JOURNAL OF ALGEBRA 131, 40–73 (1990)

On the Vanishing of Group Cohomology

D. J. BENSON

Mathematical Institute,
24–29 St. Giles, Oxford OX1 3LB, Great Britain

J. F. CARLSON*

Department of Mathematics, University of Georgia,
Athens, Georgia 30602

AND

G. R. ROBINSON

Department of Mathematics,
University of Manchester, Institute of Science and Technology,
P.O. Box 88, Manchester M60 1QD, Great Britain

Communicated by Leonard Scott

Received November 27, 1988

DEDICATED TO WALTER FEIT ON THE OCCASION OF HIS 60TH BIRTHDAY

1. INTRODUCTION

One of the fundamental questions in the cohomology of finite groups is the following. Given a module M for a finite group G, for what values of n do we have $H^n(G, M) = 0$? This paper consists of two related contributions to the study of this question.

Our first theorem is best stated in terms of Tate cohomology.

THEOREM 1.1. *If R is a commutative ring of coefficients, G a finite group, and M an RG-module, and if $\hat{H}^n(G, M) \neq 0$ for some n, then $\hat{H}^n(G, M) \neq 0$ for infinitely many values of n, positive and negative.*

Secondly, we address the question of which groups have the property that every non-projective module in the principal block has cohomology. We provide some evidence for the following conjecture.

* Partially supported by a grant from the NSF.

0021-8693/90 $3.00

Conjecture 1.2. Suppose G is a finite group and k is a field of characteristic p. Then the following are equivalent.

(i) The centaliser of every element of order p in G is p-nilpotent (i.e., has a normal p-complement).

(ii) For every non-projective module M in the principal block $B_0(kG)$, $H^n(G, M) \neq 0$ for some (and hence for infinitely many) n.

We are able to prove this conjecture as long as p is odd, and more generally we can prove that condition (ii) implies condition (i) (Proposition 5.3 and Theorem 9.5):

THEOREM 1.3. *If p is odd then Conjecture* 1.2 *is true.*

We are also able to prove that Condition (i) is equivalent to Condition (ii) for non-projective trivial source modules M in the principal block, and that it is equivalent to Condition (ii) with M replaced by its Green correspondent. More precisely, we shall prove the following.

THEOREM 1.4. *Suppose G is finite group and k is a field of characteristic p. We have the implications* (A) $\Leftrightarrow$ (A′) $\Leftrightarrow$ (A″) $\Rightarrow$ (B) $\Leftrightarrow$ (B′) $\Rightarrow$ (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (D′) $\Leftrightarrow$ (E) $\Rightarrow$ (F) *among the following statements.*

(A) *Every finitely generated module in $B_0(kG)$ is a trivial homology module* (*or TH module*; *for the definition, see Section* 2).

(A′) *Every simple module in $B_0(kG)$ is a direct summand of a TH module.*

(A″) *Every trivial source module in $B_0(kG)$ is a direct summand of a TH module.*

(B) *For every finitely generated non-projective module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some n.*

(B′) *For every finitely generated non-projective periodic module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some n.*

(C) *For every non-projective trivial source module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some n.*

(D) *The centraliser of every element of order p in G is p-nilpotent.*

(D′) *The centraliser of every non-trivial p-subgroup of G is p-nilpotent.*

(E) *For every non-projective indecomposable module M in $B_0(kG)$, with vertex D and Green correspondent $f(M)$, we have $H^n(N_G(D), f(M)) \neq 0$ for some n.*

(F) *The Cartan matrix of $B_0(kG)$ has only one non-unit principal* *divisor. In other words, the Brauer characters of the projective modules and the trivial module span all Brauer characters in the principal block.*

If p is odd then (D) $\Rightarrow$ (B).

We also give some evidence for the implication (F) $\Rightarrow$ (A'). Namely, we show that for groups with dihedral Sylow subgroups in characteristic 2, as well as for A_6 in characteristic 3 and $PSL_2(8)$ in characteristic 2, every simple module in the principal block is a TH module. To prove this more generally, a new idea seems to be necessary.

In Section 10 we give an interpretation of the above conjecture in terms of varieties. Namely, we define two subsets of the cohomology variety V_G which we call the nucleus Y_G and the representation theoretic nucleus Θ_G, and we conjecture that these are equal. We prove the containment $\Theta_G \supseteq Y_G$.

We shall prove that for any module M in the principal block, the variety of M is contained in the union of Θ_G and the variety defined by the annihilator of $H^*(G, M)$.

Finally, we formulate a conjecture for arbitrary finite groups which generalises the conjectured implication (F) $\Rightarrow$ (A') discussed above. This is given in terms of nuclear homology modules, which are a generalisation of trivial homology modules. For the sake of a non-trivial example, we prove this conjecture for the Mathieu group M_{11} in characteristic 2.

This work grew out of an attempt to study a well known question in group cohomology. If S is a simple module in the principal block $B_0(kG)$, is it true that for some value of $n > 0$ we have $H^n(G, S) \neq 0$? At present no example is known where this fails. For other contributions to the study of this question, see Linnell [16] and Linnell and Stammbach [17, 18].

2. Trivial Homology Modules

We begin with the definition. Let R be a commutative ring of coefficients and G a finite group.

Definition 2.1. An RG-module M is said to be a *trivial homology module* or *TH module* if there exists a finite complex

$$C_r \xrightarrow{\theta_r} C_{r-1} \longrightarrow \cdots \xrightarrow{\theta_2} C_1 \xrightarrow{\theta_1} C_0$$

of RG-modules and homomorphisms such that each C_i is a projective RG-module, $H_i(C_*)$ is a direct sum of copies of the trivial RG-module R for $i > 0$, and $H_0(C_*) \cong M$.

The following theorem was proved in [1, Lemma 6.2].

THEOREM 2.2. *The trivial RG-module R is a TH module. Furthermore, given any positive integer B, there is a complex* (C_*, θ) *as in the definition, with* $H_0(C_*) \cong H_r(C_*) \cong R$ *and* $H_i(C_*) = 0$ *for* $1 \leqslant i \leqslant B$.

Actually, in [1] it is shown that such a complex exists only for $R = \mathbf{Z}$, the rational integers. However, if C_* is such a complex of $\mathbf{Z}G$-modules then $R \otimes_{\mathbf{Z}} C_*$ is a complex of RG-modules with the desired properties.

THEOREM 2.3. *Let M be an R-projective RG-module. Let* (C_*, θ) *be a finite complex of projective RG-modules. Then there exists a cohomology spectral sequence whose* E_2 *term is the Tate cohomology*

$$E_2^{st} = \widehat{\operatorname{Ext}}{}^s_{RG}(H_t(C_*), M)$$

and which converges to zero.

Proof. Let (F_*, ∂) be a doubly infinite projective resolution

$$(F_*, \partial): \quad \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \to \cdots$$

of the trivial RG-module R. That is, the sequence is exact, each F_i is projective, and $\partial_0(F_0) \cong R \cong \operatorname{Ker}(\partial_{-1})$. We wish to investigate the cohomology of the total complex of the double complex $\operatorname{Hom}_{RG}(F_* \otimes_R C_*, M)$. We have two coboundary maps on this complex, namely $(1 \otimes \theta)^*$ and $(\partial \otimes 1)^*$, arising from the boundaries θ on C and ∂ on F, respectively. Consequently we get two spectral sequences converging to the cohomology of the total complex, according to which of the two coboundary maps we use first. Now note that the complex $(F_* \otimes C_t, \partial \otimes 1)$ is a doubly infinite projective resolution of the projective module C_t. Hence it is totally split and its cohomology with any coefficients is zero. Therefore the cohomology of the total complex is also zero.

On the other hand, $(F_s \otimes C_*, 1 \otimes \theta)$ is also a totally split complex since F_s is a projective module and C_t is R-projective. However, this one has cohomology

$$H_t(F_s \otimes C_*, 1 \otimes \theta) = F_s \otimes H_t(C_*, \theta).$$

The splitting shows that the E_1 term of the spectral sequence is

$$E_1^{st} \cong \operatorname{Hom}_{RG}(F_s \otimes H_t(C_*), M).$$

To get the E_2 term we take homology with respect to the map induced by ∂ on F_*. But $(F_* \otimes H_t(C_*), \partial \otimes 1)$ is a projective resolution of $H_t(C_*)$. So E_2^{st} is as given. ∎

Remark. The construction given above is in some sense an algebraic analogue of the equivariant cohomology spectral sequence

$$H^s(BG; H^t(X; R)) \Rightarrow H_G^{s+t}(X; R)$$

for a finite G-CW-complex X, in the case where the action is free, so that $H_G^*(X; R) \cong H^*(X/G; R)$. Use of Tate cohomology gets rid of the finite E_∞ term.

We are now ready to prove Theorem 1.1. We state it again here in a slightly stronger form.

THEOREM 2.4. *Given a finite group G, there exists a positive integer r such that for any commutative ring R of coefficients and any RG-module M, if $\hat{H}^n(G, M)=0$ for $r+1$ consecutive values of n then $\hat{H}^n(G, M)=0$ for all n positive and negative.*

Proof. We first prove this in the case where M is R-projective. Let

$$(C_*, \theta): C_r \to \cdots \to C_1 \to C_0$$

be a complex of projective RG-modules as in Theorem 2.2 displaying R as a TH module, and consider the spectral sequence described in Theorem 2.3. Suppose that $\hat{H}^n(G, M)$ is not always zero, but is zero for $r+1$ consecutive values of n, say m, $m+1, ..., m+r$, with either $\hat{H}^{m-1}(G, M)$ or $\hat{H}^{m+r+1}(G, M)$ non-zero. Now the top and bottom rows of this spectral sequence are

$$E_2^{sr} \cong \mathrm{Ext}^s_{RG}(H_r(C_*), M) \cong \hat{H}^s(G, M)$$
$$E_2^{s0} \cong \widehat{\mathrm{Ext}}^s_{RG}(H_0(C_*), M) \cong \hat{H}^s(G, M),$$

and so either $E_2^{m-1,r} \neq 0$ or $E_2^{m+r+1,0} \neq 0$. However, $E_2^{st}=0$ for $m \leqslant s \leqslant m+r$. The differential d_k on the E_k term takes E_k^{st} to $E_k^{s+k,t-k+1}$, and $d_k=0$ for $k>r+1$. So whether $E_2^{m-1,r}$ or $E_2^{m+r+1,0}$ is non-zero, this group can never be killed at any stage in the spectral sequence. This contradicts the fact that the spectral sequence converges to zero.

We now deal with the general case where M is not necessarily R-projective. In this case, we first remark that since $\hat{H}^n(G, M)$ does not change when the coefficient ring R is replaced by $\mathbf{Z}$, we may as well assume R is $\mathbf{Z}$. Note at this stage that we have not assumed that M is finitely generated. Now if

$$0 \to M' \to F \to M \to 0$$

is a short sequence with F a free $\mathbf{Z}G$-module, then M' is $\mathbf{Z}$-free [12, Theorem I.5.1]. Since $\hat{H}^n(G, F)=0$ (see, for example, [3, VI.5.3]),

$\hat{H}^n(G, M) \cong \hat{H}^{n+1}(G, M')$ and so the theorem for M follows from the theorem for M'. ∎

Remark. The proof of the above theorem gives explicit values of r in terms of the generators of the cohomology ring (via the proof of Theorem 2.1 in [1]). Better constants r may be found by refining the argument given.

COROLLARY 2.5. *$H^*(G, M) \neq 0$ if and only if $\hat{H}^*(G, M) \neq 0$.* ∎

COROLLARY 2.6. *Suppose R is a Noetherian ring and M is a finitely generated RG-module with the property that some power of every element of positive degree in $H^*(G, R)$ annihilates $H^*(G, M)$ (by cup product). Then $H^*(G, M) = 0$.*

Proof. By Evens [8, Theorem 6.1, Corollary 6.2], $H^*(G, M)$ is finitely generated as a module over $H^*(G, R)$, and the latter is a finitely generated ring over R. So if some power of every element of positive degree in $H^*(G, R)$ annihilates $H^*(G, M)$, then $H^*(G, M)$ is non-zero only in finitely many degrees. Thus by Theorem 2.4, $H^*(G, M) = 0$. ∎

PROPOSITION 2.7. *Suppose M_1 is a TH RG-module and M_2 is any RG-module. Then either* $\operatorname{Ext}^n_{RG}(M_1, M_2) = 0$ *for all n or $\hat{H}^n(G, M_2) \neq 0$ for infinitely many n, positive and negative.*

Proof. As in the proof of Theorem 2.4, we may reduce to the case where M_2 is R-projective. In this case, let

$$(C_*, \theta): C_r \to \cdots \to C_1 \to C_0$$

be a complex of projective RG-modules displaying M_1 as a TH module, and consider the spectral sequence

$$\widehat{\operatorname{Ext}}^s_{RG}(H_t(C_*), M_2) \Rightarrow 0$$

described in Theorem 2.3. If $\hat{H}^n(G, M_2) = 0$ for all n, then the only possible non-zero row of this spectral sequence is

$$E_2^{s0} = \widehat{\operatorname{Ext}}^s_{RG}(M_1, M_2).$$

Since the spectral sequence converges to zero, this row too must be zero. Thus if $\widehat{\operatorname{Ext}}^n_{RG}(M_1, M_2) \neq 0$ for some n, then $\hat{H}^n(G, M_2) \neq 0$ for some n and hence, by Theorem 2.4, for infinitely many n, positive and negative. ∎

COROLLARY 2.8. *Suppose M is a non-projective TH module. Then $\hat{H}^n(G, M) \neq 0$ for infinitely many n, positive and negative.*

Proof. Apply the above proposition with $M_1 = M_2 = M$. By [5], for M non-projective, $\widehat{\mathrm{Ext}}^*_{RG}(M, M) \neq 0$. We can also see this by noting that there is a non-trivial element in $\widehat{\mathrm{Ext}}^0_{RG}(M, M)$ and using Theorem 2.4. ∎

This proves the implication $(A) \Rightarrow (\mathrm{B})$ in Theorem 1.4.

3. Properties of Trivial Homology Modules

The basic constructions here depend on the fact that if

$$(C_*, \theta):\ \cdots \to C_r \to \cdots \to C_0$$

is a complex of projective modules and

$$(D_*, \phi):\ \cdots \to D_r \to \cdots \to D_0$$

is exact then any map $H_0(C) \to H_0(D)$ extends to a chain map $(C, \theta) \to (D, \phi)$. For a proof of this well known fact, see, for example, [12, Theorem IV.4.1].

Lemma 3.1. *Suppose that M is a TH module and B is a positive integer. Then there exists a complex (C_*, θ) as in the definition, with $H_i(C_*) = 0$ for $1 \leqslant i \leqslant B$.*

Proof. We prove this by induction on B. Suppose

$$(C_*, \theta): 0 \to C_r \to \cdots \to C_0 \to 0$$

is such a complex with $H_i(C_*) = 0$ for $1 \leqslant i \leqslant B-1$. Let (D_*, ϕ) be a complex as in Theorem 2.2 with $H_i(D_*) = 0$ for $1 \leqslant i \leqslant r - B$. Then we may form a homomorphism of complexes

$$\begin{array}{ccccccc}
0 \to & C_r & \to \cdots \to & C_{B+1} & \to & C_B & \to \cdots \to C_0 \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to \cdots \to & D_{r-B} \otimes H_B(C_*) & \to \cdots \to & D_1 \otimes H_B(C_*) & \to & D_0 \otimes H_B(C_*) & \to 0
\end{array}$$

By adding a suitable exact complex of projective modules to (C_*, θ), we may convert this into a surjective homomorphism of complexes. By the long exact sequence of homology, the kernel of this is a complex with the desired properties. ∎

Proposition 3.2. *Suppose that $0 \to M_1 \to M_2 \to^{\sigma} M_3 \to 0$ is a short exact sequence of RG-modules and two of the terms are TH modules. Then so is the third.*

Proof. We first treat the case where M_2 and M_3 are TH modules. In this case, let

$$(C_*, \theta): 0 \to C_r \to \cdots \to C_0 \to 0$$

be a complex displaying M_2 as a TH module. Using Lemma 3.1, we choose a complex

$$(D_*, \phi): 0 \to D_s \to \cdots \to D_0 \to 0$$

displaying M_3 as a TH module, and with $H_i(D_*) = 0$ for $1 \leqslant i \leqslant r$. The given homomorphism $M_2 \to M_3$ now extends to a map of complexes

$$\begin{array}{ccccccccccc} & & & & 0 & \longrightarrow & C_r & \longrightarrow \cdots \longrightarrow & C_0 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_s & \longrightarrow \cdots \longrightarrow & & & D_r & \longrightarrow \cdots \longrightarrow & D_0 & \longrightarrow & 0 \end{array}$$

Just as in the proof of Lemma 3.1, we now add an exact sequence of projective modules to (C_*, θ) to make the map surjective, and take the kernel. The long exact sequence of homology shows that this kernel displays M_1 as a TH module.

To treat the remaining cases, we make the observation that if $0 \to M' \to P \to M \to 0$ is a short exact sequence with P projective then M' is a TH module if and only if M is.

If M_1 and M_2 are TH modules, we choose a projective module P and a surjection $\nu: P \to M_3$ with kernel M_3'. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & P \oplus M_1 & \longrightarrow & P \oplus M_2 & \xrightarrow{(\nu, \sigma)} & M_3 & \longrightarrow & 0 \end{array}$$

we see that there is a short exact sequence $0 \to M_3' \to P \oplus M_1 \to M_2 \to 0$. Since $P \oplus M_1$ and M_2 are TH modules, so is M_3', and hence so is M_3.

If on the other hand M_1 and M_2 are TH modules, we choose a projective module P that maps onto M_2 with kernel M_2'. Composing with the surjection $M_2 \to M_3$, we see that P also maps onto M_3, and we write M_3' for the kernel. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

we see that there is a short exact sequence $0 \to M_2' \to M_3' \to M_1 \to 0$. Since M_1 and M_3' are TH modules, so is M_2', and hence so is M_2. ▮

PROPOSITION 3.3. *If M_1 and M_2 are TH kG-modules, where k is a field, then so is $M_1 \otimes_k M_2$.*

Proof. Let

$$(C_*, \theta): 0 \to C_r \to \cdots \to C_0 \to 0$$

and

$$(D_*, \phi): 0 \to D_s \to \cdots \to D_0 \to 0$$

be complexes displaying M_1 and M_2 as TH modules. We augment these complexes to

$$C_*': 0 \to C_r \to \cdots \to C_0 \to M_1 \to 0$$

and

$$D_*': 0 \to D_s \to \cdots \to D_0 \to M_2 \to 0$$

and form the tensor product

$$C_*' \otimes D_*': 0 \to C_r \otimes D_s \to \cdots \to C_0 \otimes M_2 \oplus M_1 \otimes D_0 \to M_1 \otimes M_2 \to 0.$$

By the Künneth theorem this complex has all its homology a sum of trivial modules, and so its truncation at $M_1 \otimes M_2$ is a complex displaying $M_1 \otimes M_2$ as a TH module. ▮

Remark. This works equally well when k is replaced by a hereditary commutative coefficient ring R.

PROPOSITION 3.4. *Suppose M is a TH kG-module, where k is a field. Then so is $M^* = \mathrm{Hom}_k(M, k)$.*

Proof. Let

$$(C_*, \theta): 0 \to C_r \to \cdots \to C_0 \to 0$$

be a complex of projective modules displaying M as a TH module. First we form the dual complex

$$(C_*', \theta'): 0 \to C_0' \to \cdots \to C_{-r}' \to 0$$

with $C_{-r}' = (C_r)^*$. Now we move copies of k in the homology from negative degree to positive degree one at a time as follows. Choose a complex

$$(D_* \phi): 0 \to D_s \to \cdots \to D_0 \to 0$$

giving k as a TH module, with $H_i(D_*)=0$ for $1 \leqslant i \leqslant r$. Then a non-zero map $H_{-r}(C'_*) \to H_0(D_*)$ extends to a map of complexes

$$\begin{array}{ccccccccccc} & & & & 0 & \longrightarrow & C'_0 & \longrightarrow \cdots \longrightarrow & C'_{-r} & \longrightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_s & \longrightarrow \cdots \longrightarrow & & & D_r & \longrightarrow \cdots \longrightarrow & D_0 & \longrightarrow & 0 \end{array}$$

We add an exact sequence of projective modules to (C'_*, θ') to make the map surjective, and take the kernel. The long exact sequence of homology shows that there is one less copy of k in negative degree in the resulting kernel. Continuing in this way we delete copies of k one at a time from negative degree and add copies in positive degree. Eventually we arrive at a complex displaying M^* as a TH module. ∎

THEOREM 3.5. *Let k be a field and G a finite group. Suppose every simple module in $B_0(kG)$ is a TH module. Then every finitely generated module in $B_0(kG)$ is a TH module. Thus every finitely generated non-projective module M in $B_0(kG)$ satisfies $\hat{H}^n(G, M) \neq 0$ for infinitely many n, positive and negative.*

Proof. This follows easily from Corollary 2.8 and Proposition 3.2. ∎

LEMMA 3.6. *Suppose M_1 and $M_2 \oplus M_3$ are TH modules and $\theta: M_1 \to M_2$ is surjective. Then $M_2 \oplus \mathrm{Ker}(\theta)$ is a TH module.*

Proof. This applies by applying Proposition 3.2 to the short exact sequence

$$0 \to M_2 \oplus \mathrm{Ker}(\theta) \to M_1 \oplus M_2 \oplus M_3 \xrightarrow{\left(\begin{smallmatrix}\theta & 0 & 0 \\ 0 & 0 & 1\end{smallmatrix}\right)} M_2 \oplus M_3 \to 0,$$

in which the last two terms are TH modules. ∎

PROPOSITION 3.7. *A module M is a direct summand of a TH module if and only if $M \oplus \Omega(M)$ is a TH module, where $\Omega(M)$ is the kernel of a map $P \to M$ with P projective.*

Proof. Take $M_2 = M$ and $M_1 = P$ in the above lemma. ∎

COROLLARY 3.8. *Suppose M is a non-projective direct summand of a TH module. Then $\hat{H}^n(G, M) \neq 0$ for infinitely many n, positive and negative.*

Proof. By the proposition, $M \oplus \Omega(M)$ is a TH module. But

$$\hat{H}^n(G, M \oplus \Omega(M)) \cong \hat{H}^n(G, M) \oplus \hat{H}^{n-1}(G, M). \quad ∎$$

This proves the implication (A′) ⇒ (B) of Theorem 1.4.

4. Examples in Which All Modules Are TH Modules

We start off by looking at p-groups.

Proposition 4.1. *Let k be a field of characteristic p and G a finite p-group. Then every finitely generated kG-module is a TH module.*

Proof. This follows from Theorems 3.5 and 2.2. ∎

Corollary 4.2. *Let k be a field of characteristic p and G a finite p-nilpotent group. Then every finitely generated module in $B_0(kG)$ is a TH module.* ∎

Theorem 4.3. *Suppose R is the ring of integers in an algebraic number field or its localisation or completion at a prime ideal lying above p, and G is a finite p-group. Then every finitely generated RG-module is a summand of a TH module.*

Proof. First we deal with modules of finite length. For this purpose it is sufficient to deal with a simple module of finite length. Such a module is of the form $R/\wp$ for some prime ideal $\wp$ lying above p. Since $\wp$, regarded as an RG-module, is a summand of a sum of copies of R, it is a summand of a TH module, and hence by Proposition 3.2 so is $R/\wp$.

We now deal with the general finitely generated module. Let F be the field of fractions of R, and first suppose F is a splitting field. Then the left regular representation is $FG \cong \bigoplus_i n_i S_i$, where $n_i = \dim(S_i)$. Thus the left regular representation of RG contains a submodule L of finite index with $L \cong \bigoplus_i n_i L_i$ and $FL_i \cong S_i$. Since RG/L is a torsion module of finite length, and hence a summand of a TH module, Proposition 3.2 shows that L is a summand of a TH module, and hence each L_i is a summand of a TH module. But now any finitely generated RG-module has a submodule isomorphic to a sum of modules of the form L_i, with finite quotient, and so another application of Proposition 3.2 completes this case.

In case F is not a splitting field, we extend to a splitting field, form the required complex there, and then restrict back. The restriction of the trivial module R is a sum of trivial modules, so this proves the theorem in this case. ∎

In the case of a *field* of coefficients, in order that every simple module in the principal block be a TH module, it is certainly necessary that the characters of the trivial module and the projectives in the principal block additively generate all characters in the principal block. This is equivalent to the condition that the Cartan matrix of the principal block have only one principal divisor which is not a unit. We now prove a partial converse to this.

PROPOSITION 4.4. *Suppose that the Cartan matrix of $B_0(kG)$ has only one non-unit principal divisor. If every simple module in $B_0(kG)$ is a summand of a TH module, then every simple module in $B_0(kG)$ is a TH module.*

Proof. Suppose P is a projective module in $B_0(kG)$, and Q is a maximal submodule of P. Applying Lemma 3.6 with P in place of M_1 and P/Q in place of M_2, we see that $Q \oplus P/Q$ is a TH module. If Q' is a maximal submodule of Q, then the same procedure shows that $Q' \oplus Q/Q' \oplus P/Q$ is a TH module. Continuing this way down a composition series for P, we see that a semisimple module with the same composition factors as P is a TH module.

Now by the condition on the Cartan matrix, given any simple module S, there are modules M_1 and M_2, each a sum of projectives in the principal block and copies of the trivial module, such that the Brauer character of M_1 is the same as that of $M_2 \oplus S$. By the above argument, semisimple modules M_1' and M_2' with the same composition factors as M_1 and M_2 are TH modules. The proposition now follows by applying Proposition 3.2 to the short exact sequence $0 \to M_2' \to M_1' \to S \to 0$. ▮

The following proposition, which also proves the implication (D) ⇒ (F) in Theorem 1.4, facilitates checking the conditions of Proposition 4.4.

PROPOSITION 4.5. *Suppose that the centraliser of every element of order p in G is p-nilpotent. Then the Cartan matrix of $B_0(kG)$ has only one non-unit principal divisor.*

Proof. This is well known, but for the convenience of the reader we include a proof.

We shall show that if ϕ is a Brauer character in the principal block, then $\phi - \phi(1) \cdot 1$ is the character of a virtual projective (i.e., an integral combination of characters of projective modules), where 1 denotes the trivial character. For then the $\mathbf{Z}$-module generated by the Brauer characters in $B_0(kG)$ has a basis of the form $1 = \phi_1, \phi_2 - \phi_2(1) \cdot 1, \ldots, \phi_l - \phi_l(1) \cdot 1$, where the ϕ_i are the irreducible Brauer characters in the principal block, and this proves the proposition.

To show that $\phi - \phi(1) \cdot 1$ is the character of a virtual projective, it suffices to show that the class function θ given by $\phi - \phi(1) \cdot 1$ on p-regular elements and zero on p-singular elements is a generalised character. For then, since it vanishes on p-singular elements, it must be a virtual projective.

If y is a p'-element of G, we now show that $\phi(y) \equiv \phi(1) \pmod{|C_G(y)|_p}$. If x is a non-trivial p-element centralising y, then for any irreducible character ψ in $B_0(C_G(x))$ we have $\psi(x) = \psi(xy)$ since $C_G(x)$ is p-nilpotent. Thus by Brauer's second and third main theorems, it follows that if we let

$\tilde{\phi}$ be an ordinary generalised character in B_0 lifting ϕ (the decomposition map is surjective) then $\tilde{\phi}(xy) = \tilde{\phi}(x)$. If Q is a Sylow p-subgroup of $C_G(y)$ and we define ϕ' on Q by $\phi'(x) = \tilde{\phi}(xy)$, then ϕ' is an algebraic integer combination of characters of Q. Thus $(\phi(y) - \phi(1))/|Q| = (\phi' - \tilde{\phi}|_Q, 1)_Q$ is an algebraic integer.

It now follows, for example, from [21, Lemma 2(iii)], that θ is a generalised character of G, so that $\phi - \phi(1) \cdot 1$ is the character of a virtual projective and we are done. ∎

THEOREM 4.6. *Suppose k is a field of characteristic two and G is a finite group with dihedral Sylow two-subgroups. Then every simple module in $B_0(kG)$ is a TH module. Thus every non-projective module M in $B_0(kG)$ satisfies $\hat{H}^n(G, M) \neq 0$ for infinitely many n, positive and negative.*

Proof. We use the structure of the projective indecomposable modules given in Erdmann [7]. According to the classification by Gorenstein and Walter, we have three possibilities for $G/O_{2'}(G)$.

(i) $PSL_2(q) \leqslant G/O_{2'}(G) \leqslant P\Gamma L_2(q)$, q odd.

We first treat the case where $PSL_2(q) \leqslant G/O_{2'}(G)$ but $PGL_2(q) \nleqslant G/O_{2'}(G)$. In this case, if $q \equiv 1 \pmod 4$ then there is a short exact sequence

$$0 \to L \oplus M \to \Omega^2 k \to k \to 0,$$

where L and M are the non-trivial simple modules in the principal block. Thus by Propositions 3.2 and 4.4 every simple module in the principal block is a TH module. If, on the other hand, $q \equiv 3 \pmod 4$, then there is a short exact sequence

$$0 \to L \oplus M \to \Omega^{-1} k \to k \to 0$$

so that again every simple module in the principal block is a TH module.

In case $PGL_2(q) \leqslant G/O_{2'}(G)$, if $q \equiv 1 \pmod 4$ then there is a short exact sequence

$$0 \to M \oplus M' \to \Omega^2 k \to k \to 0,$$

where M is the non-trivial simple module in the principal block and M' is a uniserial module. If $q \equiv 3 \pmod 4$ then there is a short exact sequence

$$0 \to M_1 \oplus M_2 \to \Omega^{-1} k \to k \to 0,$$

where M_1 is an extension of M by k, M_2 is an extension of k by M, and M is the non-trivial simple module in the principal block. Thus, by

Proposition 3.2, $M \oplus M$ is a TH module, and hence, by Proposition 4.4, every simple module in the principal block is a TH module.

(ii) $G/O_{2'}(G) \cong A_7$.

In this case, the two non-trivial simple modules in the principal block are L of dimension 14 and M of dimension 20. There is a short exact sequence

$$0 \to M \oplus L' \to \Omega^2 k \to k \to 0,$$

where L' is an extension of L by itself. Thus M is a summand of a TH module. Now $\Omega(L')$ has only one composition factor not isomorphic to k or M, and this is isomorphic to L. Thus every simple module in the principal block is a TH module.

(iii) $G/O_{2'}(G)$ is isomorphic to a Sylow two subgroup of G. This case is trivial as there is only one simple module in the principal block, namely k. ∎

Our next example is the group A_6 in characteristic three. The modules $\Omega^n(k)$ in this case were computed in [2]. From the information given there, we can see that there is a short exact sequence

$$0 \to k \to \Omega^3(k) \to X \oplus Y \to 0,$$

where X and Y are modules with three composition factors. The unique top composition factor of each is the trivial module, and we denote by $\tilde{X}$ and $\tilde{Y}$ the submodules with two composition factors. Thus $X \oplus Y$ and $\tilde{X} \oplus \tilde{Y}$ are TH modules, by Proposition 3.2. There is then an exact sequence

$$0 \to L \oplus M \to \Omega^{-1}(X \oplus Y) \to \tilde{X}^* \oplus \tilde{Y}^* \to 0,$$

where L and M are the two three dimensional simple modules. Thus by another application of Proposition 3.2, $L \oplus M$ is a TH module. Finally, the short exact sequence

$$0 \to L \oplus M \to \tilde{X} \oplus \tilde{Y} \to N \oplus N \to 0$$

shows that $N \oplus N$ is a TH module. An application of Proposition 4.4 now shows that every simple module in the principal block is a TH module, so that every non-projective module in the principal block has cohomology. Note that this is an example of wild representation type, so that there is no chance of proving this by classifying the indecomposables. This example also shows that the same holds for A_7 in characteristic three, since $e_0 \cdot k_{A_6}\uparrow^{A_7} \cong k_{A_7}$, and every module is a direct summand of a module induced from A_6.

Our final example is one of rank three. We write $2^3:7$ for the split extension of an elementary abelian group of order eight by an automorphism of order seven. This group has seven simple modules over a large enough field of characteristic two, all in the principal block, which we denote by $S_0 = k$, $S_1, ..., S_6$ with the notation chosen so that the second Loewy layer of the projective cover of S_i is $S_{i+1} \oplus S_{i+2} \oplus S_{i+4}$, with the indices taken modulo seven. In this case it turns out (and this is quite a hard calculation, which we shall omit for the sake of brevity) that there are short exact sequences

$$0 \to X \to \Omega^5(k) \to k \oplus k \oplus k \oplus \Omega^{-1}(k) \oplus \Omega^{-1}(k) \oplus \Omega^{-1}(k) \to 0$$

$$0 \to \Omega^{-2}(k) \to X \to Y \to 0$$

in such a way that Y has $S_1 \oplus S_2 \oplus S_4$ as a direct summand. We now dualise and use Proposition 4.4 to deduce that every simple module is a TH module. Since $e_0 \cdot k_{2^3:7}\uparrow^{PSL_2(8)} \cong k_{PSL_2(8)}$, it follows that the same is true for $PSL_2(8)$ in characteristic two.

5. Generating Modules with no Cohomology

We now give a general method for producing non-projective indecomposable modules M in $B_0(kG)$ for which $\hat{H}^n(G, M) = 0$ for all values of n.

LEMMA 5.1. *Let H be any subgroup of G and M_0 be any indecomposable kH-module which is* not *in $B_0(kH)$. Then for any summand M of $M_0\uparrow^G$ and all values of n we have $\hat{H}^n(G, M) = 0$.*

Proof. Let e_0 be the principal block idempotent for kG. By Shapiro's lemma we have

$$\hat{H}^n(G, e_0 \cdot M_0\uparrow^G) = \hat{H}^n(G, M_0\uparrow^G) \cong \hat{H}^n(H, M_0) = 0.$$

Thus $e_0 \cdot M_0\uparrow^G$ is a module with no cohomology, and so M also has no cohomology. ∎

Here are some examples of this phenomenon with M non-projective. We write D^λ for the simple $k\Sigma_n$-module corresponding to the p-regular partition λ, where p is the characteristic of k, as in James [15].

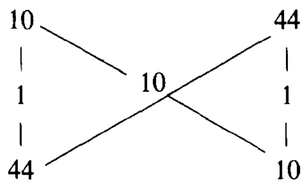
PROPOSITION 5.2. *Let M_0 denote the module $D^{((r-1)p,\, p-1)}$ for Σ_{rp-1}, and let $M = e_0 \cdot M_0\uparrow^{\Sigma_{rp}}$. Then M is a non-projective uniserial module of length 3 with composition factors $D^{((r-1)p,\, p)}$, $D^{(rp-1,\, 1)}$, $D^{((r-1)p,\, p)}$, and with no cohomology.*

Proof. The character of M is easily seen to be as given, be referring to James [15, Theorem 24.15]. Since $D^{(rp-1,\,1)}\downarrow_{\Sigma_{rp-1}}$ does not involve $D^{((r-1)p,\, p-1)}$, the Nakayama relations show that the module M is not a direct sum. Since it is self dual it must be uniserial. It is clearly not projective. ∎

Another example may be obtained by inducing a 10 dimensional simple module from $PSL_2(11)$ to M_{11} in characteristic two. This simple module is in a block of defect one and is periodic of period one. Since $10 \otimes 5 \cong 10 \oplus (\text{projective})$ for $PSL_2(11)$, we have

$$\begin{aligned} 10\uparrow^{M_{11}} \oplus (\text{projective}) &= (10 \otimes 5)\uparrow^{M_{11}} \\ &= ((10_{M_{11}})\downarrow_{PSL_2(11)} \otimes 5)\uparrow^{M_{11}} \\ &= 10 \otimes 5\uparrow^{M_{11}} \\ &= 10 \otimes (16 \oplus 44) \\ &= 10 \otimes 44 \oplus (\text{projective}) \end{aligned}$$

and so this "explains" why $\mathrm{Ext}^*_{M_{11}}(10, 44) = 0$. The diagram for this module $e_0 \cdot 10_{PSL_2(11)}\uparrow^{M_{11}} \cong e_0 \cdot (10 \otimes 44)$ is



Can such an induced module ever have a simple summand in the principal block? Such a simple module would be a counterexample to the well known conjecture that every simple module in the principal block must have cohomology, but it seems hard to construct a counterexample in this way.

The following proposition is a more systematic way of using the above technique.

PROPOSITION 5.3. *Suppose G has a non-trivial p-subgroup P such that $C_G(P)$ is not p-nilpotent. Then there is a non-projective trivial source kG-module M in the principal block with vertex P, such that $\hat{H}^n(G, M) = 0$ for all values of n.*

Proof. Since $C_G(P)$ is not p-nilpotent, we can find a p'-element $y \in C_G(P)$ with $y \notin O_{p'} C_G(P) = O_{p'} N_G(P)$. The intersection of the kernels of the simple modules in $B_0(kN_G(P))$ is $O_{p',p} N_G(P)$, which does not contain y, and so we may choose a simple module S in $B_0(kN_G(P))$ for which y is not in the kernel of the action on S.

Thus $S\downarrow_{\langle y\rangle \times P}$ contains some one dimensional submodule M_0 on which y acts non-trivially, and so M_0 is not in $B_0(\langle y\rangle \times P)$. By the Nakayama relations we have

$$\mathrm{Hom}_{N_G(P)}(M_0\uparrow^{N_G(P)}, S) \cong \mathrm{Hom}_{\langle y\rangle \times P}(M_0, S\downarrow_{\langle y\rangle \times P}) \neq 0$$

and so $M_0\uparrow^{N_G(P)}$ has an indecomposable summand M_1 in the principal block.

Since P is in the kernel of M_0, it is also in the kernel of M_1, which therefore has vertex exactly P. The Green correspondent M of M_1 is then a non-projective summand of $M_0\uparrow^G$ in the principal block. Since M_0 is not in the principal block, we have $\hat{H}^n(G, M) = 0$ for all values of n by Lemma 5.1. ∎

This proposition proves the implications (C) ⇒ (D) and (E) ⇒ (D) in Theorem 1.4.

6. Periodic Modules

In this section, we show that if all periodic modules in the principal block have cohomology, then so do all non-projective modules. We use this to analyse the case of a normal vertex.

We shall need to use the language of varieties for modules, and our terminology will be the same as in [1]. In particular, we denote by V_G the maximal ideal spectrum of the cohomology ring $H^{\cdot}(G, k) = H^*(G, k)$ if $p = 2$ and of the even cohomology ring $H^{\cdot}(G, k) = H^{\mathrm{ev}}(G, k)$ if p is odd. For a module M, $V_G(M)$ is the closed homogeneous subvariety of V_G defined by the annihilator of $\mathrm{Ext}^*_{kG}(M, M)$. If $\zeta \neq 0$ is an element of $H^{2n}(G, k)$ represented by a homomorphism $\hat{\zeta}\colon \Omega^{2n}(k) \to k$, we denote its kernel by L_ζ. The following proposition, which is the same as [1, Proposition 3.1], summarises some of the properties of these varieties. The proofs are a culmination of the work of many people.

Proposition 6.1. *Let M and N be kG-modules.*

(i) $V_G(M) = \{0\}$ *if and only if M is projective.*

(ii) $\dim V_G(M)$ *is equal to the complexity of M.*

(iii) $V_G(M \oplus N) = V_G(M) \cup V_G(N)$.

(iv) $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.

(v) $V_G(\Omega^n(M)) = V_G(M)$ *for all n.*

(vi) *For $\zeta \in H^{2n}(G, k)$, $V_G(L_\zeta) = V_G(\langle\zeta\rangle)$, the hypersurface of V_G determined by ζ.*

(vii) *The dimension of* $V_G = V_G(k)$ *is equal to the p-rank of* G (p *is the characteristic of* k).

LEMMA 6.2. *If* M_1 *is a TH kG-module and* $\hat{H}^n(G, M_2) = 0$ *for all* n, *then* $\hat{H}^n(G, M_1 \otimes M_2) = 0$ *for all* n.

Proof. This follows from Propositions 2.7 and 3.4 since

$$\hat{H}^n(G, M_1 \otimes M_2) \cong \widehat{\mathrm{Ext}}^n_{kG}(M_1^*, M_2). \quad \blacksquare$$

LEMMA 6.3. *Suppose* M_1 *is an indecomposable kG-module and* M_2 *is a TH module. Then every non-projective summand of* $M_1 \otimes M_2$ *is in the same block as* M_1.

Proof. Tensor M_1 with a complex displaying M_2 is a TH module. The part in any block other than that of M_1 is projective and exact except in degree zero, so the homology in degree zero is projective. ∎

THEOREM 6.4. *Suppose that* M *is a module in* $B_0(kG)$ *with variety* $V_G(M) \neq \{0\}$, *and* $H^n(G, M) = 0$ *for all* n. *If* V *is a closed homogeneous subset of* $V_G(M)$, *then there is a module* M' *in* $B_0(kG)$ *with* $\hat{H}^n(G, M') = 0$ *for all* n, *and* $V_G(M') = V$.

Proof. Choose elements of cohomology $\zeta_1, ..., \zeta_s$ with

$$V = V_G(M) \cap V_G(\langle \zeta_1 \rangle) \cap \cdots \cap V_G(\langle \zeta_s \rangle),$$

and let M' be the non-projective part of $M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}$. By Proposition 3.2, each L_{ζ_i} is a TH module, and hence by Proposition 3.3 so is $L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}$, so that by Lemma 6.2, $\hat{H}^n(G, M') = 0$ for all n. We have

$$\begin{aligned} V_G(M') &= V_G(M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}) \\ &= V_G(M) \cap V_G(L_{\zeta_1}) \cap \cdots \cap V_G(L_{\zeta_s}) \\ &= V_G(M) \cap V_G(\langle \zeta_1 \rangle) \cap \cdots \cap V_G(\langle \zeta_s \rangle) \\ &= V. \end{aligned}$$

By Lemma 6.3, M' is in the principal block. ∎

COROLLARY 6.5. *If there is a non-projective module in the principal block with no cohomology, then there is a non-projective periodic module in the principal block with no cohomology.*

Proof. This is the case $\dim V = 1$ of the above theorem. ∎

This completes the proof of the implication (B) $\Leftrightarrow$ (B′) in Theorem 1.4.

LEMMA 6.6. *Let G be a finite group with $O_p(G) \neq 1$ and $O_{p'}(G) = 1$. Suppose the centraliser of every element of order p in $O_p(G)$ is p-nilpotent. Then every non-trivial p'-element in G acts fixed point freely (i.e., fixing only the identity element) on $O_p(G)$.*

Proof. If a non-trivial p'-element y centralises an element x of order p in $O_p(G)$, then $\langle y \rangle \times \langle x \rangle$ acts by conjugation on $O_p(G)$. Since $C_G(x)$ is p-nilpotent, y acts trivially on $C_{O_p(G)}(x)$. So by Thompson's $A \times B$ lemma [10, Theorem 5.3.4], $y \in C_G(O_p(G))$. Since $C_G(O_p(G))$ is p-nilpotent, it follows that $y \in O_{p'}(G) = 1$. ∎

LEMMA 6.7. *Suppose a cyclic p'-group T acts on a p-group P. If L is a line through the origin in V_P which is invariant under the action of T, and on which T acts faithfully, then there is an element $\zeta \in H^{2n}(P, k)$ for some n with the following properties.*

(i) $L \cap V_P(\langle \zeta \rangle) = \{0\}$.

(ii) *The one dimensional subspace $\langle \zeta \rangle$ of $H^{2n}(P, k)$ is T-invariant and affords a faithful one dimensional representation of T.*

Proof. The line L corresponds to a T-invariant ideal $I \subseteq H^{\cdot}(P, k)$. The map

$$H^{\cdot}(P, k) \to H^{\cdot}(P, k)/I = k[L]$$

splits as a map of graded kT-modules, since the order of T is coprime to p. Let ρ be such a splitting.

Since T acts faithfully on L, T also acts faithfully on $k[L]$, which is a polynomial ring in one generator in some degree. A generator of this polynomial ring therefore spans a one dimensional space in $H^{2n}(P, k)/I$ for some n, on which T acts faithfully, and therefore its image under ρ has the desired properties. ∎

PROPOSITION 6.8. *Suppose the centraliser of every element of order p in $O_p(G)$ is p-nilpotent. If M is an indecomposable periodic module in $B_0(kG)$, whose vertex is contained in $O_p(G)$, then $\hat{H}^n(G, M) \neq 0$ for some n.*

Proof. First, we remark that since $O_{p'}(G)$ is the kernel of $B_0(kG)$, we may assume $O_{p'}(G) = 1$. If $O_p(G) = 1$, the proposition is trivial, so we may assume $O_p(G) \neq 1$, so that the previous lemma applies.

We now apply Clifford theory with respect to the subgroup $O_p(G)$. Let M_0 be an indecomposable $kO_p(G)$-module such that M is a summand of $M_0\uparrow^G$. Letting G_0 be the inertia group of M_0, namely $G_0 = \{ g \in G \mid g \otimes M_0 \cong M_0 \}$, so that $O_p(G) \subseteq G_0 \subseteq G$, Clifford theory tells us that M is induced from G_0 [13, Theorem VII.9.6]. Thus by Shapiro's lemma we may assume $G = G_0$, so that M_0 is inertial.

Now the variety $V_{O_p(G)}(M_0)$ is (projectively) connected [6], and hence is a single line through the origin in $V_{O_p(G)}$. Since M_0 is inertial, this line must be invariant under the action of $G/O_p(G)$. Let E be a minimal elementary abelian subgroup of $O_p(G)$ such that $M_0{\downarrow}_E$ is not projective. Then E is well defined up to conjugacy in $O_p(G)$, and since M_0 is inertial, every G-conjugate of E is $O_p(G)$-conjugate to E. Thus $G = O_p(G) \cdot N_G(E)$ by the Frattini argument. Moreover, $V_E(M_0)$ is a union of lines, and $N_{O_p(G)}(E)$ acts transitively on these lines, as does $N_G(E)$. Thus if L is one such line, $G = O_p(G) \cdot \mathrm{Stab}_G(L)$, where $\mathrm{Stab}_G(L)$ is the subgroup of $N_G(E)$ stabilising L (setwise). It follows that $\mathrm{Stab}_G(L)/C_G(E)$ has one dimensional faithful representation (on L) in characteristic p, so that it is a cyclic p'-group. Now $C_G(E) \subseteq O_p(G)$ by Lemma 6.6, so that since $G = O_p(G) \cdot \mathrm{Stab}_G(L)$, it follows that $G/O_p(G)$ is a cyclic p'-group which acts faithfully on the line L.

We now apply Lemma 6.7 to see that there is an element $\zeta \in H^{2n}(O_p(G), k)$ for some n, such that T acts faithfully on the one dimensional subspace generated by ζ, and $L \cap V_{O_p(G)}(\langle\zeta\rangle) = \{0\}$. Thus $\langle\zeta\rangle$ is represented by a homomorphism $\Omega^{2n}(k) \to \varepsilon$ of kG-modules, where ε is a one dimensional representation whose kernel is $O_p(G)$. Denote by L_ζ the kernel, so that there is a short exact sequence

$$0 \to L_\zeta \to \Omega^{2n}(k) \to \varepsilon \to 0.$$

Since

$$\begin{aligned} V_{O_p(G)}(L_\zeta \otimes M) &= V_{O_p(G)}(L_\zeta \otimes M_0) \\ &= V_{O_p(G)}(L_\zeta) \cap V_{O_p(G)}(M_0) \\ &= V_{O_p(G)}(\langle\zeta\rangle) \cap V_{O_p(G)}(M_0) \\ &= \{0\}, \end{aligned}$$

it follows that $L_\zeta \otimes M$ is a projective kG-module, and so the short exact sequence

$$0 \to L_\zeta \otimes M \to \Omega^{2n}(k) \otimes M \to \varepsilon \otimes M \to 0$$

shows that $\Omega^{2n}(M) \cong \varepsilon \otimes M$. Since there is some non-zero element of $\hat{H}^0(G, \varepsilon^r \otimes M)$ for some r (M must have *some* top composition factor!), we deduce that

$$\begin{aligned} \hat{H}^{-2nr}(G, M) &\cong \hat{H}^0(G, \Omega^{2nr}(M)) \\ &\cong \hat{H}^0(G, \varepsilon^r \otimes M) \\ &\neq 0. \quad \blacksquare \end{aligned}$$

THEOREM 6.9. *Suppose the centraliser of every element of order p in G is p-nilpotent. Then for every non-projective indecomposable module M in $B_0(kG)$ with vertex D and Green correspondent $f(M)$, we have $\hat{H}^n(N_G(D), f(M)) \neq 0$ for some n.*

Proof. Since the Green correspondent of a module in the principal block is again in the principal block, we may replace G by $N_G(D)$. By Corollary 6.5, we may assume M is periodic. We now apply Proposition 6.8. ∎

This completes the proof of the implication (D) $\Rightarrow$ (E) in Theorem 1.4.

7. TRIVIAL SOURCE MODULES

The main obstacle to proving the implication (D) $\Rightarrow$ (B) of Theorem 1.4, and hence Conjecture 1.2, using Theorem 6.9 is that all the cohomology of the Green correspondent of a periodic module may actually be in the image of the trace map (transfer) from proper subgroups of the vertex. When the module has trivial source, this cannot happen, as we shall see in the next theorem. This theorem gives the implication (D) $\Rightarrow$ (C) of Theorem 1.4.

THEOREM 7.1. *Let G be a finite group in which the centraliser of every element of order p is p-nilpotent. Suppose M is a trivial source module in $B_0(kG)$ with vertex D and Green correspondent $f(M)$. If $H^*(N_G(D), f(M)) \neq 0$, then $H^*(G, M) \neq 0$.*

Proof. We must show that if $H^*(N_G(D), f(M)) \neq 0$ then there are elements which are not in the image of the trace map from proper subgroups of D, since then the theorem follows from the Green correspondence. Now D is in the kernel of the action of $N_G(D)$ of $f(M)$, and $f(M)$ is projective as a module for $N_G(D)/D$, since M is a trivial source module. The theorem now follows from the next lemma.

LEMMA 7.2. *Suppose M is an indecomposable trivial source module in $B_0(kG)$ whose vertex $D \neq 1$ is normal in G. Let Z be the subgroup of $Z(D)$ generated by elements of order p, and suppose that $C_G(Z)$ is p-nilpotent. Then*

$$H^*(G, M) \Big/ \sum_{H < D} \mathrm{tr}_{H,G} H^*(H, M) \neq 0.$$

Proof. We regard $k(G/D) = k_D\uparrow^G$ as a G/D-G/D-bimodule. Then $H^*(G, k_D\uparrow^G)$ is a right $k(G/D)$-module, and the Shapiro isomorphism

$$H^*(G, k_D\uparrow^G) \xrightarrow{\mathrm{Sh}} H^*(D, k)$$

is a right $k(G/D)$-module isomorphism. Let e be a primitive idempotent in $k(G/D)$ with $M \cong k(G/D) \cdot e$. Then

$$\begin{aligned} H^*(G, M) &= H^*(G, k_D\uparrow^G \cdot e) \\ &= H^*(G, k_D\uparrow^G) \cdot e \\ &= H^*(D, k) \cdot e \subseteq H^*(D, k). \end{aligned}$$

Letting $\pi: k_D\uparrow^G\downarrow_D \to k_D$ be the natural kD-module homomorphism, for any proper subgroup H of D we have the following diagram, which the reader will easily check is commutative:

$$\begin{tikzcd}
H^*(H, M) \arrow[r, hook] \arrow[d, "\mathrm{tr}"] & H^*(H, k_D\uparrow^G\downarrow_H) \arrow[d, "\mathrm{tr}"] \arrow[dr, "\pi_*"] & & \\
H^*(D, M) \arrow[r, hook] \arrow[d, "\mathrm{tr}"] & H^*(D, k_D\uparrow^G\downarrow_D) \arrow[d, "\mathrm{tr}"] \arrow[dr, "\pi_*"] & H^*(H, k) \arrow[d, "\mathrm{tr}"] \arrow[dr, "0"] & \\
H^*(G, M) \arrow[r, hook] & H^*(G, k_D\uparrow^G) \arrow[r, "\mathrm{Sh}"] & H^*(D, k) \arrow[r, "\mathrm{res}"] & H^*(Z, k)
\end{tikzcd}$$

The composite of transfer from $H^*(H, k)$ to $H^*(D, k)$ and restriction from $H^*(D, k)$ to $H^*(Z, k)$ is zero because Z is central, by the Mackey formula. It thus suffices to produce an element $\zeta \in H^*(D, k)$ with $\mathrm{res}_{D,Z}(\zeta \cdot e) \neq 0$, so that $\zeta \cdot e \in H^*(G, M)$ cannot be a sum of tranfers from proper subgroups of D.

To construct such an element ζ, we argue as follows. We know that $C_G(Z)$ is p-nilpotent, and since $O_{p'}C_G(Z) \leqslant O_{p'}(G)$ is in the kernel of $B_0(kG)$, we may assume without loss of generality that $O_{p'}C_G(Z) = 1$, so that $C_G(Z)$ is a p-group. So every simple $k(G/D)$-module appears as a composition factor of $H^n(Z, k)$ for some n. The point is that $G/C_G(Z)$ acts faithfully on Z and hence also on a polynomial subring $E^*(Z)$ of $H^*(Z, k)$.

Now for any n, there is a $k(G/C_G(Z))$-module homomorphism $H^n(Z, k) \to H^{p^a n}(Z, k)$, where $p^a = |D : Z|$, given by $\xi \to \xi^{p^a} = \mathrm{res}_{D,Z}\, \mathrm{norm}_{Z,D}(\xi)$, where $\mathrm{norm}_{Z,D}$ is the Evens norm map. This map is injective on $E^*(Z)$, and so every simple $k(G/C_G(Z))$-module is a composition factor of $\mathrm{res}_{D,Z}(H^n(D, k))$ for some n. Since $\mathrm{res}_{D,Z}$ is a $k(G/C_G(Z))$-module homomorphism, it follows that there exists an element $\zeta \in H^*(D, k)$ such that $\mathrm{res}_{D,Z}(\zeta \cdot e) = \mathrm{res}_{D,Z}(\zeta) \cdot e \neq 0$. ∎

We now prove the implication $(\mathrm{A}'') \Rightarrow (\mathrm{A}')$ of Theorem 1.4.

THEOREM 7.3. *Suppose that every permutation module in $B_0(kG)$ is a direct summand of a TH module. Then every finitely generated module in $B_0(kG)$ is a direct summand of a TH module.*

481/131/1-5

Proof. Every module induced from a p-subgroup has a filtration by permutation modules, since every module for the p-subgroup has a filtration by one dimensional modules. Thus by Proposition 3.2, every module induced from a p-subgroup is a summand of a TH module. But every module is a direct summand of a module induced from its vertex. ∎

Proof of Theorem 1.4. The implications (A) $\Rightarrow$ (A′), (A) $\Rightarrow$ (A″), (B) $\Rightarrow$ (B′), and (B) $\Rightarrow$ (C) are trivial. The implication (A) $\Rightarrow$ (B) is Corollary 2.8, (A′) $\Rightarrow$ (B) is Corollary 3.8, (A″) $\Rightarrow$ (A′) is Theorem 7.3, (B′) $\Rightarrow$ (B) is Corollary 6.5, (C) $\Rightarrow$ (D) and (E) $\Rightarrow$ (D) follow from Proposition 5.3, (D) $\Rightarrow$ (E) is Theorem 6.9, (D) + (E) $\Rightarrow$ (C) is Theorem 7.1, and (D) $\Rightarrow$ (F) is Proposition 4.5. The implication (D) $\Leftrightarrow$ (D′) follows easily from the fact that any subgroup of a p-nilpotent group is p-nilpotent.

The implications so far give (A″) $\Rightarrow$ (A′) $\Rightarrow$ (F). But by Proposition 4.4, (A′) and (F) together imply (A).

The final statement about odd primes will be proved in the next two sections. ∎

8. A Stable Equivalence of Module Categories

In this and the next section, we shall prove Conjecture 1.2 for p odd. The basic idea is that if the centralisers of p-elements are p-nilpotent then we can find a suitable subgroup H of G with the properties that the module categories of the principal blocks of H and G are equivalent modulo projectives, and with H of such a tight shape that we can prove the theorem there.

In this section, we provide a proof of an unpublished theorem of M. Broué, which enables us to descend to a suitable subgroup, and in the next section we go through the group theory required to find a subgroup of the right shape. The proof of Broué's theorem presented here is perhaps closer in spirit to a proof by Alperin of the same result (also unpublished).

Definition 8.1. A subgroup H of a finite group G is said to be *weakly p-embedded* if $H \neq G$, $p \mid |H|$, and $N_G(Q) = N_H(Q)\, O_{p'} N_G(Q)$ (note that $O_{p'} N_G(Q) = O_{p'} C_G(Q)$) whenever $Q \neq 1$ is a p-subgroup of H.

Note in particular that if H is weakly p-embedded in G then H contains a Sylow p-subgroup of G. Also by Alperin's fusion theorem [10, Section 7.2], H controls strong p-fusion in G and so by the stable element method [14, Section X.12] we have $H^*(G, k) \cong H^*(H, k)$. We shall see that the stronger hypothesis is sufficient to guarantee that $H^*(G, M) \cong H^*(H, M)$ for any module M in the principal block.

The following is a variant of a result of Nagao [19].

LEMMA 8.2. *Let M be an indecomposable module in a block B of kG, and let $Q \neq 1$ be a p-subgroup of G. Let X be an indecomposable summand of $M\downarrow_{N_G(Q)}$ such that $X\downarrow_Q$ has a summand with vertex Q. Then X lies in a block b of $kN_G(Q)$ for which $b^G = B$.*

Proof. Let e be the central idempotent in kG corresponding to B. We may write $e = e_1 + e_2$, where $e_1 = \mathrm{Br}_Q(e)$ is an idempotent in $Z(kN_G(Q))$ and e_2 is an idempotent in $\sum_{P<Q} \mathrm{Tr}_{P,Q}(kG)^P$.

We have $M\downarrow_{N_G(Q)} = e_1 \cdot M\downarrow_{N_G(Q)} \oplus e_2 \cdot M\downarrow_{N_G(Q)}$. Let $E = \mathrm{End}_k(M)$ and let $\sigma : kG \to E$ be the homomorphism given by the action of G on M. Then $\sigma(e_2) \in \sum_{P<Q} \mathrm{Tr}_{P,Q}(E^P)$, so that by Rosenberg's lemma any primitive idempotent of E^Q occurring in the decomposition of $\sigma(e_2)$ is in $\mathrm{Tr}_{P,Q}(E^P)$ for some $P < Q$. Thus every summand of $e_2 \cdot M\downarrow_Q$ has vertex properly contained in Q, and so X is a summand of $e_1 \cdot M\downarrow_{N_G(Q)} = \mathrm{Br}_Q(e) \cdot M\downarrow_{N_G(Q)}$ and is hence in a block b with $b^G = B$. ∎

THEOREM 8.3 (M. Broué). *Suppose H is weakly p-embedded subgroup of G, and let M be an indecomposable module in $B_0(kG)$ with non-trivial vertex $P \subseteq H$. Then we have the following.*

(i) *There is an indecomposable module $f(M)$ in $B_0(kH)$, also with vertex P, such that $M\downarrow_H = f(M) \oplus (\text{projective})$.*

(ii) *$f(M)\uparrow^G \cong M \oplus (\text{projective}) \oplus (\text{modules outside the principal block})$.*

Proof. (i) Suppose $Q \neq 1$ is a vertex of an indecomposable summand of $M\downarrow_H$. By [4; 20, Corollary 1.4], there is a bijection between indecomposable summands of $M\downarrow_H$ with vertex Q and indecomposable summands of $M\downarrow_{N_H(Q)}$ with vertex Q.

Let T be an indecomposable summand of $M\downarrow_{N_H(Q)}$ with vertex Q. Let X be an indecomposable summand of $M\downarrow_{N_G(Q)}$ such that T is a summand of $X\downarrow_{N_H(Q)}$. Then $X\downarrow_Q$ has a summand with vertex Q (namely a source of T) and so by the lemma and Brauer's third main theorem, X is in $B_0(kN_G(Q))$. Since H is weakly p-embedded in G, $N_G(Q) = N_H(Q)\, O_{p'}N_G(Q)$. Since $O_{p'}N_G(Q)$ is the kernel of $B_0(N_G(Q))$, we thus have $T = X\downarrow_{N_H(Q)}$, and X has vertex Q. Again applying [4; 20, Corollary 1.4], we see that $Q = P$, and X is the Green correspondent of M. Taking $f(M)$ to be the Green correspondent of T yields the result.

(ii) Let Q be a non-trivial p-subgroup of H, and consider the indecomposable summands of $f(M)\uparrow^G\downarrow_{N_G(Q)}$ in the principal block of $kN_G(Q)$.

Since H is weakly p-embedded in G, H controls strong p-fusion in G, and so $Q \leqslant H^g$ if and only if $g \in HN_G(Q)$. It follows using Mackey decomposition that the indecomposable summands of $f(M)\uparrow^G\downarrow_{N_G(Q)}$ with vertex Q all occur in $f(M)\downarrow_{N_H(Q)}\uparrow^{N_G(Q)}$. We claim that there can be such a summand with vertex Q in the principal block of $kN_G(Q)$ if and only if Q is H-conjugate to P, in which case there is just one such summand, namely X (the Green correspondent of M).

Now for any indecomposable $kN_H(Q)$-module U, $O_{p'}N_H(Q)$ acts either without fixed points or trivially on U. If $O_{p'}N_H(Q)$ acts without fixed points on U, then $O_{p'}N_G(Q)$ acts without fixed points on $U\uparrow^{N_G(Q)}$, so $U\uparrow^{N_G(Q)}$ has no summands in $B_0(kN_G(Q))$. If $O_{p'}N_H(Q)$ acts trivially on U, then the fixed points of $O_{p'}N_G(Q)$ on $U\uparrow^{N_G(Q)}$ have the same dimension as U. In this case, since $N_G(Q)=N_H(Q)\,O_{p'}N_G(Q)$, we may extend U to a $kN_G(Q)$-module $\tilde{U}$ which is a summand of $U\uparrow^{N_G(Q)}$. Hence $\tilde{U}$ is the unique indecomposable summand of $U\uparrow^{N_G(Q)}$ on which $O_{p'}N_G(Q)$ acts trivially. This summand has the same vertex as U, and it lies in $B_0(kN_G(Q))$ if and only if U lies in $B_0(N_H(Q))$.

It follows that $f(M)\downarrow_{N_H(Q)}\uparrow^{N_G(Q)}$ has no summand in $B_0(kN_G(Q))$ with vertex Q unless $f(M)\downarrow_{N_H(Q)}$ has a summand with vertex Q in $B_0(kN_H(Q))$. If there is such a summand, then Q is a vertex for $f(M)$ and the Green correspondent of $f(M)$ is the unique such summand. Thus we may suppose $Q=P$, in which case this summand is isomorphic to T, and X is the unique indecomposable summand of $T\uparrow^{N_G(Q)}$ in $B_0(kN_G(Q))$ with vertex Q. The result now follows by another application of [4; 20, Corollary 1.4]. ∎

COROLLARY 8.4. *Suppose H is a weakly p-embedded subgroup of G and M is a kG-module. Then the restriction map is an isomorphism*

$$\mathrm{res}_{G,H}\colon H^*(G, M) \to H^*(H, M) = H^*(H, f(M)).$$

Proof. To deduce this from the theorem and Shapiro's lemma, we must show that if M is an indecomposable module not in $B_0(kG)$ then $M\downarrow_H$ has no non-projective summand in $B_0(kH)$. If $M\downarrow_H$ has a summand in $B_0(kH)$ with vertex $Q\neq 1$ then $M\downarrow_{N_H(Q)}$ has a summand T in $B_0(N_H(Q))$ with vertex Q. Lemma 8.2 and Brauer's third main theorem show that this cannot happen. ∎

9. Some Group Theory

For p odd, and under the assumption that the centraliser of every element of order p is p-nilpotent, we shall show that there is a suitable weakly p-embedded subgroup H, for which we can prove that evey non-projective module in the principal block has cohomology. We begin with some lemmas.

LEMMA 9.1. *Suppose $C_G(x)$ is p-nilpotent whenever x is an element of order p in G. Then the same is true for every section of G.*

Proof. This is clear for subgroups, so it suffices to consider quotients $\bar{G}=G/N$ for N a normal subgroup of G. We proceed by induction on $|G|$. We may assume that $O_{p'}(G)=1$.

Suppose that q is a prime divisor of $|N|$ and that a Sylow q-subgroup Q of N is not normal. Then by the Frattini argument $G=N\cdot N_G(Q)$ and $G/N\cong N_G(Q)/N_N(Q)$. Since $N_G(Q)$ is a proper subgroup of G we are done by induction. This shows that N is nilpotent. Since $O_{p'}(N)=1$ this implies that N is a p-group.

Now $C_G(N)$ is p-nilpotent by hypothesis, and so it is a p-group, as $O_{p'}(G)=1$. So $C_G O_p(G)\leqslant C_G(N)\leqslant O_p(G)$, and hence G is p-constrained. Thus by a well known result of Bender (see, for example, [14, Lemma X.1.6]), $O_{p'}C_G(x)\leqslant O_{p'}(G)=1$ and hence $C_G(x)$ is a p-group whenever x is an element of order p in G. Since N is a p-group, for y a non-trivial p'-element of G we have $C_{\bar{G}}(\bar{y})=\overline{C_G(y)}$ (where the bar denotes passage from G to $\bar{G}=G/N$), and so $C_{\bar{G}}(\bar{y})$ is a p'-group. Thus whenever $\bar{x}$ is a non-trivial element of order p in $\bar{G}$, $C_{\bar{G}}(\bar{x})$ is a p-group. This completes the proof of the lemma. ∎

PROPOSITION 9.2. *Suppose that p is odd and that the centraliser of every element of order p in G is p-nilpotent. Let $H=N_G(ZJ(S))$, with S a Sylow p-subgroup of G (here, $J(S)$ is the Thompson subgroup; see, for example, [10, Sect. 8.2]). Then H is weakly p-embedded in G.*

Proof. We first claim that H controls strong p-fusion in G. If not, then by a theorem of Glauberman [9, Theorem B], $\mathrm{Qd}(p)$ is involved in G, where $\mathrm{Qd}(p)$ denotes the semi-direct product $(\mathbf{Z}/p\times\mathbf{Z}/p)\colon SL_2(p)$ with the natural action. But this contradicts Lemma 9.1 since the centraliser of an element of order p in $\mathrm{Qd}(p)$ but not in $O_p\,\mathrm{Qd}(p)$ has order $2p^2$ and is not p-nilpotent.

Now since H controls strong p-fusion in G, whenever Q is a non-trivial p-subgroup of H, we have $N_G(Q)=N_H(Q)\,C_G(Q)$. We claim that $N_H(Q)$ contains a Sylow p-subgroup of $N_G(Q)$. For if T is a Sylow p-subgroup of G containing a Sylow p-subgroup of $N_G(Q)$ then $T^g=S$ for some $g\in G$. Both Q and Q^g are in S, and since H controls strong p-fusion in G, $Q^g=Q^h$ for some $h\in H$, and hence $N_G(Q)^g=N_G(Q)^h$. Thus $S^{h^{-1}}$ contains a Sylow p-subgroup of $N_G(Q)$, hence so does H.

It now follows that we have $N_G(Q)=N_H(Q)\,O_{p'}N_G(Q)$ since $C_G(Q)$ is p-nilpotent. ∎

In order to make use of this, we investigate the structure of $N_G(ZJ(S))$ under these hypotheses.

LEMMA 9.3. *Suppose that p is odd and that the centraliser of every element of order p in G is p-nilpotent. Let $H = N_G ZJ(S)$, with S a Sylow p-subgroup of G, and let $\bar{H} = H/O_{p'}(H)$. Then one of the following occurs.*

(i) *$\bar{H}$ is a Frobenius group whose Frobenius kernel is its Sylow p-subgroup.*

(ii) *$|\bar{H}|$ is odd and $O^p(\bar{H})$ is a Frobenius group whose kernel is its Sylow p-subgroup and whose Frobenius complement is cyclic. Also $\bar{H}/O^p(\bar{H})$ is cyclic.*

($O^p(G)$ denotes the smallest normal subgroup of G for which the quotient is a p-group.)

Proof. Let $\bar{Q} = O_p(\bar{H})$. Then by Lemma 6.6, $\bar{H}/\bar{Q}$ is faithfully represented as a group of linear transformations on $\bar{Q}/\Phi(\bar{Q})$, in which all non-trivial p'-elements act without non-trivial fixed points. If $p \nmid |\bar{H} : \bar{Q}|$, then we are in Case (i), so we may suppose that $p \mid |\bar{H} : \bar{Q}|$.

By the argument used in the proof of Lemma 9.1, the centraliser in $\bar{H}$ of a non-trivial p-element is a p-group, and the same holds in $\bar{H}/\bar{Q}$. Now if $\bar{H}$ contains an involution $\bar{t}$ then since $\bar{t}$ acts on $\bar{Q}/\Phi(\bar{Q})$ without non-trivial fixed points, it must act as minus the identity and so $\bar{t}\bar{Q}$ is central in $\bar{H}/\bar{Q}$. Since $p \mid |\bar{H} : \bar{Q}|$ this is a contradiction, and so $\bar{H}/\bar{Q}$ has odd order, so by the Feit–Thompson theorem it is solvable.

Let $L = \bar{H}/\bar{Q}$. Then an element of order p in L cannot have a non-trivial fixed point on $O_{p'}(L)$ since its centraliser is a p-group. Thus by a theorem of Thompson [10, Theorem 10.2.1] $O_{p'}(L)$ is nilpotent. Also, whenever $r \neq p$ is a prime divisor of $|L|$, the Sylow r-subgroup of L is cyclic, because otherwise, as r is odd, the Sylow r-subgroup contains $\mathbf{Z}/r \times \mathbf{Z}/r$ and so some non-trivial r-element would centralise a non-trivial element of $\bar{Q}$. This forces $O_{p'}(L)$ to be cyclic. Since L is solvable and $O_p(L) = 1$, $C_L O_{p'}(L) \leqslant O_{p'}(L)$ (see, for example, [10, Theorem 6.3.2]), so that $L/O_{p'}(L)$ is abelian, since it acts faithfully on a cyclic group. Since the centraliser of an element of order p is a p-group, $L/O_{p'}(L)$ is a cyclic p-group. ∎

Before we prove the main theorem of this section, we need a cohomological lemma.

LEMMA 9.4. *Let M be a kG-module and H a normal subgroup of G of index a power of p. If $H^*(H, M) \neq 0$ then $H^*(G, M) \neq 0$.*

Proof. Examine the spectral sequence

$$E_2^{st} = H^s(G/H, H^t(H, M)) \Rightarrow H^{s+t}(G, M).$$

Choose n minimal with $H^n(H, M) \neq 0$. Then $E_2^{0n} = \mathrm{Hom}_{G/N}(k, H^n(H, M))$

$\neq 0$ as G/H is a p-group. Since $E_2^{st}=0$ for $t<n$, it follows that $E_\infty^{0n}\neq 0$, *and so* $H^n(G, M)\neq 0$. ∎

THEOREM 9.5. *Suppose that p is odd and that the centraliser of every element of order p in G is p-nilpotent. If M is a non-projective module in* $B_0(kG)$ *then* $H^*(G, M)\neq 0$.

Proof. Let $H=N_G ZJ(S)$, with S a Sylow p-subgroup of G. By Proposition 9.2, H is weakly p-embedded in G. By Theorem 8.3 and its corollary we have

$$H^*(G, M)\cong H^*(H, f(M))\cong H^*(H/O_{p'}(H), f(M))$$

and $f(M)$ is in $B_0(kH)$. Thus by Lemma 9.3 it suffices to prove the theorem in the case where G has one of the two structures listed there. In the first case the theorem follows from Corollary 6.5 and Proposition 6.8. In the second case, we have the following situation. G has a normal subgroup $H=O^p(G)$ with G/H a non-trivial cyclic p-group. H is a Frobenius group with kernel $O_p(G)$ and $G/O_p(G)$ is a Frobenius group with kernel a cyclic p'-group. M is an indecomposable non-projective kG-module. Note that kG has only one block, as does kH in this situation. If $M\downarrow_H$ is non-projective, we may apply Lemma 9.4, and then the theorem follows from Corollary 6.5 and Proposition 6.8 as before. So suppose $M\downarrow_H$ is projective. Set $N=O_p(G)$, and let S be a Sylow p-subgroup of G. Then $S\cap S^g=N$ for $g\in G\backslash S$.

We choose M_0 an indecomposable summand of $M\downarrow_S$ such that M is a summand of $M_0\uparrow^G$, and write $M_0\uparrow^G=M\oplus X$. Then $M_0\downarrow_N$ is a summand of $M\downarrow_N$, so $M_0\downarrow_N$ is projective. Now $M_0\uparrow^G\downarrow_S=M_0\oplus Y$, where Y is projective relative to N since $S\cap S^g=N$ for $g\in G\backslash S$. Hence $X\downarrow_S$ is projective relative to N, so that X is projective relative to N. On the other hand, $X\downarrow_N$ is a summand of

$$M_0\uparrow^G\downarrow_N=\bigoplus_{g\in G/S}(g\otimes M_0)\downarrow_N,$$

which is projective, as $M_0\downarrow_N$ is projective. Hence $X\downarrow_N$ is projective, and since X is projective relative to N it follows that X is projective. Hence for every $n>0$,

$$H^n(G, M)=H^n(G, M_0\uparrow^G)=H^n(S, M_0)\neq 0. \quad ∎$$

Remark. Gorenstein [11] has classified the finite groups in which the centraliser of every involution is 2-nilpotent. It may be that the best way to prove Conjecture 1.2 in case $p=2$ is to go through the cases in this theorem individually. We have already initiated this process in Section 4 by treating the case where the Sylow 2-subgroups are dihedral. An example which we have not been able to tackle by the methods presented here is the split extension $(\mathbf{Z}/2)^{2n} : SL_2(2^n)$, where the normal subgroup is formed from the natural module by restricting the field of coefficients to $\mathbf{F}_2$.

If G has a connected split (B, N) pair of Lie rank greater than two of characteristic p, then there is a non-projective module M in $B_0(kG)$ with $H^*(G, M) = 0$. The argument goes as follows. Suppose false. For $p = 2$, we must have that $C_G(t)$ is 2-nilpotent for each involution t, by Proposition 5.3. By standard properties of (B, N) pairs, this forces the centraliser of every involution of $G/Z(G)$ to be a 2-group. The classification of such groups implies that $G/Z(G)$ has Lie rank at most two.

For p odd, the arguments of this section show that there is a p-local subgroup N which controls strong fusion. By the Borel–Tits theorem, we may assume that $N \leqslant P$ for some parabolic subgroup $P \geqslant B$. Then P also controls strong fusion, and so we may from now on assume $N = P$. Now whenever P_J is a parabolic subgroup containing B, we have $P_J = N_G(U_J) = C_G(U_J)\, N_N(U_J)$ as N controls fusion. But $C_G(U_J) \leqslant Z(G) \times Z(U_J)$ by standard properties of (B, N) pairs. Thus $C_G(U_J) \leqslant N$ (as $P = N_G(U)$ for some p-subgroup U, and as $U_J \leqslant B \leqslant P$). Hence $P_J \leqslant P$. Now, however, $\langle P_J \mid J \subset I \rangle \leqslant P$, whereas if G has rank greater than one, then $G = \langle P_J \mid J \subset I \rangle$ (as the right hand group contains all generating reflections of W).

10. Varieties and the Nucleus

In this section, we investigate the varieties of modules in the principal block of an arbitrary finite group.

Definition 10.1. The *nucleus* Y_G is the subvariety of V_G given as the union of the images of the maps $\mathrm{res}^* : V_{C_G(H)} \to V_G$ induced by $\mathrm{res} : H^{\cdot}(G, k) \to H^{\cdot}(C_G(H), k)$, where H runs over the set of subgroups of G for which $C_G(H)$ is not p-nilpotent.

The *representation theoretic nucleus* Θ_G is the subset of V_G given as the union of the varieties $V_G(M)$ as M runs over modules in $B_0(kG)$ with $H^n(G, M) = 0$ for all n. (Note that by Theorem 6.4, it suffices to consider periodic modules in this definition.)

Theorem 10.2. $\Theta_G \supseteq Y_G$.

Proof. If H is a subgroup of G for which $C_G(H)$ is not p-nilpotent, then Proposition 5.3 provides a module M in $B_0(kG)$ with $H^n(G, M) = 0$ for all n, and

$$V_G(M) = \mathrm{Im}(\mathrm{res}^* : V_{C_G(H)} \to V_G). \quad \blacksquare$$

Conjecture 10.3. $\Theta_G = Y_G$.

Remark. It does not even seem to be clear that Θ_G is a variety.

DEFINITION 10.4. If M is a kG-module, then we define $X_G(M)$ to be the closed homogeneous subvariety of V_G defined by the annihilator in $H^{\cdot}(G, k)$ of $H^*(G, M) = \mathrm{Ext}^*_{kG}(k, M)$. Note that this is the same as the annihilator of $\widehat{\mathrm{Ext}}^*_{kG}(k, M)$ by Theorem 2.4, and hence also equal to the annihilator of $\mathrm{Ext}^*_{kG}(M, k) = \mathrm{Ext}^*_{kG}(k, M^*)$ since by Tate duality

$$\widehat{\mathrm{Ext}}^n_{kG}(k, M) \cong (\widehat{\mathrm{Ext}}^{-n}_{kG}(M, k))^*.$$

The following two lemmas give the elementary properties of the variety $X_G(M)$.

LEMMA 10.5. (i) $X_G(M) \subseteq V_G(M)$.

(ii) $V_G(M) = \bigcup_{S \text{simple}} X_G(M \otimes S)$.

(iii) $X_G(M) = \{0\}$ *if and only if* $H^*(G, M) = 0$.

Proof. The first two parts of this are well known, and can be deduced, for example, from the discussion in [5]. The third part follows from Corollary 2.6. ∎

LEMMA 10.6. *If M is a TH module then we have the following.*

(i) $X_G(M) = V_G(M)$.

(ii) *If M' is any module then* $X_G(M \otimes M') \subseteq V_G(M) \cap X_G(M')$.

Proof. Let

$$(C_*, \theta) \colon C_r \to \cdots \to C_1 \to C_0$$

be a complex of projective kG-modules displaying M as a TH module.

(i) Consider the spectral sequence

$$E_2^{st} = \widehat{\mathrm{Ext}}^s_{kG}(H_t(C_*), M) \Rightarrow 0$$

of Theorem 2.3. If $\zeta \in H^*(G, k)$ annihilates $\widehat{\mathrm{Ext}}^*_{kG}(k, M)$ then ζ annihilates E_2^{st} whenever $t > 0$. Since multiplication by ζ commutes with the differentials, this means that some power of ζ must annihilate $E_2^{*0} = \widehat{\mathrm{Ext}}^*_{kG}(M, M)$ in order that the spectral sequence can abut to zero. This forces $V_G(M) \subseteq X_G(M)$. The reverse inclusion is Lemma 10.5(i).

(ii) Consider the spectral sequence

$$E_2^{st} = \widehat{\mathrm{Ext}}^s_{kG}(H_t(C_*) \otimes M', k) \Rightarrow 0.$$

If $\zeta \in H^*(G, k)$ annihilates $\widehat{\operatorname{Ext}}^*_{kG}(M', k)$ then ζ annihilates E_2^{st} whenever $t > 0$. So some power of ζ must annihilate $E_2^{*0} = \widehat{\operatorname{Ext}}^*_{kG}(M \otimes M', k)$. This forces $X_G(M \otimes M') \subseteq X_G(M')$. The other inclusion is given by $X_G(M \otimes M') \subseteq V_G(M \otimes M') \subseteq V_G(M)$ using Lemma 10.5(i) and Proposition 6.1(iv). ∎

Remark. It seems likely that we always have equality in part (ii) of the above lemma, but we have been unable to prove this except in the case where $M = L_\zeta$.

THEOREM 10.7. *For any module M in $B_0(kG)$,*

$$V_G(M) \subseteq X_G(M) \cup \Theta_G.$$

Proof. Suppose $V_G(M) \nsubseteq X_G(M) \cup \Theta_G$. Then there is a non-zero closed homogeneous subvariety (for example, a line)

$$\{0\} \neq V = V_G(M) \cap V_G(\langle \zeta_1 \rangle) \cap \cdots \cap V_G(\langle \zeta_s \rangle)$$

with $V \cap (X_G(M) \cup \Theta_G) = \{0\}$. Now on the one hand we have

$$\begin{aligned} V_G(M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}) &= V_G(M) \cap V_G(\langle \zeta_1 \rangle) \cap \cdots \cap V_G(\langle \zeta_s \rangle) \\ &= V \\ &\nsubseteq \Theta_G \end{aligned}$$

so that by definition of Θ_G we have $H^*(G, M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}) \neq 0$. On the other hand we have

$$\begin{aligned} X_G(M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}) &= X_G(M) \cap X_G(\langle \zeta_1 \rangle) \cap \cdots \cap X_G(\langle \zeta_s \rangle) \\ &\subseteq X_G(M) \cap V = \{0\} \end{aligned}$$

so that by Lemma 10.5(iii) we have $H^*(G, M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}) = 0$. This contradiction completes the proof of the theorem. ∎

DEFINITION 10.8. We say a kG-module M is a *nuclear homology module* or *NH module* if there exists a finite complex

$$C_r \xrightarrow{\theta_r} C_{r-1} \longrightarrow \cdots \xrightarrow{\theta_2} C_1 \xrightarrow{\theta_1} C_0$$

of kG-modules and homomorphisms such that the following conditions hold.

(i) Each C_i is a projective kG-module.

(ii) For $i>0$, $H_i(C_*)$ is a direct sum of copies of the trivial kG-module k and modules M' in the principal block with $V_G(M')\subseteq Y_G$.

(iii) $H_0(C_*)\cong M$.

PROPOSITION 10.9. (i) *Suppose M is an NH module and B is a positive integer. Then there exists a complex (C_*, θ) as in the definition, with $H_i(C_*)=0$ for $1\leqslant i\leqslant B$.*

(ii) *Suppose $0\to M_1\to M_2\to M_3\to 0$ is a short exact sequence of RG-modules and two of the terms are NH modules. Then so is the third.*

(iii) *If M_1 and M_2 are NH modules then so is $M_1\otimes_R M_2$.*

(iv) *If M is an NH module, then so is $M^*=\mathrm{Hom}_k(M, k)$.*

(v) *Suppose every simple module in $B_0(kG)$ is an NH module. Then every finitely generated module in $B_0(kG)$ is an NH module.*

(vi) *Suppose M_1 and $M_2\oplus M_3$ are NH modules and $\theta\colon M_1\to M_2$ is surjective. Then $M_2\oplus\mathrm{Ker}(\theta)$ is a TH module.*

(vii) *A module M is a direct summand of an NH module if and only if $M\oplus\Omega(M)$ is an NH module.*

Proof. The proofs of these are the same as the proofs of the corresponding results in Section 3. ∎

Conjecture 10.10. Every module in $B_0(kG)$ is a nuclear homology module.

PROPOSITION 10.11. *If M is an NH module and $H^*(G, M)=0$ then $V_G(M)\subseteq Y_G$.*

Proof. Let

$$(C_*, \theta)\colon C_r\to\cdots\to C_0$$

be a complex displaying M as an NH module. Consider the spectral sequence

$$E_2^{st}=\widehat{\mathrm{Ext}}_{kG}^s(H_t(C_*), M)\Rightarrow 0$$

of Proposition 2.3. Suppose $\zeta\in\mathrm{Ker}(\mathrm{res}_{G,C_G(H)}\colon H^*(G,k)\to H^*(H,k))$ for every subgroup H with $C_G(H)$ not p-nilpotent (i.e., ζ vanishes on Y_G). Then by the definition of an NH module, ζ annihilates E_2^{st} whenever $t>0$. Thus some power of ζ annihilates $E_2^{*0}=\widehat{\mathrm{Ext}}_{kG}^*(M, M)$. This proves that $V_G(M)\subseteq Y_G$. ∎

COROLLARY 10.12. *Conjecture* 10.10 *implies Conjecture* 10.3. ∎

EXAMPLE. We shall prove Conjecture 10.10 in case G is the Mathieu group M_{11} and k is a field of characteristic 2. The simple modules in the principal block in this case are k, M of dimension 44, and N of dimension 10. The minimal resolution of k as a kG-module was described in [2]. By examining $\Omega^4(k)$, we see using Proposition 10.9 that

$$\begin{array}{c} M \\ | \\ M \end{array} \oplus \begin{array}{c} M \\ | \\ k \\ | \\ N \end{array}$$

is a TH module, and hence an NH module. Now Y_G is the one dimensional subvariety of V_G corresponding to the (unique) conjugacy class of involutions, and $V_G(N) = Y_G$, so that N is an NH module. Thus by Proposition 10.9(ii),

$$\begin{array}{c} M \\ | \\ M \end{array} \oplus M$$

is an NH module. By stripping all copies of k and N from P_k, we see that $M \oplus M$ is an NH module, and hence so is M by the above. Thus by Proposition 10.9(v) every finitely generated module in the principal block is an NH module, so by Corollary 10.12 we have $\Theta_G = Y_G$. In particular, every non-projective,. non-periodic module has cohomology, since the variety cannot be in Y_G for dimensional reasons.

ACKNOWLEDGMENTS

This paper was written while the first two authors were visiting the University of Manchester Institute of Science and Technology, for a representation theory symposium funded by the SERC. We thank the SERC and all those involved in making this meeting possible.

REFERENCES

1. D. J. BENSON AND J. F. CARLSON, Complexity and multiple complexes, *Math. Z.* **195** (1987), 221–238.
2. D. J. BENSON AND J. F. CARLSON, Diagrammatic methods for modular representations and cohomology, *Comm. Algebra* **15** (1987), 53–121.
3. K. S. BROWN, "Cohomology of groups," Graduate Texts in Mathematics, Vol. 87, Springer-Verlag, Berlin/New York, 1982.

4. D. W. BURRY AND J. F. CARLSON, Restrictions of modules to local subgroups, *Proc. Amer. Math. Soc.* **84** (1982), 181–184.
5. J. F. CARLSON, Complexity and Krull dimension, Lecture Notes in Mathematics, Vol. 903, Springer-Verlag, Berlin/New York, 1981.
6. J. F. CARLSON, The variety of an indecomposable module is connected, *Invent. Math.* **77** (1984), 291–299.
7. K. ERDMANN, Principal blocks of groups with dihedral Sylow 2-subgroups, *Comm. Algebra* **5** (1977), 665–694.
8. L. EVENS, The cohomology ring of a finite group, *Trans. Amer. Math. Soc.* **101** (1961), 224–239.
9. G. GLAUBERMAN, A characteristic subgroup of a p-stable group, *Canad. J. Math.* **20** (1968), 1101–1135.
10. D. GORENSTEIN, "Finite Groups," Chelsea, New York, 1968.
11. D. GORENSTEIN, Finite groups the centralisers of whose involutions have normal 2-complements, *Canad. J. Math.* **21** (1969), 335–357.
12. P. J. HILTON AND U. STAMMBACH, "A Course in Homological Algebra," Graduate Texts in Mathematics, Vol. 4, Springer-Verlag, Berlin/New York, 1971.
13. B. HUPPERT AND N. BLACKBURN "Finite Groups II," Grundlehren der Mathematischen Wissenschaften, Vol. 242, Springer-Verlag, Berlin/New York, 1982.
14. B. HUPPERT AND N. BLACKBURN, "Finite Groups III," Grundlehren der Mathematischen Wissenschaften, Vol. 243, Springer-Verlag, Berlin/New York, 1982.
15. G. D. JAMES, "The Representation Theory of the Symmetric Groups," Springer Lecture Notes in Mathematics, Vol. 682, Springer-Verlag, Berlin/New York, 1978.
16. P. A. LINNELL, Cohomology of finite soluble groups, *J. Algebra* **107** (1987), 53–62.
17. P. A. LINNELL AND U. STAMMBACH, The block structure and Ext of p-soluble groups, *J. Algebra* **108** (1987), 280–282.
18. P. A. LINNELL AND U. STAMMBACH, The cohomology of p-constrained groups, *J. Pure Appl. Algebra* **49** (1987), 273–279.
19. H. NAGAO, A proof of Brauer's theorem on generalized decomposition numbers, *Nagoya Math. J.* **22** (1963), 73–77.
20. L. PUIG, Pointed groups and construction of characters, *Math. Z.* **176** (1981), 265–292.
21. G. R. ROBINSON, Blocks, isometries, and sets of primes, *Proc. London Math. Soc.* **51** (1985), 432–448.