A Note on Certain Generating Functions for the Generalized Bessel Polynomials

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The main object of the present paper is to prove a general theorem on bilinear, bilateral, or mixed multilateral generating functions for the generalized Bessel polynomials. It is pointed out that this theorem and some other earlier results on the subject would provide unifications (and generalizations) of numerous generating functions which were obtained recently by using group-theoretic techniques. Some interesting applications of these general results (and their natural connections with the classical Laguerre polynomials) are also indicated.

I. INTRODUCTION AND MAIN RESULTS

Over four decades ago, Krall and Frink [6] initiated a systematic study of what are now well-known in the literature as the Bessel polynomials. In their terminology, the generalized Bessel polynomials \( y_n(x, \alpha, \beta) \) are defined by

\[
y_n(x, \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} \binom{x + n + k - 2}{k} k! \left( \frac{x}{\beta} \right)^k
\]
and the simple Bessel polynomials \( y_n(x) \) are defined by

\[
y_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{n+k}{k} \right) k! \left( \frac{x}{2} \right)^k = y_n(x, 2, 2). \tag{1.2}
\]

These polynomials are orthogonal and satisfy a second-order differential equation. Thus they may be viewed as an additional member of the family of classical orthogonal polynomials, the other members being the Jacobi, Hermite, and Laguerre polynomials, and indeed also such special cases of the Jacobi polynomials as the Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, and the Tchebycheff polynomials of the first and second kinds (see, for details, Rainville [8] and Szegö [12]).

The importance of the Bessel polynomials lies in the fact that they arise naturally in a number of seemingly diverse contexts; e.g., in connexion with the solution of the wave equation in spherical polar co-ordinates [6], in network synthesis and design [3], in the representation of the energy spectral functions for a family of isotropic turbulence fields [10], and so on. For further information and details about these polynomials and their applications, we refer the reader to the excellent monograph by Grosswald [4].

In terms of the generalized hypergeometric function \( \,_{v}F_{u} \) (with \( u \) numerator and \( v \) denominator parameters) defined by

\[
\,_{v}F_{u}(x_1, \ldots, x_u; \beta_1, \ldots, \beta_v; z) = \sum_{n=0}^{\infty} \frac{(x_1)_n \cdots (x_u)_n}{(\beta_1)_n \cdots (\beta_v)_n n!} \frac{z^n}{n!}, \tag{1.3}
\]

where \( (\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda) \), the Bessel polynomials in (1.1) and (1.2) can be expressed as

\[
y_n(x, \alpha, \beta) = \,_{2}F_{0}(-n, n+1; -; -x/\beta) \tag{1.4}
\]

and

\[
y_n(x) = \,_{2}F_{0}(-n, n+1; -; -\frac{1}{2}x), \tag{1.5}
\]

respectively. For these hypergeometric polynomials, a considerable number of linear, bilinear, bilateral, and mixed multilateral generating functions are known (cf., e.g., Srivastava and Manocha [11]). Of our concern here are the following classes of bilateral or mixed multilateral generating functions for the Bessel polynomials, which were first proven by Srivastava [9, pp. 228–229, Corollaries 1 and 2] and which have since been reproduced in the latest treatise on the subject by Srivastava and Manocha [11, p. 421, Corollaries 1 and 2]:
THEOREM 1 (cf. Srivastava [9]). Corresponding to a non-vanishing function $\Omega_q(z_1, ..., z_s)$ of $s$ complex variables $z_1, ..., z_s$ ($s \in \mathbb{N} = \{1, 2, 3, \ldots\}$) and involving a complex parameter $\mu$, called the order, let

$$A^{(1)}_{m, p, q}[x; \xi_1, ..., \xi_s; \eta] = \sum_{n=0}^{\infty} a_n y_{m + qn}(x, \alpha - qn, \beta) \Omega_{\mu + pn}(\xi_1, ..., \xi_s) \frac{\eta^n}{(qn)!} \quad (a_n \neq 0; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p, q \in \mathbb{N}), \quad (1.6)$$

where $\alpha$ and $\beta$ ($\neq 0$) are complex parameters, and the coefficients $a_n$ are assumed to be non-vanishing in order for the function on the left-hand side to be non-null. Suppose also that

$$M^{p, \mu}_{n, q}(\xi_1, ..., \xi_s; \eta) = \sum_{k=0}^{[n+q]} \binom{n}{qk} a_k \Omega_{\mu + pk}(\xi_1, ..., \xi_s) \eta^k, \quad (1.7)$$

where, as usual, $[\lambda]$ represents the greatest integer less than or equal to $\lambda$.

Then

$$\sum_{n=0}^{\infty} y_{m+n}(x, \alpha - n, \beta) M^{p, \mu}_{n, q}(\xi_1, ..., \xi_s; \eta) \frac{r^n}{n!}$$

$$= \left(1 - \frac{x}{\beta}\right)^{-t - m} \exp(t)$$

$$\cdot A^{(1)}_{m, p, q} \left[ \left[ \frac{\beta x}{\beta - x t}; \xi_1, ..., \xi_s; \eta^t \right] \quad (|t| < |\beta/x|), \quad (1.8)$$

provided that each member of (1.8) exists.

THEOREM 2 (cf. Srivastava [9]). Under the hypotheses of Theorem 1, let

$$A^{(2)}_{m, p, q}[x; \xi_1, ..., \xi_s; \eta] = \sum_{n=0}^{\infty} a_n y_{m + qn}(x) \Omega_{\mu + pn}(\xi_1, ..., \xi_s) \frac{\eta^n}{(qn)!} \quad (a_n \neq 0; m \in \mathbb{N}_0; p, q \in \mathbb{N}), \quad (1.9)$$

Suppose also that the function

$$M^{p, \mu}_{n, q}(\xi_1, ..., \xi_s; \eta)$$

is defined, as before, by (1.7).
Then

\[
\sum_{n=0}^{\infty} y_{m+n}(x) M_{\alpha, \beta}^{(\rho)}(\xi_1, \ldots, \xi_s; \eta) \frac{r^n}{n!} \\
= (1 - 2\alpha t)^{-(\alpha + 1)} \exp \left( x^{-1} \{ 1 - (1 - 2\alpha t)^{1/2} \} \right) \\
\cdot A_{m, p, q}^{(2)}(x(1 - 2\alpha t)^{-1/2}, \xi_1, \ldots, \xi_s; \eta r(1 - 2\alpha t)^{-(\alpha + 1)})
\]  

(1.10)

provided that each member of (1.10) exists.

A closer examination of the assertions (1.8) and (1.10) would reveal the remarkable fact that, in spite of the relationship (1.2), Theorem 2 cannot be deduced as a special case of Theorem 1. In fact, Theorem 2 happens to yield the most general family of bilateral or mixed multilateral generating functions for the simple Bessel polynomials which has been found so far. Numerous obvious special cases of Theorem 2 are being rediscovered in the literature, especially by group-theoretic means. For example, we may cite the main result of Choudhury and Mukherjee [2, p. 59, Theorem 1] which actually is a very specialized version of Theorem 2 above when

\[ p = q = s = 1. \]

Motivated by some other results of Choudhury [1] on the generalized Bessel polynomials, analogous to those asserted by Theorem 1 above, we present here a rather elementary proof (without using the group-theoretic techniques employed in [1]) of the following interesting generalization of Theorem 1:

**Theorem 3.** Under the hypotheses of Theorem 1, let

\[
A_{m, p, q}^{(3)}(x; \xi_1, \ldots, \xi_s; \eta) = \sum_{n=0}^{\infty} a_n y_{m+n}(x, \alpha + (\rho - 1)qn, \beta) \\
\cdot \Omega_{m+n}(\xi_1, \ldots, \xi_s; \eta) \frac{r^n}{(qn)!} \\
(a_n \neq 0; m \in \mathbb{N}_0; p, q \in \mathbb{N}),
\]

(1.11)

where \( \rho \) is a complex parameter. Suppose also that

\[ \Phi_{m, p, q}^{(n)}(x; \xi_1, \ldots, \xi_s; \eta) \]

is a polynomial of degree \([n/q]\) in \( \eta \) (with coefficients depending upon \( \alpha, \beta, \mu, \rho, n, p, q, x \) as well as on \( \xi_1, \ldots, \xi_s \)) defined by

\[
\Phi_{m, p, q}^{(n)}(x; \xi_1, \ldots, \xi_s, \eta) = \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k y_{m+n}(x, \alpha - n + \rho qk, \beta) \\
\cdot \Omega_{m+n}(\xi_1, \ldots, \xi_s; \eta r^k).
\]

(1.12)
Then
\[
\sum_{n=0}^{\infty} \Phi_{n,p,q}^{x,\mu,\rho}(x; \xi_1, \ldots, \xi_s; \eta) \frac{t^n}{n!} = \left(1 - \frac{xt}{\beta}\right)^{1-x-m} \exp(t) \cdot A^{(3)}_{m,p,q} \left[ \frac{\beta x}{\beta - xt} ; \xi_1, \ldots, \xi_s ; \eta \left(1 - \frac{xt}{\beta}\right)^{-pq} \right] \quad (|t| < |\beta/\beta|), \quad (1.13)
\]

provided that each member of (1.13) exists.

2. Proof of Theorem 3

For the sake of convenience, let \( S \) denote the first member of the assertion (1.13) of Theorem 3. Then, by substituting for the polynomials
\[
\Phi_{n,p,q}^{x,\mu,\rho}(x; \xi_1, \ldots, \xi_s; \eta)
\]
from (1.12) into the left-hand side of (1.13), we obtain
\[
S = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\lfloor \eta \rfloor} a_k \cdot \frac{\eta^k}{(n-qk)! (qk)!} \cdot \Omega_{\mu+pk} (\xi_1, \ldots, \xi_s ; \frac{(\eta t)^k}{(qk)!})
\]
\[
= \sum_{k=0}^{\infty} a_k \cdot \frac{\eta^k}{(n-qk)! (qk)!} \cdot \Omega_{\mu+pk} (\xi_1, \ldots, \xi_s ; \frac{(\eta t)^k}{(qk)!})
\]
\[
= \sum_{n=0}^{\infty} y_{m+qk+n}(x, x-n+\rho pk, \beta) \frac{t^n}{n!} \quad (\Omega_{\mu+pk}(\xi_1, \ldots, \xi_s) \neq 0), \quad (2.1)
\]

where we have inverted the order of the double summation involved.

The inner series in (2.1) can be summed by appealing to the familiar generating function (cf., e.g., [11, p. 419, Equation 8.4(8)]):
\[
\sum_{n=0}^{\infty} y_{m+n}(x, x-n, \beta) \frac{t^n}{n!} = \left(1 - \frac{xt}{\beta}\right)^{1-x-m} \exp(t) \cdot y_m \left(\frac{\beta x}{\beta - xt} ; x, \beta\right) \quad (m \in \mathbb{N}_0; |t| < |\beta/\beta|)
\]
\[
(2.2)
\]
with, of course, \( m \) and \( x \) replaced by \( m + qk \) and \( x + (\rho - 1) qk \), respectively \((k \in \mathbb{N}_0, \rho \in \mathbb{N})\), and we thus find from (2.1) that

\[
\mathcal{S} = \left(1 - \frac{xt}{\beta}\right)^m \exp(t) \sum_{k = 0}^{\infty} a_k \mu_m + qk \left(\frac{\beta x}{\beta - xt}, x + (\rho - 1) qk, \beta\right) \Omega_{\mu, \rho k}(\xi_1, ..., \xi_s)
\]

\[
\eta^{\rho} \left(1 - \frac{xt}{\beta}\right)^{-\rho} \quad (|t| < |\beta/x|), \quad (2.3)
\]

which, in view of the definition (1.11), is precisely the second member of the assertion (1.13) of Theorem 3.

This evidently completes the proof of Theorem 3 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 3 holds true (at least as a relation between formal power series) for those values of the various parameters and variables involved for which each member of the assertion (1.13) exists.

3. FURTHER REMARKS AND APPLICATIONS

Just as we remarked above about Theorem 2, Theorem 3 provides a very deep generalization of all those classes of bilateral and mixed multilateral generating functions for the generalized Bessel polynomials which can be deduced as (known or new) consequences of Theorem 1.

In its special case when \( \rho = 0 \), Theorem 3 reduces immediately to Theorem 1. Moreover, Theorem 3 with

\[
m = 0, \quad \rho = q = 1, \quad \text{and} \quad \Omega_{\rho}(\xi_1, ..., \xi_s) = 1
\]

would yield one of the two main results of Chongdar [1, p. 151, Theorem 1]. His other main result [1, p. 151, Theorem 2] corresponds to the special case of Theorem 3 above when

\[
m = 0 \quad \text{and} \quad \rho = p = q = s = 1.
\]

It should be noted that our proof of Theorem 3, based upon the familiar result (2.2), is markedly different from the group-theoretic derivations of the aforementioned special cases of Theorem 3 by Chongdar [1].

For each suitable choice of the coefficients \( a_k \) \((k \in \mathbb{N}_0)\), if the multi-variable function

\[
\Omega_{\rho}(\xi_1, ..., \xi_s) \quad (s > 1)
\]
is expressed as an appropriate product of several simpler functions, each of the results considered here (including, for example, Theorem 3 above) can be applied to derive various families of mixed multilateral generating functions for the generalized Bessel polynomials. We choose to leave the details involved in these applications of Theorem 3 as an exercise for the interested reader.

Our demonstration of Theorem 3, presented in Section 2, was based upon the generating function (2.2) for the generalized Bessel polynomials. In his proof of Theorem 2, on the other hand, Srivastava [9] made use of the known result (cf., e.g., [7, p. 50, Eq. (12)])

\[
\sum_{n=0}^{\infty} y_{m+n}(x) \frac{t^n}{n!} = (1 - 2x t)^{i(m+\frac{1}{2})} \cdot \exp \left( \frac{1 - (1 - 2x t)^{1/2}}{x} \right) y_n \left( \frac{x}{1 - (1 - 2x t)^{1/2}} \right) \quad (m \in \mathbb{N}_0; \ |x| < \frac{1}{2}),
\]

(3.1)

involving the simple Bessel polynomials defined by (1.2).

Observe that the generating function (3.1) cannot be deduced from the seemingly more general result (2.2) by setting \( \alpha = \beta = 2 \). Naturally, therefore, Theorem 2 is not contained in Theorem 3, as we remarked earlier.

In terms of the classical Laguerre polynomials \( L_n^{(\alpha)}(x) \) defined by

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n-k} \frac{(-x)^k}{k!}
\]

\[
= \binom{n + \alpha}{n} \quad _{1}F_{1}(-n; \alpha + 1; x),
\]

(3.2)

we can easily find from (1.4) and the identity [11, p. 42, Eq. 1.4(3)]:

\[
\begin{align*}
\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!} & = (x_1, \ldots, x_u; \beta_1, \ldots, \beta_v; z) \\
& = (\frac{(-z)^n}{(\beta_1)_{n} \cdots (\beta_v)_{n}}) \\
& \cdot _{v+1}F_{u} \left( -n, 1 - \beta_1 - n, \ldots, 1 - \beta_v - n; \right) \\
& \left( 1 - x_1 - \ldots - x_u - n; \frac{(-1)^{u+v}}{z} \right)
\end{align*}
\]

(3.3)

that

\[
y_n(x, \alpha, \beta) = n! \left( -\frac{x}{\beta} \right)^n \frac{\beta}{x} L_n^{(\alpha-2n)} \left( \frac{\beta}{x} \right).
\]

(3.4)
which, for \( z = \beta = 2 \), yields the relationship:

\[
y_n(x) = n! \left( \frac{x^n}{2} \right) L_n^{1-2n}(x)^{1/2}.
\]  

Making use of this last relationship (3.5), which incidentally follows also from the known results [11, p. 42, Eq. 1.4(4), and p. 75, Eq. 1.9(1)], Hansen [5, p. 320, Eq. (48.19.14)] recorded the generating function (3.1) in its equivalent form:

\[
\sum_{n=0}^{\infty} \binom{m+n}{n} L_n^{2m-2n+1}(x) t^n = (1 + 4t)^{-1/2} \exp \left( \frac{x[1 - (1 + 4t)^{1/2}]}{2} \right) L_n^{2m-1}(x(1 + 4t)^{1/2}) \quad (m \in \mathbb{N}_0; |t| < \frac{1}{4}),
\]

involving the classical Laguerre polynomials defined by (3.2).

By appealing to the generating function (3.6) instead of (3.1), we can obtain an interesting analogue of Theorem 2 for Laguerre polynomials. However, in view of the relationship (3.5), this indicated analogue would simply be equivalent to Theorem 2 itself.

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