NOTE

Another Combinatorial Determinant

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1. INTRODUCTION

The following result, a special case of a theorem of Mina [1], was recently given an elegant proof by Wilf [2].

**Theorem 1.** Let \( f = 1 + a_1 x + a_2 x^2 + \cdots \) be a formal power series, and define a matrix \( c \) by

\[
    c_{i,j} = \left[ x^j \right] f^i \quad (i, j \geq 0).
\]

(Here \( \left[ x^j \right] f \) denotes the coefficient of \( x^j \) in \( f \).) Then

\[
    \det((c_{i,j})_{i,j=0}^n) = a_i^{n+1/2} \quad (n = 0, 1, 2, \ldots).
\]

The purpose of this paper is to show that this result remains essentially unchanged if we take powers in the sense of composition, instead of multiplication.

**Theorem 2.** Let \( f = x + b_1 x^2 + b_2 x^3 + \cdots \) be a formal power series, and define \( f^{(0)} = x \) and \( f^{(i)} = f(f^{(i-1)}) \) for \( i > 0 \). Define a matrix \( c \) by

\[
    c_{i,j} = \left[ x^{i+1} \right] f^{(j)} \quad (i, j \geq 0).
\]

Then

\[
    \det((c_{i,j})_{i,j=0}^n) = 1! 2! \cdots n! b_i^{n+1/2} \quad (n = 0, 1, 2, \ldots).
\]

In fact, we prove both of these theorems at once, by formulating and proving a common generalization. In both theorems, each row of the
matrix is obtained from the previous row by applying a certain transformation of power series: in Theorem 1, the transformation is
\[ t \mapsto t(1 + a_1 x + a_2 x^2 + \cdots), \]
while in Theorem 2, the transformation is
\[ t \mapsto t + b_1 t^2 + b_2 t^3 + \cdots. \]
This suggests that more generally, we should consider transformations of the form
\[ t \mapsto f(t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{m,n} t^m x^n. \]

**Theorem 3.** Let \( f(t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{m,n} t^m x^n \) be a formal power series in two variables \( t \) and \( x \), and assume \( b_{1,0} = 1 \). Define \( f^{(0)}(t) = t \) and \( f^{(i)}(t) = f(f^{(i-1)}(t)) \) for \( i > 0 \). Define a matrix \( c \) by
\[ c_{i,j} = [x^{j+1}] f^{(i)}(x) \quad (i, j \geq 0). \]

Then
\[ \det((c_{i,j})_{i,j=0}^{n}) = \prod_{k=1}^{n} \left( \sum_{m=0}^{k} m! \binom{k+1}{m+1} b_{2,0}^m b_{k,1}^{k-m} \right) \quad (n = 0, 1, 2, \ldots), \]
where \( \binom{a}{b} \) is the Stirling number of the second kind (the number of partitions of \( x \) labeled objects into \( y \) nonempty sets).

**Proof.** Following [2], we prove that the matrix
\[ b_{i,j} = (-1)^{i+j} \binom{i}{j} \quad (i, j \geq 0) \]
has the property that \( bc \) is upper triangular. Put
\[ g_i(t) = \sum_k (-1)^{i+k} \binom{i}{k} f^{(k)}(t); \]
then \( (bc)_{i,j} = [x^i] g_j(x) \), and the theorem will follow from the fact that
\[ g_i(t) = \sum_{j=0}^{i} j! \binom{i+1}{j+1} b_{2,0}^j b_{i+1-j}^{i+1} t^{i+1} x^{i-j} + \text{higher-order terms}, \]
which we prove by induction on \(i\) (the case \(i = 0\) being true by definition). If we write \(g_{i+1}(t) = \sum_{m,n} k_{m,n} t^{m+1}x^n\) (so in particular, \(k_{m,n} = 0\) for \(m + n < i - 1\) and \(k_{m,n} = m! \sum_{m+n+1} b_{m,n}^2 \) for \(m + n = i - 1\)), then

\[
g_i(t) = g_{i+1}(f(t)) - g_{i+1}(t)
\]

\[
= \sum_{m,n} k_{m,n} x^n(f(t))^{m+1} - t^{m+1}
\]

\[
= \sum_{m,n} k_{m,n} x^n (m+1)(f(t)/t - 1) + (m+1) (f(t)/t - 1)^2 + \ldots
\]

\[
= \sum_{m+n=i} (m+1) k_{m,n} t^{m+1}x^n(b_{2,0}t + b_{1,1}x) + \text{higher-order terms}
\]

\[
= \sum_{m+n=i} t^{m+1}x^n[(m+1) b_{1,1} k_{m,n-1} + mb_{2,0} k_{m-1,n}]
\]

+ higher-order terms

\[
= \sum_{m+n=i} t^{m+1}x^n b_{2,0} \left\{ \binom{i}{m} + (m+1) \binom{i}{m+1} \right\} + \text{higher-order terms,}
\]

as desired.

It should be pointed out that the full version of Mina’s theorem, which states that

\[
\det \left( \left( \frac{d^j}{dx^j} f(x) \right)_{x=0} \right)^n = 1! 2! \ldots n! f^r(x)^{n+1/2},
\]

does not appear to admit an analogous generalization. The difficulty seems to be that while Mina’s theorem follows from applying Theorem 1 to the Taylor expansion of \(f(x)/f(t)\) at \(t\) for each \(t \in \mathbb{R}\), Theorem 2 can only be applied to the Taylor expansion of \(f\) at its fixed points.

REFERENCES

1. L. Mina, Formole generali delle derivate successive d’una funzione espresse mediante quelle della sua inversa, Giornale di Mat. 43 (1904), 196–212.