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Jackson theorem in L_p, $0 < p < 1$, for functions on the sphere

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Abstract

The best approximation of functions in $L_p(S^{d-1})$, $0 < p < 1$ by spherical harmonic polynomials is shown to be bounded by a modulus of smoothness recently introduced by the second author.

Keywords: Spherical harmonic polynomials; Modulus of smoothness

1. Introduction

The space H_k of spherical harmonic polynomials of degree k on the unit sphere

$$S^{d-1} \equiv \{\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : |x|^2 = x_1^2 + \cdots + x_d^2 = 1\}$$

$(d \geq 3)$ is the collection of restrictions to S^{d-1} of the homogeneous harmonic polynomials of degree k. The dimension of H_k is $\binom{d+k-1}{k} - \binom{d+k-3}{k-2}$ (see [9, p. 140]). The space H_k can also be described by

$$H_k = \{\varphi : \widetilde{\Delta}\varphi = -k((k+d-2)\varphi)\} \tag{1.1}$$

where $\widetilde{\Delta}$ is the Laplace–Beltrami operator given by

$$\widetilde{\Delta} f(x) = \Delta f\left(\frac{x}{|x|}\right) \quad \text{for } x \in S^{d-1},\ \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}. \tag{1.2}$$

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The moduli of smoothness usually used for functions on the sphere ($d \geq 3$) involves averages and hence are not defined for $f \in L_p(S^{d-1})$ when $0 < p < 1$. We use here the modulus of smoothness introduced in [4] and given by

$$\omega(f,t)_p = \sup\{\|\Delta_\rho f\|_{L_p(S^{d-1})} : \rho \in SO(d), \rho x \cdot x \geq \cos t \text{ for all } x \in S^{d-1}\} \tag{1.3}$$

where $SO(d)$ is the class of orthogonal matrices whose determinant equals 1 and where

$$\Delta_\rho f(x) = f(\rho x) - f(x). \tag{1.4}$$

We will prove for $0 < p < 1$ the Jackson-type inequality

$$E_n(f)_p \leq C\,\omega\left(f, \frac{1}{n}\right)_p \tag{1.5}$$

where

$$E_n(f)_p \equiv \inf\left\{\|f - \varphi\|_{L_p(S^{d-1})} : \varphi \in \text{ span}\left(\bigcup_{k=0}^{n-1} H_k\right)\right\}. \tag{1.6}$$

In [4] higher moduli of smoothness, $\omega^r(f,t)_p$ were defined, and we conjecture that the more general Jackson-type inequality

$$E_n(f)_p \leq C\,\omega^r(f,t)_p \tag{$*$}$$

is valid for $d \geq 3$ with $\omega^r(f,t)_p$ given in [4]. The Jackson-type inequality ($*$) was proved for $L_p(S^{d-1})$, $1 \leq p \leq \infty$ in [5] and for many other Banach spaces of functions in [3].

For the proof we will construct the operators $O_{n,\rho} f(x)$ from $f \in L_p(S^{d-1})$ to span $\left(\bigcup_{k=0}^{2n} H_k\right)$ which will not necessarily exist for all $\rho \in SO(d)$; for each $f \in L_p(S^{d-1})$, however, $O_{n,\rho} f(x)$ will exist for almost all ρ with respect to the Haar measure of $SO(d)$ and will have a specific bound on a set of positive measure of $SO(d)$. It will be shown that $O_{n,\rho} f(x)$ has de la Vallée Poussin-type properties.

This type of proof for a Jackson-type result was applied by Runovskii (see [7,8]) to approximation by trigonometric polynomials on $T = [0, 2\pi)$ and on the torus T^d. Of course, the situation here is much more involved as we have to replace a simple formula on T or on T^d by using kth degree geometric design or Marcinkiewicz-type cubature formulae. Moreover, as elements of $SO(d)$ do not commute, we could not repeat our process and obtain ($*$) from (1.5).

In Section 2 we will construct the operators $O_{n,\rho} f$. Some preliminary results will be given in Section 3 and the Jackson-type result (1.5) will be proved in Section 4. In Section 5 the open conjecture ($*$) is presented with a prize for its proof.

2. Definition of the operators $O_{n,\rho} f(x)$

In this section we define the class of operators $O_{n,\rho} f(x)$ which for some ρ (almost all ρ) maps f into the space of spherical harmonic polynomials of degree $2n$. We first define the kernels $W_n(x \cdot y)$ by

$$W_n(x \cdot y) = \sum_{m=0}^{\infty} \eta\left(\frac{m}{n}\right)\left\{\sum_{j=1}^{d_m} Y_{m,j}(x) Y_{m,j}(y)\right\} \tag{2.1}$$

where $Y_{m,j}$ is any basis of orthonormal elements of H_m and $\eta(u) \in C^\infty[0,\infty)$, $\eta(u) = 1$ for $0 < u \le 1$ and $\eta(u) = 0$ for $u \ge 2$. We recall that the zonal function $Z_x^{(m)}(y)$ satisfies (see [9, 143–149])

$$Z_x^{(m)}(y) = \sum_{j=1}^{d_m} Y_{m,j}(x)Y_{m,j}(y) = c_{m,d}P_m^\lambda(x\cdot y), \quad \lambda = \frac{d-2}{2},$$

where $\{P_m^\lambda(t)\}_{m=0}^\infty$, the ultraspherical polynomials, are the orthogonal system of polynomials on $[-1,1]$ with respect to the weight $(1-t^2)^{\frac{d-3}{2}}$. Therefore, (2.1) can be rewritten as

$$W_n(x\cdot y) = \sum_{m=0}^\infty \eta\left(\frac{m}{n}\right)Z_x^{(m)}(y) = \sum_{m=0}^\infty \eta\left(\frac{m}{n}\right)c_{m,d}P_m^\lambda(x\cdot y), \quad \lambda = \frac{d-2}{2}.$$

It was shown in [1, Lemma 3.3] that

$$|W_n(\cos\theta)| \le J(\ell)n^{d-1}(1+n\theta)^{-\ell} \tag{2.2}$$

for any integer ℓ. Clearly, we have

$$\int_{S^{d-1}} |W_n(x\cdot y)|^p \mathrm{d}y = C_d \int_0^\pi |W_n(\cos\theta)|^p (\sin\theta)^{d-2}\mathrm{d}\theta.$$

Therefore, for sufficiently large ℓ (and any ℓ satisfying $d < \ell p$ will do) the inequality (2.2) combined with straightforward computation implies

$$\int_{S^{d-1}} |W_n(x\cdot y)|^p \mathrm{d}y \le A(p,d)n^{(p-1)(d-1)}, \quad 0 < p \le 1. \tag{2.3}$$

We set $|S^{d-1}| \equiv \int_{S^{d-1}} \mathrm{d}y$.

For homogeneous spherical polynomials of degree $2k$ there is a set G_k, $G_k = \{x_{k,i} : x_{k,i} \in S^{d-1}\}$ which satisfies the cubature formula

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}} \varphi(y)\mathrm{d}y = \sum_{s=1}^{N_k} \lambda_s \varphi(x_{k,s}) \quad \text{for all } \varphi \in \operatorname{span}\left(\bigcup_{\ell=0}^{2k} H_\ell\right) \tag{2.4}$$

with $0 \le \lambda_s$ and hence $\sum_{s=1}^{N_k} \lambda_s = 1$. It is also known (see for instance [6, Proposition 2, p. 204]) that

$$\binom{d+k-1}{k} \le N_k \le \binom{d+2k-1}{2k}, \quad N_k \equiv |G_k|. \tag{2.5}$$

Therefore, $N_k \approx k^{d-1}$; that is, $A(d)^{-1}k^{d-1} \le N_k \le A(d)k^{d-1}$. The set G_k is sometimes called spherical geometric design (when $\lambda_s = N_k^{-1}$). We note that in many works in functional analysis the gap between the lower and upper estimates of $N_k \equiv N_k(d)$ seems bigger, but the reason is that in those works the dimension d is allowed to tend to infinity and k is fixed, while in our work k grows and d remains fixed (but arbitrary). A less abstract proof of the existence of G_k satisfying (2.4) and $N_k \approx k^{d-1}$ can be found in [2] where conditions on the relative location of $x_{k,i}$ and on the size of λ_i are given.

We observe that $\rho G_k = \{\rho x_{k,i} : x_{k,i} \in G_k\}$ also satisfies the cubature formula (2.4) as $\varphi(\rho x) \in \text{span}\left\{\bigcup_{\ell=0}^{2k} H_\ell\right\}$ if $\varphi(x) \in \text{span}\left(\bigcup_{\ell=0}^{2k} H_\ell\right)$, and also $\int_{S^{d-1}} \varphi(\rho x)\mathrm{d}x = \int_{S^{d-1}} \varphi(x)\mathrm{d}x$.

We define $O_{n,\rho} f$ by

$$O_{n,\rho} f(x) = |S^{d-1}| \sum_{x_{k,i} \in G_k} \lambda_i f(\rho x_{k,i}) W_n(x \cdot \rho x_{k,i}) \tag{2.6}$$

for some k satisfying $\frac{3}{2} n \le k < 2n$. Clearly, given a specific ρ (2.6) is not always defined for all $f \in L_p(S^{d-1})$ even when $1 \le p \le \infty$. In the next section we will show that for every $f \in L_p(S^{d-1})$, with $0 < p \le 1$, $O_{n,\rho} f(x)$ is defined and bounded in $L_p(S^{d-1})$ for almost all ρ i.e. a.e. with respect to the Haar measure of $SO(d)$. Obviously, $O_{n,\rho} f(x)$ (when defined) maps f into the space span $\left\{\bigcup_{\ell=0}^{2n} H_\ell\right\}$. We will further show that for $f \in \text{span}\left\{\bigcup_{\ell=0}^{n} H_\ell\right\}$, $O_{n,\rho} f(x) = f(x)$ for all ρ.

3. Properties of $O_{n,\rho} f(x)$

We first prove an estimate of $O_{n,\rho} f(x)$. In this paper the Haar measure on $SO(d)$ is normalized by $\int_{SO(d)} \mathrm{d}\rho = 1$.

Theorem 3.1. *For* $0 < p \le 1$

$$\left\{\int_{SO(d)} \int_{S^{d-1}} |O_{n,\rho} f(x)|^p \mathrm{d}x\, \mathrm{d}\rho\right\}^{1/p} \le C_p(d) \|f\|_{L_p(S^{d-1})}. \tag{3.1}$$

Proof. For $0 < p \le 1$ we write, using (2.3),

$$\begin{aligned}\int_{SO(d)} \int_{S^{d-1}} |O_{n,\rho} f(x)|^p \mathrm{d}x\, \mathrm{d}\rho &\le \int_{SO(d)} \int_{S^{d-1}} \sum_{x_{k,i} \in G_k} \lambda_i^p |f(\rho x_{k,i})|^p |W_n(x \cdot \rho x_{k,i})|^p \mathrm{d}x\, \mathrm{d}\rho \\ &\le A(p,d) \int_{SO(d)} n^{(p-1)(d-1)} \sum_{x_{k,i} \in G_k} \lambda_i^p |f(\rho x_{k,i})|^p \mathrm{d}\rho \\ &= A(p,d) \|f\|_{L_p(S^{d-1})}^p n^{(p-1)(d-1)} \sum_{x_{k,i} \in G_k} \lambda_i^p.\end{aligned}$$

Using the Hölder inequality and recalling $\frac{3}{2} n \le k \le 2n$, we have

$$\sum_{i=1}^{|G_k|} \lambda_i^p \le \left(\sum_{i=1}^{|G_k|} \lambda_i\right)^p \left(\sum_{i=1}^{|G_k|} 1\right)^{1-p} = |G_k|^{1-p} \approx (n^{d-1})^{1-p},$$

which completes the proof of (3.1) for $0 < p \le 1$. □

Therefore, for any $f \in L_p(S^{d-1})$ and any n there exists ρ (depending on f and n) such that

$$\|O_{n,\rho} f\|_{L_p(S^{d-1})} \le C_p(d) \|f\|_{L_p(S^{d-1})}. \tag{3.2}$$

Furthermore, for any $A > 1$ and $0 < p \le 1$

$$m\left(\rho : \|O_{n,\rho} f\|_{L_p(S^{d-1})}^p < A C_p(d)^p \|f\|_{L_p(S^{d-1})}^p\right) \ge 1 - \frac{1}{A}. \tag{3.3}$$

Clearly, when $f \in C(S^{d-1})$, $O_{n,\rho}f(x)$ is defined for all ρ. We now show that $O_{n,\rho}$ is the identity on $\varphi \in \operatorname{span}\left(\bigcup_{\ell=0}^{n} H_\ell\right)$.

Theorem 3.2. *For* $\varphi \in \operatorname{span}\left(\bigcup_{\ell=0}^{n} H_\ell\right)$

$$O_{n,\rho}\varphi(x) = \varphi(x). \tag{3.4}$$

Proof. As $O_{n,\rho}\varphi(x) \in \operatorname{span}\left(\bigcup_{\ell=0}^{2n} H_\ell\right)$ for any $\varphi \in C(S^{d-1})$, we have only to show for $Y_{\ell,j}$ with $0 \le \ell \le n$ that for $m \le 2n$

$$\int_{S^{d-1}} \left(O_{n,\rho}Y_{\ell,j}(x)\right) Y_{m,j_1}(x)\mathrm{d}x = \begin{cases} 1 & \ell = m,\ j = j_1 \\ 0 & \text{otherwise.} \end{cases}$$

In the above we assume that for each H_r $\{Y_{r,i}\}$ is a fixed orthonormal basis.

We note that $Y_{\ell,j_1}(x)Y_{m,j_2}(x) \in \operatorname{span}\left(\bigcup_{r=0}^{\ell+m} H_r\right)$ following the fact that it is a homogeneous polynomial of degree $m + \ell$ and using [9, Th. 2.1, p. 139].

When $\ell \ne m$ $(m \le 2n)$ or $\ell = m$ and $j \ne j_1$

$$\begin{aligned} \int_{S^{d-1}} \left(O_{n,\rho}Y_{\ell,j}(x)\right) Y_{m,j_1}(x)\mathrm{d}x &= |S^{d-1}|\eta\left(\frac{m}{n}\right) \sum_{i=1}^{|G_k|} \lambda_i Y_{\ell,j}(\rho x_{k,i}) Y_{m,j_1}(\rho x_{k,i}) \\ &= \eta\left(\frac{m}{n}\right) \int_{S^{d-1}} Y_{\ell,j}(y) Y_{m,j_1}(y)\mathrm{d}y = 0. \end{aligned}$$

When $\ell = m$ and $j = j_1$, then $\eta\left(\frac{m}{n}\right) = 1$ and

$$\begin{aligned} \int_{S^{d-1}} \left(O_{n,\rho}Y_{\ell,j}(x)\right) Y_{\ell,j}(x)\mathrm{d}x &= |S^{d-1}| \sum_{i=1}^{k} \lambda_i Y_{\ell,j}(\rho x_{k,i}) Y_{\ell,j}(\rho x_{k,i}) \\ &= \int_{S^{d-1}} Y_{\ell,j}(y) Y_{\ell,j}(y)\mathrm{d}y = 1. \quad \square \end{aligned}$$

If $f \in L_p(S^{d-1})$ and for some pair (n^*, ρ^*), $O_{n^*,\rho^*}f$ is defined and belongs to $L_p(S^{d-1})$, then for $\varphi \in C(S^{d-1})$ $O_{n^*,\rho^*}(f + \varphi)$ is defined, and as $O_{n,\rho}\varphi(x)$ is continuous in ρ and hence $O_{n,\rho}\varphi(\cdot) \in L_p(S^{d-1})$ (for any n and ρ), $O_{n^*,\rho^*}(f + \varphi) \in L_p(S^{d-1})$. In particular, the above holds if $\varphi \in \operatorname{span}\left(\bigcup_{\ell=0}^{n-1} H_\ell\right)$.

The spherical harmonic polynomial $\varphi_n \in \operatorname{span}\left(\bigcup_{\ell=0}^{n-1} H_\ell\right)$ is the best approximant from span $\left(\bigcup_{\ell=0}^{n-1} H_\ell\right)$ to f in $L_p(S^{d-1})$ if

$$\|f - \varphi_n\|_{L_p(S^{d-1})} = E_n(f)_{L_p(S^{d-1})} \equiv \inf\left\{\|f - \varphi\|_{L_p(S^{d-1})} : \varphi \in \operatorname{span}\left(\bigcup_{k=0}^{n-1} H_k\right)\right\}. \tag{3.5}$$

From the above we can derive:

Corollary 3.3. *For $f \in L_p(S^{d-1})$ and φ_n given by (3.5)*

$$m\left(\rho : \|O_{n,\rho}(f-\varphi_n)(\cdot)\|^p_{L_p(S^{d-1})} \le AC_p(d)^p\|f-\varphi_n\|^p_{L_p(S^{d-1})}\right)$$
$$\equiv m\left(E(f,n,p,A)\right) \ge 1-\frac{1}{A}, \tag{3.6}$$

and for $\rho \in E(f,n,p,A)$

$$\|O_{n,\rho}f\|^p_{L_p(S^{d-1})} \le \left(AC_p(d)^p+2\right)\|f\|^p_{L_p(S^{d-1})}. \tag{3.7}$$

Proof. As $f-\varphi_n \in L_p(S^{d-1})$, (3.6) is a corollary of (3.2). Furthermore, using Theorem 3.2,

$$O_{n,\rho}f(x) = O_{n,\rho}(f-\varphi_n)(x) + O_{n,\rho}\varphi_n(x) = O_{n,\rho}(f-\varphi_n)(x)+\varphi_n(x),$$

and hence for $\rho \in E(f,n,p,A)$

$$\begin{aligned}\|O_{n,\rho}f(\cdot)\|^p_{L_p(S^{d-1})} &\le AC_p(d)^p\|f-\varphi_n\|^p_{L_p(S^{d-1})} + \|\varphi_n\|^p_{L_p(S^{d-1})}\\ &\le AC_p(d)^p\|f\|^p_{L_p(S^{d-1})} + 2\|f\|^p_{L_p(S^{d-1})}. \quad \square\end{aligned}$$

4. Jackson-type result for $r = 1$

In this section we establish the Jackson-type inequality for $d \ge 3$, $0 < p < 1$ and $r = 1$.

Theorem 4.1. *For $f \in L_p(S^{d-1})$ with $d \ge 3$ we have*

$$E_n(f)_p \le C\omega\left(f, \frac{1}{n}\right)_p, \quad p > 0 \tag{4.1}$$

where $E_n(f)_p$ and $\omega(f,t)_p$ are given by (1.6) and (1.3) respectively.

Proof. As a result stronger than (4.1) is given for $1 \le p \le \infty$ in [5], we have to prove our theorem only for $0 < p < 1$. We note that $E_{2n}(f)_p \le \|f - O_{n,\rho}f\|_p$, and as $\omega\left(f,\frac{1}{n}\right)_p \le 2^{1/p}\omega\left(f,\frac{1}{2n}\right)_p$ (using the consideration in [4, 192–193]), it is sufficient to prove that for some ρ and for any $n \ge 1$ the inequality $\|f - O_{n,\rho}f\|_p \le C\omega\left(f,\frac{1}{n}\right)_p$ to obtain (4.1) for $n > 1$. Therefore, it is enough to show that

$$\int_{SO(d)}\int_{S^{d-1}} |f(x) - O_{n,\rho}f(x)|^p \mathrm{d}x\,\mathrm{d}\rho \le C\omega\left(f,\frac{1}{n}\right)^p_p \tag{4.2}$$

for some positive C to establish (4.1) for $n > 1$.

We now write

$$\begin{aligned}&\int_{SO(d)}\int_{S^{d-1}} |f(x) - O_{n,\rho}f(x)|^p \mathrm{d}x\,\mathrm{d}\rho\\ &\quad\le |S^{d-1}|^p\int_{SO(d)}\int_{S^{d-1}}\sum_{i=1}^{|G_k|}\lambda_i^p|f(x)-f(\rho x_{k,i})|^p|W_n(x\cdot\rho x_{k,i})|^p\mathrm{d}x\,\mathrm{d}\rho\\ &\quad= |S^{d-1}|^p\sum_{i=1}^{|G_k|}\lambda_i^p\int_{SO(d)}\int_{S^{d-1}}|f(x)-f(\rho x_{k,i})|^p|W_n(x\cdot\rho x_{k,i})|^p\mathrm{d}x\,\mathrm{d}\rho.\end{aligned}$$

Clearly,

$$\int_{SO(d)} \int_{S^{d-1}} |f(x) - f(\rho x_{k,i})|^p |W_n(x \cdot \rho x_{k,i})|^p \mathrm{d}x\, \mathrm{d}\rho$$
$$= \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \int_{S^{d-1}} |f(x) - f(y)|^p |W_n(x \cdot y)|^p \mathrm{d}x\, \mathrm{d}y.$$

Therefore, following the arguments in the proof of Theorem 3.1, we need only show that

$$I = \int_{S^{d-1}} \int_{S^{d-1}} |f(x) - f(y)|^p |W_n(x \cdot y)|^p \mathrm{d}x\, \mathrm{d}y$$
$$\le C_1 n^{(p-1)(d-1)} \omega\left(f, \frac{1}{n}\right)_p^p. \tag{4.3}$$

We split the double integral of (4.3) into two parts, I_1 and I_2 ($I = I_1 + I_2$), dealing with the regions $x \cdot y \ge \frac{1}{\sqrt{2}}$ and $x \cdot y < \frac{1}{\sqrt{2}}$ respectively, and show that both yield the estimate required for I.

For I_2 we have $|W_n(x \cdot y)| \le J(\ell) \frac{n^{d-1}}{(1+\frac{n\pi}{4})^\ell} \approx n^{d-1-\ell}$, and hence

$$I_2 \le C_2 n^{(d-1-\ell)p} \int_{S^{d-1}} \int_{S^{d-1}} |f(x) - f(y)|^p \mathrm{d}x\, \mathrm{d}y$$
$$= C_2 |S^{d-1}| n^{(d-1-\ell)p} \int_{SO(d)} \int_{S^{d-1}} |f(x) - f(Qx)|^p \mathrm{d}x\, \mathrm{d}Q$$
$$\le C_2 |S^{d-1}| n^{(d-1-\ell)p} \omega(f, \pi)_p^p$$
$$\le C_3 n^{(d-1-\ell)p} n \omega\left(f, \frac{1}{n}\right)_p^p,$$

which for ℓ sufficiently large, will yield the appropriate estimate. To evaluate I_1 we write

$$I_1 = \int_{S^{d-1}} \int_0^{\pi/4} \int_{x \cdot y = \cos\theta} |f(x) - f(y)|^p |W_n(\cos\theta)|^p \mathrm{d}\gamma(x)\, \mathrm{d}\theta\, \mathrm{d}y$$

where $\mathrm{d}\gamma$ is the Lebesgue measure on the set $\{x : x \cdot y = \cos\theta\}$. For $g \in L_1(S^{d-1})$ we define

$$S_\theta\,(g(\cdot), y) \equiv \frac{1}{m_\theta} \int_{x \cdot y = \cos\theta} g(x) \mathrm{d}\gamma(x), \quad S_\theta(1, y) = 1$$

and as $|f(x) - f(y)|^p$ belongs to $L_1(S^{d-1})$ for almost all y,

$$I_1 = \int_{S^{d-1}} \int_0^{\pi/4} m_\theta S_\theta \left(|f(\cdot) - f(y)|^p, y\right) |W_n(\cos\theta)|^p\, \mathrm{d}\theta\, \mathrm{d}y$$
$$\le C \int_{S^{d-1}} \int_0^{\pi/4} S_\theta \left(|f(\cdot) - f(y)|^p, y\right) \theta^{d-2} \frac{n^{(d-1)p}}{(1+n\theta)^{\ell p}}\, \mathrm{d}\theta\, \mathrm{d}y. \tag{4.4}$$

At this point the proofs for even and odd dimensions d diverge.

For even d we use the transformation $\tau_\theta = Q^{-1} M_\theta Q$ where

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta & & & \bigcirc \\ -\sin\theta & \cos\theta & & & \\ & & \ddots & & \\ & \bigcirc & & \cos\theta & \sin\theta \\ & & & -\sin\theta & \cos\theta \end{pmatrix}$$

and recall (see [3, 174–175]) that

$$S_\theta\left(|f(\cdot) - f(y)|^p, y\right) = \int_{SO(d)} |f(Q^{-1} M_\theta Q y) - f(y)|^p \mathrm{d}Q.$$

We may now use the Fubini theorem on (4.4) and write

$$\begin{aligned} I_1 &\le C \int_0^{\pi/4} \left\{ \int_{SO(d)} \int_{S^{d-1}} |f(Q^{-1} M_\theta Q y) - f(y)|^p \mathrm{d}y\, \mathrm{d}Q \right\} \frac{\theta^{d-2} n^{(d-1)p}}{(1+n\theta)^{\ell p}}\, \mathrm{d}\theta \\ &\le C \int_0^{\pi/4} \omega(f,\theta)_p^p \frac{\theta^{d-2} n^{(d-1)p}}{(1+n\theta)^{\ell p}}\, \mathrm{d}\theta. \end{aligned}$$

Following [4, Th.2.3], we have for $0 < p < 1$

$$\omega(f,\theta)_p^p \le (1+n\theta)\omega\left(f, \frac{1}{n}\right)_p^p,$$

and hence for ℓ large enough

$$I_1 \le C_1 n^{(d-1)(p-1)} \omega\left(f, \frac{1}{n}\right)_p^p.$$

For odd d we use the transformation $\tau^*_\theta = Q^{-1} M^*_\theta Q$ where

$$M^*_\theta = \begin{pmatrix} \cos\theta & \sin\theta & & & \bigcirc & \\ -\sin\theta & \cos\theta & & & & \\ & & \ddots & & & \\ & \bigcirc & & \cos\theta & \sin\theta & \\ & & & -\sin\theta & \cos\theta & \\ & & & & & 1 \end{pmatrix}$$

and define (see [3, (4.3)])

$$A_\theta\left(|f(\cdot) - f(y)|^p, y\right) \equiv \int_{SO(d)} |f(Q^{-1} M^*_\theta Q y) - f(y)|^p \mathrm{d}Q.$$

As $Q^{-1} M^*_\theta Q y \cdot y = M^*_\theta Q y \cdot Q y = M^*_\theta z \cdot z \ge \cos\theta$, we have

$$\begin{aligned} \int_{S^{d-1}} A_\theta\left(|f(\cdot) - f(y)|^p, y\right) \mathrm{d}y &= \int_{SO(d)} \int_{S^{d-1}} |f(Q^{-1} M^*_\theta Q y) - f(y)|^p \mathrm{d}y\, \mathrm{d}Q \\ &\le \omega(f,\theta)_p^p. \end{aligned} \tag{4.5}$$

Using [3, Lemma 4.2], we have

$$\frac{1}{t}\int_{t/\sqrt{2}}^{t} S_\theta\left(|f(\cdot)-f(y)|^p, y\right)\theta^{d-2}(1+n\theta)^{-\ell p}\mathrm{d}\theta$$
$$\le C(\ell,p)t^{d-2}(1+nt)^{-\ell p}A_t\left(|f(\cdot)-f(y)|^p, y\right) \tag{4.6}$$

with $C(\ell, p)$ independent of t, n, y and f. We now use (4.5) and (4.6) to estimate I_1 (for odd d). Recalling $\frac{\pi}{4} < 1$, we write

$$\begin{aligned}
I_1 &\le C_1 n^{p(d-1)}\sum_{j=0}^{\infty}2^{-j(d-1)/2}(1+n2^{-j/2})^{-\ell p}\\
&\quad\times\int_{S^{d-1}}2^{j/2}\int_{2^{-(j+1)/2}}^{2^{-j/2}}S_\theta\left(|f(\cdot)-f(y)|^p, y\right)\mathrm{d}\theta\,\mathrm{d}y\\
&\le Cn^{p(d-1)}\sum_{j=0}^{\infty}2^{-j(d-1)/2}(1+n2^{-j/2})^{-\ell p}\int_{S^{d-1}}A_{2^{-j/2}}\left(|f(\cdot)-f(y)|^p, y\right)\mathrm{d}y\\
&= C_2 n^{p(d-1)}\sum_{j=0}^{\infty}2^{-j(d-1)/2}(1+n2^{-j/2})^{-\ell p}\omega(f,2^{-j/2})_p^p\\
&\equiv\left\{\sum_{j=0}^{j_0-1}+\sum_{j=j_0}^{\infty}\cdots\right\}\equiv J_1+J_2.
\end{aligned}$$

Selecting j_0 so that $2^{-j_0/2}\le\frac{1}{n}<2^{-(j_0-1)/2}$, we estimate J_2 by

$$\begin{aligned}
J_2 &\le C_2 n^{p(d-1)}\omega(f,2^{-j_0/2})_p^p\sum_{j=j_0}^{\infty}2^{-j(d-1)/2}\\
&\le C_3 n^{(p-1)(d-1)}\omega\left(f,\frac{1}{n}\right)_p^p.
\end{aligned}$$

To estimate J_1 we use [4, Theorem 2.3,(2.4)] and write

$$\begin{aligned}
J_1 &\le C_4 n^{p(d-1)}\sum_{j=0}^{j_0-1}2^{-j(d-1)/2}\left(1+2^{(j_0-j-1)/2}\right)^{-\ell p}\left(1+2^{(j_0-j)/2}\right)\omega\left(f,2^{-j_0/2}\right)_p^p\\
&\le C_5 n^{p(d-1)}2^{-j_0(d-1)/2}\omega\left(f,\frac{1}{n}\right)_p^p\sum_{j=0}^{j_0-1}2^{-(j-j_0)(d-\ell p)/2}\\
&\le C_6 n^{(p-1)(d-1)}\omega\left(f,\frac{1}{n}\right)_p^p
\end{aligned}$$

provided that ℓ is large enough.

We proved (4.1) for $n \ge n_0$ for some fixed n_0. To prove (4.1) for $n < n_0$ it is sufficient to show that there exists a constant c such that $\|f-c\|_p \le C_1\omega(f,1)_p$, since then

$$E_n(f)_p \le E_1(f)_p \le \|f-c\|_p \le C_1\omega(f,1)_p,$$

and using [4, (2.4)],

$$\omega(f,1)_p \le C(n_0,p)\omega\left(f,\frac{1}{n}\right)_p \quad \text{for} \quad 1\le n<n_0.$$

We now observe that there exists y_0 satisfying

$$\begin{aligned}\int_{S^{d-1}} |f(y_0) - f(x)|^p \mathrm{d}x &\le \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \int_{S^{d-1}} |f(y) - f(x)|^p \mathrm{d}x\, \mathrm{d}y \\ &= \int_{SO(d)} \int_{S^{d-1}} |f(\rho x) - f(x)|^p \mathrm{d}x\, \mathrm{d}y \\ &\le \omega(f, \pi)_p^p \le C_2 \omega(f, 1)_p^p,\end{aligned}$$

and choose $f(y_0) = c$ to complete the proof. □

5. Conclusion

In the effort to prove for $E_n(f)_{L_p(S^{d-1})}$ of (1.6) (and (3.5)) and $\omega^r(f, t)_{L_p(S^{d-1})}$ of [4] that

$$E_n(f)_{L_p(S^{d-1})} \le C\omega^r\left(f, \frac{1}{n}\right)_{L_p(S^{d-1})} \tag{$*$}$$

for all $f \in L_p(S^{d-1})$, all $d \ge 3$, all integers r and $0 < p < 1$, this paper has made a small contribution (i.e. $(*)$ for $r = 1$). A prize of 100$ CAD will be given by the second author to the person (or group) who is the first to prove $(*)$ for all $d \ge 3$, all integers r and $p > 0$.

Partial results of the above conjecture will receive a nod of approval and a virtual pat on the back.

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