1]$ *satisfies* (P5) *and* $I(x, 1) = 1$ *for all* $x \in [0, 1]$, *then the mapping* T *determined by* (6) *is residual to* I.

3. Triple I method based on pointwise sustaining degrees

It is noticed that the α in (4) is a constant. As a general extension, we now replace it by $\alpha(u, v)$ for $u \in U$ and $v \in V$, where $\alpha(u, v) \in [0, 1]$, and thus (4) becomes as follows

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \ge \alpha(u, v), \quad u \in U, v \in V. \tag{7}$$

Inequality (7) can be interpreted as that the sustaining degree of $A \to B$ to $A^* \to B^*$ at point $(u, v) (\in U \times V)$ should be greater than or equal to $\alpha(u, v)$. Based on this, we introduce the concept of pointwise sustaining degree between fuzzy sets as follows.

Definition 3.1. Let $\to$ be an implication on $[0, 1]$, W a nonempty set and $C, D \in F(W)$, we call $C(w) \to D(w)$ the sustaining degree of C to D at point $w \in W$, or pointwise sustaining degree of C to D, and denote it by $\mathrm{sust}(C, D)(w)$.

According to Definition 3.1, inequality (7) can be rewritten as

$$\mathrm{sust}(A \to B, A^* \to B^*)(u, v) \ge \alpha(u, v), \quad (u, v) \in U \times V.$$

Proposition 3.1. *Suppose that* $\to$ *is an implication on* $[0, 1]$ *satisfying* (P5) *and* (P), *and its residual mapping* T *is nondecreasing in its the second component and associative. For any* $A, B, C \in F(W)$ *and* $w \in W$, *if* $\mathrm{sust}(A, B)(w) \ge \alpha(w)$, $\mathrm{sust}(B, C)(w) \ge \beta(w)$, *then* $\mathrm{sust}(A, C)(w) \ge T(\beta(w), \alpha(w))$, *where* $\alpha(w), \beta(w) \in [0, 1]$.

Proof. Since $\to$ satisfies (P5) and (P), we know from Theorem 2.1 that residuation condition (5) holds. From the conditions $\mathrm{sust}(A, B)(w) \ge \alpha(w)$ and $\mathrm{sust}(B, C)(w) \ge \beta(w)$, i.e., $\alpha(w) \le A(w) \to B(w)$ and $\beta(w) \le B(w) \to C(w)$, we have $T(\alpha(w), A(w)) \le B(w)$ and $T(\beta(w), B(w)) \le C(w)$. Further, since T is nondecreasing in its second component and associative, we obtain

$$C(w) \ge T(\beta(w), B(w)) \ge T(\beta(w), T(\alpha(w), A(w))) = T(T(\beta(w), \alpha(w)), A(w)).$$

So, $T(\beta(w), \alpha(w)) \le A(w) \to C(w)$, i.e., $\mathrm{sust}(A, C)(w) \ge T(\beta(w), \alpha(w))$. □

3.1. $\alpha(u, v)$-triple I method for FMP

If the implication $\to$ in FMP is nondecreasing in its second component, then the maximum of (3) equals

$$M(u, v) = (A(u) \to B(v)) \to (A^*(u) \to 1). \tag{8}$$

So, in order to guarantee the inequality (7) holds, we always assume in this subsection that implication I satisfies (P8) and $\alpha(u, v) \le M(u, v)$ for all $u \in U$ and $v \in V$. Under the above assumptions, we now generalize Wang's α-triple I principle for FMP (see [1–4]) as follows.

$\alpha(u, v)$-**triple I principle for FMP:** The conclusion $B^*(\in F(V))$ of FMP problem (1) is the smallest fuzzy subset of V satisfying (7).

This B^* is called $\alpha(u, v)$-triple I solution of (1). Especially, if $\alpha(u, v) = M(u, v)$ for all $u \in U$ and $v \in V$, then we call B^* triple I solution of (1).

Remark 3.1. The above concept of triple I solution is different from the one in [3], where the maximum $M(u, v)$ was limited to 1.

Theorem 3.1. *Suppose that an implication I satisfies* (P5) *and* (P)*, then the $\alpha(u, v)$-triple I solution $B^*(\in F(V))$ of FMP* (1) *can be expressed as follows:*

$$B^*(v) = \sup_{u \in U} T(T(\alpha(u, v), I(A(u), B(v))), A^*(u)), \quad v \in V \tag{9}$$

where T is the mapping residual to I.

Proof. Since implication I satisfies (P5) and (P), from Theorem 2.1 we know that I and T satisfy the residuation condition (5). Thus, it follows from (5) that the condition (7) is equivalent to the following condition

$$T(T(\alpha(u, v), I(A(u), B(v))), A^*(u)) \le B^*(v), \quad u \in U, v \in V. \tag{10}$$

Then, (9) follows from the fact that B^* is the smallest fuzzy subset satisfying (10). $\square$

Remark 3.2. The formula (9) in the case of $\alpha(u, v) \equiv \alpha$ (constant) for all $u \in U$ and $v \in V$ was first given in [3], where the implication I was required to be regular, which is a stronger condition than the one given above.

The conditions in Theorem 3.1 guarantee the formula (9) can be suitable for more implications. For instance, the following implications $I_1 - I_{10}$ (see, e.g. [4,11]) can be used to formula (9) since they satisfy (P5) and (P), although they are not regular since $I_1 - I_6$ do not satisfy (P1) and $I_7 - I_{10}$ do not satisfy (P4),

$$I_1(x, y) = (1 - x) \vee y, \quad \text{(Kleene–Dienes implication)},$$
$$I_2(x, y) = (1 - x) + xy, \quad \text{(Reichenbach implication)},$$
$$I_3(x, y) = y^x \quad (\text{ assume } 0^0 = 1) \text{ (Yager implication)},$$
$$I_4(x, y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1, \end{cases} \qquad I_5(x, y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y = 1 \\ 0, & \text{otherwise}, \end{cases}$$
$$I_6(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0, \end{cases} \qquad I_7(x, y) = \begin{cases} 1, & \text{if } x \le y \\ 1 - x, & \text{if } x > y, \end{cases}$$
$$I_8(x, y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{if } x > y, \end{cases} \quad \text{(Gaines–Rescher implication)},$$
$$I_9(x, y) = \begin{cases} 1, & \text{if } x \le y \\ \dfrac{\lg x}{\lg y}, & \text{if } x > y > 0 \\ 0, & \text{if } x > y = 0, \end{cases} \qquad I_{10}(x, y) = \begin{cases} 1, & \text{if } x \le y \\ \dfrac{1 - x}{1 - y}, & \text{if } x > y \end{cases}$$

where $x, y \in [0, 1]$, $\vee = \max$ and $\wedge = \min$.

Especially, if we take $\alpha(u, v) = M(u, v)$ in (9), then the triple I solution B^* of FMP (1) can be expressed as follows:

$$B^*(v) = \sup_{u \in U} T(T(M(u, v), I(A(u), B(v))), A^*(u)), \quad v \in V. \tag{11}$$

Further, if $M(u, v) \equiv 1$ for all $u \in U$ and $v \in V$, then the triple I solution B^* of FMP (1) is as follows:

$$B^*(v) = \sup_{u \in U} T(T(1, I(A(u), B(v))), A^*(u)), \quad v \in V. \tag{12}$$

Again, if $T(1, y) = y$ holds for all $y \in [0, 1]$, then (12) becomes as follows:

$$B^*(v) = \sup_{u \in U} T(I(A(u), B(v)), A^*(u)), \quad v \in V. \tag{13}$$

For a residual pair (T, I), we know that $T(1, y) = y$ holds for all $y \in [0, 1]$ if and only if I satisfies (P1) (see [4]), and $M(u, v) = 1$ for all $u \in U$ and $v \in V$ follows from (P1). So we have the following result.

Corollary 3.1. *Under the same condition as Theorem 3.1, if the implication I satisfies* (P1)*, then B^* determined by (13) is the triple I solution of FMP (1), where T is the mapping residual to I.*

Remark 3.3. The formula (13) was first given in [3], where the implication I was required to be regular.

The above Corollary 3.1 guarantees the formula (13) can be suitable for more implications. For instance, the implications $I_7 - I_{10}$ can be used to formula (13) since they satisfy (P1), (P5) and (P), although they are not regular.

Example 3.1. Let $U = V = [0, 1]$, $A(u) = \frac{u+1}{3}$, $B(v) = 1 - v$, $A^* = 1 - u$, and $\alpha(u, v) = \frac{6+v-u}{9}$, where $u, v \in [0, 1]$. Assume that the implication employed in FMP is Kleene–Dienes implication I_1 defined as above. We now calculate the $\alpha(u, v)$-triple I solution B^* of FMP according to the formula (9).

$$\begin{aligned} I_1(A(u), B(v)) &= (1 - A(u)) \vee B(v) = \frac{2-u}{3} \vee (1 - v) \\ &= \begin{cases} \dfrac{2-u}{3}, & \text{if } \dfrac{2-u}{3} \geq 1 - v, \text{ i.e., } u - 3v \leq -1 \\ 1 - v, & \text{if } \dfrac{2-u}{3} < 1 - v, \text{ i.e., } u - 3v > -1. \end{cases} \end{aligned}$$

The mapping T_1 residual to I_1 is as follows:

$$T_1(u, v) = \begin{cases} 0, & \text{if } u + v \leq 1 \\ u, & \text{if } u + v > 1, \end{cases} \quad u, v \in [0, 1].$$

According to (9), we obtain the $\alpha(u, v)$-triple I solution of FMP as follows, for any $v \in [0, 1]$,

$$\begin{aligned} B^*(v) &= \sup_{u \in [0,1]} T_1(T_1(\alpha(u, v), I_1(A(u), B(v))), A^*(u)) \\ &= \left(\sup \left\{ T_1 \left(T_1 \left(\frac{6+v-u}{9}, \frac{2-u}{3} \right), A^*(u) \right) \middle| u \in [0, 1], u - 3v \leq -1 \right\} \right) \\ &\quad \vee \left(\sup \left\{ T_1 \left(T_1 \left(\frac{6+v-u}{9}, 1 - v \right), A^*(u) \right) \middle| u \in [0, 1], u - 3v > -1 \right\} \right) \\ &= \left(\sup \left\{ T_1 \left(\frac{6+v-u}{9}, 1 - u \right) \middle| u \in [0, 1], u - 3v \leq -1, 4u - v < 3 \right\} \right) \\ &\quad \vee \left(\sup \left\{ T_1 \left(\frac{6+v-u}{9}, 1 - u \right) \middle| u \in [0, 1], u - 3v > -1, u + 8v < 6 \right\} \right) \\ &= \left(\sup \left\{ \frac{6+v-u}{9} \middle| u \in [0, 1], u - 3v \leq -1, 4u - v < 3, 10u - v < 6 \right\} \right) \end{aligned}$$

$$
\begin{aligned}
&\quad \vee \left(\sup \left\{ \frac{6+v-u}{9} \middle| u \in [0,1], u - 3v > -1, u + 8v < 6, 10u - v < 6 \right\} \right) \\
&= \left(\sup \left\{ \frac{6+v-u}{9} \middle| u \in [0,1], u - 3v \le -1, 10u - v < 6 \right\} \right) \\
&\quad \vee \left(\sup \left\{ \frac{6+v-u}{9} \middle| u \in [0,1], u - 3v > -1, 10u - v < 6 \right\} \right) \\
&= \sup \left\{ \frac{6+v-u}{9} \middle| u \in [0,1], 10u - v < 6 \right\} \\
&= \frac{6+v}{9}.
\end{aligned}
$$

The above last step is due to that $0 \in \{u \in [0,1] | 10u - v < 6\}$ for any $v \in [0,1]$ and $\frac{6+v-u}{9}$ is nonincreasing in the variable u.

3.2. *$\alpha(u, v)$-triple I method for FMT*

In this subsection, we focus on the FMT problem (2). First, in order to guarantee (7) holds in our discussion, we always assume in this subsection that $\alpha(u, v)$ is not greater than the maximum $N(u, v)$ of (3). Especially, if the employed implication I in FMT is nonincreasing in its first and nondecreasing in its second component, then $N(u, v) = (A(u) \to B(v)) \to (0 \to B^*(v))$.

According to the α-triple I principle for FMT (see [1–4]), we have the following.

$\alpha(u, v)$-**triple I principle for FMT:** The conclusion $A^*(\in F(U))$ of FMT problem (2) is the largest fuzzy subset of U satisfying (7).

This A^* is called $\alpha(u, v)$-triple I solution of (2). Especially, if $\alpha(u, v)$ takes the maximum $N(u, v)$ of (3), then we call the A^* triple I solution of (2).

Remark 3.4. The above concept of triple I solution is different from the one in [3], where the maximum $N(u, v)$ was limited to 1.

Theorem 3.2. *Suppose that an implication I satisfies* (P2), (P5) *and* (P)*, then the $\alpha(u, v)$-triple I solution $A^*(\in F(U))$ of FMT* (2) *can be expressed as follows:*

$$A^*(u) = \inf_{v \in V} I(T(\alpha(u, v), I(A(u), B(v))), B^*(v)), \quad u \in U \tag{14}$$

where T is the mapping residual to I.

Proof. Since an implication I satisfies (P5) and (P), it follows from Theorem 2.1 that (T, I) is a residual pair, i.e., the residuation condition (5) holds. Thus, it follows from (5) and (P2) that the condition (7) is equivalent to the condition

$$A^*(u) \le I(T(\alpha(u, v), I(A(u), B(v))), B^*(v)), \quad u \in U, v \in V. \tag{15}$$

Then, (14) follows from the fact that A^* is the largest fuzzy subset satisfying (15). $\square$

Remark 3.5. Notice that $I(x, I(y, z)) = I(T(x, y), z)$ $(\forall x, y, z \in [0, 1])$ holds for any regular implication I [7], where T is the left-continuous t-norm residual to I. So we have that if the implication I in (14) is regular, then formula (14) becomes as follows:

$$A^*(u) = \inf_{v \in V} I(\alpha(u, v), I(I(A(u), B(v)), B^*(v))), \quad u \in U. \tag{16}$$

The above formula (16) in the case of $\alpha(u, v) = \alpha$(constant) for all $u \in U$ and $v \in V$ is the result obtained in [5].

The following is several examples of the implications satisfying (P2), (P5) and (P), for any $x, y \in [0, 1]$,

$$I_{11}(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ \dfrac{y}{x}, & \text{if } x > 0, \end{cases} \quad \text{(Goguen implication)},$$

$$I_{12}(x,y)=\begin{cases}1, & \text{if } x\le y\\ y, & \text{if } x>y,\end{cases}\quad \text{(Gödel implication)},$$

$$I_{13}(x,y)=\begin{cases}1, & \text{if } x\le y\\ 1-x+y, & \text{if } x>y,\end{cases}\quad \text{(Łukasiewicz implication)}.$$

$$I_{14}(x,y)=\begin{cases}1, & \text{if } x\le y\\ (1-x)\vee y, & \text{if } x>y.\end{cases}\quad (I_0 \text{ implication}),$$

Corollary 3.2. *Suppose that the implication I in FMT (2) is normal w.r.t. an involutive negation n, then the $\alpha(u,v)$-triple I solution of FMT (2) can be expressed as follows:*

$$A^*(u)=\inf_{v\in V} S(S(n(\alpha(u,v)),n(I(A(u),B(v)))),B^*(v)),\quad u\in U \tag{17}$$

where S is the dual of the t-norm T residual to I w.r.t. the negation n.

Proof. Since the implication I is normal, it is regular and contrapositive w.r.t. an involutive negation n. Then, from Theorem 1 in [7], we know that $I(x,y)=n(T(x,n(y)))$, i.e., $I(x,y)=S(n(x),y)$ for all $x,y\in[0,1]$. So (17) follows from (14). □

Remark 3.6. If we take $n=n_s$ and $\alpha(u,v)\equiv\alpha$ (constant) for all $u\in U$ and $v\in V$, then (17) is the formula established by Wang and Fu in [3], where the proof was much more complicated than the one given above.

Especially, if we take $\alpha(u,v)=N(u,v)$ (the maximum of (3)) in (14), then the triple I solution A^* of FMT (2) can be expressed as follows:

$$A^*(u)=\inf_{v\in V} I(T(N(u,v),I(A(u),B(v))),B^*(v)),\quad u\in U. \tag{18}$$

Similar to the induction process of Corollary 3.1, we have the following.

Corollary 3.3. *Under the same condition as Theorem 3.2, if the implication I satisfies* (P1)*, then the triple I solution A^* of FMT (2) is as follows:*

$$A^*(u)=\inf_{v\in V} I(I(A(u),B(v)),B^*(v)),\quad u\in U. \tag{19}$$

Implications $I_{11}-I_{14}$ are four examples satisfying the condition of Corollary 3.3. Particularly, if the implication I is normal w.r.t. an involutive negation n, then triple I method (19) for FMT (2) can also be expressed in the following way:

$$A^*(u)=\inf_{v\in V} S(n(I(A(u),B(v))),B^*(v)),\quad u\in U. \tag{20}$$

This is because $I(x,y)=S(n(x),y)$ holds for all $x,y\in[0,1]$ (see [7]), where S is the dual of T residual to I w.r.t. the negation n.

Remark 3.7. The formula (20) in the case of $n=n_s$ is the result given in [3], where it was deduced in another way.

Example 3.2. Let $U=V=[0,1]$, $A(u)=\frac{u+1}{3}$, $B(v)=1-v$, $B^*(v)=\frac{2}{3}$, and $\alpha(u,v)=\frac{6+v-u}{9}$, where $u,v\in[0,1]$. Assume that the implication employed in FMT is I_{14} defined as above. We now calculate the $\alpha(u,v)$-triple I solution A^* of FMT according to (14).

$$I_{14}(A(u),B(v))=I_{14}\left(\frac{u+1}{3},1-v\right)=\begin{cases}1, & \text{if } \dfrac{u+1}{3}\le 1-v,\ \text{i.e., } u+3v\le 2\\ \dfrac{2-u}{3}\vee(1-v), & \text{if } \dfrac{u+1}{3}>1-v,\ \text{i.e., } u+3v>2.\end{cases}$$

The mapping T_{14} residual to I_{14} is as follows:

$$T_{14}(u,v)=\begin{cases}0, & \text{if } u+v\le 1\\ u\wedge v, & \text{if } u+v>1,\end{cases}\qquad u,v\in[0,1].$$

According to (14), we obtain the $\alpha(u,v)$-triple I solution of FMT as follows, for any $u\in[0,1]$,

$$\begin{aligned}A^*(u)&=\inf_{v\in[0,1]} I_{14}(T_{14}(\alpha(u,v),I_{14}(A(u),B(v))),B^*(v))\\&=\left(\inf\left\{I_{14}\left(T_{14}\left(\frac{6+v-u}{9},1\right),\frac{2}{3}\right)\middle|\, v\in[0,1],u+3v\le 2\right\}\right)\\&\quad\wedge\left(\inf\left\{I_{14}\left(T_{14}\left(\frac{6+v-u}{9},\frac{2-u}{3}\vee(1-v)\right),\frac{2}{3}\right)\middle|\, v\in[0,1],u+3v>2\right\}\right).\end{aligned}$$

Taking into account $T_{14}(\frac{6+v-u}{9},\frac{2-u}{3}\vee(1-v))\le\frac{2-u}{3}\vee(1-v)\le\frac{2}{3}$ for $v\in\{v\in[0,1]|u+3v>2\}$, we have

$$\begin{aligned}A^*(u)&=\inf\left\{I_{14}\left(\frac{6+v-u}{9},\frac{2}{3}\right)\middle|\, v\in[0,1],u+3v\le 2\right\}\\&=\inf\left\{\left(1-\frac{6+v-u}{9}\right)\vee\frac{2}{3}\middle|\, v\in[0,1],u+3v\le 2,v-u>0\right\}\\&=\frac{2}{3}.\end{aligned}$$

4. Conclusion

The concept of pointwise sustaining degrees has been first introduced and the triple I principles of FMP and FMT problems have been improved. More general expressions of triple I method for FMP and FMT have been established under weaker conditions. Thus, on the one hand, the existing unified forms of triple I method have been generalized to new forms and their application field has also been enlarged since the new forms cover more implications. On the other hand, these new expressions of triple I method will bring convenience to analyze the reasoning methods algebraically and to make further study to their logical foundation problems.

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