Discrete Mathematics 97 (1991) 357–369
North-Holland

The spectrum of maximal sets of one-factors

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Received 7 March 1990

In memory of Egmont Köhler.

Abstract

Rees, R. and W.D. Wallis, The spectrum of maximal sets of one-factors, Discrete Mathematics 97 (1991) 357–369.

A set $\{F_i\}$ of disjoint one-factors on n vertices is maximal if the complement of the graph $\bigcup F_i$ has no one-factor. We determine the spectrum of pairs $\{(n, k)$: there exists a maximal set of k one-factors on n vertices$\}$.

1. Introduction

A graph $G = (V, E)$ consists of a non-empty set V of vertices together with a collection E of unordered pairs of distinct vertices from V, these pairs being called edges. Two vertices are said to be adjacent if and only if there is an edge joining them. The *degree* of a vertex is the number of edges to which it belongs; a graph is called *regular* if and only if every vertex has the same degree. If $G = (V, E)$ and $H = (V, E')$ are graphs then the union $G \cup H$ of G and H is the graph $(V, E \cup E')$. The complement $\bar{G}$ of G is the graph $(V, \bar{E})$ where $\bar{E} = \{(x, y): (x, y) \notin E\}$. In particular then if G is any graph on n vertices, $G \cup \bar{G}$ is the complete graph K_n.

A *matching* is a vertex-disjoint collection of edges; a *one-factor* is a matching which covers the vertices of G (or, equivalently, a 1-regular spanning subgraph of G). A pair of one-factors will be called *disjoint* if they have no edges in common. A one-factorization of G is a collection of pairwise disjoint one-factors which partitions the edge set of G. In order to have one-factorization, G must have an

0012-365X/91/$03.50

even number of vertices and must be regular, but these necessary conditions are not sufficient. Petersen [7] observed that if a regular graph of degree 3 has an edge whose removal disconnects the graph (a 'bridge') then it has no one-factorization; and there are bridgeless regular graphs without one-factorizations (the Petersen graph is an example on 10 vertices).

It has been widely conjectured (see [3, 6]) that every regular graph G of degree d on $2m$ vertices has a one-factorization 'provided d is large enough'—where 'large enough' usually means that d is approximately m or bigger. Various forms of this conjecture have been called 'the one-factorization conjecture'. In considering this there has been some interest in the case where the complement of G has a one-factorization, or equivalently in discussing whether a set of pairwise disjoint one-factors can be embedded in a one-factorization of a complete graph.

A set of one-factors of K_{2m} is called *premature* [8] if they are edge-disjoint but cannot be extended to a one-factorization; a premature set is *maximal* [4] if it cannot be extended by adding even one more factor. In other words, a collection $\{F_i\}$ of mutually disjoint one-factors on a set V of $2m$ vertices will be called a *maximal* set if the graph $\overline{\bigcup F_i}$ contains no one-factor. The problem with which we are herein concerned is to determine for which integers $0 < k < 2m$ does there exist a maximal set $F_1, \ldots, F_k$ of precisely k one-factors on a set of $2m$ vertices. We are able to give a complete solution to this problem.

Theorem 1.0. *Let n be a positive even integer. There exists a maximal set of k mutually disjoint one-factors on n vertices if and only if either*

(i) $2 \cdot \lfloor n/4 \rfloor + 1 \leqslant k \leqslant n-1$ *and k is odd, or*

(ii) $\frac{1}{3}(2n+4) \leqslant k \leqslant n-4$ *and k is even.*

We direct the reader to [0] for a general reference on graph theory, and to [6] for a specific discussion of and survey on one-factors and one-factorizations.

2. Preliminary results

In this section we review some of the basic material which we will need to prove Theorem 1.0.

Lemma 2.1. *If m is odd then a maximal set in K_{2m} contains at least m one-factors. If m is even, a maximal set in K_{2m} contains at least $m+1$ one-factors.*

The bulk of the proof of Lemma 2.1 is a straightforward application of Dirac's Theorem. Showing that when m is even K_{2m} cannot contain a maximal set of m one-factors is a little more involved, and we refer the reader to [4].

It is easy to see that there cannot exist a maximal set of $n-2$ one-factors on n vertices; the complement of the union of these one-factors is itself a one-factor. To establish the lower bound $k \geqslant \frac{1}{3}(2n+4)$ in condition (ii) of Theorem 1.0 we will need the following result from [9].

Lemma 2.2. *Let G be a regular graph of odd valency d on n vertices. If G has no one-factor then $n \geqslant 3d+7$.*

Proof. Since G has no one-factor Tutte's Theorem implies that there is some w-set W of vertices in G whose deletion creates at least $w+2$ odd components (since n is even, w and the number of odd components must have the same parity). Since d is odd, G itself cannot have any odd components, whence $w \geqslant 1$. Let us call an odd component of $G-W$ large if it has more than d vertices, and small otherwise. Clearly any odd component of $G-W$ is joined to W by at least one edge; in the case of a small component it is not difficult to see that there must be at least d edges joining it to W (since the graph G is regular of valency d). Thus if we let α be the number of large components and β be the number of small components of $G-W$, we have

$$\alpha+\beta \geqslant w+2 \tag{2.1}$$

and

$$\alpha+d\beta \leqslant wd. \tag{2.2}$$

Since d is odd, each large component of $G-W$ has at least $d+2$ vertices. Therefore

$$n \geqslant w+(d+2)\alpha+\beta. \tag{2.3}$$

Now α is nonnegative, so inequality (2.2) implies $\beta \leqslant w$, so that in turn inequality (2.1) implies $\alpha \geqslant 2$; but applying (2.2) and (2.1) again we have in fact $\alpha \geqslant 3$. Recalling that $w \geqslant 1$ inequality (2.3) now gives $n \geqslant 3d+7$.

This completes the proof of Lemma 2.2. □

Now suppose that we have a maximal set of k one-factors on n vertices, where $n-k$ is even. Setting $d=n-k-1$ and applying Lemma 2.2 we see that $n \geqslant 3(n-k-1)+7$, which simplifies to $k \geqslant \frac{1}{3}(2n+4)$.

We summarize the foregoing discussion.

Lemma 2.3. *The conditions* (i), (ii) *of Theorem* 1.0 *are necessary in order that there exist a maximal set of k one-factors on n vertices.*

Showing that condition (i) is in fact sufficient is quite simple, and we dispense of this case now.

Theorem 2.4 [4]. *If n is a positive even integer, $2\lfloor n/4\rfloor+1 \leqslant k \leqslant n-1$ and $n-k$ is odd then there is a maximal set of k mutually disjoint one-factors on n vertices.*

Proof. Take the vertex set $\mathbf{Z}_k \cup \{a_i\colon 1 \leqslant i \leqslant n-k\}$, and develop the following one-factor modulo k:

$$\begin{array}{rl}
a_1, 0 & \frac{1}{2}(n-k-1)+1,\ k-\frac{1}{2}(n-k-1)-1 \\
a_2, 1 & \frac{1}{2}(n-k-1)+2,\ k-\frac{1}{2}(n-k-1)-2 \\
a_3, k-1 & \vdots \\
a_4, 2 & \frac{1}{2}(k-1),\ \frac{1}{2}(k+1) \\
\vdots & \\
a_{n-k-1}, \frac{1}{2}(n-k-1) & \\
a_{n-k}, k-\frac{1}{2}(n-k-1) &
\end{array}$$

The edges in the right hand column are used only when $k \neq n/2$. These edges represent $k-n/2$ pairs in a starter on $\mathbf{Z}_k$.

The complement of the union of these k one-factors has as one of its components a K_{n-k} (on the symbols $\{a_i\colon 1 \leqslant i \leqslant n-k\}$); since $n-k$ is odd our k one-factors constitute a maximal set. □

Dealing with the case where k is even is considerably more difficult (only the case $k=n-4$ has previously been solved, see [2]), and it is to this case that the remainder of the paper is devoted.

Let $F_1, \ldots, F_k$ be a maximal set of one-factors on n vertices. The complement of $\bigcup F_i$ is a regular graph of valency $n-1-k$; we will call the number $d=n-1-k$ the *deficiency* of the maximal set. For the sake of brevity we will call $F_1, \ldots, F_k$ a *d-set* in K_n. From Lemma 2.2 it remains to be shown that for each odd integer $d \geqslant 3$ and each even integer $n \geqslant 3d+7$ there exists a d-set in K_n.

We will need three preliminary results, the first of which involves the notion of a sub-one-fractorization. A one-factorization F of a graph G is a decomposition of the edge set of G into disjoint one-factors. If H is an induced subgraph of G then a one-factorization F' of H is called a *sub-one-factorization* of F provided that for each one-factor $f' \in F'$ there is a one-factor $f \in F$ such that $f' \subseteq f$. The following is well known (see e.g. [6]).

Lemma 2.5. *If m and n are even integers with $n \geqslant 2m$ then the complete graph K_n admits a one-factorization containing a sub-one-factorization of some $K_m \subseteq K_n$.*

Corollary 2.6. *If there is a d-set in K_m then there is a d-set in K_n for each even integer $n \geqslant 2m$.*

Proof. Take a one-factorization $\{F_1, F_2, \ldots, F_{n-1}\}$ on K_n containing a sub-one-factorization $\{F'_1, F'_2, \ldots, F'_{m-1}\}$ on K_m (where for each $i=1,2,\ldots,m-1$ $F'_i \subseteq F_i$). Replace the one-factors on K_m by a d-set $\{f_1, f_2, \ldots, f_{m-1-d}\}$. Then the one-factors in

$$\{f_1 \cup (F_1 - F_1'), f_2 \cup (F_2 - F_2'), \ldots, f_{m-1-d} \cup (F_{m-1-d} - F'_{m-1-d})\}$$
$$\cup \{F_m, F_{m+1}, \ldots, F_{n-1}\}$$

form a d-set in K_n. □

The second result is a special case of a theorem of Folkman and Fulkerson [5, Theorem 4.2]. (For an elementary proof see e.g. [1].)

Lemma 2.7. *Let G be a graph with $e = c \cdot k$ edges, where $c \geq \chi(G)$ (=edge coloring number of G). Then the edge set of G admits a decomposition into c matchings, each with k edges.*

Finally, we will make use of the following result concerning edge-colorings in complete bipartite graphs.

Lemma 2.8. *Let $K_{m,n}$ be the complete bipartite graph with bipartition $[X, Y]$ where $|X| = m$, $|Y| = n$, $m \leq n$. Let $Y_1, Y_2, \ldots, Y_n$ be any collection of m-subsets of Y such that each vertex $y \in Y$ is contained in exactly m of the Y_js. Then there is an edge-decomposition of $K_{m,n}$ into matchings $M_1, M_2, \ldots, M_n$ where for each $j = 1, 2, \ldots, n$ M_j is a matching (with m edges) from X to Y_j.*

Proof. (J.A. Bondy, personal communication.) Let A be the $(0, 1)$-incidence matrix of the design with point set Y and blocks $Y_1, Y_2, \ldots, Y_n$. Then A is an $n \times n$ matrix with constant row and column sum m, and so by Hall's Theorem we can write A as a sum

$$A = P_1 + P_2 + \cdots + P_m$$

of permutation matrices. For each $j = 1, 2, \ldots, n$ the matching M_j is defined as follows: Take the jth column in A and let $y_{i_1}, y_{i_2}, \ldots, y_{i_m}$ be the vertices (in Y) indexing the rows in which a 1 occurs. For each $k = 1, 2, \ldots, m$ there is a unique permutation matrix P_k with a 1 in the (i_k, j) position. Now set $M_j = \{(x_k, y_{i_k})$: $k = 1, 2, \ldots, m$; $x_k \in X\}$. It is readily verified that the matchings $M_1, M_2, \ldots, M_n$ form an edge-decomposition of $K_{m,n}$. □

3. Maximal sets of odd deficiency

In this section we will establish that for each odd integer $d \geq 3$ and each even integer $n \geq 3d + 7$ there is a d-set in K_n. From the discussion following Theorem 2.4 this will, together with Theorem 2.4, establish Theorem 1.0.

We begin by defining a family of graphs which we shall use repeatedly throughout the sequel. For each odd integer $d \geq 3$ let $\mathcal{G}_d$ be the following graph on $3d+7$ vertices:

Vertex set: $((\mathbf{Z}_{d+1} \cup \{a\}) \times \{1, 2, 3\}) \cup \{\infty\}$

Edge set: all edges (x, y) where $x \in (\mathbf{Z}_{d+1} \cup \{a\}) \times \{i\}$ and $y \in (\mathbf{Z}_{d+1} \cup \{a\}) \times \{j\}$, $i \neq j$; additionally, the edges

$$0_1 1_1,\ 2_1 3_1, \ldots, (d-3)_1 (d-2)_1,\ (d-1)_1 a_1,\ d_1 a_1;$$

$$0_2 1_2,\ 2_2 3_2, \ldots, (d-3)_2 (d-2)_2,\ (d-1)_2 a_2,\ d_2 a_2;$$

$$0_3 1_3;\ 2_3 3_3,\ 3_3 4_3, \ldots, \left(\frac{d-1}{2}\right)_3 \left(\frac{d+1}{2}\right)_3, \left(\frac{d+3}{2}\right)_3 \left(\frac{d+5}{2}\right)_3,$$

$$\left(\frac{d+5}{2}\right)_3 \left(\frac{d+7}{2}\right)_3, \ldots, (d-1)_3 d_3,\ a_3 2_3,\ a_3 d_3;$$

$$\infty 0_3,\ \infty 1_3,\ \infty\left(\frac{d+1}{2}\right)_3,\ \infty\left(\frac{d+3}{2}\right)_3;$$

and all edges ∞i_1, ∞i_2 where $i \in \mathbf{Z}_{d+1}$.

The graphs $\mathcal{G}_3$, $\mathcal{G}_5$ and $\mathcal{G}_d$ are illustrated in Figs. 1–3.

Note that $\mathcal{G}_d$ is a $(2d+6)$-regular graph on $3d+7$ vertices. Furthermore, $\bar{\mathcal{G}}_d$ has no one-factor (removing ∞ leaves three odd components).

Lemma 3.1. *For each odd integer $d \geq 3$ the graph $\mathcal{G}_d$ has a one-factorization.*

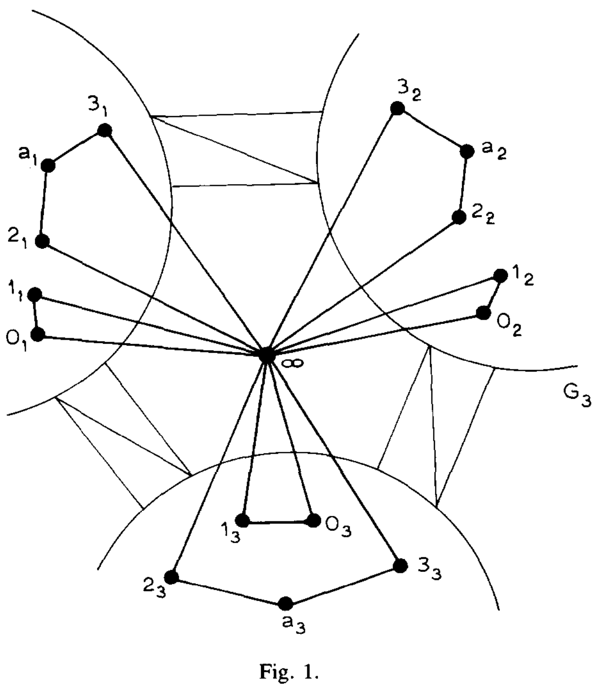
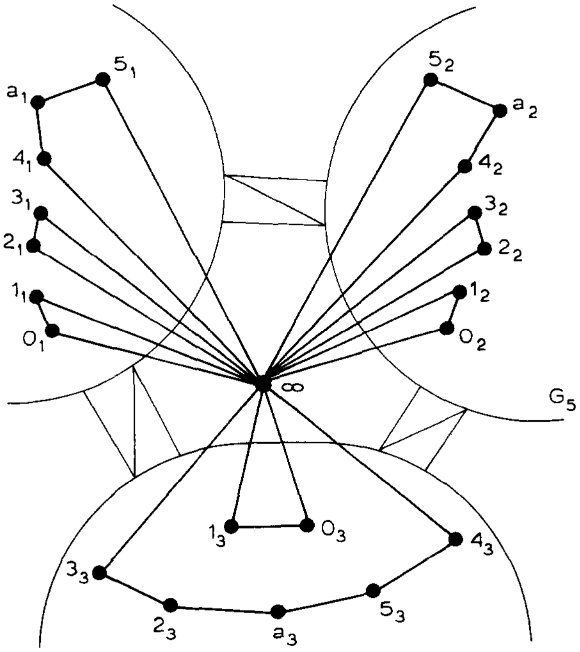
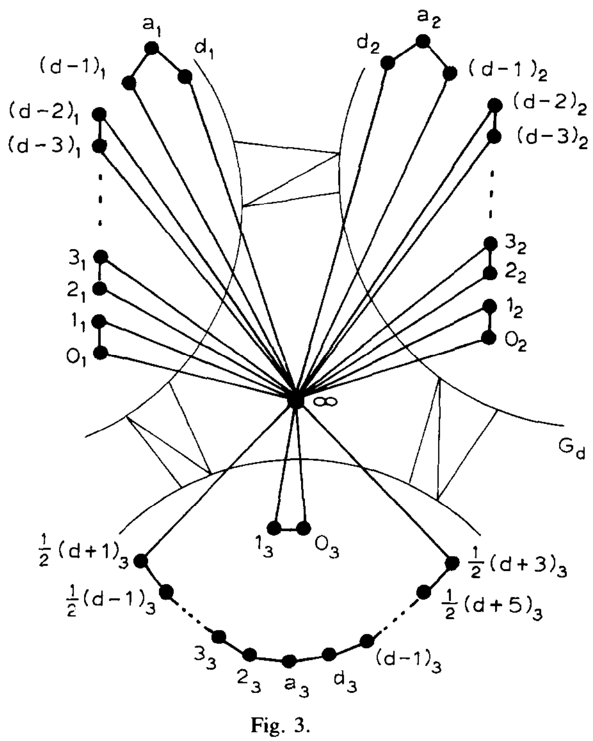


Fig. 1.



5_1
a_1
4_1
3_1
2_1
1_1
0_1
5_2
a_2
4_2
3_2
2_2
1_2
0_2
∞
G_5
1_3
0_3
4_3
3_3
2_3
a_3
5_3

Fig. 2.



a_1
d_1
$(d-1)_1$
$(d-2)_1$
$(d-3)_1$
3_1
2_1
1_1
0_1
a_2
d_2
$(d-1)_2$
$(d-2)_2$
$(d-3)_2$
3_2
2_2
1_2
0_2
∞
G_d
$\frac{1}{2}(d+1)_3$
$\frac{1}{2}(d-1)_3$
1_3
0_3
$\frac{1}{2}(d+3)_3$
$\frac{1}{2}(d+5)_3$
3_3
2_3
a_3
d_3
$(d-1)_3$

Fig. 3.

Proof. We consider two cases.

Case (i): $d \equiv 3$ *modulo* 4, $d \geqslant 7$.

Develop each of the following two one-factors modulo $d+1$:

(I)

$$
\begin{array}{ccccc}
1_1 1_2 & \tfrac{1}{2}(d+7)_1 3_2 & 2_2 0_3 & \tfrac{1}{2}(d+5)_1 \tfrac{1}{2}(d+1)_3 & \infty \tfrac{1}{2}(d+1)_1 \\
3_1 5_2 & \tfrac{1}{2}(d+11)_1 7_2 & 4_2 1_3 & \tfrac{1}{2}(d+9)_1 \tfrac{1}{2}(d+3)_3 & a_3 \tfrac{1}{2}(d+3)_1 \\
5_1 9_2 & \tfrac{1}{2}(d+15)_1 11_2 & 6_2 2_3 & \vdots & a_1 d_2 \\
\vdots & \vdots & \vdots & 0_1 \tfrac{1}{4}(3d-1)_3 & a_2 d_3 \\
\tfrac{1}{2}(d-1)_1 (d-2)_2 & d_1 (d-4)_2 & (d-1)_2 \tfrac{1}{2}(d-3)_3 & \vdots & \\
 & & 0_2 \tfrac{1}{2}(d-1)_3 & \tfrac{1}{2}(d-3)_1 (d-1)_3 &
\end{array}
$$

(II)

$$
\begin{array}{ccccc}
0_1 1_2 & \tfrac{1}{2}(d+5)_1 3_2 & \tfrac{1}{2}(d+5)_2 0_3 & 1_1 \tfrac{1}{2}(d+1)_3 & \infty \tfrac{1}{2}(d+1)_2 \\
2_1 5_2 & \tfrac{1}{2}(d+9)_1 7_2 & \tfrac{1}{2}(d+9)_2 1_3 & 3_1 \tfrac{1}{2}(d+3)_3 & a_1 \tfrac{1}{2}(d-1)_3 \\
4_1 9_2 & \tfrac{1}{2}(d+13)_1 11_2 & \tfrac{1}{2}(d+13)_2 2_3 & 5_1 \tfrac{1}{2}(d+5)_3 & a_2 \tfrac{1}{2}(d+1)_1 \\
\vdots & \vdots & \vdots & \vdots & a_3 d_2 \\
\tfrac{1}{2}(d-3)_1 (d-2)_2 & (d-1)_1 (d-4)_2 & 0_2 \tfrac{1}{4}(d-3)_3 & d_1 d_3 & \\
 & & \vdots & & \\
 & & \tfrac{1}{2}(d-3)_2 \tfrac{1}{2}(d-3)_3 & &
\end{array}
$$

The remaining edges in $\mathscr{G}_d$ can be arranged into two hamiltonian cycles, viz:

$d=7$ $\quad 0_1 1_1 0_2 2_1 1_2 3_1 2_2 1_3 0_3 \infty 4_3 3_3 2_3 3_2 4_1 5_1 4_2 5_2 6_1 a_1 7_1 6_2 5_3 6_3 7_3 a_3 a_2 7_2 0_1;$

$\quad 0_1 6_2 a_2 a_1 a_3 2_2 3_1 2_1 1_3 \infty 5_3 6_1 4_2 3_3 4_1 2_2 3_2 5_1 4_3 5_2 7_1 6_3 7_2 1_1 0_3 1_2 0_2 7_3 0_1.$

$d \geqslant 11$ $\quad 0_1 1_1 0_2 2_1 1_2 3_1 2_2 1_3 0_3 \infty [\tfrac{1}{2}(d+1)_3 \tfrac{1}{2}(d-1)_3 \cdots 3_3 2_3] 3_2 [4_1 5_1 4_2 5_2 \cdots$

$\tfrac{1}{2}(d+1)_1 \tfrac{1}{2}(d+3)_1 \tfrac{1}{2}(d+1)_2 \tfrac{1}{2}(d+3)_2]$

$\tfrac{1}{2}(d+5)_1 \tfrac{1}{2}(d+7)_1 \tfrac{1}{2}(d+5)_2 \tfrac{1}{2}(d+3)_3 \tfrac{1}{2}(d+5)_3$

$\tfrac{1}{2}(d+7)_2 \tfrac{1}{2}(d+9)_1 [\tfrac{1}{2}(d+11)_1 \tfrac{1}{2}(d+9)_2 (d+7)_3 \tfrac{1}{2}(d+9)_3$

$\tfrac{1}{2}(d+11)_2 \tfrac{1}{2}(d+13)_1 \cdots$

$(d-2)_1 (d-3)_2 (d-4)_3 (d-3)_3 (d-2)_2$

$(d-1)_1]^* a_1 d_1 (d-1)_2 (d-2)_3 (d-1)_3 d_3 a_3 a_2 d_2 0_1;$

$0_1 (d-1)_2 a_2 a_1 a_3 2_3 3_1 2_1 1_3 \infty [\tfrac{1}{2}(d+3)_3 \tfrac{1}{2}(d+5)_1$

$\tfrac{1}{2}(d+1)_2 \tfrac{1}{2}(d-1)_3 \tfrac{1}{2}(d+1)_1 \tfrac{1}{2}(d-3)_2$

$\cdots 3_3 4_1 2_2][3_2 5_1 4_3 5_2 7_1 6_3 \cdots \tfrac{1}{2}(d+3)_2 \tfrac{1}{2}(d+7)_1 \tfrac{1}{2}(d+5)_3][\tfrac{1}{2}(d+7)_3 \tfrac{1}{2}(d+9)_1$

$\tfrac{1}{2}(d+5)_2 \tfrac{1}{2}(d+7)_2 \tfrac{1}{2}(d+11)_1 \tfrac{1}{2}(d+9)_3 \cdots (d-2)_3$

$(d-1)_1(d-3)_2(d-2)_2d_1(d-1)_3]$

$d_2 1_1 0_3 1_2 0_2 d_3 0_1.$

* Omit this sequence when $d = 11$.

Case (ii): $d \equiv 1$ *modulo* 4, $d \geqslant 5$.
Develop each of the following two one-factors modulo $d+1$:

$$
\text{(I)}\quad
\begin{array}{cccc}
1_1 1_2 & 2_2 0_3 & \frac{1}{2}(d+7)_1\frac{1}{2}(d+3)_3 & \\
3_1 5_2 & 4_2 1_3 & \frac{1}{2}(d+11)_1\frac{1}{2}(d+5)_3 & \infty\frac{1}{2}(d+3)_1 \\
5_1 9_2 & 6_2 2_3 & \frac{1}{2}(d+15)_1\frac{1}{2}(d+7)_3 & a_1(d-2)_2 \\
\vdots & \vdots & \vdots & a_2\frac{1}{2}(d+1)_3 \\
(d-2)_1(d-6)_2 & (d-1)_2\frac{1}{2}(d-3)_3 & \frac{1}{2}(d-1)_1 d_3 & a_3 d_1 \\
 & 0_2\frac{1}{2}(d-1)_3 & &
\end{array}
$$

$$
\text{(II)}\quad
\begin{array}{cccc}
1_1 2_2 & \frac{1}{2}(d+5)_2 0_3 & 0_1\frac{1}{2}(d-1)_3 & \\
3_1 6_2 & \frac{1}{2}(d+9)_2 1_3 & 2_1\frac{1}{2}(d+1)_3 & \infty\frac{1}{2}(d+1)_2 \\
5_1 10_2 & \frac{1}{2}(d+13)_2 2_3 & 4_1\frac{1}{2}(d+3)_3 & a_1 d_3 \\
\vdots & \vdots & \vdots & a_2 d_1 \\
(d-2)_1(d-5)_2 & \frac{1}{2}(d-3)_2\frac{1}{2}(d-3)_3 & (d-1)_1(d-1)_3 & a_3(d-1)_2
\end{array}
$$

The remaining edges in $\mathscr{G}_d$ can be arranged into two hamiltonian cycles, viz.:

$d = 5$ $\quad 0_1 1_1 0_2 2_1 1_2 3_1 2_2 1_3 0_3 \infty 3_3 2_3 3_2 4_1 a_1 5_1 4_2 a_2 a_3 5_3 4_3 5_2 0_1;$

$0_1 4_2 3_3 4_1 2_2 3_2 5_1 4_3 \infty 1_3 2_1 3_1 2_3 a_3 a_1 a_2 5_2 1_1 0_3 1_2 0_2 5_3 0_1.$

$d \geqslant 9$ $\quad 0_1 1_1 0_2 2_1 1_2 3_1 2_2 1_3 0_3 \infty[\frac{1}{2}(d+1)_3\frac{1}{2}(d-1)_3 \cdots 3_3 2_3]3_2[4_1 5_1 4_2 5_2 \cdots$

$\frac{1}{2}(d+3)_1\frac{1}{2}(d+5)_1\frac{1}{2}(d+3)_2\frac{1}{2}(d+5)_2]\frac{1}{2}(d+3)_3$

$\frac{1}{2}(d+5)_3[\frac{1}{2}(d+7)_1\frac{1}{2}(d+9)_1$

$\frac{1}{2}(d+7)_2\frac{1}{2}(d+9)_2\frac{1}{2}(d+7)_3\frac{1}{2}(d+9)_3 \cdots (d-3)_1(d-2)_1(d-3)_2$

$(d-2)_2(d-3)_3(d-2)_3]^*(d-1)_1$

$a_1 d_1(d-1)_2 a_2 a_3 d_3(d-1)_3 d_2 0_1;$

$0_1(d-1)_2[(d-2)_3(d-1)_3 d_1(d-2)_2(d-1)_1(d-3)_2(d-4)_3$

$(d-3)_3(d-2)_1 \cdots \frac{1}{2}(d+5)_3$

$\frac{1}{2}(d+7)_3\frac{1}{2}(d+9)_1\frac{1}{2}(d+5)_2\frac{1}{2}(d+7)_1\frac{1}{2}(d+3)_2][\frac{1}{2}(d+1)_3\frac{1}{2}(d+3)_1\frac{1}{2}(d-1)_2$

$\frac{1}{2}(d-3)_3\frac{1}{2}(d-1)_1\frac{1}{2}(d-5)_2 \cdots 3_3 4_1 2_2][3_2 5_1 4_3 5_2 7_1 6_3 \cdots \frac{1}{2}(d+1)_2$

$\frac{1}{2}(d+5)_1\frac{1}{2}(d+3)_3]\infty$

$1_3 2_1 3_1 2_3 a_3 a_1 a_2 d_2 1_1 0_3 1_2 0_2 d_3 0_1.$

* Omit this sequence when $d = 9$.

The value $d = 3$ remains to be dealt with. Develop each of the following one-factors modulo 4:

$$\begin{matrix} 1_11_2 & \infty 2_1 \\ 0_21_3 & a_13_2 \\ 2_20_3 & a_23_3 \\ 0_12_3 & a_33_1 \end{matrix}; \qquad \begin{matrix} 0_11_2 & \infty 2_2 \\ 0_20_3 & a_11_3 \\ 1_12_3 & a_22_1 \\ 3_13_3 & a_33_2 \end{matrix}.$$

The remaining edges in $\mathscr{G}_3$ can be arranged into two hamiltonian cycles:

$$0_11_10_22_11_20_31_32_23_12_3\infty 3_3a_3a_1a_23_20_1;$$

$$0_12_2a_2a_32_33_21_10_3\infty 1_32_1a_13_11_20_23_30_1.$$

This completes the proof of Lemma 3.1. □

Theorem 3.2 *Let d be an odd integer, $d \geqslant 3$. If $n = 3d + 7$ or n is even and $n \geqslant 6d + 14$, then there is a d-set in K_n.*

Proof. Apply Lemma 3.1 and the remark preceding it. Then apply Corollary 2.6. □

It remains to be shown that for each even integer n, $3d + 9 \leqslant n \leqslant 6d + 12$, there exists a d-set in K_n. We begin by considering the case $n \geqslant 4d + 8$.

Theorem 3.3 *For each odd integer $d \geqslant 3$ and each even integer n, $4d + 8 \leqslant n \leqslant 6d + 12$, there exists a d-set in K_n.*

Proof. Let H be the graph $\mathscr{G}_d \vee K_{n-(3d+7)}$ (where $\vee$ denotes the usual join function). Let $t = n - (3d + 7)$; then $d + 1 \leqslant t \leqslant 3d + 5$. Let $\{F_1, F_2, \ldots, F_{2d+6}\}$be the one-factorization of $\mathscr{G}_d$ constructed in Lemma 3.1, and let $\{F'_1, F'_2, \ldots, F'_{t-(d+1)}\}$ be a collection of $t - (d + 1)$ mutually disjoint one-factors in K_t. We construct our d-set as follows.

For each $i = 1, 2, \ldots, t - (d + 1)$ take the one-factors $E_i = F_i \cup F'_i$.

There remain in $\mathscr{G}_d$ the one-factors $F_{t-d}, F_{t-d+1}, \ldots, F_{2d+6}$ which among them contain $(3d + 7) \cdot \frac{1}{2}(2d + 6 - (t - d) + 1) = (3d + 7) \cdot \frac{1}{2}(3d + 7 - t)$ edges. Apply Lemma 2.7 with $c = 3d + 7$ to partition this set of edges into $3d + 7$ matchings $M_1, M_2, \ldots, M_{3d+7}$, each with $\frac{1}{2}(3d + 7 - t)$ edges. For each $j = 1, 2, \ldots, 3d + 7$ the matching M_j covers all but a set S_j of t vertices; since the union of all matchings M_j is a regular graph of valency $3d + 7 - t$ each vertex in $\mathscr{G}_d$ is contained in exactly t of the sets S_j. Now we apply Lemma 2.8 to construct a set $M'_1, M'_2, \ldots, M'_{3d+7}$ of matchings on H where each M'_j matches the vertices of S_j to the vertices of K_t. For each $j = 1, 2, \ldots, 3d + 7$ take the one-factor $E_j = M_j \cup M'_j$.

The set of one-factors $\{E_i: 1 \leqslant i \leqslant t - (d + 1)\} \cup \{E_j: 1 \leqslant j \leqslant 3d + 7\}$ then consistutes a maximal set (the complement of their union cannot contain a one-factor, since we have noted previously that $\bar{\mathscr{G}}_d$ contains no one-factor), and so a d-set as required. □

Finally, we turn our attention to the case where $3d+9 \leq n \leq 4d+6$. Let $d \geq 3$ be given, and for each $w = 3, 5, \ldots, d$ let $\mathcal{G}_d^w$ be the graph obtained from $\mathcal{G}_d$ as follows: First we remove the vertex 0_3. Next, we consider the subgraph $\mathcal{H}_d$ of $\mathcal{G}_d$ spanned by the vertices $((\mathbf{Z}_{d+1} - \{0, 1\}) \times \{3\}) \cup \{a_3\}$ (these are precisely the vertices which are *not* adjacent to 0_3 in $\mathcal{G}_d$); this subgraph forms a d-cycle minus an edge. Relabel the vertices of $\mathcal{H}_d$ with the elements of $\mathbf{Z}_d$ so that the edges form the path $0, 2, 4, \ldots, d-2$. Now we add $(w-1) \cdot \frac{1}{2}(d-w)$ new edges to this subgraph: for each $i = 0, 1, \ldots, w-2$ add the edges in the set

$$E(i) = \left\{(i-j, i+j) \colon \frac{w+1}{2} \leq j \leq \frac{d-1}{2}\right\}.$$

Note that if $w = d$ then $E(i) = \emptyset$ for each $i = 0, 1, \ldots, d-2$ (whence no new edges are being added).

For each $i = 0, 1, \ldots, w-2$ let $V(i)$ be the $d-w+1$ vertices in the set

$$\{i\} \cup \left\{i-j, i+j \colon \frac{w+1}{2} \leq j \leq \frac{d-1}{2}\right\}$$

(i.e. the vertex i together with the vertices covered by the edges in $E(i)$), and let $V(w-1)$ be the $d-w+1$ vertices in the set $\{w-1, w, \ldots, d-1\}$.

Lemma 3.4. *The graph $\mathcal{G}_d^w$ has $3d+6$ vertices and has edge-coloring number $\chi(\mathcal{G}_d^w) \leq 2d+5+w$.*

Proof. The graph $\mathcal{G}_d$ has edge-coloring number $2d+6$ (Lemma 3.1). Since each set of edges $E(i)$ forms a matching in $\mathcal{G}_d^w$ and there are $w-1$ of these matchings it follows that $\chi(\mathcal{G}_d^w) \leq 2d+6+w-1 = 2d+5+w$. $\square$

The graph $\mathcal{G}_d^w$ has

$$\frac{(3d+7)(2d+6)}{2} - (2d+6) + (w-1) \cdot \tfrac{1}{2}(d-w) = (2d+5+w) \cdot \left(\frac{3d+6-w}{2}\right)$$

edges. From Lemma 3.4 we can now apply Lemma 2.7 (with $c = 2d+5+w$) to construct an edge decomposition of $\mathcal{G}_d^w$ into matchings $M'_1, M'_2, \ldots, M'_{2d+5+w}$, each with $(3d+6-w)/2$ edges. For each $j = 1, 2, \ldots, 2d+5+w$ let S'_j be the vertices in $\mathcal{G}_d^w$ which are *not* covered by the matching M'_j.

Let us now turn our attention to the sets $V(i)$ defined prior to Lemma 3.4. These sets form the rows in the following $w \times (d-w+1)$ array written on $\mathbf{Z}_d$ (where $w = d$, the array consists of the first column only):

$$\begin{array}{ccccc}
0 & \frac{1}{2}(w+1) & \frac{1}{2}(w+3) & \cdots & d-\frac{1}{2}(w+1) \\
1 & \frac{1}{2}(w+3) & \frac{1}{2}(w+5) & \cdots & d-\frac{1}{2}(w-1) \\
\vdots & \vdots & \vdots & & \vdots \\
w-2 & \frac{1}{2}(3w-3) & \frac{1}{2}(3w-1) & \cdots & \frac{1}{2}(w-5) \\
w-1 & w & w+1 & \cdots & d-1.
\end{array}$$

A simple rearrangement of the symbols in the first column and last row (namely, $k \to k + (w-1)/2 \pmod d$) transforms the above array into one which contains each symbol at most once in each column; for each $i = 1, 2, \ldots, d - w + 1$ let S_i be the set of vertices in $\mathcal{G}_d^w$ represented by the ith column in the transformed array.

Lemma 3.5. *The sets S_j', $j = 1, 2, \ldots, 2d + 5 + w$ together with S_i, $i = 1, 2, \ldots, d - w + 1$ form a collection of $3d + 6$ w-sets with the property that each vertex in $\mathcal{G}_d^w$ is contained in exactly w of them.*

Proof. Let x be a vertex in $\mathcal{G}_d^w$. If x is not a vertex of $\mathcal{H}_d$ then x has degree $2d + 5$ in $\mathcal{G}_d^w$ and so will be contained in w of the sets S_j'. Since each S_i is a subset of the vertices of $\mathcal{H}_d$, x will not be contained in any of these sets.

Now let x be a vertex of $\mathcal{H}_d$. Then x has degree $2d + 6 + k$ in $\mathcal{G}_d^w$ where k is the number of matchings $E(i)$ that cover x. Hence, x is *not* contained in $2d + 5 + w - (2d + 6 + k) = w - k - 1$ of the matchings M_j' and thus is in $w - k - 1$ of the sets S_j'. Now $k = \#$ matchings $E(i)$ that cover $x = \#$ (sets $V(i)$ that contain $x) - 1 = \#$ (sets S_i that contain $x) - 1$; that is, x is contained in $k + 1$ of the sets S_i. In total then x is contained in $w - k - 1 + k + 1 = w$ sets. That each set S_i, S_j' contains w vertices follows from their definitions. $\square$

We are now in a position to prove the following.

Theorem 3.6. *For each odd integer $d \geq 3$ and each even integer n with $3d + 9 \leq n \leq 4d + 6$ there is a d-set in K_n.*

Proof. Let $w = n - (3d + 6)$; then w is odd and $3 \leq w \leq d$. Take the graph $\mathcal{G}_d^w \vee \bar{K}_w$. From Lemma 2.8 and Lemma 3.5 the edges joining $\mathcal{G}_d^w$ to $\bar{K}_w$ can be arranged into $3d + 6$ matchings $M_1, M_2, \ldots, M_{3d+6}$ where for each $j = 1, \ldots, 2d + 5 + w$ M_j is matching from S_j' to $\bar{K}_w$. Recalling that S_j' is the set of vertices not covered by the matching M_j' in $\mathcal{G}_d^w$, we now form the graph $\boldsymbol{H}$ by taking as its edges the union of the one-factors $M_j' \cup M_j$ on $\mathcal{G}_d^w \vee \bar{K}_w$, for $j = 1, \ldots, 2d + 5 + w$. Then $\bar{\boldsymbol{H}}$ is a d-regular graph on $n = 3d + 6 + w$ vertices. It remains to be shown that $\bar{\boldsymbol{H}}$ has no one-factor. To see this, note that the graph $\boldsymbol{H}$ can be considered as being constructed from $\mathcal{G}_d$ by replacing the vertex 0_3 in $\mathcal{G}_d$ by w new vertices $x_1, x_2, \ldots, x_w$, replacing each edge $(v, 0_3)$ in $\mathcal{G}_d$ by w new edges (v, x_1), (v, x_2), $\ldots$, (v, x_w) and then adding some edges to the subgraph spanned by the vertices of $\mathcal{H}_d \cup \{x_1, x_2, \ldots, x_w\}$. In particular then (referring back to the definition of $\mathcal{G}_d$) removing the vertex ∞ from $\bar{\boldsymbol{H}}$ will create an odd component on the vertices $(\mathbf{Z}_{d+1} \cup \{a\}) \times \{1\}$ and a second odd component on the vertices $(\mathbf{Z}_{d+1} \cup \{a\}) \times \{2\}$ (just as happened in $\overline{\mathcal{G}_d}$). By Tutte's Theorem $\bar{\boldsymbol{H}}$ therefore contains no one-factor. $\square$

Collecting the results of Theorems 3.2, 3.3 and 3.6 we get the main result of this section.

Theorem 3.7. *For each odd integer $d \geq 3$ and each even integer $n \geq 3d + 7$ there is a d-set in K_n.*

Conclusion

Theorems 2.4 and 3.7 together give us the spectrum for maximal sets of one-factors (Theorem 1.0): If n is a positive even integer then a maximal set of k mutually disjoint one-factors on n vertices exists if and only if either $2 \cdot \lfloor n/4 \rfloor + 1 \leq k \leq n - 1$ and k is odd, or $\frac{1}{3}(2n + 4) \leq k \leq n - 4$ and k is even.

Acknowledgments

The authors would like to thank J.A. Bondy for his proof of Lemma 2.8. The first author is supported in part by NSERC grant P 36507.

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