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Equivalence of rational representations of behaviors

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ABSTRACT

This article deals with the equivalence of representations of behaviors of linear differential systems. In general, the behavior of a given linear differential system has many different representations. In this paper we restrict ourselves to kernel and image representations. Two kernel representations are called equivalent if they represent one and the same behavior. For kernel representations defined by polynomial matrices, necessary and sufficient conditions for equivalence are well known. In this paper, we deal with the equivalence of *rational representations*, i. e. kernel and image representations that are defined in terms of *rational matrices*. As the first main result of this paper, we will derive a new condition for the equivalence of rational kernel representations of possibly noncontrollable behaviors. Secondly we will derive conditions for the equivalence of rational representations of a given behavior in terms of the polynomial modules generated by the rows of the rational matrices. We will also establish conditions for the equivalence of rational image representations. Finally, we will derive conditions under which a given rational kernel representation is equivalent to a given rational image representation.

1. Introduction

In this article, we deal with the issue of equivalence of representations of a given behavior with the emphasis on rational representations. In the behavioral approach, a mathematical model of a phenomenon is viewed as a restricted subset of all possible outcomes. More precisely, a mathematical model is defined as a pair $(\mathfrak{U}, \mathfrak{B})$, with $\mathfrak{U}$ the *universum*, with *outcomes* as its elements, and $\mathfrak{B}$ the behavior. A dynamical system is viewed as a mathematical model in which the objects of interest are functions of time: the universum $\mathfrak{U}$ is a function space. The behavior $\mathfrak{B}$ of the dynamical system is the set of all time trajectories in $\mathfrak{U}$ that are compatible with the laws of the system. More precisely, a dynamical system Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T}$ a subset of $\mathbb{R}$, called the *time axis*, $\mathbb{W}$ a set called the *signal space*, and $\mathfrak{B}$ a subset of $\mathbb{W}^{\mathbb{T}}$ (the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$) called the *behavior* (see [1]). In the context of linear, finite-dimensional, time-invariant systems this leads to the concept of *linear differential system*. A linear differential system is defined to be a system whose behavior is equal to the set of solutions of a finite number of higher order, linear, constant coefficient differential equations. This set of differential equations is then called a representation of the behavior, often called a *kernel representation*. One and the same behavior admits many different kernel representations. In addition to kernel representations, *controllable* linear differential systems can be represented in many ways as the image of a differential operator. Traditionally, kernel and image representations of linear differential systems involve *polynomial matrices*. Recently, in [2], the concept of *rational representation* was defined and elaborated, extending the class of representations to kernel, latent variable, and image representations involving *rational* matrices (see Sections 3, 5 and 6 of [2], respectively). The motivation for this comes from the fact that in systems and control, representations of dynamical systems often involve (rational) transfer matrices. In order to be able to fit such representations into the behavioral framework in a natural way, the notion of rational representations of behaviors needed to be formalized. Related material on rational representations of behaviors can be found in [3–5] and, in an input–output framework, in [6–8].

As noted above, a given linear differential system admits many different representations. Two representations are called *equivalent* if they represent one and the same behavior. The issue of equivalence of representations of behaviors has been studied before, in an input–output framework in [9–14], and in a behavioral framework in [1,15–17]. In the present paper, we will study the equivalence of kernel representations and image representation in terms of rational matrices. In particular, we consider the question how the rational matrices appearing in equivalent rational kernel representations and rational image representations are related.

The outline of this article is as follows. In the remainder of this section we will introduce the notation, and review some basic material on polynomial and rational matrices. In Section 2 we will review linear differential systems and their polynomial and rational kernel and image representations. In Section 3 we formally

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state the main problems addressed in this paper. In Section 4 we review the problem of equivalence of polynomial kernel representations. We establish new results here, and obtain, for two given polynomial kernel representations, separate conditions under which their controllable parts are equal, and their sets of autonomous parts are equal. Combining these conditions, we reobtain the well known "classical" result on the equivalence of polynomial kernel representations. In Section 5 we will apply these results to obtain up to now unknown conditions under which rational representations of possibly uncontrollable behaviors are equivalent. In Section 6 we deal with the module characterization of equivalence of rational kernel representations of a given behavior. In Section 7 we consider the equivalence of image representations. Finally in Section 8 we deal with the question of under what conditions kernel representations are equivalent to image representations.

As announced, first a few words about the notation and nomenclature used. We use the standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. We use $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc. for the real linear spaces of vectors and matrices with components in $\mathbb{R}$. $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathbb{R}(\xi)$ will denote the field of real rational functions in the indeterminate ξ. $\mathbb{R}[\xi]$ will denote the ring of polynomials in the indeterminate ξ with real coefficients. We will use $\mathbb{R}(\xi)^{\mathtt{n}}$, $\mathbb{R}(\xi)^{\mathtt{n}\times\mathtt{m}}$, $\mathbb{R}[\xi]^{\mathtt{n}}$, $\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{m}}$, etc. for the spaces of vectors and matrices with components in $\mathbb{R}(\xi)$, and $\mathbb{R}[\xi]$ respectively. If one, or both, dimensions are unspecified, we will use the notation $\mathbb{R}(\xi)^{\bullet\times\mathtt{m}}$, $\mathbb{R}(\xi)^{\mathtt{n}\times\bullet}$, $\mathbb{R}[\xi]^{\bullet\times\bullet}$ or $\mathbb{R}(\xi)^{\bullet\times\bullet}$, etc. Elements of $\mathbb{R}(\xi)^{\mathtt{n}\times\mathtt{m}}$ are called *real rational matrices*, elements of $\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{m}}$ are called *real polynomial matrices*. A square non-singular polynomial matrix U is called *unimodular* if the determinant of U is a non-zero constant.

To conclude this section we state the following well known facts that are used ubiquitously in the analysis in the rest of this paper (see Theorem 6.3-16 and Section 6.5.2 from [18]).

Proposition 1.1. *Let $R \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{q}}$ be a full row rank polynomial matrix. Then there exist unimodular polynomial matrices U and V such that $R = U\begin{bmatrix}D & 0\end{bmatrix}V$, where $D = \mathrm{diag}(z_1, z_2, \ldots, z_{\mathtt{p}})$, $z_1, z_2, \ldots, z_{\mathtt{p}}$ are monic polynomials obeying the division property $z_i \mid\mid z_{i+1}$, $i = 1, 2, \ldots, \mathtt{p}-1$. The polynomial matrix $\begin{bmatrix}D & 0\end{bmatrix}$ is called the Smith form of R.*

Proposition 1.2. *Let $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{q}}$ be a full row rank rational matrix. Then there exist unimodular polynomial matrices U and V such that $G = U\Pi^{-1}\begin{bmatrix}D & 0\end{bmatrix}V$, where $D = \mathrm{diag}(z_1, z_2, \ldots, z_{\mathtt{p}})$ and $\Pi = \mathrm{diag}(\pi_1, \pi_2, \ldots, \pi_{\mathtt{p}})$. Here, $z_1, z_2, \ldots, z_{\mathtt{p}}$ are monic polynomials obeying the division property $z_i \mid\mid z_{i+1}$, $i = 1, 2, \ldots, \mathtt{p}-1$ and $\pi_1, \pi_2, \ldots, \pi_{\mathtt{p}}$ are monic polynomials obeying the division property $\pi_{i+1} \mid\mid \pi_i$, $i = 1, 2, \ldots, \mathtt{p}-1$. Also z_i and π_i are coprime for $i = 1, 2, \ldots, \mathtt{p}$. The rational matrix $\Pi^{-1}\begin{bmatrix}D & 0\end{bmatrix}$ is called the Smith–McMillan form of G.*

2. Linear differential systems

In this section we will review the basic material on linear differential systems and their polynomial and rational representations.

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{\mathtt{w}}$ is the signal space, and the *behavior* $\mathfrak{B}$ is a linear subspace of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. For any such system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ there exists a real polynomial matrix R with $\mathtt{w}$ columns, i.e. $R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$, such that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}. \tag{1}$$

Such a system is called a *linear differential system*. The set of all linear differential systems with $\mathtt{w}$ variables is denoted by $\mathfrak{L}^{\mathtt{w}}$. The representation (1) of the behavior $\mathfrak{B}$ is called a *polynomial kernel representation* of $\mathfrak{B}$, and often we write $\mathfrak{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. If R has $\mathtt{p}$ rows, then the polynomial kernel representation is said to be *minimal* if every polynomial kernel representation of $\mathfrak{B}$ has at least $\mathtt{p}$ rows. A given polynomial kernel representation, $\mathfrak{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, is minimal if and only if the polynomial matrix R has full row rank (see [1], Theorem 3.6.4). The number of rows in any minimal polynomial kernel representation of $\mathfrak{B}$, denoted by $\mathtt{p}(\mathfrak{B})$, is called the *output cardinality* of $\mathfrak{B}$. This number corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$. For a detailed exposition of polynomial representations of behaviors, we refer to [1].

Recently, in [2], a meaning was given to the equation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$, where $R(\xi)$ is a given real *rational* matrix. In order to do this, we need the concept of left coprime factorization over $\mathbb{R}[\xi]$.

Definition 2.1. Let G be a real rational matrix. The pair of real polynomial matrices (P, Q) is called a *left coprime factorization of G over $\mathbb{R}[\xi]$* if

1. $\det(P) \neq 0$,
2. $G = P^{-1}Q$,
3. the matrix $\begin{bmatrix}P(\lambda) & Q(\lambda)\end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$.

A meaning to the equation

$$G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0, \tag{2}$$

with $R(\xi)$ a real rational matrix is then given as follows: Let (P, Q) be a left coprime factorization of R over $\mathbb{R}[\xi]$. Then we define:

Definition 2.2. Let $w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Then we define w to be a solution of (2) if it satisfies the differential equation $Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$.

This space of solutions is independent of the particular left coprime factorization. Indeed, if $R = P_1^{-1}Q_1$ is a second left coprime factorization then by [18], Theorem 6.5-4, there exists a unimodular U such that $P_1 = UP$ and $Q_1 = UQ$. Hence from Theorem 3.6.2 in [1], $\ker\left(Q_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \ker\left(Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Thus, (2) represents the uniquely determined linear differential system $\Sigma = \left(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \ker\left(Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)\right) \in \mathfrak{L}^{\mathtt{w}}$.

Since the behavior $\mathfrak{B}$ of the system Σ is the central item, often we will speak about the system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (instead of $\Sigma \in \mathfrak{L}^{\mathtt{w}}$). If a behavior $\mathfrak{B}$ is represented by $G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ $\left(\text{or} : \mathfrak{B} = \ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)\right)$, with $G(\xi)$ a real rational matrix, then we call this a *rational kernel representation of* $\mathfrak{B}$. If G has $\mathtt{p}$ rows, then the rational kernel representation is called *minimal* if every rational kernel representation of $\mathfrak{B}$ has at least $\mathtt{p}$ rows. It can be shown that a given rational kernel representation $\mathfrak{B} = \ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ is minimal if and only if the rational matrix G has full row rank. As in the polynomial case, every $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits a minimal rational kernel representation. It follows immediately from Definition 2.2 that the number of rows in any minimal rational kernel representation of $\mathfrak{B}$ is equal to the number of rows in any minimal polynomial kernel representation of $\mathfrak{B}$, and therefore equal to $\mathtt{p}(\mathfrak{B})$, the output cardinality of $\mathfrak{B}$. In general, if $\mathfrak{B} = \ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ is a rational kernel representation, then $\mathtt{p}(\mathfrak{B}) = \mathrm{rank}(G)$. This follows immediately from the corresponding result for polynomial kernel representations (see [1]).

Before proceeding, we recall the concepts of autonomous behavior and controllable behavior. We state the following definitions from [1]:

Definition 2.3. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called autonomous if for all $w_1, w_2 \in \mathfrak{B}$, $w_1(t) = w_2(t)$ for $t \leq 0$ implies $w_1(t) = w_2(t)$ for all t.

Definition 2.4. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. It is called controllable if for any two trajectories $w_1, w_2 \in \mathfrak{B}$, there exists a $t_1 \geq 0$ and a trajectory $w \in \mathfrak{B}$ with the property that $w(t) = w_1(t)$ for $t \leq 0$, and $w(t) = w_2(t - t_1)$ for $t \geq t_1$.

We denote the set of all autonomous linear differential systems with $\mathtt{w}$ variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$ and the set of all controllable linear differential systems with $\mathtt{w}$ variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$.

It is well known that a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is controllable if and only if there exists a positive integer $\mathtt{l}$ and a real polynomial matrix $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{l}}$ such that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \text{ s.t. } w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\}. \quad (3)$$

The representation (3) is called a *polynomial image representation* of $\mathfrak{B}$ because $\mathfrak{B}$ is written as the image of the differential operator $M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$. In this case we will write $\mathfrak{B} = \mathrm{im}\left(M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. It can be shown that the polynomial matrix M can be chosen of full column rank. Also, M has full column rank if and only if the number of columns is equivalent to $\mathtt{m}(\mathfrak{B})$, the input cardinality of $\mathfrak{B}$. This number is equal to $\mathtt{w} - \mathtt{p}(\mathfrak{B})$, and equals the number of inputs in any input–output representation of $\mathfrak{B}$. Even more, M can be chosen to be right prime over $\mathbb{R}[\xi]$, equivalently, $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. In that case, in (3) the latent variable ℓ is uniquely determined by the manifest variable w, and the image representation is called *observable*.

In [2], also the concept of *rational image representation* was introduced. We will give a brief review here. Let $H(\xi)$ be a real rational matrix. We will first give a meaning to the equation

$$w = H\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell. \quad (4)$$

Of course (4) should be interpreted as

$$\left(I - H\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)\begin{pmatrix} w \\ \ell \end{pmatrix} = 0,$$

in the context of (2). If $H = D^{-1}N$ is a left coprime factorization over $\mathbb{R}[\xi]$ then $D^{-1}(D - N)$ is a left coprime factorization of $(I - H)$ and therefore (w, ℓ) satisfies (4) if and only if $D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$. For a given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, the representation

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\bullet}) \text{ s.t. } w = H\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\}, \quad (5)$$

with $H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$, is called a *rational image representation*. In that case, we write $\mathfrak{B} = \mathrm{im}\left(H\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. It was shown in [2] that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits a rational image representation if and only if it is controllable. Like for polynomial image representations, the rational matrix H can then be chosen of full column rank, and it has full column rank if and only if the number of columns is equal to the input cardinality $\mathtt{m}(\mathfrak{B})$.

3. Problem formulation

In this section, we shall state the main problems addressed in this paper.

Problem 1. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be minimal rational kernel representations. Find necessary and sufficient conditions on G_1 and G_2 so that $\mathfrak{B}_1 = \mathfrak{B}_2$.

Problem 2. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}{}_{\mathrm{contr}}$. Let $H_1, H_2 \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$ have full column rank. Let $\mathfrak{B}_1 = \mathrm{im}\left(H_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \mathrm{im}\left(H_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be rational image representations. Find necessary and sufficient conditions on H_1 and H_2 so that $\mathfrak{B}_1 = \mathfrak{B}_2$.

Problem 3. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$. Let $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$ have full row rank and $H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$ have full column rank. Let $\mathfrak{B}_1 = \ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \mathrm{im}\left(H\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be a rational kernel and image representation respectively. Find necessary and sufficient conditions on G and H so that $\mathfrak{B}_1 = \mathfrak{B}_2$.

4. Equivalence of polynomial kernel representations

In this section, we discuss the equivalence of polynomial kernel representations from a slightly different perspective compared to that discussed in [1], and arrive at conditions which we shall use in addressing the issue of equivalence of rational kernel representations.

Given a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, it can be decomposed into the direct sum of its controllable part $\mathfrak{B}_{\mathrm{contr}}$, and an autonomous part $\mathfrak{B}_{\mathrm{aut}}$, i.e. $\mathfrak{B} = \mathfrak{B}_{\mathrm{aut}} \oplus \mathfrak{B}_{\mathrm{contr}}$. This is dealt with in detail in [1]. It is proved in [1] that for a given behavior, the controllable part is unique. It is also shown in [1] that for a given behavior, an autonomous part is not unique. Let

$$\mathcal{A}(\mathfrak{B}) = \{\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}} \mid \mathcal{P} \oplus \mathfrak{B}_{\mathrm{contr}} = \mathfrak{B}\} \quad (6)$$

denote the set of all autonomous direct summands of $\mathfrak{B}_{\mathrm{contr}}$ in $\mathfrak{B}$. The following lemma expresses the equality of behaviors in terms of equality of the controllable parts and equality of the sets of autonomous parts.

Lemma 4.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if*

1. $\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$ *and*
2. $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$.

Proof (*Only If*). : This part of the proof is obvious.

(*If*): Let $\mathcal{P}_1 \in \mathcal{A}(\mathfrak{B}_1)$. Then we have $\mathfrak{B}_1 = \mathcal{P}_1 + \mathfrak{B}_{1,\mathrm{contr}} = \mathcal{P}_1 + \mathfrak{B}_{2,\mathrm{contr}}$. Since also $\mathcal{P}_1 \in \mathcal{A}(\mathfrak{B}_2)$ the latter equals $\mathfrak{B}_2$. □

Kernel representations of the behaviors in $\mathcal{A}(\mathfrak{B})$ are discussed in [1]. For the sake of completeness, we shall state the following lemma, which describes kernel representations of the controllable as well as the autonomous parts of a given behavior.

Lemma 4.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be a minimal polynomial kernel representation, and let U and V be unimodular polynomial matrices such that $R = U\begin{bmatrix} D & 0 \end{bmatrix}V$, where $\begin{bmatrix} D & 0 \end{bmatrix}$ is the Smith form of R. Then we have:*

1. $\mathfrak{B}_{\mathrm{contr}} = \ker\left(\begin{bmatrix} I & 0 \end{bmatrix}V\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, *and*
2. $\mathcal{P} \in \mathcal{A}(\mathfrak{B})$ *if and only if* $\mathcal{P} = \ker\left(\begin{bmatrix} D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & 0 \\ 0 & I \end{bmatrix}W\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, *for some unimodular polynomial matrix W, satisfying* $\begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \end{bmatrix}W$.

This characterization of the autonomous parts of a given behavior is also dealt with in [1], in Excercise 5.6. It can be verified that W mentioned above takes the form $W = \begin{bmatrix} I & 0 \\ W_3 & W_4 \end{bmatrix}$, where W_3 is any polynomial matrix of appropriate dimensions, and W_4 is any unimodular polynomial matrix.

Remark 4.3. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be a minimal polynomial kernel representation, and let U and V be unimodular polynomial matrices such that $R = U\begin{bmatrix} D & 0 \end{bmatrix}V$, where $\begin{bmatrix} D & 0 \end{bmatrix}$ is the Smith form of R. From the above lemma and the structure of W, a unimodular polynomial matrix, it is clear that $\mathcal{P} \in \mathcal{A}(\mathfrak{B})$ if and only if it admits a minimal polynomial kernel representation

$$\ker\left(\begin{bmatrix} D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & 0 \\ F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & S\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}V\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right), \quad (7)$$

where F is an arbitrary polynomial matrix of appropriate dimensions, and S is an arbitrary unimodular polynomial matrix.

Equivalence of polynomial kernel representations has been dealt with before in [1]. We recall the following well known result given as Theorem 3.6.2 in [1]:

Proposition 4.4. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $R_1, R_2 \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ be such that $R_1\left(\frac{d}{dt}\right)w = 0$ and $R_2\left(\frac{d}{dt}\right)w = 0$ are minimal polynomial kernel representations of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular polynomial matrix U such that $R_1 = UR_2$.*

In order to proceed, we have the following lemma:

Lemma 4.5. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$ be a minimal polynomial kernel representation. Let $R = FR'$ be any factorization of R such that $F \in \mathbb{R}[\xi]^{\mathtt{p}(\mathfrak{B})\times\mathtt{p}(\mathfrak{B})}$ is square and non-singular, and $R' \in \mathbb{R}[\xi]^{\mathtt{p}(\mathfrak{B})\times\mathtt{w}}$ such that $R'(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Then $\mathfrak{B}_{\mathrm{contr}} = \ker\left(R'\left(\frac{d}{dt}\right)\right)$.*

Proof. Let U and V be unimodular polynomial matrices such that $R = U\begin{bmatrix} D & 0\end{bmatrix}V$, where $\begin{bmatrix} D & 0\end{bmatrix}$ is the Smith form of R.

It is clear that $\mathfrak{B}_{\mathrm{contr}} = \ker\left(\begin{bmatrix} I & 0\end{bmatrix}V\left(\frac{d}{dt}\right)\right)$ (from Lemma 4.2). Now let $R = FR'$ be any factorization of R such that $R'(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$ and F is square and non-singular. We have following identities:

$$\begin{aligned}\begin{bmatrix} UD & 0\end{bmatrix} &= FR'V^{-1}\\ &= F\begin{bmatrix} R'_{11} & R'_{12}\end{bmatrix},\\ &= \begin{bmatrix} FR'_{11} & FR'_{12}\end{bmatrix},\end{aligned}$$

where $\begin{bmatrix} R'_{11} & R'_{12}\end{bmatrix} := R'V^{-1}$. This implies $FR'_{12} = 0$, and hence $R'_{12} = 0$. Since $R'(\lambda)V^{-1}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$ and R'_{11} is square, we must have that R'_{11} is a unimodular polynomial matrix. Therefore $R' = U'\begin{bmatrix} I & 0\end{bmatrix}V$ and $\ker\left(R'\left(\frac{d}{dt}\right)\right) = \ker\left(U'\left(\frac{d}{dt}\right)\begin{bmatrix} I & 0\end{bmatrix}V\left(\frac{d}{dt}\right)\right) = \mathfrak{B}_{\mathrm{contr}}$ (from Proposition 4.4). □

The following theorem is the main result of this section. It expresses equality of the controllable parts of two behaviors in terms of their polynomial kernel representations, and it gives additional conditions under which the sets of autonomous parts are also equal.

Theorem 4.6. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(R_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(R_2\left(\frac{d}{dt}\right)\right)$ be minimal polynomial kernel representations. Then*

(a) *$\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$ if and only if there exist square nonsingular polynomial matrices M and N such that*

$$MR_1 = NR_2. \tag{8}$$

(b) *Assume that $\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$. Then for any pair of square nonsingular polynomial matrices M, N such that (8) holds, we have $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$ if and only if $M^{-1}N$ is a unimodular polynomial matrix.*

Proof. Let U_i and V_i be unimodular polynomial matrices such that $R_i = U_i\begin{bmatrix} D_i & 0\end{bmatrix}V_i$, where $\begin{bmatrix} D_i & 0\end{bmatrix}$ is the Smith form of R_i, for $i = 1, 2$. From Lemma 4.2, we have $\mathfrak{B}_{1,\mathrm{contr}} = \ker\left(\begin{bmatrix} I & 0\end{bmatrix}V_1\left(\frac{d}{dt}\right)\right)$, and $\mathfrak{B}_{2,\mathrm{contr}} = \ker\left(\begin{bmatrix} I & 0\end{bmatrix}V_2\left(\frac{d}{dt}\right)\right)$.

(a) *(Only if):* Since $\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$, by Proposition 4.4, there exists a unimodular polynomial matrix U such that $\begin{bmatrix} I & 0\end{bmatrix}V_1 = U\begin{bmatrix} I & 0\end{bmatrix}V_2$ holds.

Consequently, $D_1\begin{bmatrix} I & 0\end{bmatrix}V_1 = D_1UD_2^{-1}D_2\begin{bmatrix} I & 0\end{bmatrix}V_2$. It is easy to seethat there exist square nonsingular polynomial matrices $\tilde{M}$ and $\tilde{N}$, such that $\tilde{M}^{-1}\tilde{N} = D_1UD_2^{-1}$. Therefore we have $\tilde{M}D_1\begin{bmatrix} I & 0\end{bmatrix}V_1 = \tilde{N}D_2\begin{bmatrix} I & 0\end{bmatrix}V_2$. Define $M := \tilde{M}U_1^{-1}$ and $N := \tilde{N}U_2^{-1}$. Then we have $MR_1 = NR_2$.
(If): Let $G_1 = MR_1 = NR_2$. Then we have

$$G_1 = MU_1D_1\begin{bmatrix} I & 0\end{bmatrix}V_1 = NU_2D_2\begin{bmatrix} I & 0\end{bmatrix}V_2. \tag{9}$$

Further, from Lemma 4.5, it is evident that $\left(\ker\left(G_1\left(\frac{d}{dt}\right)\right)\right)_{\mathrm{contr}} = \ker\left(\begin{bmatrix} I & 0\end{bmatrix}V_1\left(\frac{d}{dt}\right)\right) = \ker\left(\begin{bmatrix} I & 0\end{bmatrix}V_2\left(\frac{d}{dt}\right)\right)$. Therefore $\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$.

(b) *(Only if):* Assume $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$. As $\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$, from Lemma 4.1 it is clear that $\mathfrak{B}_1 = \mathfrak{B}_2$. Therefore from Proposition 4.4, we have

$$R_1 = UR_2, \tag{10}$$

where U is a unimodular polynomial matrix. Further we have

$$R_1 = M^{-1}NR_2. \tag{11}$$

Since R_1 and R_2 are minimal kernel representations, R_1 and R_2 have full row rank. Therefore from (10) and (11) it is clear that $U = M^{-1}N$.
(If): As $\mathfrak{B}_{1,\mathrm{contr}} = \mathfrak{B}_{2,\mathrm{contr}}$, from Proposition 4.4, we have $\begin{bmatrix} I & 0\end{bmatrix}V_1 = \tilde{U}\begin{bmatrix} I & 0\end{bmatrix}V_2$, where $\tilde{U}$ is a unimodular polynomial matrix, and it can be checked that $V_1V_2^{-1}$ takes the form $V_1V_2^{-1} = \begin{bmatrix} \tilde{V}_{11} & 0\\ \tilde{V}_{21} & \tilde{V}_{22}\end{bmatrix}$, where $\tilde{V}_{11}, \tilde{V}_{22}$ are unimodular polynomial matrices. Further, we have $MU_1\begin{bmatrix} D_1 & 0\end{bmatrix}V_1 = NU_2\begin{bmatrix} D_2 & 0\end{bmatrix}V_2$. Define $M' = MU_1$ and $N' = NU_2$. Then we have $\begin{bmatrix} D_1 & 0\end{bmatrix}V_1 = M'^{-1}N'\begin{bmatrix} D_2 & 0\end{bmatrix}V_2$. Now, consider any $\mathcal{P} \in \mathcal{A}(\mathfrak{B}_1)$, then from Remark 4.3, we know that there exists a square nonsingular polynomial matrix F_1, and a unimodular polynomial matrix S_1, such that $\mathcal{P} = \ker\left(\begin{bmatrix} D_1\left(\frac{d}{dt}\right) & 0\\ F_1\left(\frac{d}{dt}\right) & S_1\left(\frac{d}{dt}\right)\end{bmatrix}V_1\left(\frac{d}{dt}\right)\right)$. We have

$$\begin{aligned}\begin{bmatrix} D_1 & 0\\ F_1 & S_1\end{bmatrix}V_1 &= \begin{bmatrix} M'^{-1}N'\begin{bmatrix} D_2 & 0\end{bmatrix}\\ \begin{bmatrix} F_1 & S_1\end{bmatrix}V_1V_2^{-1}\end{bmatrix}V_2\\ &= \begin{bmatrix} M'^{-1}N'\begin{bmatrix} D_2 & 0\end{bmatrix}\\ \begin{bmatrix} \tilde{F}_1 & S_1\tilde{V}_{22}\end{bmatrix}\end{bmatrix}V_2\\ &= \begin{bmatrix} M'^{-1}N' & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} D_2 & 0\\ \tilde{F}_1 & S_1\tilde{V}_{22}\end{bmatrix}V_2.\end{aligned}$$

It is easy to see that $\begin{bmatrix} M'^{-1}N' & 0\\ 0 & I\end{bmatrix}$ and $S_1\tilde{V}_{22}$ are unimodular polynomial matrices. From Proposition 4.4, we have $\mathcal{P} = \ker\left(\begin{bmatrix} D_2\left(\frac{d}{dt}\right) & 0\\ \tilde{F}_1\left(\frac{d}{dt}\right) & S_1\left(\frac{d}{dt}\right)\tilde{V}_{22}\left(\frac{d}{dt}\right)\end{bmatrix}V_2\left(\frac{d}{dt}\right)\right)$. From Remark 4.3 it is clear that $\mathcal{P} \in \mathcal{A}(\mathfrak{B}_2)$. The reverse inclusion is obvious. □

Evidently, from the above theorem we have the following corollary:

Corollary 4.7. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(R_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(R_2\left(\frac{d}{dt}\right)\right)$ be minimal polynomial kernel representations. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exist square and nonsingular polynomial matrices M, N such that $MR_1 = NR_2$ and $M^{-1}N$ is a unimodular polynomial matrix.*

Obviously, Corollary 4.7 is a restatement of Proposition 4.4. However, in combination with Theorem 4.6 it shows the origin of the unimodular matrix U. The corollary has been derived in two stages. Firstly, it has been shown that equality of the controllable parts of a given behavior is equivalent to the existence of square and non-singular matrices M and N. Secondly, unimodularity of $M^{-1}N$ has been shown to be equivalent to equality of the sets of autonomous parts of the behavior.

5. Equivalence of rational kernel representations

In this section we address the question of equivalence of minimal rational kernel representations. We will first recall the concepts of polynomial and rational annihilators of a given behavior from [2], Section 7.

Definition 5.1. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$.

1. $n \in \mathbb{R}[\xi]^{1\times\mathtt{w}}$ is called a *polynomial annihilator* of $\mathfrak{B}$ if $n\left(\frac{d}{dt}\right) w = 0$ for all $w \in \mathfrak{B}$.
2. $n \in \mathbb{R}(\xi)^{1\times\mathtt{w}}$ is called a *rational annihilator* of $\mathfrak{B}$ if $n\left(\frac{d}{dt}\right) w = 0$ for all $w \in \mathfrak{B}$.

We denote the set of polynomial annihilators of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ by $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$ and the set of rational annihilators of $\mathfrak{B}$ by $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)}}$. It is a well known result that for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$ is a finitely generated submodule of the $\mathbb{R}[\xi]$-module $\mathbb{R}[\xi]^{1\times\mathtt{w}}$. Moreover, if $\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$ is a polynomial kernel representation, then this submodule is generated by the rows of R. In the context of rational representations one needs to impose controllability:

Theorem 5.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)}}$ is a subspace of the $\mathbb{R}(\xi)$-linear vector space $\mathbb{R}(\xi)^{1\times\mathtt{w}}$ if and only if $\mathfrak{B}$ is controllable. If $G\left(\frac{d}{dt}\right) w = 0$ is a minimal rational kernel representation of $\mathfrak{B}$, then the rows of G form a basis of $(\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$, the rational annihilators of the controllable part of $\mathfrak{B}$.*

Proof. The first statement is the content of Statement 1 of Theorem 11 in [2]. Let $G = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$ of G. Then $\mathfrak{B} = \ker\left(Q\left(\frac{d}{dt}\right)\right)$ is a minimal polynomial kernel representation. Let $Q = UD\begin{bmatrix} I & 0\end{bmatrix}V$ be the Smith form of Q. Then from Lemma 4.2 we have $\mathfrak{B}_{\text{contr}} = \ker\left(\begin{bmatrix} I & 0\end{bmatrix}V\left(\frac{d}{dt}\right)\right)$. Let $n \in (\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$. Let $n = u^{-1}v$ be a left coprime factorization of n over $\mathbb{R}[\xi]$. Then by definition we have $v\left(\frac{d}{dt}\right) w = 0$ for all $w \in \mathfrak{B}_{\text{contr}}$. Thus, by Definition 5.1, $v \in (\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}[\xi]}}$. Consequently, there exists a $l \in \mathbb{R}[\xi]^{1\times\bullet}$ such that $v = l\begin{bmatrix} I & 0\end{bmatrix}V$. Hence

$$
\begin{aligned}
n &= u^{-1}v \\
&= u^{-1}l\begin{bmatrix} I & 0\end{bmatrix}V \\
&= (u^{-1}lD^{-1}U^{-1}P)(P^{-1}UD\begin{bmatrix} I & 0\end{bmatrix}V) \\
&= (u^{-1}lD^{-1}U^{-1}P)(P^{-1}Q) \\
&= (u^{-1}lD^{-1}U^{-1}P)G.
\end{aligned}
$$

Define $m := u^{-1}lD^{-1}U^{-1}P$. Then we have $n = mG$. Thus, n is a $\mathbb{R}(\xi)$-linear combination of the rows of G. Since n was arbitrary, the rows of G span the subspace $(\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$ of the $\mathbb{R}(\xi)$-linear vector space $\mathbb{R}(\xi)^{1\times\mathtt{w}}$. Finally, as $\mathfrak{B} = \ker\left(G\left(\frac{d}{dt}\right)\right)$ is a minimal rational kernel representation, the rows of G are linearly independent over $\mathbb{R}(\xi)$. We conclude then that these rows form a basis of $(\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$. □

Remark 5.3. It follows immediately from the previous theorem for any behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ we have $\dim(\mathfrak{B}^{\perp_{\mathbb{R}(\xi)}}) = \mathtt{p}(\mathfrak{B})$, i.e. the dimension of the linear space of rational annihilators of a controllable behavior is equal to the output cardinality of $\mathfrak{B}$.

The following theorem is an immediate consequence of Theorem 5.2. It gives necessary and sufficient conditions for the controllable parts of two behaviors to be equal in terms of the rational kernel representations.

Theorem 5.4. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{d}{dt}\right)\right)$ be minimal rational kernel representations. Then the following statements are equivalent:*

(a) $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$.
(b) *There exists a nonsingular rational matrix W such that $G_1 = WG_2$.*
(c) *There exist nonsingular polynomial matrices M and N such that $MG_1 = NG_2$.*

Proof. The equivalence of (b) and (c) is obvious. We first prove the implication (a) ⇒ (b). As $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$ we have $(\mathfrak{B}_{1,\text{contr}})^{\perp_{\mathbb{R}(\xi)}} = (\mathfrak{B}_{2,\text{contr}})^{\perp_{\mathbb{R}(\xi)}} =: \mathfrak{T}$. From Theorem 5.2, the rows of G_1 and G_2 both form a basis for the subspace $\mathfrak{T}$ of $\mathbb{R}(\xi)^{1\times\mathtt{w}}$. Then, from basic linear algebra, there exists a square, nonsingular rational matrix W such that $G_1 = WG_2$.

Conversely, assume $G_1 = WG_2$. Let $G_1 = P_1^{-1}Q_1$ and $G_2 = P_2^{-1}Q_2$ be left coprime factorizations over $\mathbb{R}[\xi]$ of G_1 and G_2. Let $W = M^{-1}N$ be a left coprime factorization over $\mathbb{R}[\xi]$ of W. Then both M and N are nonsingular. By definition we have $\mathfrak{B}_1 = \ker\left(Q_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(Q_2\left(\frac{d}{dt}\right)\right)$. Then,

$$
\begin{aligned}
G_1 = WG_2 &\iff P_1^{-1}Q_1 = M^{-1}NP_2^{-1}Q_2 \\
&\iff Q_1 = P_1M^{-1}NP_2^{-1}Q_2.
\end{aligned}
$$

Now factorize $P_1M^{-1}NP_2^{-1} = \tilde{M}^{-1}\tilde{N}$. Then we have $\tilde{M}Q_1 = \tilde{N}Q_2$. From Theorem 4.6,(a) follows. □

Evidently, the above theorem only gives a necessary condition on G_1 and G_2 for the associated behaviors to be equal. Again however, we would like to obtain conditions that are necessary *and* sufficient. As shown in Corollary 4.7, in case of polynomial kernel representations, Statement 3 of Theorem 5.4 together with unimodularity of $M^{-1}N$ serves the purpose. Hence, a first guess is to check whether this also holds true for rational representations. However, the following simple counterexample shows that this is not the case.

Example 5.5. $G_1(\xi) = 1$ and $G_2(\xi) = \frac{1}{\xi}$. These are equivalent representations since they both represent the $\{0\}$-behavior. For all M, N such that $MG_1 = NG_2$, we have $M^{-1}N = \frac{1}{\xi}$, which is not even a polynomial.

In order to proceed we need following definition:

Definition 5.6. A *greatest common left divisor (gcld)* of two polynomial matrices $P, Q \in \mathbb{R}[\xi]^{\mathtt{m}} \times \bullet$ is any square polynomial matrix D such that $P = DP_1$ and $Q = DQ_1$, and such that for all square polynomial matrices D_1 satisfying $P = D_1P_1$ and $Q = D_1Q_1$ there exists a polynomial matrix F such that $D = D_1F$.

For given polynomial matrices P and Q, we denote by $\text{gcld}(P, Q)$ any greatest common left divisor (gcld) of P and Q. If $\begin{bmatrix} P & Q\end{bmatrix}$ has full row rank, then any gcld must be a non-singular polynomial matrix. In that case any two gcld's are related by post-multiplication with a unimodular polynomial matrix.

Now, the following theorem is the first main result of this paper. The theorem states that the additional conditions on M and N so that the sets of autonomous parts of $\ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\ker\left(G_2\left(\frac{d}{dt}\right)\right)$ are also equal involve the greatest common left divisor matrices $\text{gcld}(M, MG_1)$ and $\text{gcld}(N, NG_2)$. More precisely:

Theorem 5.7. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{d}{dt}\right)\right)$ be minimal rational kernel representations. Assume $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$. Then we have $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$ if and only if there exist square nonsingular polynomial matrices M and N such that $MG_1 = NG_2$, $MG_1 = NG_2$ is a polynomial matrix, and $\text{gcld}(M, MG_1)^{-1}\,\text{gcld}(N, NG_2)$ is a unimodular polynomial matrix.*

Proof (*Only if*). Let U_i and V_i be unimodular polynomial matrices such that $G_i = U_i \Pi_i^{-1} \begin{bmatrix} D_i & 0 \end{bmatrix} V_i$, where $\Pi_i^{-1} \begin{bmatrix} D_i & 0 \end{bmatrix}$ is the Smith–McMillan form of G_i, for $i = 1, 2$.

Assume $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$. Then from Remark 4.3, $\mathcal{P} \in \mathcal{A}(\mathfrak{B}_1)$ admits a polynomial kernel representation

$$\ker\left(\begin{bmatrix} D_1\left(\frac{d}{dt}\right) & 0 \\ F_1\left(\frac{d}{dt}\right) & S_1\left(\frac{d}{dt}\right) \end{bmatrix} V_1\left(\frac{d}{dt}\right)\right),$$

similarly it also admits apolynomial kernel representation

$$\ker\left(\begin{bmatrix} D_2\left(\frac{d}{dt}\right) & 0 \\ F_2\left(\frac{d}{dt}\right) & S_2\left(\frac{d}{dt}\right) \end{bmatrix} V_2\left(\frac{d}{dt}\right)\right),$$

where F_1, F_2 are arbitrary polynomial matrices of appropriate dimensions and S_1, S_2 are unimodular polynomial matrices. From Proposition 4.4, there exists a U, a unimodular polynomial matrix, such that

$$\begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 = U \begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2.$$

Using the assumption that $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$, it can be verified that U must be of the form $U = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix}$, where U_{11} and U_{22} are unimodular polynomial matrices. Therefore we have

$$\begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2,$$

which implies

$$\begin{bmatrix} \Pi_1 U_1^{-1} U_1 \Pi_1^{-1} \begin{bmatrix} D_1 & 0 \end{bmatrix} V_1 \\ \begin{bmatrix} F_1 & S_1 \end{bmatrix} V_1 \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Pi_2 U_2^{-1} U_2 \Pi_2^{-1} \begin{bmatrix} D_2 & 0 \end{bmatrix} V_2 \\ \begin{bmatrix} F_2 & S_2 \end{bmatrix} V_2 \end{bmatrix}.$$

Define $M := \Pi_1 U_1^{-1}$ and $N := U_{11} \Pi_2 U_2^{-1}$. Then we have

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G_1 \\ \begin{bmatrix} F_1 & S_1 \end{bmatrix} V_1 \end{bmatrix} = \begin{bmatrix} N & 0 \\ U_{21} \Pi_2 U_2^{-1} & U_{22} \end{bmatrix} \begin{bmatrix} G_2 \\ \begin{bmatrix} F_2 & S_2 \end{bmatrix} V_2 \end{bmatrix}.$$

It is evident from the above equation that MG_1 and NG_2 are polynomial matrices and that $MG_1 = NG_2$. Define $L := MG_1 = NG_2$. Then we have $R_1 := \text{gcld}(M, L) = I$, and similarly $R_2 := \text{gcld}(N, L) = U_{11}$. Hence, it is evident that $R_1^{-1} R_2 = U_{11}$ is a unimodular polynomial matrix.

(If): Assume that $L := MG_1 = NG_2$ is a polynomial matrix. Let $\text{gcld}(M, L) =: R_1$ and $\text{gcld}(N, L) =: R_2$. Let $G_1 = P_1^{-1} Q_1$ and $G_2 = P_2^{-1} Q_2$ be left coprime factorizations of G_1 and G_2 over $\mathbb{R}[\xi]$. Obviously we have $P_1^{-1} Q_1 = M^{-1} L$ and $P_2^{-1} Q_2 = N^{-1} L$. Hence, from [18], Lemma 6.5-5, there exist square nonsingular polynomial matrices $\tilde{R}_1, \tilde{R}_2$ such that $\tilde{R}_1 \begin{bmatrix} P_1 & Q_1 \end{bmatrix} = \begin{bmatrix} M & L \end{bmatrix}$ and $\tilde{R}_2 \begin{bmatrix} P_2 & Q_2 \end{bmatrix} = \begin{bmatrix} N & L \end{bmatrix}$. Further, using the left primeness of $\begin{bmatrix} P_i & Q_i \end{bmatrix}$, it can be verified that $\tilde{R}_1$ and $\tilde{R}_2$ are gcld's of $\begin{bmatrix} M & L \end{bmatrix}$and $\begin{bmatrix} N & L \end{bmatrix}$ respectively. Also, since M and N are square nonsingular polynomial matrices, $\begin{bmatrix} M & L \end{bmatrix}$ and $\begin{bmatrix} N & L \end{bmatrix}$ have full row rank. Consequently, we have that R_1 and R_2 are nonsingular. Hence, there exist unimodular polynomial matrices U_1 and U_2 such that $R_1 = \tilde{R}_1 U_1$ and $R_2 = \tilde{R}_2 U_2$. Define $\tilde{M} := R_1 U_1$, $\tilde{N} := R_2 U_2$. Then we have $\tilde{M} Q_1 = \tilde{N} Q_2$ and $\tilde{M}^{-1} \tilde{N} = U$, which is a unimodular polynomial matrix. Therefore, from Theorem 4.6, we have $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$. □

The following corollary is the second main result of this paper. It gives necessary and sufficient conditions on the rational matrices G_1 and G_2 for $\ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\ker\left(G_2\left(\frac{d}{dt}\right)\right)$ to be equal. In fact, by combining Theorems 5.4 and 5.7 we immediately obtain:

Corollary 5.8. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{d}{dt}\right)\right)$ be minimal rational kernel representations. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exist square and nonsingular polynomial matrices M, N such that*

(a) $MG_1 = NG_2$,
(b) $MG_1 = NG_2$ *is a polynomial matrix and*
(c) $\text{gcld}(M, MG_1)^{-1}\text{gcld}(N, NG_2)$ *is a unimodular polynomial matrix.*

Corollary 5.8 is illustrated below in the following examples.

Example 5.9. $G_1(\xi) = 1$, $G_2(\xi) = \frac{1}{\xi}$ represent the same behavior:

1. $MG_1 = NG_2$ with $N(\xi) = \xi$, $M(\xi) = 1$ nonsingular polynomial,
2. $MG_1 = NG_2 = 1$ is polynomial and $\text{gcld}(N, NG_2) = \text{gcd}(\xi, 1) = 1$, $\text{gcld}(M, MG_1) = \text{gcd}(1, 1) = 1$.

Example 5.10. $G_1(\xi) = (\xi \;\; \xi)$, $G_2(\xi) = \left(\frac{1}{\xi} \;\; \frac{1}{\xi}\right)$ do not represent the same behavior:

1. their controllable parts are the same: $MG_1 = NG_2$ with $N(\xi) = \xi^2$, $M(\xi) = 1$ nonsingular polynomial,
2. for any M, N such that $MG_1 = NG_2$ we must have $N(\xi) = \xi^2 M(\xi)$. Hence $\text{gcld}(M, MG_1) = \text{gcd}(M, \xi M, \xi M) = M$, while $\text{gcld}(N, NG_2) = \text{gcd}(\xi^2 M, \xi M, \xi M) = \xi M$.

Remark 5.11. We note that, in the case that G_1 and G_2 are polynomial matrices, Corollary 5.8 immediately yields Corollary 4.7. Indeed, in that case $\text{gcld}(M, MG_1) = M$ and $\text{gcld}(N, NG_2) = N$ so condition (b) becomes: $M^{-1}N$ is a unimodular polynomial matrix.

According to Corollary 5.8, in order to check the equivalence of rational representations, we need to check for the existence of square and nonsingular polynomial matrices M and N that satisfy (a), (b) and (c) in Corollary 5.8. The algorithm below achieves this objective.

Algorithm 1. Let $G_1, G_2 \in \mathbb{R}(\xi)^{\mathtt{k} \times \mathtt{w}}$ and let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{d}{dt}\right)\right)$ be minimal kernel representations. Then,

1. Solve $G_1 = WG_2$ for $W \in \mathbb{R}(\xi)^{\mathtt{k} \times \mathtt{k}}$. If there exists no solution, declare $\mathfrak{B}_1 \neq \mathfrak{B}_2$, else continue further.
2. Find a left coprime factorization of W over $\mathbb{R}[\xi]$. Let it be $W = M^{-1}N$, where M, N are square and nonsingular polynomial matrices.
3. Find a left coprime factorization of MG_1 over $\mathbb{R}[\xi]$. Let it be $MG_1 = P^{-1}Q$ where $P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}$. Then $PMG_1 = PNG_2 = Q$ is a polynomial matrix.
4. Find $L := \text{gcld}(PM, Q)^{-1}\text{gcld}(PN, Q)$. If L is a unimodular polynomial matrix, declare $\mathfrak{B}_1 = \mathfrak{B}_2$, else declare $\mathfrak{B}_1 \neq \mathfrak{B}_2$.

Before elaborating on the above algorithm, we state an alternative algorithm to check the equivalence of rational representations of a given behavior.

Algorithm 2. Let $G_1, G_2 \in \mathbb{R}(\xi)^{\mathtt{k} \times \mathtt{w}}$ and let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{d}{dt}\right)\right)$ be minimal kernel representations. Then,

1. Find a left coprime factorization of G_1 over $\mathbb{R}[\xi]$. Let it be $G_1 = P_1^{-1}Q_1$, where $P_1, Q_1 \in \mathbb{R}[\xi]^{\bullet \times \bullet}$.

2. Find a left coprime factorization of G_2 over $\mathbb{R}[\xi]$. Let it be $G_2 = P_2^{-1}Q_2$, where $P_2, Q_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.
3. Solve $Q_1 = UQ_2$ for U, where U is a unimodular polynomial matrix. If a solution exists, declare $\mathfrak{B}_1 = \mathfrak{B}_2$, else declare $\mathfrak{B}_1 \neq \mathfrak{B}_2$.

The first algorithm has two advantages. Firstly, in case the behaviors $\mathfrak{B}_1, \mathfrak{B}_2$ are not equal, it is already declared in Step-1, without actually proceeding to left coprime factorizations. Secondly, it finds in Step-1 whether the controllable parts of the behavior are equal for the given kernel representations.

6. A module characterization of equivalence of rational representations

In this section, we will give conditions for the equivalence of rational representations of a given behavior in terms of the polynomial modules generated by the rows of the rational matrices. In Section 5, the polynomial and rational annihilators of a given behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ have been introduced and discussed. For a given behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, with rational representation $\mathfrak{B} = \ker\left(G\left(\frac{d}{dt}\right)\right)$, we will now first establish the relation between the $\mathbb{R}[\xi]$-module generated by the rows of the rational matrix G, and the module of polynomial annihilators of $\mathfrak{B}$. In case of polynomial kernel representations, the following proposition is well known.

Proposition 6.1. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$ be a minimal polynomial kernel representation. Let $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$ denote the module of polynomial annihilators of $\mathfrak{B}$. Then the rows of R form a basis for $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$.*

In the following, for a given rational matrix $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$, we will denote by $\langle G\rangle_{\mathbb{R}[\xi]}$ the set of all linear combinations of the rows of G using coefficients from the polynomial ring $\mathbb{R}[\xi]$. Clearly this set is a $\mathbb{R}[\xi]$-module. The intersection $\langle G\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}}$ consists of all linear combinations of the rows of G using coefficients from $\mathbb{R}[\xi]$ that are polynomial vectors. Clearly, this intersection is a $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$. The following theorem states that this intersection module is in fact equal to the module of polynomial annihilators of $\mathfrak{B}$.

Theorem 6.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$. Let $\mathfrak{B} = \ker\left(G\left(\frac{d}{dt}\right)\right)$ be a minimal rational kernel representation. Then $\langle G\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}} = \mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$.*

Proof. Let $G = P^{-1}Q$ be a left coprime factorization of G over $\mathbb{R}[\xi]$. We first prove the following inclusion: $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}} \subseteq \langle G\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}}$. By Definition 2.2, $\mathfrak{B} = \ker\left(Q\left(\frac{d}{dt}\right)\right)$. From Proposition 6.1, we know that the rows of Q form a basis for $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$. Let $n \in \mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$. Then there exists a polynomial row vector l such that $n = lQ$. Hence

$$\begin{aligned} n &= lQ \\ &= lPP^{-1}Q \\ &= lPG \\ &= mG, \end{aligned}$$

where $m = lP$ is a polynomial row vector. Therefore $n \in \langle G\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}}$.

We now prove the converse inclusion, $\langle G\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}} \subseteq \mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$. Let U and V be unimodular polynomial matrices such that $G = U\Pi^{-1}\begin{bmatrix} D & 0\end{bmatrix}V$, where $\Pi^{-1}\begin{bmatrix} D & 0\end{bmatrix}$ is the Smith–McMillan form of G_i. Define $P := \Pi U^{-1}$ and $Q := \begin{bmatrix} D & 0\end{bmatrix}V$. Then $G = P^{-1}Q$ is a left coprime factorization and $\mathfrak{B} = \ker\left(Q\left(\frac{d}{dt}\right)\right)$. Let $l \in \langle G\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}}$. Then we have $l = nG$ for some polynomial row vector n. Define $l' := lV^{-1}$. Then

$$\begin{aligned} l' &= nU\Pi^{-1}\begin{bmatrix} D & 0\end{bmatrix}, \quad \text{where } l' := lV^{-1} \\ &= \tilde{n}\Pi^{-1}\begin{bmatrix} D & 0\end{bmatrix}, \quad \text{where } \tilde{n} := nU \\ &= \tilde{n}\begin{bmatrix} \operatorname{diag}\left(\frac{z_1}{\pi_1}, \ldots, \frac{z_k}{\pi_k}\right) & 0\end{bmatrix}. \end{aligned} \tag{12}$$

Write $l' = \begin{bmatrix} l_1 & l_2 & \cdots & l_{\mathtt{w}}\end{bmatrix}$ and $\tilde{n} = \begin{bmatrix} \tilde{n}_1 & \tilde{n}_2 & \cdots & \tilde{n}_k\end{bmatrix}$. From Eq. (12) we have $l'_i = \tilde{n}_i\frac{z_i}{\pi_i}$, for $i = 1, 2, \ldots, k$ and, $l'_i = 0$ for $i > k$. As (z_i, π_i) are coprime, there exists $m_i \in \mathbb{R}[\xi]$ such that $\tilde{n}_i = m_i\pi_i$, for $k = 1, 2, \ldots, k$. Let $m = \begin{bmatrix} m_1 & m_2 & \cdots & m_k\end{bmatrix}$. Then $n = \tilde{n}U^{-1} = m\Pi U^{-1} = mP$ so $l = nG = mPG = mQ$. Therefore $l \in \mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$. □

By combining Theorems 6.2 and 5.7 we finally get the following complete characterization of the equivalence of rational kernel representations:

Theorem 6.3. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathfrak{B}_1 = \ker\left(G_1\left(\frac{d}{dt}\right)\right)$ and $\mathfrak{B}_2 = \ker\left(G_2\left(\frac{d}{dt}\right)\right)$ be minimal rational kernel representations. Then following statements are equivalent:*

1. $\mathfrak{B}_1 = \mathfrak{B}_2$.
2. $\mathfrak{B}_1^{\perp_{\mathbb{R}[\xi]}} = \mathfrak{B}_2^{\perp_{\mathbb{R}[\xi]}}$.
3. *There exists square non-singular polynomial matrices M and N such that*
   (a) $MG_1 = NG_2$,
   (b) *$MG_1 = NG_2$ is a polynomial matrix and*
   (c) *$\text{gcld}(M, MG_1)^{-1}\text{gcld}(N, NG_2)$ is a unimodular polynomial matrix.*
4. $\langle G_1\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}} = \langle G_2\rangle_{\mathbb{R}[\xi]} \cap \mathbb{R}[\xi]^{1\times\mathtt{w}}$.

7. Equivalence of rational image representations

In this section we will address the issue of equivalence of rational image representations. In particular, we will establish a solution to Problem 2 as stated in Section 3. We first recall the following fact on polynomial and rational image representations of behaviors (see Theorem 9 in [2]).

Theorem 7.1. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is controllable,*
2. *$\mathfrak{B}$ admits a polynomial image representation $\mathfrak{B} = \operatorname{im}\left(M\left(\frac{d}{dt}\right)\right)$, with $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ of full column rank.*
3. *$\mathfrak{B}$ admits a polynomial image representation $\mathfrak{B} = \operatorname{im}\left(M\left(\frac{d}{dt}\right)\right)$ with $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ right prime over $\mathbb{R}[\xi]$,*
4. *$\mathfrak{B}$ admits a rational image representation with $\mathfrak{B} = \operatorname{im}\left(H\left(\frac{d}{dt}\right)\right)$ with $H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$ of full column rank.*

In the sequel, the following result will be useful. The result states that right coprime factorization of a rational image representation leads to a polynomial image representation.

Lemma 7.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $H \in R(\xi)^{\mathtt{w}\times\bullet}$ be such that $\mathfrak{B} = \operatorname{im}\left(H\left(\frac{d}{dt}\right)\right)$. Let $H = MP^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then $\mathfrak{B} = \operatorname{im}\left(M\left(\frac{d}{dt}\right)\right)$.*

Proof. Let $H = D^{-1}N$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then we have

$$\ker\left(\begin{bmatrix} D\left(\frac{d}{dt}\right) & N\left(\frac{d}{dt}\right)\end{bmatrix}\right) = \operatorname{im}\left(\begin{bmatrix} M\left(\frac{d}{dt}\right) \\ -P\left(\frac{d}{dt}\right)\end{bmatrix}\right).$$

Thuswe obtain

$$\begin{aligned}\mathfrak{B} &= \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \text{ s.t. } D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\} \\ &= \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell, \ell' \text{ s.t. } \begin{pmatrix} w \\ \ell \end{pmatrix} = \begin{pmatrix} M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ -P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{pmatrix} \ell' \right\} \\ &= \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell' \text{ s.t. } w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell' \right\}. \quad \square\end{aligned}$$

In order to proceed, we will now first study the question under which conditions two polynomial image representations are equivalent, i.e. represent the same behavior.

Theorem 7.3. 1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $M_1, M_2 \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ have full column rank, and let $\mathfrak{B}_1 = \operatorname{im}\left(M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \operatorname{im}\left(M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a square nonsingular rational matrix R such that $M_2 = M_1R$.*

2. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $M_1, M_2 \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ be right prime over $\mathbb{R}[\xi]$, and let $\mathfrak{B}_1 = \operatorname{im}\left(M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \operatorname{im}\left(M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular polynomial matrix U such that $M_2 = M_1U$.*

Proof. We first prove the 'only if' part of Statement 2. By right primeness, both $M_1(\lambda)$ and $M_2(\lambda)$ have full column rank for all $\lambda \in \mathbb{C}$, so correspond to observable image representations. From $\mathfrak{B}_1 = \mathfrak{B}_2$ it follows that also the orthogonal complements coincide, i.e. $\mathfrak{B}_1^\perp = \mathfrak{B}_2^\perp$ (see [19]). By observability we have $\mathfrak{B}_i^\perp = \ker\left(M_i^\sim\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, where $M_i^\sim(\xi) := M_i^\top(-\xi)$ $(i = 1, 2)$. By Proposition 4.4 there exists a unimodular polynomial matrix V such that $M_2^\sim = VM_1^\sim$. This implies $M_2 = M_1U$, with $U := V^\sim$ again unimodular.

Next, we prove the 'only if' part of Statement 1. Both M_1 and M_2 have full column rank. Hence, we can factorize $M_i = \overline{M}_iR_i$, with $\overline{M}_i$ right prime over $\mathbb{R}[\xi]$ and R_i a nonsingular polynomial matrix $(i = 1, 2)$. By nonsingularity, $R_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is surjective, and therefore $\operatorname{im}\left(M_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(\overline{M}_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ $(i = 1, 2)$. Consequently, $\mathfrak{B}_1 = \mathfrak{B}_2$ implies $\operatorname{im}\left(\overline{M}_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(\overline{M}_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Then, by the 'only if' part of Statement 2, there exists a unimodular polynomial matrix U such that $\overline{M}_2 = \overline{M}_1U$. This implies $M_2 = M_1R$, with $R := R_1^{-1}UR_2$.

Finally we prove the 'if' part of Statement 1. Assume that $M_2 = M_1R$ with R a nonsingular rational matrix. Let $R = KL^{-1}$ be a right coprime factorization of R over $\mathbb{R}[\xi]$. Then we have $M_2L = M_1K$, with K and L nonsingular polynomial matrices. Again by the surjectivity of $L\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ and $K\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, we obtain $\mathfrak{B}_1 = \operatorname{im}\left(M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)K\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)L\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \mathfrak{B}_2$. This also proves the 'if' part of Statement 2. $\square$

Next, we consider controllable behaviors represented by rational image representations.

Theorem 7.4. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $H_1, H_2 \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$ have full column rank and let $\mathfrak{B}_1 = \operatorname{im}\left(H_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \operatorname{im}\left(H_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a square nonsingular rational matrix R such that $H_2 = H_1R$.*

Proof. Let $H_i = M_iP_i^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then by Lemma 7.2, $\mathfrak{B}_i = \operatorname{im}\left(M_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ $(i = 1, 2)$. By Theorem 7.3, $\mathfrak{B}_1 = \mathfrak{B}_2$ implies that there exists a nonsingular rational matrix $\overline{R}$ such that $M_2 = M_1\overline{R}$. Thus $H_2 = H_1R$, with $R := P_1\overline{R}P_2^{-1}$ nonsingular. Conversely, if $H_2 = H_1R$ then $M_2 = M_1P_1^{-1}RP_2$. Then, by Theorem 7.3, $\operatorname{im}\left(M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ so $\mathfrak{B}_1 = \mathfrak{B}_2$. $\square$

8. Equivalence of rational kernel and image representations

So far, we have derived necessary and sufficient conditions under which the kernels of two rational differential operators represent one and the same behavior and conditions under which the images of two rational differential operators represent the same behavior. In the present section, we will derive necessary and sufficient conditions for behaviors given as the kernel of a rational differential operator and the image of a rational differential operator to be equal.

Theorem 8.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $\mathfrak{B}_1 = \ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \operatorname{im}\left(H\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be a minimal rational kernel representation and a rational image representation of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively, where $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{w}}$ has full row rank and $H \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{m}}$ has full column rank. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if $GH = 0$ and $\mathtt{p} + \mathtt{m} = \mathtt{w}$.*

Proof. Let $G = P^{-1}Q$ and $H = ND^{-1}$ be left and right coprime factorizations of G and H, respectively over $\mathbb{R}[\xi]$. Then from Definition 2.2 and Lemma 7.2, we have $\mathfrak{B}_1 = \ker\left(Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ and $\mathfrak{B}_2 = \operatorname{im}\left(N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. As $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$, we recall that $\mathfrak{B}_1^{\perp_{\mathbb{R}(\xi)}}$ and $\mathfrak{B}_2^{\perp_{\mathbb{R}(\xi)}}$ are subspaces of the $\mathbb{R}(\xi)$-linear vector space $\mathbb{R}(\xi)^{1\times\mathtt{w}}$ (see Theorem 5.2) with $\dim(\mathfrak{B}_1^{\perp_{\mathbb{R}(\xi)}}) = \mathtt{p}$ and $\dim(\mathfrak{B}_2^{\perp_{\mathbb{R}(\xi)}}) = \mathtt{w}-\mathtt{m}$.

(Only if): As $\mathfrak{B}_1 = \mathfrak{B}_2$ we have $\mathtt{p} = \mathtt{w} - \mathtt{m}$. Further, since $\ker\left(Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) = \operatorname{im}\left(N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, we have $Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}-\mathtt{m}})$. Consequently, we have $QN = 0$ so $GH = 0$.

(If) We have $GH = 0$ if and only if $P^{-1}QND^{-1} = 0$ if and only if $QN = 0$. In order to proceed we first prove $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$. Let $w \in \mathfrak{B}_2$, then there exists an $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}-\mathtt{m}})$ such that $w = N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$. Then $Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$ (since $Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = 0$). Therefore $w \in \mathfrak{B}_1$, so $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$. Finally $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$ implies that $\mathfrak{B}_1^{\perp_{\mathbb{R}(\xi)}} \subseteq \mathfrak{B}_2^{\perp_{\mathbb{R}(\xi)}}$. By using the assumption that $\mathtt{p} = \mathtt{w} - \mathtt{m}$, the dimensions of these two subspaces are equal, so we must have $\mathfrak{B}_1^{\perp_{\mathbb{R}(\xi)}} = \mathfrak{B}_2^{\perp_{\mathbb{R}(\xi)}}$. Therefore we conclude that $\mathfrak{B}_1 = \mathfrak{B}_2$. $\square$

9. Conclusions

In this paper we have dealt with the equivalence of representations of a given behavior with emphasis on rational representations. We have obtained new necessary and sufficient conditions for the equivalence of polynomial kernel representations, illustrating the origin of the unimodular matrix appearing in equivalent polynomial kernel representations. As the first major contribution of this paper, we have obtained necessary and sufficient conditions for the equivalence of rational kernel representations of controllable as well as uncontrollable behaviors. As the second major contribution of this paper, we also have derived conditions for the equivalence of rational representations of a given behavior in terms of the polynomial modules generated by the rows of the rational matrices. Further, we have obtained necessary and sufficient conditions for the equivalence of image representations in the context of both polynomial and rational representations. Finally, we have obtained necessary and sufficient conditions for the equality of behaviors defined as the kernel of a rational differential operator and the image of a rational differential operator.

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