On the instability of differential systems
with varying delay

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Abstract

The unstable properties of the null solution of the nonautonomous delay system \( x'(t) = A(t)x(t) + B(t)x(t - r_{1}(t)) + f(t, x(t), x(t - r_{2}(t))) \) are examined; the nonconstant delays \( r_{1}, r_{2} \) are assumed to be continuous bounded functions. The case \( A = \text{constant} \) is reviewed, where a theorem, recalling the Perron instability theorem for ordinary differential equations, is obtained.

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1. Introduction

The study of unstable properties of the linear nonautonomous delay system

\[
x'(t) = A(t)x(t) + B(t)x(t - r(t)),
\]

has received a little attention in the study of functional equations [6,8,9]. A classical result assures [8] that the autonomous system (that is \( A(t), B(t) \) and \( r(t) \) are constant) is unstable if the algebraic equation

\[
\det(\lambda I - A - Be^{-r\lambda}) = 0
\]

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has a root with a positive real part. This method of investigation cannot be applied to the nonautonomous Eq. (1). A remarkable result on this subject follows from the following example due to Zverkin [5]:

\[ x'(t) = a(t)x(t - 1.5\pi), \quad t \geq 0, \]

where

\[ a(t) = \begin{cases} 
0, & 0 \leq t \leq 1.5\pi, \\
-\cos t, & 1.5\pi < t \leq 3\pi, \\
1, & t > 3\pi.
\end{cases} \]

The solutions of this equation are stable for the initial time \( t_0 \in [0, \frac{3}{2}\pi] \); on the contrary, they are unstable for \( t_0 > 3\pi \). This situation shows how different the stability of nonautonomous delay equations can be in comparison with ordinary differential equations.

In this paper, we are interested in nonautonomous delay equations. We will consider the general nonlinear equation

\[ x'(t) = A(t)x(t) + B(t)x(t - r_1(t)) + f(t, x(t), x(t - r_2(t))), \quad t \geq 0, \tag{2} \]

in which \( f(t, 0, 0) = 0 \), the delay functions \( r_1(t), r_2(t) \) are assumed to be continuous and bounded: \( r_1, r_2 : [0, \infty) \rightarrow [0, \sigma], \) and the functions \( A(t), B(t) \) are continuous on the interval \( [t_0 - \sigma, \infty) \).

The fundamental tool of this paper is the Coppel’s instability theorem for nonautonomous ordinary differential equations [3], which is reproduced here for further references.

**Theorem A.** Assume that the continuous function \( f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies

\[ |f(t, y)| \leq \gamma|y|, \]

where \( \gamma \) is a constant. Let \( P \) be a projection matrix, \( P \neq I \), such that

\[ \int_0^t |\Psi(t)P\Psi^{-1}(s)| \, ds + \int_0^\infty |\Psi(t)(I - P)\Psi^{-1}(s)| \, ds \leq K, \quad \forall t, \tag{3} \]

where \( \Psi \) is the fundamental matrix of the linear equation

\[ x'(t) = A(t)x(t). \tag{4} \]

Under these conditions, if \( K\gamma < 1 \), then the null solution of equation

\[ y'(t) = A(t)y(t) + f(t, y(t)), \quad f(t, 0) = 0, \]

is unstable.

We point out the fact that the Perron instability theorem of ordinary differential equations [1] follows from this result. The aim of this paper is to extend Theorem A to the delay system (2).
2. Definitions and notations

The notation $|x|$ stands for a fixed norm of a vector $x$ in the linear space $\mathbb{R}^n$. The corresponding matrix norm of a matrix $A$ will be denoted by $|A|$. Throughout, we will denote $J(t_0) = [t_0, \infty)$, $J_\sigma(t_0) = [t_0 - \sigma, \infty)$; for an interval $I$, we will denote by $C(I)$ the space of continuous and bounded functions defined on $I$; if $x \in C(J_\sigma(t_0))$ and $t \geq t_0$, then $x_t \in C([\sigma, 0])$ will denote the function $x_t(s) = x(t + s), s \in [-\sigma, 0]$. In the space $C([-\sigma, 0])$, the following norm will be used

$$|\phi|_\sigma = \max_{s \in [-\sigma, 0]} |\phi(s)|.$$

$x(t; t_0, \phi)$ will denote a solution of the initial value problem

$$\begin{cases}
  x'(t) = A(t)x(t) + B(t)x(t - r_1(t)) + f(t, x(t), x(t - r_2(t))), & t \geq t_0, \\
  x_{t_0} = \phi, & \phi \in C([-\sigma, 0]).
\end{cases}$$

For $x \in C(J_\sigma(t_0))$, let $|x|_\infty := \sup_{t \in J_\sigma(t_0)} |x(t)|$.

**Definition 1.** We shall say that the zero solution of Eq. (2) is stable on the axis $[0, \infty)$, iff for each $\varepsilon > 0$, and every $t_0 \geq 0$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that $|\phi| < \delta$ implies $|x(t; t_0, \phi)| < \varepsilon, \forall t \geq t_0$.

3. $B$ and $\gamma$ are bounded

We will assume that the function $f(t, x, y)$ is continuous and satisfies

$$|f(t, x, y)| \leq \gamma(t) \max\{|x|, |y|\}, \quad |x| \leq \rho, |y| \leq \rho, \quad (H1)$$

where $\gamma$ is a continuous function and $\rho$ is a positive number. Let us define the following operators:

$$U_{t_0}[x](t) = \int_{t_0}^{t} \Psi(s)P\Psi^{-1}(s)B(s)x(s - r_1(s)) \, ds$$

$$- \int_{t}^{\infty} \Psi(s)(I - P)\Psi^{-1}(s)B(s)x(s - r(s)) \, ds,$$

$$V_{t_0}[x](t) = \int_{t_0}^{t} \Psi(s)P\Psi^{-1}(s)f(s, x(s), x(s - r_2(s))) \, ds$$

$$- \int_{t}^{\infty} \Psi(s)(I - P)\Psi^{-1}(s)f(s, x(s), x(s - r_2(s))) \, ds,$$
and
\[ T_{t_0} = U_{t_0} + V_{t_0}. \]
The operators \( T_{t_0}, U_{t_0}, V_{t_0} \) are applied to functions of the space \( C(J_\sigma(t_0)) \).

If the functions \( B(t), \gamma(t) \) are bounded, then the following estimates are consequences of condition (3)
\[ |U_{t_0}[x](t)| \leq K|B|_{t_0}^\infty|x|_{t_0}^\infty, \quad \forall t \geq t_0, \tag{5} \]
\[ |V_{t_0}[x](t)| \leq K|\gamma|_{t_0}^\infty|x|_{t_0}^\infty, \quad \forall t \geq t_0. \tag{6} \]

**Theorem 1.** If the functions \( B, \gamma \) are bounded; (H1), (3) are fulfilled with \( P \neq I \), then the null solution of Eq. (2) is unstable on the interval \([0, \infty)\) if
\[ K(|B|_{t_0}^\infty + |\gamma|_{t_0}^\infty) < 1, \tag{7} \]
for some \( t_0 \geq 0 \).

**Proof.** Let us assume the contrary: for a given positive \( \varepsilon < \rho \) and \( t_0 \geq 0 \) there exists a positive \( \delta = \delta(\varepsilon, t_0) \), such that \( |\phi|_\sigma < \delta \) implies
\[ |x(t; t_0, \phi)| < \varepsilon, \quad \forall t \geq t_0. \]
Let us define
\[ y(t) = x(t; t_0, \phi) - T_{t_0}[x(\cdot; t_0, \phi)](t), \quad t \geq t_0, \]
where the initial time \( t_0 \) satisfies (7) and the initial value function \( \phi \) satisfies
\[ 0 \neq |\phi|_\sigma < \delta, \quad |\phi|_\sigma = |\phi(0)|, \quad P\phi(0) = 0. \tag{8} \]
From the estimates (5), (6), the function \( y(t) \) is well defined, bounded and it is easy to verify that it satisfies Eq. (4) for \( t \geq t_0 \).

From the definition of function \( y \) we obtain
\[ y(t_0) = \phi(0) + (I - P) \int_{t_0}^{\infty} \Psi^{-1}(s)B(s)x(s - r_1(s)) \, ds \]
\[ + (I - P) \int_{t_0}^{\infty} \Psi^{-1}(s)f(s, x(s), x(s - r_2(s))) \, ds \in (I - P)[V], \]
implying \( Py(t_0) = 0 \). From Lemma 3.2 in [4] we obtain \( (I - P)y(t_0) = 0 \). Hence \( y(t) = 0, \forall t \geq t_0 \). But in this case \( x(\cdot; t_0, \phi) \) satisfies the integral equation
\[ x(t, t_0, \phi) = U_{t_0}[x(\cdot, t_0, \phi)](t) + V_{t_0}[x(\cdot, t_0, \phi)](t), \quad t \geq t_0. \]
Using again the estimates (5), (6), we obtain
\[
\sup_{t \in [t_0, \infty)} |x(t, t_0, \phi)| \leq (K |B|_{t_0}^{\infty} + K |\gamma|_{t_0}^{\infty}) |x(\cdot, t_0, \phi)|_{t_0}^{\infty}.
\]

But the choice of the function \(\phi\) by condition (8) implies
\[
|x(\cdot, t_0, \phi)|_{t_0}^{\infty} \leq (K |B|_{t_0}^{\infty} + K |\gamma|_{t_0}^{\infty}) |x(\cdot, t_0, \phi)|_{t_0}^{\infty}.
\]

From (7) we obtain
\[
|x(\cdot, t_0, \phi)|_{t_0}^{\infty} = 0 \iff x(t; t_0, \phi) = 0, \quad t \geq t_0.
\]
This last is contradictory with \(x(t_0; t_0, \phi) = \phi(0) \neq 0\). \(\square\)

Theorem 1 extends the result of Theorem A to the delay equation (2). Nevertheless, by means of this theorem, the influence of the delay functions on the instability of the null solution of Eq. (2) cannot be appreciated. We study this situation in the forthcoming section.

4. \(B\) and/or \(\gamma\) are not bounded

We write the Eq. (2) in the equivalent form
\[
x'(t) = (A(t) + B(t))x(t) + B(t)(x(t - r_1(t)) - x(t)) + f(t, x(t), x(t - r_2(t))), \quad f(t, 0, 0) = 0. \quad (9)
\]
Throughout, \(\Phi\) will denote the fundamental matrix of the equation
\[
y'(t) = (A(t) + B(t))y(t). \quad (10)
\]

Following paper [3], we will assume the existence of a projection matrix \(P\), such that
\[
\int_0^t |\Phi(t) P \Phi^{-1}(s)| \, ds + \int_t^\infty |\Phi(t) (I - P) \Phi^{-1}(s)| \, ds \leq K, \quad \forall t \geq 0. \quad (11)
\]
In this section we will assume, instead of condition (H1), the hypothesis
\[
|f(t, x, y)| \leq \gamma(t) |x - y|, \quad |x|, |y| \leq \rho, \quad (H2)
\]
for some positive \(\rho\). Regarding Eq. (9), for \(t \geq t_0\), let us define the operators:
\[
\tilde{U}_{t_0}[x](t) = \int_{t_0}^t \Phi(t) P \Phi^{-1}(s) B(s) (x(s - r_1(s)) - x(s)) \, ds
\]
\[
- \int_t^\infty \Phi(t) (I - P) \Phi^{-1}(s) B(s) (x(s - r(s)) - x(s)) \, ds,
\]
\[
\tilde{\mathcal{V}}_{t_0}[x](t) = \int_{t_0}^{t} \Phi(t) \Phi^{-1}(s) f\left(s, x(s), x(s - r_2(s))\right) ds
\]

\[
- \int_{t}^{\infty} \Phi(t)(I - P) \Phi^{-1}(s) f\left(s, x(s), x(s - r_2(s))\right) ds,
\]

and

\[
\tilde{T}_{t_0} = \tilde{\mathcal{U}}_{t_0} + \tilde{\mathcal{V}}_{t_0}.
\]

Operators \(\tilde{T}_{t_0}, \tilde{\mathcal{U}}_{t_0}, \tilde{\mathcal{V}}_{t_0}\) are applied to functions of the space \(C(J_{\sigma}(t_0))\). In this section we will use the results of Schauder and Tychonoff [4] on an appropriate functional space that is described below. For a function \(x \in C(J_{\sigma}(t_0))\), the functions \(\tilde{\mathcal{U}}_{t_0}[x], \tilde{\mathcal{V}}_{t_0}[x]\) are not defined on \([t_0 - \sigma, t_0]\). We complete this definition in the following manner:

\[
\tilde{\mathcal{U}}_{t_0}[x](t) = \tilde{\mathcal{U}}_{t_0}[x](t_0), \quad \tilde{\mathcal{V}}_{t_0}[x](t) = \tilde{\mathcal{U}}_{t_0}[x](t_0), \quad t \in [t_0 - \sigma, t_0].
\]

Following the ideas in [2,7], we will study the operators \(\tilde{\mathcal{U}}_{t_0}, \tilde{\mathcal{V}}_{t_0}, \tilde{T}_{t_0}\) on the closed subspace \(\mathcal{M}\) of \(C(J_{\sigma}(t_0))\) satisfying

\[
\left|x(t) - x(t')\right| \leq M \beta(t)(t - t')|x|_{t_0}^{\infty}, \quad t - \sigma \leq t' \leq t, \quad t' \geq t_0,
\]

where

\[
\beta(t) = \max\{1, |(A + B)|_{\sigma}\}, \quad t \geq t_0.
\]

**Lemma 1.** Under condition (11), if \((I - P)y_0 = 0\), then \(\Phi y_0 \in \mathcal{M}\).

**Proof.** From Lemma 3.1 in [4], the function \(\Phi y_0\) is bounded. We have to verify the property (M). If \(t - \sigma \leq t' \leq t, \quad t' \geq t_0\), then

\[
|\Phi(t)y_0 - \Phi(t')y_0| = \left|\int_{t'}^{t} (A(s) + B(s))\Phi(s)y_0 ds\right|
\]

\[
\leq \int_{t'}^{t} |A(s) + B(s)||\Phi(s)y_0| ds
\]

\[
\leq M \beta(t)(t - t')|\Phi y_0|_{t_0}^{\infty},
\]

from whence the proof of the lemma is accomplished. \(\square\)

By \(B_\rho\), we will denote the closed ball of radius \(\rho > 0\), centered at \(x = 0\) in the space \(\mathcal{M}\).
Lemma 2. Let us assume that conditions (H2), (11) and
\[ 5K \left( |\beta r_1 B|_t^\infty + |\beta r_2 \gamma|_t^\infty \right) < 1 \quad (12) \]
are satisfied, then for all \( 0 < \rho_1 < \rho \) we have \( \tilde{T}_{t_0} : B_{\rho_1} \to B_{\rho_1} \).

Proof. From the definition of \( \tilde{U}_{t_0} \), for any \( x \in C(J_\sigma(t_0)) \) we have
\[
|\tilde{U}_{t_0}[x](t)| \leq KC \int_{t_0}^t \Phi(t) P \Phi^{-1}(s) \|B(s)\| \|x(s - r_1(s)) - x(s)\| ds
+ \int_{t}^{\infty} \Phi(t)(I - P) \Phi^{-1}(s) \|B(s)\| \|x(s - r_1(s)) - x(s)\| ds
\leq K |\beta r B|_t^\infty |x|_t^\infty.
\]
This implies
\[
|\tilde{U}_{t_0}[x]|_0^\infty \leq K |\beta r_1 B|_t^\infty |x|_t^\infty.
\]
Similarly, we have
\[
|\tilde{V}_{t_0}[x](t)|_0^\infty \leq K |\gamma r_2 \beta|_t^\infty |x|_t^\infty, \quad t \geq t_0 - \sigma.
\]
From these estimates it follows \( |\tilde{T}_{t_0}[x]|_0^\infty \leq \rho_1 \), if \( |x|_0^\infty \leq \rho_1 \).

Now, we will prove that \( \tilde{T}_{t_0}[x] \) satisfies condition (M). Let \( t - \sigma \leq t' \leq t \), \( t' \geq t_0 \). We write:
\[
\tilde{U}_{t_0}[x](t) - \tilde{U}_{t_0}[x](t') = I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 = \int_{t'}^t \Phi(t) P \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds,
\]
\[
I_2 = \int_{t'}^t \Phi(t')(I - P) \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds,
\]
\[
I_3 = \int_{t_0}^{t} \left[ \Phi(t) - \Phi(t') \right] P \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds,
\]
\[
I_4 = \int_{t}^{\infty} \left[ \Phi(t') - \Phi(t) \right] (I - P) \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds.
\]
We may write \( I_1, I_2 \) in the form
\begin{align*}
I_1 &= \int_{t'}^t \Phi(s) P \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds \\
&\quad + \int_{t'}^t \int_{s}^t (A + B)(u) \Phi(u) du P \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds,
I_2 &= \int_{t'}^t \Phi(s) (I - P) \Phi^{-1}(s) B(s) \left( x(s) - x(s - r_1(s)) \right) ds \\
&\quad + \int_{t'}^t \int_{s}^t (A + B)(u) \Phi(u) du (I - P) \Phi^{-1}(s) B(s) \\
&\quad \times \left( x(s) - x(s - r_1(s)) \right) ds.
\end{align*}

From condition (11) we have
\begin{align*}
|I_1 + I_2| &\leq 2 K \beta(t) r_1 \beta B_{l_0} \infty (t - t') |x|_{l_0} \infty \\
&\quad + \int_{t'}^t \left| B(s) \left( x(s) - x(s - r_1(s)) \right) \right| ds.
\end{align*}
\begin{equation}
\leq 3 \beta(t) r_1 \beta B_{l_0} \infty (t - t') |x|_{l_0} \infty .
\end{equation}

For \( s \leq t' \leq t \), we may write the identity
\begin{align*}
\left( \Phi(t) - \Phi(t') \right) P \Phi^{-1}(s) = \int_{t'}^t (A + B)(u) \Phi(u) P \Phi^{-1}(s) du
\end{align*}
implying
\begin{equation}
\left| \left( \Phi(t) - \Phi(t') \right) P \Phi^{-1}(s) \right| \leq \beta(t) \int_{t'}^t \left| \Phi(u) P \Phi^{-1}(s) \right| du.
\end{equation}

For \( t' \leq t \leq s \) we write
\begin{align*}
\left( \Phi(t) - \Phi(t') \right) (I - P) \Phi^{-1}(s)
\end{align*}
\begin{equation}
= \int_{t'}^t (A + B)(u) \Phi(u)(I - P) \Phi^{-1}(s) du.
\end{equation}
From (14) we have

$$|I_3| \leq \beta(t) \int_{t_0}^{t'} \left( \int_{t_0}^{t} |\Phi(u) P \Phi^{-1}(s)| \, du \right) |B(s)| \beta(s) r_1(s) \, ds |x|_{t_0}^{\infty}.$$ 

The Fubini theorem on iterative integration gives

$$|I_3| \leq \beta(t) |\beta r_1 B|_{t_0}^{\infty} \int_{t_0}^{t} \left( \int_{t_0}^{t'} |\Phi(u) P \Phi^{-1}(s)| \, ds \right) \, du |x|_{t_0}^{\infty}.$$ 

In analogous form, using (15) we obtain

$$|I_4| \leq \beta(t) |\beta r_1 B|_{t_0}^{\infty} \int_{t_0}^{t} \left( \int_{t_0}^{\infty} |\Phi(u) (I - P) \Phi^{-1}(s)| \, ds \right) \, du |x|_{t_0}^{\infty}.$$ 

From (11) we obtain

$$|I_2 + I_3| \leq 2K \beta(t) |r_1 B|_{t_0}^{\infty}. \quad (16)$$

The estimates (13), (16) imply

$$\left| \tilde{U}_{t_0} [x](t) - \tilde{U}_{t_0} [x](t') \right| \leq 5K \beta(t) |\beta r_1 B|_{t_0}^{\infty} (t - t') |x|_{t_0}^{\infty}. \quad (17)$$

For the operator $\tilde{V}_{t_0}$ we write:

$$\tilde{V}_{t_0} [x](t) - \tilde{V}_{t_0} [x](t') = J_1 + J_2 + J_3 + J_4, \quad t - \sigma \leq t' \leq t, t' \geq t_0,$$

where

$$J_1 = \int_{t'}^{t} \Phi(t) P \Phi^{-1}(s) f(s, x(s), x(s - r_2(s))) \, ds,$$

$$J_2 = \int_{t'}^{t} \Phi(t')(I - P) \Phi^{-1}(s) f(s, x(s), x(s - r_2(s))) \, ds,$$

$$J_3 = \int_{t_0}^{t} \left[ \Phi(t) - \Phi(t') \right] P \Phi^{-1}(s) f(s, x(s), x(s - r_2(s))) \, ds,$$

$$J_4 = \int_{t}^{\infty} \left[ \Phi(t) - \Phi(t') \right] (I - P) \Phi^{-1}(s) f(s, x(s), x(s - r_2(s))) \, ds.$$ 

By a similar reasoning to the deduction of the estimate (17) we may write

$$\left| \left[ \tilde{V}_{t_0} [x](t) - \tilde{V}_{t_0} [x](t') \right] \right| \leq 5K \beta(t) |\beta r_2 |_{t_0}^{\infty} (t - t') |x|_{t_0}^{\infty}. \quad (18)$$
From (17), (18), $\tilde{T}_{t_0} = \tilde{U}_{t_0} + \tilde{V}_{t_0}$ and (12) we obtain that the function $\tilde{T}_{t_0}[x]$ satisfies condition (M).

**Theorem 2.** Let us assume (H2), condition (11), with $P \neq I$, and (12) for some $t_0 \geq 0$. Then the zero solution of Eq. (2) is unstable on $[0, \infty)$.

**Proof.** Let us assume that the null solution of Eq. (2) is stable on $[0, \infty)$. Then for every $t_0 \geq 0$ and $\varepsilon < \rho$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x(t_0, \Phi(t_0))| < \varepsilon, \forall t \geq t_0$. Let $\rho_1$ be a small number such that

$$0 < \rho_1 < \min\{\rho, \delta\},$$

and let $\eta > 0$ be a small number satisfying

$$\eta + 4K(|\beta r_1 B|_{t_0}^\infty + |\beta r_2 \gamma|_{t_0}^\infty) \rho_1 \leq \rho_1.$$

We fix an initial condition $y_0$ satisfying

$$Py_0 = 0, \quad |\Phi(t)y_0| \leq \eta, \quad \forall t \geq t_0 - \sigma. \quad (19)$$

Let us consider the integral equation $x = \mathcal{F}[x]$, where the operator $\mathcal{F}$ is defined by

$$\mathcal{F}[x](t) = \Phi(t)y_0 + \tilde{T}_{t_0}[x](t), \quad t \geq t_0 - \sigma.$$

From Lemma 2 and the choice of number $\eta$, we have $\mathcal{F} : B_{\rho_1} \to B_{\rho_1}$. Moreover, this operator satisfies all the conditions of the Schauder–Tychonoff theorem [4]. Consequently, we may assure the existence of $x(t)$, a fixed point of the operator $\mathcal{F}$ in the ball $B_{\rho_1} \subset \mathcal{M}$. A straightforward calculation shows that $x(t)$ is a solution of Eq. (9). Moreover, for $t \in [t_0 - \sigma, t_0]$ we have

$$|x(t)| \leq |\Phi(t)y_0| + |\tilde{T}_{t_0}[x](t)|$$

$$\leq \eta + 4K(|\beta r_1 B|_{t_0}^\infty + |\beta r_2 \gamma|_{t_0}^\infty) \rho_1 \leq \rho_1 < \delta,$$

implying that $x(\cdot)$ satisfies $|x(t)| < \varepsilon, t \geq t_0$. But the conditions of the theorem imply the conditions of Lemma 2, and therefore the function $T[x]$ is bounded. Since

$$x(t) = y(t, t_0, y_0) + \tilde{T}_{t_0}[x](t),$$

we obtain that the function $y(\cdot, t_0, y_0)$ must be bounded. But this contradicts the choice of $y_0$ with conditions (19). \qed

In many situations it is convenient to consider Theorem 2 under condition (H1) instead of condition (H2).

**Lemma 3.** Let us assume that conditions (H1), (11), with $P \neq I$, and

$$5K(|\beta r_1 B|_{t_0}^\infty + |\gamma|_{t_0}^\infty) < 1 \quad (20)$$

are satisfied, then for all $0 < \rho_1 < \rho$ we have $\tilde{T}_{t_0} : B_{\rho_1} \to B_{\rho_1}$. 

The proof is similar to that of Lemma 2. The following theorem follows in the same manner as the proof of Theorem 2.

**Theorem 3.** Let us assume (H1), (20), and \( P \neq I \) in condition (11). Then the zero solution of Eq. (2) is unstable.

We emphasize the following remarkable fact, if \( P = 0 \) in (11), then, respectively, conditions (12) and (20) can be written in the form

\[
3K \left( |\beta r_1 B|_{t_0}^\infty + |\beta r_2 \gamma|_{t_0}^\infty \right) < 1 \tag{21}
\]

and

\[
3K \left( |\beta r_1 B|_{t_0}^\infty + |\gamma|_{t_0}^\infty \right) < 1.
\]

5. Some applications

Theorem 2 can be applied to the equation

\[
x'(t) = B(t)x(t - r(t)), \quad t \geq 0.
\]  

(22)

If for some \( t_0 \geq 0 \), \( \Phi \), the fundamental matrix of system

\[
z'(t) = B(t)z(t),
\]

satisfies (11), with \( P \neq I \), then condition (12), that in this case is accomplished if

\[
5K |rB|_{t_0}^\infty \max\{1, |B|_{t_0}^\infty\} < 1,
\]

implies that the null solution of Eq. (22) is not stable on the interval \([0, \infty)\).

If \( B(t) = B \) is a constant matrix having an eigenvalue \( \lambda \) with a positive real part, then the change of variable \( x = e^{\alpha t}y \) in Eq. (22), where \( 0 < \alpha < \max\{\text{Re} \lambda: \text{Re} \lambda > 0\} \), reduces (22) to the form

\[
y'(t) = (B - \alpha I)y(t) + B\gamma(t - r(t)) + f(t, y(t - r(t))),
\]

where the function

\[
f(t, y) = B(t)(e^{-\alpha r(t)} - 1)y
\]

satisfies condition (H1) with \( \gamma'(t) = |B|\alpha r(t)e^{\alpha \sigma} \). For a small \( \alpha \) the matrix \( \Phi(t) = \exp(B - \alpha I)t \) satisfies the condition (11). The condition (20), required in Theorem 3, is satisfied for a small value of \( \sigma \). Therefore, if the constant matrix \( B \) has an eigenvalue with a positive real part and the bound \( \sigma \) of the delay \( r(t) \) satisfies

\[
5K \sigma |B| \max\{1, |B|\} < 1,
\]

then the null solution of the equation

\[
x'(t) = Bx(t - r(t)),
\]

is unstable.
Let us consider the nonlinear equation
\[ x'(t) = Ax(t) + f(t, x(t), x(t - r(t))), \quad f(t, 0, 0) = 0, \] (23)
where the matrix \( A \) is constant, with an eigenvalue \( \lambda \) satisfying \( \text{Re} \lambda > 0 \).
Performing the change of variable \( x = e^{\alpha t} y \), we obtain
\[ y'(t) = (A - \alpha I)x(t) + F(t, y(t), y(t - r(t))), \quad F(t, 0, 0) = 0, \]
where
\[ F(t, x, y) = e^{-\alpha t} f(t, e^{\alpha t} x, e^{\alpha (t - r(t))} y). \] (24)
It is clear that, if \( f(t, x, y) \) satisfies (H1), then \( F(t, x, y) \) satisfies condition (H1) with the same function \( \gamma(t) \). According to Theorem 1, if \( K|\gamma|_\infty^n t_0 < 1 \), then the null solution of equation
\[ x'(t) = Ax(t) + f(t, x(t), x(t - r(t))), \quad f(t, 0, 0) = 0, \]
is unstable. Since (H1) is a condition of local character, then we may write the following Perron instability theorem for equations with varying delays.

**Theorem 4.** If the constant matrix \( A \) has an eigenvalue with a positive real part, and the continuous function \( f(t, x, y) \) satisfies
\[ \lim_{|x|+|y|\to0} \frac{f(t, x, y)}{|x|+|y|} = 0, \]
then the null solution of Eq. (23) is unstable.

**Proof.** There exists a positive number \( \rho \) such that \( f(t, x, y) \) satisfies condition (H1) in the ball \( \max\{|x|, |y|\} < \rho \), with a constant \( \gamma \), depending on \( \rho \), satisfying \( \lim_{\rho \to 0} \gamma(\rho) = 0 \). \( \square \)

For equations with constant delay this theorem can be found in [6].

The method introduced by Coppel in [3] can successfully be applied in other situations. For example, let us consider the linear system with many bounded delays:
\[ x'(t) = Ax(t) + \sum_{k=1}^{n} B_k(t)x(t - r_k(t)) + \sum_{k=1}^{n} f_k(t, x(t), x(t - r_k(t))). \] (25)
If the matrix \( A \) has an eigenvalue \( \lambda \) with a positive real part and
\[ K\left( \sum_{k=1}^{n} |B_k|_\infty^{t_0} + \sum_{k=1}^{n} |\gamma_k|_\infty^{t_0} \right) < 1, \]
where \( K \) is the constant appearing in the condition (3), for the matrix \( \Psi(t) = \exp(A - \alpha t) \), where \( 0 < \alpha < \max\{\text{Re} \lambda: \text{Re} \lambda > 0\} \), then the null solution of
Eq. (25) is unstable. In particular, if the matrix $A$ has an eigenvalue $\lambda$ with a positive real part, and the continuous functions $f_k(t,x,y)$ satisfy
\[
\lim_{|x|+|y| \to 0} (|x| + |y|)^{-1} \sum_{k=1}^{n} |f_k(t,x,y)| = 0,
\]
then the null solution of the equation
\[
x'(t) = Ax(t) + \sum_{k=1}^{n} f_k(t,x(t),x(t-r_{n+k}(t)))
\]
(26)
is unstable.

Even Theorem 2 can be applied to Eq. (26) if all the functions $f_k(t,x,y)$ satisfy (H2). The corresponding condition (12) has the form
\[
5K \sum_{k=1}^{n} \max\{1, |A|\} |r_k y_k|_{t_0} < 1.
\]
(27)
Therefore, if the matrix $A$ has an eigenvalue with a positive real part, and condition (27) is fulfilled, then the zero solution of Eq. (26) is unstable.

The instability of the null solution will occur in the scalar equation with non-constant delay
\[
x^{(n)} + \sum_{k=1}^{n} a_{n-k} x^{(n-k)}(t) = \sum_{k=1}^{n-1} b_{n-k} x^{(n-k)}(t-r_{n-k}(t)),
\]
(28)
where $a_i, b_i$ are constant and each delay function $r_i$ is bounded by a constant $\sigma_i$, if some root of the polynomial
\[
P(\lambda) = \lambda^n + (a_{n-1} - b_{n-1})\lambda^{n-1} + \cdots + (a_0 - b_0)
\]
has a positive real part, and
\[
5K \sum_{k=0}^{n-1} \sigma_k \max\{1, |b_k|^2\} < 1,
\]
(29)
for a constant $K$ obtained from condition (11), once the linear scalar equation
\[
x^{(n)} + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_0 - b_0)x = 0
\]
is reduced to the vectorial equation (10). Condition (29) will be satisfied on the interval $[t_0, \infty)$, for a large $t_0$, if the delay functions $r_k(t)$ have the property
\[
\lim_{t \to \infty} r_k(t) = 0.
\]

More precise conditions can be given for the first-order scalar equation
\[
x'(t) = a(t)x(t) + \sum_{k=1}^{n} b_k x^{(k)}(t-r_k(t)),
\]
to which it is known [8] that

\[ a(t) \leq -\delta < 0, \quad \sum_{k=1}^{n} |b_k| < k\delta, \quad 0 < k < 1, \quad 0 \leq r_k \leq \sigma, \]

where \( \delta \) is a constant, implies that the zero solution is uniformly asymptotically stable. From Theorem 1 we may point the following result: if we assume

\[ a(t) \geq \delta > 0, \quad \sum_{k=1}^{n} |b_k| < k\delta, \quad 0 < k < 1, \quad 0 \leq r_k \leq \sigma, \]

then the zero solution of this equation is unstable. The instability of the solutions of this equation follows from Theorem 2 under conditions

\[ a(t) + \sum_{k=1}^{n} |b_k| > \delta, \quad 0 < k < 1, \quad 0 \leq r_k \leq \sigma, \]

and condition (21), which is accomplished if

\[ 3 \sum_{k=1}^{n} |r_k b_k| < 1. \]

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