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On the Growth and Deficiencies of Meromorphic Functions of Positive Integral Order

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1. INTRODUCTION

Let f be a meromorphic function of finite order λ satisfying the condition $f(0)=1$. The sequence of zeros of f, in their order of increasing magnitude, will be denoted by $\{a_n\}$; the sequence of poles by $\{b_n\}$. Let

$$n(r, a),\ N(r, a),\ m(r, a),\ T(r, f),\ M(r, f),\ \delta(a), \tag{1}$$

be the usual symbols of Nevanlinna Theory associated with f. In addition put

$$n(r) = n(r, 0) + n(r, \infty), \qquad N(r) = N(r, 0) + N(r, \infty), \tag{2}$$

and let

$$m_p(r, w) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |w(re^{i\theta})|^p \, d\theta\right\} \qquad (w = \log|f|,\ 1 \leqslant p < \infty). \tag{3}$$

Thus

$$\log M(r, f) = \lim_{p\to\infty} m_p(r, w^+) \tag{4}$$

and

$$m_1(r, w) = 2T(r, f) - N(r) \tag{5}$$

by Jensen's formula [15, p. 125] since $f(0)=1$.

When λ is not an integer, the order of each of the functions in (1), (2), and (3) is at most λ. Furthermore, comparison of the growth of any of these functions as $r\to\infty$ leads to non-trivial inequalities. The problem of obtaining best possible inequalities for limits of ratios of such functions is of great interest and difficulty. The known results about such problems may be summarized as follows:

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Denote by $\mathscr{M} = \mathscr{M}_\lambda$ the class of all memomorphic functions having finite non-integral order λ. Let $\mathscr{E}$ be the subclass of entire functions belonging to $\mathscr{M}$ and let $\mathscr{R}$ be the subclass of entire functions having only real negative zeros. Let ψ_λ be the 2π-periodic function defined by

$$\psi_\lambda(\theta) = \pi\lambda \csc \pi\lambda \cos \lambda\theta, \qquad -\pi < \theta < \pi, \tag{6}$$

and use $m_p(\psi_\lambda)$ to denote its $L_p(-\pi, \pi)$ mean.

For each fixed p $(1 \leqslant p < \infty)$ and for each class $\mathscr{O} = (\mathscr{M}, \mathscr{E}, \mathscr{R})$ of functions, define two constants

$$K(p, \mathscr{O}) = K(p, \lambda, \mathscr{O}) = \sup\{\liminf_{r\to\infty} (m_p(r, w)/N(r))\} \tag{7}$$

where $w = \log |f|$, and the sup is taken over all $f \in \mathscr{O}$;

$$k(p, \mathscr{R}) = k(p, \lambda, \mathscr{R}) = \inf\{\limsup_{r\to\infty}(m_p(r, w)/N(r))\} \tag{8}$$

where $w = \log |f|$ and the inf is taken over all $f \in \mathscr{R}$.

With the above notation, one of the problems mentioned in the previous paragraph takes the following form:

PROBLEM 1. *Determine exact values of the constants defined in* (7) *and* (8).

The first results concerning Problem 1 were obtained by Edrei and Fuchs [5]. From their results we can deduce

$$K(1, \lambda, \mathscr{M}) = m_1(\psi_\lambda) \qquad (0 < \lambda < 1). \tag{9}$$

Hellerstein and Williamson [8] obtained the second result:

$$K(1, \lambda, \mathscr{R}) = m_1(\psi_\lambda). \tag{10}$$

Miles and Shea [10] completely settled the L_2 case:

$$K(2, \lambda, \mathscr{M}) = m_2(\psi_\lambda). \tag{11}$$

Using (11), they were able to get the best bound known yet for $K(1, \lambda, \mathscr{M})$.

The L_p case was considered by Abi-Khuzam [2], who obtained

$$K(p, \lambda, \mathscr{E}) = m_p(\psi_\lambda) \qquad (0 < \lambda < 1,\ 1 \leqslant p < \infty). \tag{12}$$

Evaluation of the constant $k(p, \lambda, \mathscr{R})$ proved much easier. Thus Abi-Khuzam [2] obtained

$$k(p, \lambda, \mathscr{R}) = m_p(\psi_\lambda) \qquad (1 \leqslant p < \infty). \tag{13}$$

The special case of (13) corresponding to $p=1$, was obtained earlier by Ostrowski [13]; see also [8]. The limiting case of (12) obtained by letting $p \to \infty$, is the following old result of Valiron [16]:

$$K(\infty, \lambda, \mathscr{E}) = m_\infty(\psi_\lambda) \qquad (0 < \lambda < 1). \tag{14}$$

We remark that, in each of the results mentioned above, equality is realized by canonical products of order λ having regularly distributed real negative zeros [1].

It will be noticed that all results stated above have the common assumption that λ is not an integer. This is sort of necessary, since when λ is an integer, the function $N(r)$ may vanish identically and all previous inequalities, properly stated, will be trivial.

If $N(r)$ does not vanish, or even if $N(r)$ has order λ, the ratio $N(r)/m_p(r, w)$ may still tend to 0 as $r \to \infty$, when λ is an integer. On the other hand, results of Edrei and Fuchs [6] imply that, if λ is a positive integer, it is still possible to obtain good comparison theorems for $m_p(r, w)$; not with $N(r)$ but, at any rate, with a function closely related to the zeros and poles of f. The results of Edrei and Fuchs having been obtained under the assumption $\delta(0) + \delta(\infty) = 2$, it is natural to ask whether comparison theorems can be obtained in general.

In this paper we propose to formulate and solve the analogue of Problem 1 for integral orders by employing, instead of $N(r)$, a function $s(r)$ defined in terms of $N(r)$ only. Using $s(r)$ we shall obtain

(i) Comparison theorems for $m_p(r, w)$.

(ii) Abelian theorems for $\log |f|$, where $f \in \mathscr{R}$.

(iii) A characterization of sets of zeros and poles that have maximal deficiencies.

(iv) Some remarks about the structure of the Taylor series of functions with real negative zeros.

The function $s(r)$ mentioned above is found as follows: Starting from the Fourier series representation

$$w(re^{i\theta}) = \log |f(re^{i\theta})| = \sum_{m=-\infty}^{\infty} c_m(r)\, e^{im\theta} \tag{15}$$

where f is a meromorphic function of positive integral order λ, we find that the coefficient $c_\lambda(r)$ behaves, in many instances, very much like a certain function $s(r)$ and is in all cases dominated by $s(r)$. For $|m| \neq \lambda$, we will have at a suitably chosen sequence r_n tending to ∞

$$|c_m(r)| = o(s(r)) \qquad (r = r_n \to \infty). \tag{16}$$

We thus conclude that $w(re^{i\theta})$ behaves like $s(r)\,e^{i\lambda\theta}$ and obtain the comparison between $m_p(r, w)$ and $s(r)$. As might be expected, restrictions on the zeros and poles of f allow us to make the above analysis sharper, and we obtain new abelian theorems rather simply, as well as results on the deficiencies of functions whose zeros and poles are confined to certain sectors of the plane.

2. Statement of Results

Let f be a meromorphic function of positive integral order q. Let $\lambda(N)$ be the order of the function N defined in (2). There are four distinct cases that may arise:

(a) $\lambda(N) < q$;

(b) $\lambda(N) = q, \quad \int^{\infty} t^{-q-1} N(t)\,dt < +\infty, \quad \liminf_{r\to\infty} r^{-q} T(r, f) > 0$;

(c) $\lambda(N) = q, \quad \int^{\infty} t^{-q-1} N(t)\,dt < +\infty, \quad \liminf_{r\to\infty} r^{-q} T(r, f) = 0$;

(d) $\lambda(N) = q, \quad \int^{\infty} t^{-q-1} N(t)\,dt = \infty$.

If f satisfies one of the conditions (a) or (b), then it is easy to show that $N(r) = o(T(r, f))$ as $r \to \infty$. This is a well-known case, thoroughly investigated by Edrei and Fuchs [6]. The results of Edrei and Fuchs [6] give, among other things,

$$m_p(r, w) \sim r^q\,|c(r)|\,m_p(\cos q\theta) \qquad (r \to \infty,\ 1 \leqslant p < \infty), \tag{2.1}$$

where

$$c(r) = \alpha_q + q^{-1} \left\{ \sum_{|a_n| \leqslant r} a_n^{-q} - \sum_{|b_n| \leqslant r} b_n^{-q} \right\} \tag{2.2}$$

and α_q is defined by the Taylor expansion

$$\log f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \tag{2.3}$$

of $\log f(z)$ near 0.

This paper is devoted to the study of the remaining cases (c) and (d), and we start by modifying our definitions of the classes $\mathscr{M}$, $\mathscr{E}$, $\mathscr{R}$ of functions.

Fix the positive integer q and denote by $\mathscr{M}$ the class of all meromorphic functions f satisfying the following:

(i) $f(0) = 1$.

(ii) order $f = q$.

(iii) f satisfies exactly one of the above conditions (c) or (d).

(iv) Near 0, $\log f$ is given by (2.3).

We denote by $\mathscr{E}$ the subclass of entire functions belonging to $\mathscr{M}$ and by $\mathscr{R}$ the subclass of entire functions with only real negative zeros belonging to $\mathscr{E}$. We always write $w = \log |f|$ and retain the symbols defined in (1), (2), (3), as well as the symbol $c(r)$ defined in (2.2). The conditions (c) and (d) stated at the beginning of this section will be referred to repeatedly. We introduce, in addition, a function $s(r)$ as follows:

$$\begin{aligned} s(r) = a^*(r) &= qr^q \int_r^\infty t^{-q-1} N(t)\, dt \qquad \text{if } f \in \mathscr{M} \text{ and satisfies (c);} \\ &= a(r) = qr^q \int_0^r t^{-q-1} N(t)\, dt \qquad \text{if } f \in \mathscr{M} \text{ and satisfies (d).} \end{aligned} \tag{2.4}$$

We are now ready to state our results.

THEOREM 1. *For each $f \in \mathscr{M}$ and for each fixed p $(1 \leqslant p < \infty)$, we have*

$$\liminf_{r \to \infty} \{m_p(r, w)/s(r)\} \leqslant m_p(\cos q\theta). \tag{2.5}$$

Furthermore, this inequality is sharp.

Complementary to Theorem 1 we have

THEOREM 2. *For each $f \in \mathscr{R}$ and for each fixed p $(1 \leqslant p < \infty)$, we have*

$$\limsup_{r \to \infty} \{m_p(r, w)/s(r)\} \geqslant m_p(\cos q\theta). \tag{2.6}$$

For functions f in the class $\mathscr{R}$, it is even possible to obtain the asymptotic behavior of $\log |f|$ on a sequence of circles tending to infinity:

THEOREM 3. *Let $f \in \mathscr{R}$. Then there exists a sequence $\{x_n\}$ of real numbers increasing to infinity such that*:

(i) *if f satisfies* (c) *then, as $x_n \to \infty$,*

$$\log |f(x_n e^{i\theta})| = \{(-1)^{q+1} \cos q\theta\}\, s(x_n) + o(s(x_n)) \tag{2.7}$$

uniformly in $-\pi + \delta < \theta < \pi - \delta$, where $0 < \delta < \pi$.

(ii) *if f satisfies* (d) *then, as $x_n \to \infty$,*

$$\log |f(x_n e^{i\theta})| = \{(-1)^q \cos q\theta\} \, s(x_n) + o(s(x_n)) \tag{2.8}$$

uniformly in $-\pi + \delta < \theta < \pi - \delta$.

Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{2.9}$$

be the Taylor's series expansion about the origin of a function $f \in \mathscr{R}$. As an immediate consequence of Theorem 3 we have:

THEOREM 4. *Let $f \in \mathscr{R}$ be of order q. If f satisfies* (c) *and q is even or, if f satisfies* (d) *and q is odd, then the coefficients f_n of f change their sign infinitely often.*

Shea [14] has shown that entire functions with only real negative zeros and positive integral order have only one possible deficient value, namely zero. We extend Shea's result to meromorphic functions by allowing the zeros and poles to lie in certain sectors.

THEOREM 5. *Let $f \in \mathscr{M}$ be of order q. Assume that the zeros of f belong to the set $D_1(\eta, \varphi)$ and that its poles belong to the set $D_2(\eta, \varphi)$, where $0 \leqslant \eta < \pi/2q$, $\varphi \in [0, 2\pi)$ and*

$$\begin{aligned} D_1(\eta, \varphi) &= \bigcup_{j=0}^{q} \{z \colon |\arg z - \varphi - \pi(2j/q)| \leqslant \eta\} \\ D_2(\eta, \varphi) &= \bigcup_{j=0}^{q-1} \{z \colon |\arg z - \varphi - \pi(2j+1)/q| \leqslant \eta\}. \end{aligned} \tag{2.10}$$

Then 0 *and* ∞ *are the only possible deficient values of f.*

If, in addition, as $r \to \infty$

$$N(r) \sim r^q L(r) \tag{2.11}$$

where L is a slowly varying function, then

$$m_p(r, w) \sim r^q \, |c(r)| \, m_p(\cos q\theta) \tag{2.12}$$

and 0 *and* ∞ *have each deficiency* 1.

The implication (2.11) $\Rightarrow$ (2.12) above, generalizes an Abelian theorem in [1]. Our next result represents a stronger form of this theorem:

THEOREM 6. *Let f be as in Theorem* 5 *and let $s(r)$ and $c(r)$ be the functions defined in* (2.4) *and* (2.2). *Let the assumption* (2.11) *be replaced by the weaker assumption*

$$s(r) \sim r^q L(r) \tag{2.13}$$

where L is a slowly varying function. Then, $|c(r)|$ is a slowly varying function, (2.12) *holds true,* 0 *and* ∞ *are maximally deficient values of f and, if $f \in \mathscr{E}$,*

$$\log M(r, f) \sim r^q\,|c(r)| \qquad (r \to \infty). \tag{2.14}$$

An Abelian theorem of Bowen [3] asserts that if $f \in \mathscr{R}$, and satisfies (2.11), then $s(r)$ is regularly varying and the conclusions of our Theorem 3 hold globally. We can give a stronger result as follows:

THEOREM 7. *Let $f \in \mathscr{R}$ and assume that $s(r)$ satisfies* (2.13). *Then the conclusions of Theorem* 3 *hold true for every sequence $\{x_n\}$ tending to infinity.*

It is natural to inquire whether a condition weaker than (2.11) can still make 0 and ∞ maximally deficient values of f. Our next Theorem characterizes those sets of zeros and poles that have maximal deficiencies.

THEOREM 8. *Let f be a meromorphic function of finite order λ $(1 \leqslant \lambda < \infty)$ and satisfying $f(0) = 1$. Let $\{a_n\}$, $\{b_n\}$ be the zeros and poles of f. Put $q = [\lambda]$ and define $c(r)$ by* (2.2). *A necessary and sufficient condition for* 0 *and* ∞ *to be maximally deficient values of f is that the following two conditions hold true*:

(i) *$|c(r)|$ is a slowly varying function*;

(ii) $N(r) = o(r^q\,|c(r)|)$ $(r \to \infty)$.

In particular, under (i) *and* (ii), *f is of positive integral order q and* (2.12) *holds true.*

3. A GROWTH LEMMA

Let $f \in \mathscr{M}$ and let N be the associated function defined in (2). Recall that the order of N is a positive integer q, and put

$$b(r, m) = r^m \int_0^r t^{-m-1} N(t)\, dt \qquad (m \text{ integer}, r > 0). \tag{3.1}$$

Thus the function $a(r)$ defined in (2.4) satisfies $a(r) = qb(r, q)$.

The following Lemma is fundamental to the proofs in this paper.

LEMMA 1.

(I) *If* $\int_0^\infty t^{-q-1}N(t)\,dt=+\infty$, *then there exists a sequence* $\{r_n\}$ *increasing to infinity such that*

$$N(r)=o(a(r)) \qquad (r=r_n\to\infty); \tag{3.2}$$

$$b(r,m)=o(a(r)) \qquad (r=r_n\to\infty,\ 1\leqslant m<q,\ q\geqslant 2). \tag{3.3}$$

(II) *If* $\int_0^\infty t^{-q-1}N(t)\,dt<+\infty$ *then, in addition to a sequence* $\{r_n\}$ *satisfying* (3.2) *and* 3.3), *there exists a* (*possibly different*) *sequence* $\{t_n\}$ *increasing to infinity such that*

$$N(r)=o(a^*(r)) \qquad (r=t_n\to\infty); \tag{3.4}$$

$$b(r,m)=o(a^*(r)) \qquad (r=t_n\to\infty,\ 1\leqslant m<q,\ q\geqslant 2). \tag{3.5}$$

Proof of (I). Since $f\in\mathscr{M}$, the function N has order q and so the function $a(r)$ has order q. By a well-known result on proximate orders [9, p. 35, Theorem 16], there exists a slowly varying function L and a sequence $\{r_n\}$ increasing to infinity such that

$$a(r_n)=r_n^q L(r_n), \qquad a(r)\leqslant r^q L(r) \quad (r\geqslant 0).$$

Thus

$$\begin{aligned}\int_0^r t^{-q-1}N(t)\,dt &\leqslant L(r) \qquad \text{if} \quad r\geqslant 0\\ &= L(r) \qquad \text{if} \quad r=r_n.\end{aligned} \tag{3.6}$$

Let $\sigma>1$ be fixed. Then (3.6) and the slow variation of L imply

$$\int_0^{\sigma r} t^{-q-1}N(t)\,dt \sim \int_0^r t^{-q-1}N(t)\,dt \qquad (r=r_n\to\infty) \tag{3.7}$$

from which follows

$$\int_r^{\sigma r} t^{-q-1}N(t)\,dt = o\left(\int_0^r t^{-q-1}N(t)\,dt\right) \qquad (r=r_n\to\infty). \tag{3.8}$$

Consider now the inequalities

$$\int_{\sigma^{-1}r}^{r} t^{-q-1}N(t)\,dt \leqslant r^{-q}N(r)\,q^{-1}(\sigma^q-1)\leqslant \sigma^q\int_r^{\sigma r} t^{-q-1}N(t)\,dt. \tag{3.9}$$

Combining (3.8) and the second inequality of (3.9) we obtain in view of the definition of $b(r, q)$:

$$r^{-q}N(r) = o(r^{-q}b(r, q)) = o(r^{-q}a(r)) \qquad (r = r_n \to \infty). \tag{3.10}$$

Now (3.2) follows immediately from (3.10).

The first inequality in (3.9) and (3.10) give

$$\int_{\sigma^{-1}r}^{r} t^{-q-1}N(t)\,dt = o(r^{-q}b(r, q)) \qquad (r = r_n \to \infty), \tag{3.11}$$

and this in turn implies

$$\int_0^{\sigma^{-1}r} t^{-q-1}N(t)\,dt \sim \int_0^r t^{-q-1}N(t)\,dt \qquad (r = r_n \to \infty). \tag{3.12}$$

We now prove (3.3). Since $b(r, m) \leqslant b(r, q-1)$ for $1 \leqslant m < q$ and $q \geqslant 2$, it suffices to prove (3.3) for $m = q-1$. Starting from $r^{-q+1}b(r, q-1)$, where $q \geqslant 2$, an integration by parts together with some obvious estimates gives

$$\begin{aligned} &r^{-q+1}b(r, q-1) \\ &\quad = \int_0^r t^{-q}N(t)\,dt = r\int_0^r t^{-q-1}N(t)\,dt - \int_0^r \left(\int_0^t s^{-q-1}N(s)\,ds\right) dt \\ &\quad \leqslant r\int_0^r t^{-q-1}N(t)\,dt - (1-\sigma^{-1})\, r\int_0^{\sigma^{-1}r} s^{-q-1}N(s)\,ds. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) we conclude that

$$\limsup_{r_n\to\infty} \frac{b(r_n, q-1)}{b(r_n, q)} \leqslant \sigma^{-1} \qquad (\sigma > 1,\ q \geqslant 2). \tag{3.14}$$

Since $\sigma > 1$ was otherwise arbitrary, (3.14) implies

$$b(r, q-1) = o(b(r, q)) = o(a(r)) \qquad (r = r_n \to \infty). \tag{3.15}$$

This completes the proof of (3.3).

Proof of (II). Suppose that $\int_0^\infty t^{-q-1}N(t)\,dt < +\infty$. An easy differentiation shows that the function $r^q \int_r^\infty t^{-q-1}N(t)\,dt$ is increasing and the inequalities

$$q^{-1}N(r) < r^q \int_r^\infty t^{-q-1}N(t)\,dt < r^q \int_0^\infty t^{-q-1}N(t)\,dt,$$

imply that it has order q, since N has order q. Applying Edrei's Lemma [4,

p. 86] to $r^q \int_r^\infty t^{-q-1}N(t)\,dt$, we conclude that there exists sequences $\{r_m\}$ and $\{t_m\}$ increasing to infinity, and a positive sequence $\{\xi_m\}$ such that

$$r_m = t^m e^{\sqrt{m}}, \qquad t_m \to \infty, \qquad \xi_m \to 0 \tag{3.16}$$

and such that

$$\int_t^\infty s^{-q-1}N(s)\,ds \leqslant (1+\xi_m)\int_{r_m}^\infty s^{-q-1}N(s)\,ds \qquad (t_m \leqslant t \leqslant r_m), \tag{3.17}$$

and

$$\int_t^\infty s^{-q-1}N(s)\,ds \leqslant \left(\frac{t}{r_m}\right)^{-1/m}\int_{r_m}^\infty s^{-q-1}N(s)\,ds \qquad (t_0 \leqslant t \leqslant r_m). \tag{3.18}$$

We propose to show that the sequence $\{t_m\}$ satisfies (3.4) and (3.5).

From (3.17) and the fact that $\int_t^\infty s^{-q-1}N(s)\,ds$ is decreasing we conclude that

$$\int_{t_m}^\infty s^{-q-1}N(s)\,ds \sim \int_{r_m}^\infty s^{-q-1}N(s)\,ds \qquad (m\to\infty). \tag{3.19}$$

For $0<t<r$, the equality

$$t\int_t^\infty s^{-q-2}N(s)\,ds$$
$$= \int_t^\infty s^{-q-1}N(s)\,ds - t\int_t^\infty u^{-2}\left(\int_u^\infty s^{-q-1}N(s)\,ds\right)du, \tag{3.20}$$

together with some obvious estimates, gives

$$t\int_t^\infty s^{-q-2}N(s)\,ds$$
$$\leqslant \int_t^\infty s^{-q-1}N(s)\,ds - t\left(\int_r^\infty s^{-q-1}N(s)\,ds\right)(-r^{-1}+t^{-1}). \tag{3.21}$$

In (3.21), put $t=t_m$, $r=r_m$ and then divide by $\int_{t_m}^\infty s^{-q-1}N(s)\,ds$. In view of (3.19) and (3.16), this gives

$$t\int_t^\infty s^{-q-2}N(s)\,ds = o\left(\int_t^\infty s^{-q-1}N(s)\,ds\right) \qquad (t=t_m\to\infty). \tag{3.22}$$

Now the inequality $q^{-1}r^{-q}N(r) \leqslant r\int_r^\infty s^{-q-1}N(s)\,ds$, where $q\geqslant 2$, the definition of $a^*(r)$ and (3.22) imply (3.4).

To prove (3.5) it suffices, once again, to consider only the case $m = q - 1$, where $q \geqslant 2$. Starting from the equality

$$\int_0^t s^{-q} N(s)\, ds = -t \int_t^\infty s^{-q-1} N(s)\, ds + \int_0^t \left(\int_u^\infty s^{-q-1} N(s)\, ds \right) du, \qquad (3.23)$$

we put $t = t_m$ and apply (3.17) and (3.18). This gives

$$\begin{aligned} \int_0^{t_m} s^{-q} N(s)\, ds \leqslant & -t_m \int_{t_m}^\infty s^{-q-1} N(s)\, ds \\ & + \left\{ \int_{r_m}^\infty s^{-q-1} N(s)\, ds \right\} \left\{ \frac{m}{m-1} \left(\frac{r_m}{t_m} \right)^{1/m} t_m \right\} \\ & + o \left(\int_{r_m}^\infty s^{-q-1} N(s)\, ds \right). \end{aligned} \qquad (3.24)$$

Dividing both sides by $t_m \int_{t_m}^\infty s^{-q-1} N(s)\, ds$, and noting that, by (3.16), $(r_m/t_m)^{1/m} \to 1$ as $m \to \infty$, we obtain in view of (3.19),

$$\int_0^{t_m} s^{-q} N(s)\, ds = o \left(t_m \int_{t_m}^\infty s^{-q-1} N(s)\, ds \right) \qquad (m \to \infty). \qquad (3.25)$$

Now (3.5) follows immediately from (3.25). This finishes the proof of Lemma 1.

Remark. The conclusions of part (I) of Lemma 7 hold true if the sequence $\{r_m\}$ there is taken to be a sequence of polya peaks of $a(r)$. The use of proximate orders there appears necessary for one of the conclusions of Lemma 5 of the next section.

4. The Fourier Coefficients

In this section we present some results for the Fourier coefficients

$$c_m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(re^{i\theta})\, e^{-im\theta}\, d\theta$$

of $w(re^{i\theta}) = \log |f(re^{i\theta})|$.

Let $f \in \mathscr{M}$ and recall that $\log f(z) = \sum_{m=1}^\infty \alpha_m z^m$ for z near 0. For positive real r and integral m the function

$$b(r, m) = r^m \int_0^r t^{-m-1} N(t)\, dt, \qquad (4.1)$$

was introduced in Section 3. Using this function we define a sequence $\{\gamma_m(r)\}_{m=0}^{\infty}$ as follows:

$$\gamma_0(r) = N(r).$$
$$\gamma_m(r) = \tfrac{1}{2}m\{b(r, m) - b(r, -m)\} + N(r) \qquad (1 \leqslant m \leqslant q). \tag{4.2}$$
$$\gamma_m(r) = \tfrac{1}{2}mb(r, -m) + \tfrac{1}{2}mr^m \int_r^{\infty} t^{-m-1} N(t)\, dt - N(r) \qquad (m > q).$$

Our first lemma in this section is an important observation of Miles and Shea [10, 11].

LEMMA 2. *With the above notation, we have*

$$\begin{aligned} &|c_0(r)| \leqslant N(r). \\ &|c_m(r)| \leqslant \tfrac{1}{2}\,|\alpha_m|\, r^m + \gamma_m(r) && (1 \leqslant m \leqslant q). \\ &|c_m(r)| \leqslant \gamma_m(r) && (m \geqslant q+1). \\ &c_{-m}(r) = c_m(r) && (m > 0). \end{aligned} \tag{4.3}$$

Furthermore, the sequence $\{\gamma_m(r)\}$ *satisfies*

$$(m+1)\,\gamma_{m+1}(r) \leqslant m\gamma_m(r) \qquad (m \geqslant q+1). \tag{4.4}$$

Proof. The proof of (4.3) is given in [10, p. 380]. The proof of (4.4) is given in [181, p. 14]. Clearly, (4.4) implies that $\{\gamma_m(r)\}_{m>q}$ is, for each fixed $r > 0$, a decreasing sequence. In this connection, it may be of interest to point out that it is in fact a convex sequence.

Our next lemma concerns the coefficient $c_q(r)$.

LEMMA 3. *Let* $f \in \mathscr{M}$ *be of order* q.

(I) *If* f *satisfies* (d), *then*

$$|c_q(r)| \leqslant \tfrac{1}{2}a(r) + o(a(r)) + O(N(r)) \qquad (r \to \infty). \tag{4.5}$$

(II) *If* f *satisfies* (c), *then*

$$|c_q(r)| \leqslant \tfrac{1}{2}a^*(r) + O(N(r)) \qquad (r \to \infty). \tag{4.6}$$

Proof of (I). If f satisfies (d), then $\int_0^{\infty} t^{-q-1} N(t)\, dt = \infty$, and so, by (4.3),

$$\begin{aligned} |c_q(r)| &\leqslant o(a(r)) + \gamma_q(r) \\ &= o(a(r)) + \tfrac{1}{2}a(r) - \tfrac{1}{2}qb(r, -q) + N(r). \end{aligned} \tag{4.7}$$

Now (4.5) follows from (4.7) and the inequality

$$b(r,-q)=r^{-q}\int_0^r t^{q-1}N(t)\,dt \leqslant q^{-1}N(r).$$

Proof of (II). If f satisfies (c), then the constant α_q defined in (2.3), is given by

$$-q\alpha_q=\sum_{n=1}^{\infty} a_n^{-q}-\sum_{n=1}^{\infty} b_n^{-q} \tag{4.8}$$

both series being absolutely convergent. Now $c_q(r)$ is given by [10, p. 379]

$$\begin{aligned} c_q(r)=\frac{1}{2}\alpha_q r^q+\frac{1}{2q}\sum_{|a_n|\leqslant r}\left\{\left(\frac{r}{a_n}\right)^q-\left(\frac{\bar{a}_n}{r}\right)^q\right\} \\ -\frac{1}{2q}\sum_{|b_n|\leqslant r}\left\{\left(\frac{r}{b_n}\right)^q-\left(\frac{\bar{b}_n}{r}\right)^q\right\}. \end{aligned} \tag{4.9}$$

Thus if we use (4.8) in (4.9), do an obvious simplification, and then apply the triangle inequality as in [10], we obtain

$$|c_q(r)|\leqslant\frac{1}{2q}\int_r^{\infty}\left(\frac{r}{t}\right)^q dn(t)+\frac{1}{2q}\int_0^r\left(\frac{t}{r}\right)^q dn(t) \tag{4.10}$$

where $n(t)=n(t,0)+n(t,\infty)$. Integrating twice by parts and doing some obvious estimates we arrive at (4.6).

Further improvements of (4.5) and (4.6) are possible if the zeros and poles are suitably restricted.

LEMMA 4. *Let $f\in\mathscr{R}$ be of order q.*

(I) *If f satisfies* (d), *then*

$$|2c_q(r)-(-1)^q\,a(r)|=o(a(r))+O(N(r)) \qquad (r\to\infty). \tag{4.11}$$

(II) *If f satisfies* (c), *then*

$$|2c_q(r)-(-1)^{q+1}\,a^*(r)|=O(N(r)) \qquad (r\to\infty). \tag{4.12}$$

Proof. Since $f\in\mathscr{R}$, only the sum involving a_n appears in (4.9). Writing the sum as a Stieltjes integral we obtain

$$c_q(r)=\frac{1}{2}\alpha_q r^q+\frac{(-1)^q}{2q}\int_0^r\left\{\left(\frac{r}{t}\right)^q-\left(\frac{t}{r}\right)^q\right\}dn(t) \tag{4.13}$$

and this simplifies, as in Lemma 3, to

$$c_q(r) = -\frac{(-1)^q}{2q}\int_r^\infty \left\{ \left(\frac{r}{t}\right)^q - \left(\frac{t}{r}\right)^q \right\} dn(t) \tag{4.14}$$

if f satisfies (d).

Now (4.11) and (4.12) follow easily from (4.13) and (4.14).

We next compare the growth of the coefficients $c_m(r)$ of w with the growth of $s(r)$ at the sequences $\{r_n\}$ and $\{t_n\}$ of Lemma 1.

LEMMA 5. *Let $\{r_n\}$ and $\{t_n\}$ be the sequences obtained in Lemma* 1.

(I) *If f satisfies* (d), *then*

$$|c_m(r)| = o(a(r)) \qquad (|m| \neq q,\ r = r_n \to \infty), \tag{4.15}$$

and

$$\limsup_{r=r_n\to\infty} \frac{|c_q(r)|}{a(r)} \leqslant \frac{1}{2}. \tag{4.16}$$

(II) *If f satisfies* (c), *then*

$$|c_m(r)| = o(a^*(r)) \qquad (|m| \neq q,\ r = t_n \to \infty), \tag{4.17}$$

and

$$\limsup_{r=t_n\to\infty} \frac{|c_q(r)|}{a^*(r)} \leqslant \frac{1}{2}. \tag{4.18}$$

Proof. The assertions in (4.16) and (4.18) follow immediately from (4.5), (4.6) and (3.2) and (3.4).

For $1 \leqslant |m| < q$ and $q \geqslant 2$, the assertions (4.15) and (4.17) follow immediately from Lemma 2, (4.2) and (3.3) and (3.5).

Suppose next that $|m| > q$. By Lemma 2, we have

$$|c_m(r)| \leqslant \gamma_m(r) \leqslant \gamma_{q+1}(r). \tag{4.19}$$

By (4.2),

$$\gamma_{q+1}(r) \leqslant \frac{1}{2} N(r) + \frac{q+1}{2} r^{q+1} \int_r^\infty t^{-q-2} N(t)\, dt. \tag{4.20}$$

If f satisfies (c), then (3.4) and (3.22) applied in (4.20) imply

$$\gamma_{q+1}(r) = o\left(r^q \int_r^\infty t^{-q-1} N(t)\, dt \right) \qquad (r = t_n \to \infty). \tag{4.21}$$

For $|m| > q$, (4.17) now follows from (4.19) and (4.21).

If f satisfies (d), then (4.20) still holds true, but the integral appearing there is estimated by use of (3.6). Integrating by parts and using (3.6) we get

$$\begin{aligned} r^{q+1}\int_r^\infty s^{-q-2}N(s)\,ds &= -b(r,q)+r^{q+1}\int_r^\infty t^{-2}\left(\int_0^t s^{-q-1}N(s)\,ds\right)dt \\ &\leqslant -b(r,q)+r^{q+1}\int_r^\infty t^{-2}L(t)\,dt. \end{aligned} \tag{4.22}$$

Putting $r=r_n$ and dividing by $b(r_n,q)=(1/q)\,r_n^qL(r_n)$ we see [7, p. 273] that the left-hand side of the resulting inequality approaches $-1+1/(|-2+1+0|)=0$. From this, (3.2) and (4.20) we conclude that

$$\gamma_{q+1}(r)=o(a(r)) \qquad (r=r_n\to\infty). \tag{4.23}$$

For $|m|>q$, (4.15) now follows from (4.19) and (4.23). This finishes the proof of Lemma 5.

5. Proof of Theorem 1

Let $f\in\mathscr{M}$. Write

$$w(re^{i\theta})=\log|f(re^{i\theta})|\sim\sum_{m=-\infty}^{\infty}c_m(r)\,e^{im\theta}$$

and put

$$\tilde{w}(re^{i\theta})=w(re^{i\theta})-\sum_{|m|\leqslant q}c_m(r)\,e^{im\theta}.$$

Using the fact that the $L_p(-\pi,\pi)$ norm is a non-decreasing function of p together with Parseval's identity, (4.3) and (4.4), we obtain, when $1\leqslant p\leqslant 2$,

$$\begin{aligned} m_p(r,\tilde{w})\leqslant m_2(r,\tilde{w}) &= \left\{\sum_{|m|>q}|c_m(r)|^2\right\}^{1/2}\leqslant\left\{\sum_{|m|>q}\gamma_m^2(r)\right\}^{1/2} \\ &\leqslant\sqrt{2}\,(q+1)\,\gamma_{q+1}(r)\left\{\sum_{m=1}^{\infty}m^{-2}\right\}^{1/2}. \end{aligned} \tag{5.1}$$

If $2 < p < \infty$, we apply the Hausdorff–Young theorem together with (4.3) and (4.4) to obtain

$$m_p(r, \tilde{w}) \leqslant 2^{(1/p')}(q+1)\,\gamma_{q+1}(r) \left\{ \sum_{m=1}^{\infty} m^{-p'} \right\}^{1/p'} \tag{5.2}$$

where $1/p + 1/p' = 1$.

Let $\{u_n\}$ be the sequence defined as follows:

$$\begin{aligned} u_n &= t_n \qquad \text{if } f \text{ satisfies (c)} \\ &= r_n \qquad \text{if } f \text{ satisfies (d).} \end{aligned} \tag{5.3}$$

(This sequence will be used again and again in all subsequent proofs.)

In (5.1) and (5.2) put $r = u_n$, divide by $s(u_n)$ and let $u_n \to \infty$. By Lemma 5, it follows that for $1 \leqslant p < \infty$,

$$\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |w(re^{i\theta}) - c_q(r)\,e^{iq\theta} - \overline{c_q(r)}\,e^{-iq\theta}|^p \, d\theta \right\}^{1/p} = o(s(r)) \tag{5.4}$$

as $r = u_n \to \infty$.

If $c_q(u_n) = 0$ for all $n > n_0$, then (5.4) would imply that $m_p(r, w) = o(s(r))$ as $r = u_n \to \infty$, and the conclusion of Theorem 1 will be true. Otherwise, we may select a subsequence, also denoted by $\{u_n\}$, such that $c_q(u_n) \neq 0$ for all n. If we write $\Theta(u_n)$ for one of the determinations of the argument of $c_q(u_n)$, then (5.4) takes the form

$$\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |w(re^{i\theta}) - 2\,|c_q(r)| \cos(q\theta + \Theta(r))|^p \, d\theta \right\}^{1/p} = o(s(r)) \qquad (1 \leqslant p < \infty) \tag{5.6}$$

as $r = u_n \to \infty$.

Now Minkowski's inequality, (5.6), (4.16) and (4.18) give (2.5).

6. Proof of Theorem 2

Let $f \in \mathscr{R}$, then (5.4) holds true for $w = \log |f|$. Inside the absolute value signs in (5.4), add and subtract $(-1)^{q+1}\, a^*(r) \cos q\theta$ or $(-1)^q\, a(r) \cos q\theta$ according as f satisfies (c) or (d). If we then apply Minkowski's inequality and Lemma 4, we obtain

$$\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |w(re^{i\theta}) - (-1)^q\, a(r) \cos q\theta|^p \, d\theta \right\}^{1/p} \leqslant o(s(r)) + o(a(r)) + O(N(r)) \qquad (r = r_n \to \infty) \tag{6.1}$$

if f satisfies (d); while if f satisfies (c), we obtain

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |w(re^{i\theta}) - (-1)^{q+1} a^*(r) \cos q\theta|^p \, d\theta\right\}^{1/p}$$
$$= o(s(r)) + O(N(r)) \qquad (r = t_n \to \infty). \tag{6.2}$$

Now another application of Minkowski's inequality, (6.1), (6.2) and Lemma 1 lead to

$$m_p(r, w) \sim s(r)\, m_p(\cos q\theta) \qquad (r = u_n \to \infty,\ 1 \leqslant p < \infty). \tag{6.3}$$

This implies (2.6) and proves the sharpness of (2.5) at the same time. Note that (6.3) is much stronger than (2.6).

7. Proof of Theorem 3

Let $f \in \mathscr{R}$ and write

$$w(re^{i\theta}) = \log |f(re^{i\theta})| = \sum_{m=-\infty}^{\infty} c_m(r)\, e^{im\theta} \qquad (-\pi < \theta < \pi). \tag{7.1}$$

For $m > q$, the sequence $\{c_m(r)\}$ is given by [10]

$$c_m(r) = (-1)^{m+1}\, \gamma_m(r), \tag{7.2}$$

where $\{\gamma_m(r)\}$ is defined in (4.2).

Since, by Lemma 2, $\{\gamma_m(r)\}_{m>q}$ is a non-decreasing sequence, it follows by partial summation that

$$\left| \sum_{m>q} 2\gamma_m(r) \cos m\theta \right| < 2 \left| \csc\left(\frac{\theta}{2}\right) \right| \gamma_{q+1}(r) \qquad (0 < \theta < 2\pi). \tag{7.3}$$

From (7.3), (7.2) and $c_{-m}(r) = c_m(r)$ we get

$$\left| \sum_{|m|>q} c_m(r)\, e^{im\theta} \right| < 2 \left| \csc\left(\frac{\delta}{2}\right) \right| \gamma_{q+1}(r)$$
$$(-\pi + \delta < \theta < \pi - \delta,\ 0 < \delta < \pi). \tag{7.4}$$

Let $\{u_n\}$ be the sequence defined in (5.3). Then (7.1), (7.4) and Lemma 5 give

$$w(re^{i\theta}) - c_q(r)\, e^{iq\theta} - \overline{c_q(r)}\, e^{-iq\theta} = o(s(r)) \tag{7.5}$$

as $r = u_n$ tends to infinity, uniformly in $-\pi + \delta < \theta < \pi - \delta$. Now (2.7) and

(2.8), with $x_n = u_n$, follow immediately from (7.5), Lemma 4, and Lemma 1. This finishes the proof of Theorem 3.

Proof of Theorem 4. Let f satisfy the assumptions of Theorem 4, and suppose, to get a contradiction, that the coefficients f_n in its Taylor series (2.9) have a fixed sign from some point on, i.e., that there exist an integer $n_0 \geqslant 0$ such that

$$f_n \text{ is constant in sign for all } n \geqslant n_0. \tag{7.6}$$

Then it is easy to see that, on $|z| = r$,

$$|f(z)| \leqslant Kr^{n_0} + |f(r)| \qquad (r \geqslant 0), \tag{7.7}$$

where K is a constant independent of r.

Taking logarithms on both sides of (7.7) we obtain

$$\log |f(z)| \leqslant \log^+ Kr^{n_0} + \log^+ |f(r)| + \log 2 \qquad (0 \leqslant r = |z|). \tag{7.8}$$

Under the assumptions of Theorem 4, (2.7) and (2.8) imply

$$\log |f(x_n)| = -s(x_n) + o(s(x_n)) \qquad (x_n \to \infty). \tag{7.9}$$

Now (7.9) and (7.8) imply that the maximum modulus of f satisfies

$$M(r, f) \leqslant K_1 r^{n_0} \qquad (r = x_n,\ K_1 = \text{constant}),$$

which is impossible since f is transcendental. Hence (7.6) is false and the coefficients f_n change their sign infinitely often.

8. Proof of Theorem 5

Let $f \in \mathscr{M}$ be of order q and assume that its zeros $\{a_n\}$ and poles $\{b_n\}$ satisfy

$$\{a_n\} \subset D_1(\eta, \varphi), \qquad \{b_n\} \subset D_2(\eta, \varphi) \tag{8.1}$$

where D_1 and D_2 are defined by (2.10).

We shall prove that

$$\liminf_{r \to \infty} \frac{N(r)}{T(r, f)} = 0. \tag{8.2}$$

Once (8.2) is shown, we prove that 0 and ∞ are the only possible deficient values of f by an argument analogous to that used by Shea [14, p. 205]: We start with the classical estimate [12, p. 256]

$$\sum_{n=1}^{p} m(r, c_n) \leqslant 2T(r, f) + O(\log r) \qquad (r \to \infty, f \text{ of finite order})$$

where the c_r are any p ($\geqslant 3$) distinct complex numbers. This inequality implies that

$$\limsup_{r \to \infty} \frac{m(r, c_1) + m(r, c_2)}{T(r, f)} + \sum_{n=3}^{p} \liminf_{r \to \infty} \frac{m(r, c_n)}{T(r, f)} \leqslant 2.$$

Choosing $c_1 = 0$, $c_2 = \infty$ and using Nevanlinna's first fundamental theorem we obtain

$$2 - \liminf_{r \to \infty} \frac{N(r)}{T(r, f)} + \sum_{n=3}^{p} \delta(c_n, f) \leqslant 2$$

where $\delta(c, f)$ is the Nevanlinna deficiency of the value of c. This last inequality together with (8.2) clearly implies that $\delta(c, f) = 0$ for all values of $c \neq 0, \infty$.

Let us now prove (8.2).

If f satisfies (c) and $c_q(r)$ is its qth Fourier-coefficient, then $c_q(r)$ is given by (4.9) and simplifies, by (4.8), to

$$\begin{aligned} c_q(r) = &-\frac{1}{2q} \sum_{|a_n| > r} (r/a_n)^q - \frac{1}{2q} \sum_{|a_n| \leqslant r} (r/\bar{a}_n)^{-q} \\ &+ \frac{1}{2q} \sum_{|b_n| > r} (r/b_n)^q + \frac{1}{2q} \sum_{|b_n| \leqslant r} (r/\bar{b}_n)^{-q}. \end{aligned} \tag{8.3}$$

In view of (8.1) and (2.10), (8.3) gives

$$\begin{aligned} |c_q(r)| = |-c_q(r)\, e^{iq\varphi}| &\geqslant \frac{\cos q\eta}{2q} \left\{ \int_r^{\infty} \left(\frac{r}{t}\right)^q dn(t) + \int_0^r \left(\frac{t}{r}\right)^q dn(t) \right\} \\ &= \frac{\cos q\eta}{2} \left\{ qr^q \int_r^{\infty} t^{-q-1} N(t)\, dt + qr^{-q} \int_0^r t^{q-1} N(t)\, dt - 2N(r) \right\}. \end{aligned} \tag{8.4}$$

In (8.4) put $r = t_n$ where $\{t_n\}$ is the sequence defined in part (II) of Lemma 1. In view of (3.4) and (3.5), this gives

$$|c_q(r)| \geqslant \left(\frac{\cos q\eta}{2}\right) a^*(r)\{1 + o(1)\} \qquad (r = t_n \to \infty). \tag{8.5}$$

Now (8.5) and (3.4) give $N(t_n) = o(|c_q(t_n)|)$ which, with $|c_q(r)| \leqslant 2T(r, f) - N(r)$, gives (8.2).

Suppose next that f satisfies (d). Then

$$\operatorname{Re}(\alpha_q e^{iq\varphi}) + q\cos q\eta \int_0^r t^{-q-1}N(t)\,dt \sim q\cos q\eta \int_0^r t^{-q-1}N(t)\,dt. \tag{8.6}$$

By (4.9), (8.1) and (2.10) we have

$$\begin{aligned} |c_q(r)| &\geqslant \frac{1}{2}\operatorname{Re}(\alpha_q r^q e^{iq\varphi}) + \cos q\eta \left\{ \frac{q}{2}\int_0^r \left[\left(\frac{r}{t}\right)^q - \left(\frac{t}{r}\right)^q \right] \frac{N(t)}{t}\,dt + N(r) \right\} \\ &\sim \frac{r^q}{2}\cos q\eta \left\{ |\alpha_q| + q\int_0^r t^{-q-1}N(t)\,dt \right\} \sim \left(\frac{\cos q\eta}{2}\right) a(r) \end{aligned} \tag{8.7}$$

as r $(= r_n)$ tends to infinity through the sequence $\{r_n\}$ of Lemma 1.

By (3.2), this shows that $N(r_n) = o(|c_q(r_n)|)$ from which we conclude that (8.2) holds true. This finishes the proof of (8.2) and completes the proof of the first part of Theorem 5.

The second part of Theorem 5 will follow from Theorem 6, since the condition $N(r) \sim r^q L(r)$ implies that $s(r) \sim r^q L_1(r)$ where L and L_1 are slowly varying functions.

Proof of Theorem 6. Let $f \in \mathscr{M}$ be of order q and suppose that it satisfies (8.1). Let $s(r)$ and $c(r)$ be the functions defined in (2.4) and (2.2) and assume that (2.13) holds true.

If we introduce the notation

$$\begin{aligned} s(r; g) &= qr^q \int_0^r t^{-q-1} g(t)\,dt \\ &= qr^q \int_r^\infty t^{-q-1} g(t)\,dt, \end{aligned}$$

then an integration by parts gives

$$\begin{aligned} N(r) + s(r; N) &= s(r; n) \qquad \text{if } f \text{ satisfies (d)}, \\ -N(r) + s(r; N) &= s(r; n) \qquad \text{if } f \text{ satisfies (c)}. \end{aligned} \tag{8.8}$$

We are assuming that $s(r; N) \sim r^q L(r)$. This implies that $\int_r^{\sigma r} t^{-q-1}N(t)\,dt = o(r^{-q}s(r; N))$ for each fixed $\sigma > 1$. By (3.9) this implies that $N(r) = o(s(r; N))$, which when used in (8.8) gives

$$s(r; n) \sim s(r; N) \sim r^q L(r) \qquad (r \to \infty). \tag{8.9}$$

But then repetition of the above arguments for $s(r; n)$ shows that

$$n(r) = o(s(r; n)) = o(s(r; N)) \qquad (r \to \infty), \tag{8.10}$$

and

$$\int_r^{\sigma r} t^{-q-1} n(t)\, dt = o(s(r; n)) \qquad (r \to \infty, \sigma > 1). \tag{8.11}$$

If we take (8.4) and (8.7) into consideration, together with (8.10), it follows that

$$N(r) = o(|c_q(r)|), \qquad n(r) = o(|c_q(r)|) \qquad (r \to \infty). \tag{8.12}$$

But, by (4.9)

$$|2c_q(r) - r^q c(r)| \leqslant q^{-1} n(r).$$

Therefore

$$|c_q(r)| \sim \tfrac{1}{2} r^q \, |c(r)| \qquad (r \to \infty), \tag{8.13}$$

and

$$N(r) = o(r^q \, |c(r)|) \qquad (r \to \infty). \tag{8.14}$$

We now show that $c(r)$ is a slowly varying function.

Fix $\sigma > 1$. Then, by (2.2),

$$|c(\sigma r) - c(r)| \leqslant q^{-1} \int_r^{\sigma r} t^{-q}\, dn(t)$$

$$= q^{-1} \left\{ t^{-q} n(t) |_r^{\sigma r} + q \int_r^{\sigma r} t^{-q-1} n(t)\, dt \right\}. \tag{8.15}$$

Using (8.9) and (8.10) in (8.4) and (8.7), we see that

$$|c_q(r)| \geqslant \frac{\cos q\eta}{2} \{ s(r; n) + o(s(r; n)) \}$$

which, in view of (8.13), implies that

$$\tfrac{1}{2} r^q \, |c(r)| \geqslant \tfrac{1}{4} \cos q\eta \{ s(r; n) + o(s(r; n)) \}.$$

In (8.15), if we divide by $|c(r)|$ and take into account this last inequality and then (8.10) and (8.11) we arrive at

$$\lim_{r \to \infty} \left| \frac{c(\sigma r)}{c(r)} - 1 \right| \leqslant \lim_{r \to \infty} \frac{\sigma^{-q} n(\sigma r)}{s(r; n)}$$

$$= \lim_{r \to \infty} \frac{\sigma^{-q} n(\sigma r)}{s(\sigma r; n)} \frac{s(\sigma r; n)}{s(r; n)} = 0 \tag{8.16}$$

by (8.10) and regular variation of $s(r; n)$.

This shows that $|c(r)|$ is a slowly varying function.

To prove (2.12), note that by (8.14) and slow variation of $|c(r)|$ and properties of slowly varying functions, we have

$$|c_m(r)| = o(r^q\,|c(r)|) \qquad (r \to \infty,\ 1 \leqslant m < q`\ q \geqslant 2) \tag{8.16}$$

and

$$\gamma_{q+1}(r) = o(r^q\,|c(r)|) \qquad (r \to \infty) \tag{8.17}$$

where $\{c_m(r)\}$ are the Fourier coefficients of $w = \log|f|$, and $\gamma_{q+1}(r)$ is defined in (4.2).

Now (2.12) follows from (8.16), (8.17) and (5.1) and (5.2).

As a particular case of (2.12), we have

$$2T(r, f) - N(r) \sim r^q\,|c(r)|\, m_1(\cos q\theta),$$

which, with (8.14), implies

$$T(r, f) \sim \tfrac{1}{2} r^q\,|c(r)|\, m_1(\cos q\theta)$$

and hence

$$N(r) = o(T(r, f)) \qquad (\text{as } r \to \infty).$$

This shows that 0 and ∞ have maximal deficiencies.

The proof of (2.14) may be obtained by a very slight modification of [17, p. 510], and we omit it. This completes the proof of Theorems 6 and 5.

The proof of Theorem 7 follows immediately from (7.1), (7.4), Lemma 4 and properties of slowly varying functions.

9. Proof of Theorem 8

The necessary part in Theorem 8 is a well-known theorem of Edrei and Fuchs [6, p. 261]. For the sufficiency let f be a meromorphic function of finite order λ $(1 \leqslant \lambda < \infty)$ and assume that conditions (i) and (ii) of Theorem 8 are satisfied. We first show that λ must be an integer.

Suppose that λ is not an integer, then the order of N equals λ which is impossible since, under (i) and (ii), N has order $\leqslant q$. Thus $\lambda = q$.

If $c_q(r)$ and $c(r)$ are defined by (4.9) and (2.2), respectively, then for $\varepsilon > 0$ there exists r_0 such that

$$\begin{aligned} |2c_q(r) - r^q c(r)| &\leqslant q^{-1} n(r) \leqslant K_1 N(2r) \\ &\leqslant K_1\,\varepsilon(2r)^q\,|c(2r)| \leqslant K_2\,\varepsilon r^q\,|c(r)| \qquad (r \geqslant r_0), \end{aligned} \tag{9.2}$$

where K_1 and K_2 are constants independent of r, and we have used conditions (i) and (ii) of Theorem 8.

It follows from (9.2) that

$$2\,|c_q(r)| \sim r^q\,|c(r)| \qquad (r \to \infty)$$

so that

$$N(r) = o(|c_q(r)|) \qquad (r \to \infty). \tag{9.3}$$

Now the inequality $|c_q(r)| \leqslant 2T(r) - N(r)$ and (9.3) imply that $N(r) = o(T(r))$. Thus 0 and ∞ are maximally deficient values of f. The other conclusion of Theorem 8 now follows from (8.16), (8.17), (5.1) and (5.2).

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