JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **180**, 151–159 (1993)

A Note on Certain Generating Functions for the Generalized Bessel Polynomials

MING-PO CHEN

Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan

AND

H. M. SRIVASTAVA

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada

Submitted by J. L. Brenner

Received October 28, 1991

The main object of the present paper is to prove a general theorem on bilinear, bilateral, or mixed multilateral generating functions for the generalized Bessel polynomials. It is pointed out that this theorem and some other earlier results on the subject would provide unifications (and generalizations) of numerous generating functions which were obtained recently by using group-theoretic techniques. Some interesting applications of these general results (and their natural connections with the classical Laguerre polynomials) are also indicated. © 1993 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

Over four decades ago, Krall and Frink [6] initiated a systematic study of what are now well-known in the literature as the Bessel polynomials. In their terminology, the *generalized Bessel polynomials* $y_n(x, \alpha, \beta)$ are defined by

$$y_n(x, \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha + n + k - 2}{k} k! \left(\frac{x}{\beta}\right)^k \tag{1.1}$$

0022-247X/93 $5.00

and the *simple Bessel polynomials* $y_n(x)$ are defined by

$$y_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} k! \left(\frac{x}{2}\right)^k = y_n(x, 2, 2). \tag{1.2}$$

These polynomials are orthogonal and satisfy a second-order differential equation. Thus they may be viewed as an additional member of the family of classical orthogonal polynomials, the other members being the Jacobi, Hermite, and Laguerre polynomials, and indeed also such special cases of the Jacobi polynomials as the Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, and the Tchebycheff polynomials of the first and second kinds (see, for details, Rainville [8] and Szegö [12]). The importance of the Bessel polynomials lies in the fact that they arise naturally in a number of seemingly diverse contexts; e.g., in conection with the solution of the wave equation in spherical polar co-ordinates [6], in network synthesis and design [3], in the representation of the energy spectral functions for a family of isotropic turbulence fields [10], and so on. For further information and details about these polynomials and their applications, we refer the reader to the excellent monograph by Grosswald [4].

In terms of the generalized hypergeometric function ${}_uF_v$ (with u numerator and v denominator parameters) defined by

$${}_uF_v(\alpha_1, ..., \alpha_u; \beta_1, ..., \beta_v; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_u)_n}{(\beta_1)_n \cdots (\beta_v)_n} \frac{z^n}{n!}, \tag{1.3}$$

where $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$, the Bessel polynomials in (1.1) and (1.2) can be expressed as

$$y_n(x, \alpha, \beta) = {}_2F_0(-n, \alpha + n - 1; —; -x/\beta) \tag{1.4}$$

and

$$y_n(x) = {}_2F_0(-n, n+1; —; -\tfrac{1}{2}x), \tag{1.5}$$

respectively. For these hypergeometric polynomials, a considerable number of linear, bilinear, bilateral, and mixed multilateral generating functions are known (cf., e.g., Srivastava and Manocha [11]). Of our concern here are the following classes of bilateral or mixed multilateral generating functions for the Bessel polynomials, which were first proven by Srivastava [9, pp. 228–229, Corollaries 1 and 2] and which have since been reproduced in the latest treatise on the subject by Srivastava and Manocha [11, p. 421, Corollaries 1 and 2]:

THEOREM 1 (cf. Srivastava [9]). *Corresponding to a non-vanishing function $\Omega_\mu(\xi_1, ..., \xi_s)$ of s complex variables $\xi_1, ..., \xi_s$ ($s \in \mathbb{N} = \{1, 2, 3, ...\}$) and involving a complex parameter μ, called the order, let*

$$\Lambda^{(1)}_{m, p, q}[x; \xi_1, ..., \xi_s; \eta] = \sum_{n=0}^{\infty} a_n\, y_{m+qn}(x, \alpha - qn, \beta)\, \Omega_{\mu+pn}(\xi_1, ..., \xi_s) \frac{\eta^n}{(qn)!}$$
$$(a_n \neq 0; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p, q \in \mathbb{N}), \qquad (1.6)$$

where α and β ($\neq 0$) are complex parameters, and the coefficients a_n are assumed to be non-vanishing in order for the function on the left-hand side to be non-null. Suppose also that

$$M^{p, \mu}_{n, q}(\xi_1, ..., \xi_s; \eta) = \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k\, \Omega_{\mu+pk}(\xi_1, ..., \xi_s)\eta^k, \qquad (1.7)$$

where, as usual, $[\lambda]$ represents the greatest integer less than or equal to λ. Then

$$\sum_{n=0}^{\infty} y_{m+n}(x, \alpha - n, \beta) M^{p, \mu}_{n, q}(\xi_1, ..., \xi_s; \eta) \frac{t^n}{n!}$$
$$= \left(1 - \frac{xt}{\beta}\right)^{1-\alpha-m} \exp(t)$$
$$\cdot \Lambda^{(1)}_{m, p, q}\left[\frac{\beta x}{\beta - xt}; \xi_1, ..., \xi_s; \eta t^q\right] \qquad (|t| < |\beta/x|), \qquad (1.8)$$

provided that each member of (1.8) *exists.*

THEOREM 2 (cf. Srivastava [9]). *Under the hypotheses of Theorem* 1, *let*

$$\Lambda^{(2)}_{m, p, q}[x; \xi_1, ..., \xi_s; \eta] = \sum_{n=0}^{\infty} a_n\, y_{m+qn}(x)\, \Omega_{\mu+pn}(\xi_1, ..., \xi_s) \frac{\eta^n}{(qn)!}$$
$$(a_n \neq 0; m \in \mathbb{N}_0; p, q \in \mathbb{N}). \qquad (1.9)$$

Suppose also that the function

$$M^{p, \mu}_{n, q}(\xi_1, ..., \xi_s; \eta)$$

is defined, as before, by (1.7).

Then

$$\sum_{n=0}^{\infty} y_{m+n}(x)\, M_{n,q}^{p,\mu}(\xi_1, ..., \xi_s; \eta)\frac{t^n}{n!}$$
$$= (1-2xt)^{-(1/2)(m+1)} \exp(x^{-1}\{1-(1-2xt)^{1/2}\})$$
$$\cdot \Lambda_{m,p,q}^{(2)}[x(1-2xt)^{-1/2}; \xi_1, ..., \xi_s; \eta t^q (1-2xt)^{-(1/2)q}], \quad (1.10)$$

provided that each member of (1.10) *exists.*

A closer examination of the assertions (1.8) and (1.10) would reveal the remarkable fact that, in spite of the relationship (1.2), Theorem 2 cannot be deduced as a special case of Theorem 1. In fact, Theorem 2 happens to yield the most general family of bilateral or mixed multilateral generating functions for the *simple* Bessel polynomials which has been found so far. Numerous obvious special cases of Theorem 2 are being rediscovered in the literature, especially by group-theoretic means. For example, we may cite the *main result* of Chongdar and Mukherjee [2, p. 59, Theorem 1] which actually is a very specialized version of Theorem 2 above when

$$p = q = s = 1.$$

Motivated by some other results of Chongdar [1] on the *generalized* Bessel polynomials, analogous to those asserted by Theorem 1 above, we present here a rather elementary proof (*without* using the group-theoretic techniques employed in [1]) of the following interesting generalization of Theorem 1:

THEOREM 3. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda_{m,p,q}^{(3)}[x; \xi_1, ..., \xi_s; \eta] = \sum_{n=0}^{\infty} a_n\, y_{m+qn}(x, \alpha + (\rho-1)qn, \beta)$$
$$\cdot \Omega_{\mu+pn}(\xi_1, ..., \xi_s)\frac{\eta^n}{(qn)!}$$
$$(a_n \neq 0;\, m \in \mathbb{N}_0;\, p, q \in \mathbb{N}), \quad (1.11)$$

where ρ *is a complex parameter. Suppose also that*

$$\Phi_{n,p,q}^{\alpha,\mu,\rho}(x; \xi_1, ..., \xi_s; \eta)$$

is a polynomial of degree $[n/q]$ *in* η (*with coefficients depending upon* α, β, μ, ρ, n, p, q, x *as well as on* $\xi_1, ..., \xi_s$) *defined by*

$$\Phi_{n,p,q}^{\alpha,\mu,\rho}(x; \xi_1, ..., \xi_s; \eta) = \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k\, y_{m+n}(x, \alpha - n + \rho qk, \beta)$$
$$\cdot \Omega_{\mu+pk}(\xi_1, ..., \xi_s)\eta^k. \quad (1.12)$$

Then

$$\sum_{n=0}^{\infty} \Phi_{n,p,q}^{\alpha,\mu,\rho}(x;\xi_1,\ldots,\xi_s;\eta)\frac{t^n}{n!}$$
$$=\left(1-\frac{xt}{\beta}\right)^{1-\alpha-m}\exp(t)$$
$$\cdot \Lambda_{m,p,q}^{(3)}\left[\frac{\beta x}{\beta - xt};\xi_1,\ldots,\xi_s;\eta t^q\left(1-\frac{xt}{\beta}\right)^{-\rho q}\right] \qquad (|t|<|\beta/x|), \quad (1.13)$$

provided that each member of (1.13) *exists.*

2. Proof of Theorem 3

For the sake of convenience, let $\mathscr{S}$ denote the first member of the assertion (1.13) of Theorem 3. Then, by substituting for the polynomials

$$\Phi_{n,p,q}^{\alpha,\mu,\rho}(x;\xi_1,\ldots,\xi_s;\eta)$$

from (1.12) into the left-hand side of (1.13), we obain

$$\mathscr{S} = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/q]} a_k\, y_{m+n}(x,\alpha - n + \rho q k, \beta)$$
$$\cdot \Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\frac{\eta^k}{(n-qk)!\,(qk)!}$$
$$= \sum_{k=0}^{\infty} a_k\, \Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\frac{(\eta t^q)^k}{(qk)!}$$
$$\cdot \sum_{n=0}^{\infty} y_{m+qk+n}(x,\alpha+(\rho-1)\,qk-n,\beta)\frac{t^n}{n!} \qquad (\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\neq 0), \quad (2.1)$$

where we have inverted the order of the double summation involved.

The inner series in (2.1) can be summed by appealing to the familiar generating function (cf., e.g., [11, p. 419, Equation 8.4(8)]):

$$\sum_{n=0}^{\infty} y_{m+n}(x,\alpha-n,\beta)\frac{t^n}{n!}=\left(1-\frac{xt}{\beta}\right)^{1-\alpha-m}\exp(t)$$
$$\cdot y_m\left(\frac{\beta x}{\beta - xt},\alpha,\beta\right) \qquad (m\in\mathbb{N}_0;\ |t|<|\beta/x|) \quad (2.2)$$

with, of course, m and α replaced by $m+qk$ and $\alpha+(\rho-1)\,qk$, respectively ($k\in\mathbb{N}_0$; $q\in\mathbb{N}$), and we thus find from (2.1) that

$$\mathscr{S}=\left(1-\frac{xt}{\beta}\right)^{1-\alpha-m}\exp(t)$$
$$\cdot\sum_{k=0}^{\infty}\frac{a_k}{(qk)!}\,y_{m+qk}\left(\frac{\beta x}{\beta-xt},\alpha+(\rho-1)\,qk,\beta\right)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)$$
$$\cdot\left\{\eta t^q\left(1-\frac{xt}{\beta}\right)^{-\rho q}\right\}^k \qquad (|t|<|\beta/x|), \tag{2.3}$$

which, in view of the definition (1.11), is precisely the second member of the assertion (1.13) of Theorem 3.

This evidently completes the proof of Theorem 3 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 3 holds true (at least as a relation between formal power series) for those values of the various parameters and variables involved for which each member of the assertion (1.13) exists.

3. Further Remarks and Applications

Just as we remarked above about Theorem 2, Theorem 3 povides a very deep generalization of all those classes of bilateral and mixed multilateral generating functions for the generalized Bessel polynomials which can be deduced as (known or new) consequences of Theorem 1.

In its special case when $\rho=0$, Theorem 3 reduces immediately to Theorem 1. Moreover, Theorem 3 with

$$m=0, \qquad \rho=q=1, \qquad \text{and} \qquad \Omega_\mu(\xi_1,\ldots,\xi_s)\equiv 1$$

would yield one of the two main results of Chongdar [1, p. 151, Theorem 1]. His other main result [1, p. 151, Theorem 2] corresponds to the special case of Theorem 3 above when

$$m=0 \qquad \text{and} \qquad \rho=p=q=s=1.$$

It should be noted that our proof of Theorem 3, based upon the familiar result (2.2), is markedly different from the group-theoretic derivations of the aforementioned special cases of Theorem 3 by Chongdar [1].

For each suitable choice of the coefficients a_k ($k\in\mathbb{N}_0$), if the multivariable function

$$\Omega_\mu(\xi_1,\ldots,\xi_s) \qquad (s>1)$$

is expressed as an appropriate product of several simpler functions, each of the results considered here (including, for example, Theorem 3 above) can be applied to derive various families of mixed multilateral generating functions for the generalized Bessel polynomials. We choose to leave the details involved in these applications of Theorem 3 as an exercise for the interested reader.

Our demonstration of Theorem 3, presented in Section 2, was based upon the generating function (2.2) for the generalized Bessel polynomials. In his proof of Theorem 2, on the other hand, Srivastava [9] made use of the known result (cf., e.g., [7, p. 50, Eq. (12)])

$$\sum_{n=0}^{\infty} y_{m+n}(x)\frac{t^n}{n!}=(1-2xt)^{-(m+1)/2}$$
$$\cdot\exp\left(\frac{1-(1-2xt)^{1/2}}{x}\right)y_m\left(\frac{x}{(1-2xt)^{1/2}}\right)\qquad\left(m\in\mathbb{N}_0;\ |xt|<\frac{1}{2}\right), \tag{3.1}$$

involving the simple Bessel polynomials defined by (1.2).

Observe that the generating function (3.1) cannot be deduced from the seemingly more general result (2.2) by setting $\alpha=\beta=2$. Naturally, therefore, Theorem 2 is not contained in Theorem 3, as we remarked earlier.

In terms of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$$
$$=\binom{n+\alpha}{n}\,{}_1F_1(-n;\alpha+1;x), \tag{3.2}$$

we can easily find from (1.4) and the identity [11, p. 42, Eq. 1.4(3)]:

$${}_{u+1}F_v(-n,\alpha_1,\dots,\alpha_u;\beta_1,\dots,\beta_v;z)$$
$$=\frac{(\alpha_1)_n\cdots(\alpha_u)_n}{(\beta_1)_n\cdots(\beta_v)_n}(-z)^n$$
$$\cdot{}_{v+1}F_u\Bigg(-n,1-\beta_1-n,\dots,1-\beta_v-n;$$
$$1-\alpha_1-n,\dots,1-\alpha_u-n;\frac{(-1)^{u+v}}{z}\Bigg) \tag{3.3}$$

that

$$y_n(x,\alpha,\beta)=n!\left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right), \tag{3.4}$$

which, for $\alpha = \beta = 2$, yields the relationship:

$$y_n(x) = n!\left(-\frac{x}{2}\right)^n L_n^{(-2n-1)}\left(\frac{2}{x}\right). \tag{3.5}$$

Making use of this last relationship (3.5), which incidentally follows also from the known results [11, p. 42, Eq. 1.4(4), and p. 75, Eq. 1.9(1)], Hansen [5, p. 320, Eq. (48.19.14)] recorded the generating function (3.1) in its *equivalent* form:

$$\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(-2m-2n-1)}(x)\, t^n$$
$$= (1+4t)^{-(2m+1)/2} \exp\left(\frac{x\{1-(1+4t)^{1/2}\}}{2}\right)$$
$$\cdot L_m^{(-2m-1)}(x(1+4t)^{1/2}) \quad \left(m \in \mathbb{N}_0;\ |t| < \frac{1}{4}\right), \tag{3.6}$$

involving the classical Laguerre polynomials defined by (3.2).

By appealing to the generating function (3.6) instead of (3.1), we can obtain an interesting analogue of Theorem 2 for Laguerre polynomials. However, in view of the relationship (3.5), this indicated analogue would simply be *equivalent* to Theorem 2 itself.

ACKNOWLEDGMENTS

This work was completed during the second author's visit to the Institute of Mathematics (Academia Sinica) at Taipei while he was on sabbatical leave from the University of Victoria for the academic year 1990–1991. The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

REFERENCES

1. A. K. CHONGDAR, On generating functions of generalized Bessel polynomials, *Kobe J. Math.* **5** (1988), 151–154.
2. A. K. CHONGDAR AND M. C. MUKHERJEE, On the bilateral generating functions of simple Bessel polynomials, *Proc. Indian Acad. Sci. Math. Sci.* **98** (1988), 59–62.
3. F. GALVEZ AND J. S. DEHESA, Some open problems of generalised Bessel polynomials, *J. Phys. A: Math. Gen.* **17** (1984), 2759–2766.
4. E. GROSSWALD, "Bessel Polynomials," Springer-Verlag, Berlin/Heidelberg/New York, 1978.
5. E. R. HANSEN, "A Table of Series and Products," Prentice–Hall, Englewood Cliffs, NJ, 1975.

6. H. L. KRALL AND O. FRINK, A new class of orthogonal polynomials: The Bessel polynomials, *Trans. Amer. Math. Soc.* **65** (1949), 100–115.
7. E. B. MCBRIDE, "Obtaining Generating Functions," Springer Tracts in Natural Philosophy, Vol. 21, Springer-Verlag, New York/Heidelberg/Berlin, 1971.
8. E. D. RAINVILLE, "Special Functions," Macmillan, New York, 1960; reprinted by Chelsea, Bronx, NY, 1971.
9. H. M. SRIVASTAVA, Some bilateral generating functions for a certain class of special functions. I and II, *Nederl. Akad. Wetensch. Indag. Math.* **42** (1980), 221–233 and 234–246.
10. H. M. SRIVASTAVA, Some orthogonal polynomials representing the energy spectral functions for a family of isotropic turbulence fields, *Z. Angew. Math. Mech.* **64** (1984), 255–257.
11. H. M. SRIVASTAVA AND H. L. MANOCHA, "A Treatise on Generating Functions," Halsted Press (Horwood, Chichester); Wiley, New York/Chichester/Brisbane/Toronto, 1984.
12. G. SZEGÖ, "Orthogonal Polynomials," American Mathematidal Society Colloquium Publications, Vol. 23, 4th ed., Amer. Math. Soc., Providence, RI, 1975.