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Journal of Algebra 277 (2004) 643–657
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Normalizing isomorphisms between Burnside rings

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Received 27 June 2003
Available online 10 February 2004
Communicated by Michel Broué

Abstract

We prove that if two finite groups G and G' have isomorphic Burnside rings, then there is a normalized isomorphism between these rings, that is, a ring isomorphism $\theta : B(G) \to B(G')$ such that $\theta(G/1) = G'/1$. We use this to prove that if two finite groups have isomorphic Burnside rings, then there is a one-to-one correspondence between their families of soluble subgroups which preserves order and conjugacy class of subgroups.

Keywords: Burnside ring; Table of marks; Normalized isomorphism

1. Introduction

A very important algebraic invariant that can be associated to any finite group G is its *Burnside ring* $B(G)$, which we define in Section 2. This object has been studied from many different perspectives. As a commutative ring, much has been proved about its internal structure (see [14,15,22,23]). The Burnside ring encapsulates information about the G-sets of the group, which carry a lot of combinatorial information, and at a deeper level, it also lends itself for the analysis of more sophisticated G-sets such as G-posets, or more generally, simplicial G-sets (see Quillen's articles [17,18]). Many induction theorems have been proved about the Burnside ring using its prime spectrum and primitive idempotents (see Dress's work in [6]). The functoriality of $B(G)$ has also been exploited, and authors

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[1] Partially supported by a grant from CONACYT, Project 37260-E "Anillos de Burnside."
[2] Partially supported by a grant from PAPIIT, Project IN108802 "Combinatoria en Morelia."

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2003.10.031

such as Bouc, Thévenaz, and Webb have studied it from the point of view of Mackey functors and Green functors (see [2,21]).

A natural question to ask is whether non-isomorphic groups can have isomorphic Burnside rings. Although the answer is negative when both of the groups are abelian, or more generally, Hamiltonian (see [19]), this question has been settled by Thévenaz in [20], where he constructs infinitely many examples of non-isomorphic groups with isomorphic Burnside rings. More examples were later provided by Kimmerle and Roggenkamp in [10]. In all the known examples, the non-isomorphic groups had isomorphic *tables of marks* (defined in Section 2). It is easy to prove that groups with isomorphic tables of marks must have isomorphic Burnside rings, but it is still an open problem to determine whether groups with isomorphic Burnside rings must have isomorphic tables of marks (see [13]). It is a simple computation to prove that two groups G and G' have isomorphic tables of marks if and only if there is a ring isomorphism $\psi : B(G) \to B(G')$ such that for every subgroup U of G, $\psi(G/U)$ is of the form G'/U' for some subgroup U' of G'; in this case $|U| = |U'|$. One step towards this direction is to construct a *normalized* isomorphism between the Burnside rings, that is, a ring isomorphism $\theta : B(G) \to B(G')$ such that $\theta(G/1) = G'/1$, where 1 denotes the trivial subgroup. In this paper we prove that if two finite groups have isomorphic Burnside rings, then there exists a normalized isomorphism between them. The existence of a normalized isomorphism already implies that several invariants of the groups must be preserved, for example, the number of soluble subgroups (see Section 5), which are in an order-preserving correspondence.

In Section 2 we define Burnside rings and introduce all the basic concepts that we shall later need. In Section 3 we review some results about automorphisms of Burnside rings which were developed in Nicolson's paper [16]. Our own results are adaptations of Nicolson's ideas. In Section 4 we prove our main theorem and we apply it in Section 5 to generalize a theorem of Kimmerle's and Roggenkamp's (namely [11, Proposition 2.2]).

2. Burnside rings

In this section we introduce the basic concepts and notation that we shall use in this paper. Our presentation is very terse. For a fuller account of Burnside rings, we refer the reader to [1,3–5,9].

Let G be a finite group. A *G-set* is a finite set X where G acts on the left via a group homomorphism into the group of permutations of X. Two G-sets are *isomorphic* if there exists a bijection between them which preserves the action of G. The disjoint union and the Cartesian product of G-sets can be given naturally a structure of G-set. With these operations, the isomorphism classes of G-sets form a commutative half-ring, $B^+(G)$. Its associated ring is the *Burnside ring* of the group G, denoted by $B(G)$ (some authors write $\Omega(G)$ for the Burnside ring).

Each transitive G-set is isomorphic to a set of left cosets G/U for a subgroup U of G, and the G-sets G/U and G/T are isomorphic if and only if U and T are conjugate subgroups of G. Moreover, the family $\{G/U\}$ where U ranges over a set of representatives of conjugacy classes of subgroups of G, is a basis for $B(G)$ as an abelian group. For each subgroup U of G and every G-set X, let $\varphi_U(X)$ denote the number of elements of X which are fixed by all the elements of U, and use the same notation for the function $\varphi_U : B(G) \to \mathbb{Z}$ which is its natural extension to the Burnside ring. The following formula will be useful:

$$\varphi_U(G/T) = \frac{|N_G(U)|}{|T|}\beta(U,T),$$

where $\beta(U,T)$ is the number of subgroups of T which are G-conjugate to U. We have that $\varphi_U = \varphi_T$ if and only if U and T are conjugate. The square matrix whose entries are the numbers $\varphi_U(G/T)$, where U and T range over representatives of all the conjugacy classes of subgroups of G, is called the *table of marks* of the group G. Two groups are said to have isomorphic tables of marks if there is an ordering of their conjugacy classes of subgroups such that their resulting tables of marks are identical. Moreover, the functions φ_U induce an embedding

$$\varphi : B(G) \longrightarrow \prod_{\mathcal{C}(G)} \mathbb{Z},$$

where $\mathcal{C}(G)$ is the family of conjugacy classes of subgroups of G. The latter ring is called the *ghost ring* of G and is denoted by $\widetilde{B}(G)$. Thus, we sometimes regard the Burnside ring as a subring of the ghost ring. Since the ghost ring is a product of copies of the ring of integers, its primitive idempotents are in correspondence with the family $\mathcal{C}(G)$; for each subgroup U of G, we denote by $e_U^G = e_U$ the primitive idempotent of $\widetilde{B}(G)$ associated to U. There is an explicit formula for e_U^G in terms of the G/T (see [8]):

$$e_U^G = \frac{1}{|N_G(U)|}\sum_{T \leqslant U} \mu(T,U)|T|G/T,$$

where μ is the Möbius function of the subgroup lattice of G. We define

$$x_U^G = \big[U : D(U)\big]_0 \frac{|N_G(U)|}{|U|} e_U^G,$$

where $D(U)$ is the derived subgroup of U and n_0 denotes the product of all prime divisors of the integer n. As with φ_U, we have that $x_U^G = x_T^G$ if and only if U is conjugate to T in G. It is known that x_U^G is the least multiple of e_U^G that belongs to $B(G)$. This fact was proved by Nicolson in 1978 (see [16]) and later given different proofs by various authors such as Kratzer and Thévenaz (see [12]), or Dress and Vallejo (see [7]), who gave a very simple proof using Dress congruences.

Note also that an isomorphism between two Burnside rings $B(G)$ and $B(G')$ sends each x_U^G to some $x_{U'}^{G'}$, establishing a bijection $U \mapsto U'$ between the conjugacy classes of subgroups of G and the conjugacy classes of subgroups of G'. Since for each x_U^G we see that

$$\left(x_U^G\right)^2 = \frac{|N_G(U)|}{|U|}\big[U : D(U)\big]_0 x_U^G,$$

it follows that in this bijection

$$\frac{|N_G(U)|}{|U|}\big[U : D(U)\big]_0 = \frac{|N_{G'}(U')|}{|U'|}\big[U' : D(U')\big]_0.$$

It is easy to see that for a subgroup U of G, we have that

$$\frac{|N_G(U)|}{|U|}\big[U : D(U)\big]_0 = |G|$$

if and only if $|N_G(U)| = |G|$ and $|U| = [U : D(U)]_0$, that is, if and only if U is an abelian normal subgroup of G and the order of U is square-free. Since groups with isomorphic Burnside rings must have the same order, it follows that U is an abelian normal subgroup of G of square-free order if and only if so is U', that is, the families of abelian normal subgroups of square-free order of G and G' correspond under this bijection.

These special subgroups play a very important role in the study of the isomorphisms between two Burnside rings. The trivial subgroup is one of such special subgroups, which means that an isomorphism from $B(G)$ to $B(G')$ must send x_1^G to some $x_{U'}^{G'}$ with U' an abelian normal subgroup of G' of square-free order. Not any choice of abelian normal subgroup of G' of square-free order is possible as the image of x_1^G. In this paper we prove that the only possible choices are precisely the same U' so that $x_{U'}^{G'}$ can be the image of $x_1^{G'}$ under an *automorphism* of $B(G')$; these subgroups have been characterized by Nicolson in [16]. Hence, by composing with the inverse of such an automorphism, we shall be able to create a normalized isomorphism from $B(G)$ to $B(G')$.

There is a certain kind of reversed duality which we use when we normalize an isomorphism from $B(G)$ to $B(G')$. Just as the image of x_1^G is an $x_{U'}^{G'}$ with U' a certain abelian normal subgroup of G' of square-free order, the pre-image of $x_1^{G'}$ is an x_W^G with W an abelian normal subgroup of G of square-free order. The correspondence induced by the isomorphism establishes a bijection between the families of subgroups of W and U'. Each of these subgroups is characteristic in its parent (W or U') since it is the only subgroup of its order, and so it is also an abelian normal subgroup (of G or G' accordingly) of square-free order. This correspondence between the families of subgroups of W and U' reverses inclusions, which is to be expected, since W corresponds to 1 and 1 corresponds to U'.

3. Automorphisms of Burnside rings

In this section we quote without proof the most important results from Nicolson's paper on automorphisms of Burnside rings [16]. Our own results are adaptations of Nicolson's ideas, with the added complication that we have to work on two different rings. The gist of

Nicolson's paper is to determine when a subgroup U of a finite group G is in the orbit of the trivial subgroup under the automorphism group of the Burnside ring, that is, whether there exists an automorphism σ of $B(G)$ such that $\sigma(x_1^G) = x_U^G$. All of these results will later be used in our proofs.

The following result links divisibility in the Burnside ring with the internal structure of the lattice of subgroups of G. Denote xUx^{-1} by ${}^xU$.

Lemma 3.1 (Proposition 3.1). *Let G be a finite group and let U, T be subgroups of G. Let p be a prime number. If p divides $x_U^G + x_T^G$ in $B(G)$ with $U \neq T$, then one of the following cases holds*:

(i) *there exists $x \in G$ such that ${}^xU$ is a normal subgroup of T of index p*;
(ii) *there exists $g \in G$ such that ${}^gT$ is a normal subgroup of U of index p*;
(iii) *there exist $x, g \in G$ such that $U \cap {}^gT$ is a normal subgroup of U of index p and ${}^xU \cap T$ is a normal subgroup of T of index p.*

Remark 3.2. Note that in the previous lemma we conclude that p must divide the order of U or the order of T.

As a special case of the previous lemma, we can characterize the cyclic subgroups of order p of G by arithmetic properties.

Corollary 3.3 (Corollary 1). *Let G be a finite group and p a prime number. Let U be a nontrivial subgroup of G. Then p divides $x_U^G + x_1^G$ in $B(G)$ if and only if U has order p.*

The following result explores some of the properties of abelian normal subgroups of square-free order.

Lemma 3.4 (Lemma 3.2). *Let G be a finite group. Let U be a subgroup of G such that $|U|$ is square-free. If U has an abelian normal subgroup of prime index, then the coefficient of $G/1$ in x_U^G is $(-1)^s$, where s is the number of primes in $|U|$ (that is, the Möbius function $\mu(|U|)$).*

Next we encounter one of the "elementary" automorphisms of $B(G)$. Note that this particular automorphism has order two and that when restricted to certain subfamilies of the lattice of subgroups, it is order-preserving with respect to the partial order given by inclusion of subgroups.

Lemma 3.5 (Proposition 3.4). *Let G be a finite group and let p be a prime divisor of $|G|$. If G has a unique subgroup U of order p, then there is an automorphism σ of the Burnside ring $B(G)$ such that*:

$$\sigma\left(x_T^G\right) = \begin{cases} x_{TU}^G, & \text{if } p \text{ does not divide } |T|, \\ x_R^G, & \text{if } T = RU \text{ and } p \text{ does not divide } |R|, \\ x_T^G, & \text{otherwise.} \end{cases}$$

The case when the abelian normal subgroup has square-free order divisible by two has to be dealt with separately in the following two lemmas.

Lemma 3.6 (Part (b) of the proof of Proposition 3.5). *Let G be a finite group and U a subgroup of order* 4. *Then there exist non-trivial subgroups T_i which are not conjugate to U, and integers a_i such that* 4 *divides $2x_U^G + \sum a_i x_{T_i}^G + x_1^G$ and* 2 *divides $x_{T_i}^G + x_1^G$ for all i.*

Lemma 3.7 (Part (c) of the proof of Proposition 3.5). *Let G be a finite group and U a normal subgroup of order* 2. *If for a subgroup T of G there exist subgroups R_i of G and integers b_i such that* 4 *divides $2x_T^G + \sum b_i x_{R_i}^G + x_U^G$,* 2 *divides $x_{R_i}^G + x_U^G$ for all i, and U is not conjugate to any of the R_i, then U is a subgroup of T and T has order* 4.

The following two theorems are the core of Nicolson's article.

Theorem 3.8. *Let G be a finite group and let U be an abelian normal subgroup of G. Assume that the order of U is odd and square-free. Then G has no other subgroup of the same order as U if and only if there is an automorphism of $B(G)$ sending x_U^G to x_1^G.*

Theorem 3.9. *Let G be a finite group and let U be an abelian normal subgroup of G. Assume that the order of U is even and square-free. Then the following are equivalent*:

(i) *G has exactly one subgroup of order p for every odd prime divisor p of $|U|$, and the Sylow* 2*-subgroup of U is contained in every subgroup of G of order* 4.
(ii) *There is an automorphism of $B(G)$ sending x_U^G to x_1^G.*

The following result is proved implicitly in Nicolson's article. We quote it here with an explicit proof for the reader's convenience.

Lemma 3.10. *Let G be a finite group, let U be an abelian normal subgroup of G of square-free order, and let p be an odd prime dividing $|U|$. If T is a subgroup of G such that p divides $x_U^G + x_T^G$, then $T = O^p(U)$.*

Proof. By Lemma 3.1, we have one of the following three cases:

(i) T is conjugate to a subgroup of U of index p. Since U is an abelian normal subgroup of G of square-free order, this implies that T is $O^p(U)$.

(ii) U is a normal subgroup of T of index p. We notice that p divides the coefficient of G/U in x_U^G, namely, $|U|$. On the other hand, the coefficient of G/U in x_T^G is

$$-\big[T : D(T)\big]_0 \frac{|U|}{|T|} = -\frac{[T : D(T)]_0}{p},$$

which is not divisible by p, hence p cannot divide $x_U^G + x_T^G$, which is a contradiction.

(iii) $U \cap T$ is a normal subgroup of T of index p, and $U \cap T$ is a normal subgroup of U of index p (where we may have to replace T by a conjugate). By Lemma 3.4, the coefficient of $G/1$ in x_U^G and in x_T^G is $(-1)^s$, where s is the number of prime divisors of $|U|$. But p is an odd prime, which cannot divide $2(-1)^s$, contradicting the fact that p divides $x_U^G + x_T^G$. □

4. Isomorphisms between Burnside rings

In this section we extend Nicolson's results on automorphisms of Burnside rings to isomorphisms ψ between Burnside rings of different groups G and G'.

We begin by establishing properties of a subgroup U' of G' such that $\psi(x_1^G) = x_{U'}^{G'}$. It is curious that the information we obtain from U' has an effect on G, not on G'.

Proposition 4.1. *Let G, G' be finite groups and $\psi : B(G) \to B(G')$ an isomorphism between their Burnside rings. Assume $\psi(x_1^G) = x_{U'}^{G'}$ (so that U' is an abelian normal subgroup of G' and $|U'|$ is square-free).*

(a) *If p is a prime number which divides the order of U', then there is a normal subgroup of G of order p.*
(b) *If p is an odd prime divisor of $|U'|$, then G has a unique subgroup of order p.*

Proof. (a) Let U'_p be the Sylow p-subgroup of U' and let T, U_p be subgroups of G such that $\psi(x_T^G) = x_1^{G'}$, $\psi(x_{U_p}^G) = x_{U'_p}^{G'}$. Note that U'_p is non-trivial, which implies that T and U_p are not conjugate. Note also that T and U_p must be abelian normal subgroups of G of square-free order. Since U'_p has order p, by Corollary 3.3, p divides $x_1^{G'} + x_{U'_p}^{G'}$. The isomorphism ψ^{-1} preserves divisibility, and therefore p divides $x_T^G + x_{U_p}^G$. By Remark 3.2, we must have that p divides the order of T or the order of U_p. In either case, the Sylow p-subgroup of the appropriate subgroup is a normal subgroup of G of order p.

(b) Let Q and R be subgroups of G of order p. By Corollary 3.3, p divides $x_1^G + x_Q^G$ so that p divides $x_{U'}^{G'} + x_{Q'}^{G'}$, where $\psi(x_Q^G) = x_{Q'}^{G'}$. By Lemma 3.10, Q' is $O^p(U)$. Similarly, we construct R' and conclude that it is equal to Q', which proves that Q and R are conjugate in G. In (a) we proved that there is at least one normal subgroup of order p, which must therefore be the unique subgroup of G of order p. □

The following lemma is a partial converse to Lemma 3.1.

Lemma 4.2. *Let G be a finite group, U an abelian normal subgroup of G whose order is square-free, and P a normal subgroup of G of order p, with p a prime number which does not divide $|U|$. Then p divides $x_U^G + x_{UP}^G$.*

Proof. Note that if U is an abelian group with $|U|$ square-free, then for any subgroup T of U we have that $\mu(T, U) = \mu(|U|/|T|)$, where the latter Möbius function is the usual one defined for the ring of integers. In our case both U and UP are abelian, with square-free order. Hence, for any subgroup T of U, it follows that

$$\mu(T, UP) = \mu\big(|UP|/|T|\big) = \mu\big(|U|p/|T|\big) = -\mu\big(|U|/|T|\big) = -\mu(T, U).$$

On the other hand,

$$x_U^G = \sum_{T \leqslant U} \mu(T, U)|T|G/T,$$

$$x_{UP}^G = \sum_{T \leqslant U} \mu(T, UP)|T|G/T + \sum_{T \leqslant U} \mu(TP, UP)|TP|G/TP,$$

$$x_U^G + x_{UP}^G = p \sum_{T \leqslant U} \mu(TP, UP)|T|G/TP,$$

which is divisible by p. □

Remark 4.3. An equivalent way of stating the previous lemma is to say that p divides $x_U^G + x_{O^p(U)}^G$, where U is an abelian normal subgroup of G whose order is square-free, and p is any prime divisor of $|U|$.

The following proposition establishes a certain symmetry between the image of x_1^G and the pre-image of $x_1^{G'}$ under a ring isomorphism. The technique used here also yields as a side result a property of the Sylow 2-subgroup of the image U' of the trivial subgroup of G.

Proposition 4.4. *Let G, G' be finite groups and $\psi : B(G) \to B(G')$ an isomorphism between their Burnside rings. If $\psi(x_1^G) = x_{U'}^{G'}$ and $\psi^{-1}(x_1^{G'}) = x_T^G$, then $|U'| = |T|$. Furthermore, if $|U'|$ is even, there exists an isomorphism from $B(G)$ to $B(G')$ sending x_1^G to $x_{U_2'}^{G'}$ where U_2' is the Sylow 2-subgroup of U'.*

Proof. Let $|U'| = p_1 p_2 \ldots p_s$ be a product of s distinct primes. If $|U'|$ is even, assume that $p_1 = 2$. By Proposition 4.1, for each p_i there exists a normal subgroup U_{p_i} of G of order p_i. Let R be the abelian normal subgroup of G generated by the U_i. Note that the order of R is equal to the order of U'.

Put $T_0 = R$, $T_1 = O^{p_1}(R)$, $T_2 = O^{p_2}(T_1), \ldots, T_s = O^{p_s}(T_{s-1}) = 1$. Note that all T_i are abelian normal subgroups of G whose orders are square-free. By Remark 4.3, p_i divides $x_{T_i}^G + x_{T_{i-1}}^G$ for $i = 1, \ldots, s$, so it also divides its image under ψ, that is, p_i divides $x_{T_i'}^{G'} + x_{T_{i-1}'}^{G'}$ for some abelian normal subgroups T_i' of G' of square-free order. Note that $T_s' = U'$.

Since p_s is an odd prime that divides the order of T_s' and p_s divides $x_{T_s'}^{G'} + x_{T's-1}^{G'}$, by Lemma 3.10 we have that $T_{s-1}' = O^{p_s}(T_s')$, and in particular T_{s-1}' has order $p_1 p_2 \ldots p_{s-1}$. Now p_{s-1} is an odd prime dividing both the order of T_{s-1}' and $x_{T_{s-1}'}^{G'} + x_{T's-2}^{G'}$, so we conclude that $T'_{s-2} = O^{p_{s-1}}(T'_{s-1})$ and T'_{s-2} has order $p_1 p_2 \dots p_{s-2}$. Continuing in this fashion, we prove that T'_1 has order p_1, and in fact it is the Sylow p_1-subgroup of U'. If p_1 is odd, we can repeat this step and conclude that $T'_0 = 1$, so in this case $R = T$ and $|T| = |R| = |U'|$.

Assume now that p_1 is equal to 2. The subgroup T'_1 of G' is such that $\psi(x^G_{T_1}) = x^{G'}_{T'_1}$ where $T_1 = U_{p_2} U_{p_3} \dots U_{p_s}$. Since these primes are all odd, by Proposition 4.1, U_{p_i} is the only subgroup of G of that order. For each p_i with $i \geqslant 2$, let σ_i be an automorphism of $B(G)$ as in Lemma 3.5, and let $\sigma = \sigma_s \circ \sigma_{s-1} \circ \dots \circ \sigma_2$, so that $\sigma(x^G_1) = x^G_{T_1}$. Let Y be a subgroup of G such that $\sigma(x^G_Y) = x^G_T$. Consider the composition $\psi \circ \sigma : B(G) \to B(G')$. Note that $\psi(\sigma(x^G_1)) = \psi(x^G_{T_1}) = x^{G'}_{T'_1}$, which proves the last part of the statement.

It remains to show that $|T| = |U'|$. We have that $\psi(\sigma(x^G_Y)) = \psi(x^G_T) = x^{G'}_1$. Since $|T'_1| = 2$, by Corollary 3.3, 2 divides $x^{G'}_1 + x^{G'}_{T'_1}$, so 2 divides $x^G_Y + x^G_1$, and again by Corollary 3.3, it follows that Y has order 2. But the primes $2 = p_1, p_2, \dots, p_s$ are all different, and since $\sigma(x^G_Y) = x^G_T$, by Lemma 3.5, we have that $T = Y U_{p_2} U_{p_3} \dots U_{p_s}$, so $|T| = 2 p_2 p_3 \dots p_s = |U'|$. □

Now we can conclude that the properties that U' induced on G carry over to G'.

Corollary 4.5. *Let G, G' be finite groups and $\psi : B(G) \to B(G')$ an isomorphism between their Burnside rings. Assume $\psi(x^G_1) = x^{G'}_{U'}$, with U' of odd order. If p is a prime divisor of $|U'|$, then G' has a unique subgroup of order p. Furthermore, G' has no other subgroup of the same order as U'.*

Proof. Combining Propositions 4.4 and 4.1 for ψ^{-1}, we get the first part. The second part follows easily from the first. □

The previous corollary completes the case when U' had odd order (assuming of course Nicolson's results on automorphisms of Burnside rings). Our next proposition deals with the case when $|U'|$ is even.

Proposition 4.6. *Let G, G' be finite groups and $\psi : B(G) \to B(G')$ an isomorphism between their Burnside rings. Assume $\psi(x^G_1) = x^{G'}_{U'}$. If the order of U' is even, then the Sylow 2-subgroup of U' is a normal subgroup of G' of order 2 which is contained in all subgroups of order 4.*

Proof. By Proposition 4.4, without loss of generality we may assume that U' has order 2. Let T' be a subgroup of G' of order 4. We must show that T' contains U'. By Lemma 3.6, there exist nontrivial subgroups R'_i which are not conjugate to T', and integers a_i such that 4 divides $2x^{G'}_{T'} + \sum a_i x^{G'}_{R'_i} + x^{G'}_1$ and 2 divides $x^{G'}_{R'_i} + x^{G'}_1$ for all i. Taking ψ^{-1}, we have now that 4 divides $2x^G_T + \sum a_i x^G_{R_i} + x^G_R$ and 2 divides $x^G_{R_i} + x^G_R$ for all i, where the subgroups T, R_i, and R correspond to T', R'_i, and 1, respectively. Note that R is a normal subgroup of G and by Proposition 4.4, it has the same order as U', which is 2, so by Lemma 3.7 it follows that T has order 4. Since T has order 4, once again by Lemma 3.6 there exist appropriate subgroups K_i of G and integers b_i such that 4 divides $2x_T^G + \sum b_i x_{K_i}^G + x_1^G$ and 2 divides $x_{K_i}^G + x_1^G$ for all i. Taking ψ, we have that 4 divides $2x_{T'}^{G'} + \sum b_i x_{K_i'}^{G'} + x_{U'}^{G'}$ and 2 divides $x_{K_i'}^{G'} + x_{U'}^{G'}$ for all i. By Lemma 3.7, T' contains U'. □

Now we can prove our main result: any ring isomorphism between two Burnside rings can be normalized.

Theorem 4.7. *Let G, G' be finite groups. If their Burnside rings are isomorphic, then there exists a normalized isomorphism between them, that is, a ring isomorphism $\theta : B(G) \to B(G')$ such that $\theta(x_1^G) = x_1^{G'}$.*

Proof. Let $\psi : B(G) \to B(G')$ be an isomorphism between the two Burnside rings and let U' be the abelian normal subgroup of G' of square-free order such that $\psi(x_1^G) = x_{U'}^{G'}$. If the order of U' is odd, by Corollary 4.5, G' has no other subgroup of the same order as U'. By Theorem 3.8, there exists an automorphism α of $B(G')$ such that $\alpha(x_{U'}^{G'}) = x_1^{G'}$. Take $\theta = \alpha \circ \psi$.

If the order of U' is even, by Proposition 4.4 there exists an isomorphism ρ from $B(G)$ to $B(G')$ sending x_1^G to $x_{U_2'}^{G'}$ where U_2' is the Sylow 2-subgroup of U'. By Proposition 4.6, U_2' is a subgroup of G' of order 2 which is contained in all subgroups of order 4 of G'. By Theorem 3.9, there exists an automorphism β of $B(G')$ sending $x_{U_2'}^{G'}$ to $x_1^{G'}$. Take $\theta = \beta \circ \rho$. □

Remark 4.8. As we said before, normalizing an isomorphism between two Burnside rings is only the first step in constructing an isomorphism that preserves tables of marks. We believe this is the first part of an induction process, and that by composing with suitable automorphisms we shall reach the desired isomorphism.

5. Applications

In this section we generalize a result about automorphisms of Burnside rings to isomorphisms thereof. We shall use without proof the following lemma, which is Claim 2 in the proof of [11, Proposition 2.2].

Lemma 5.1. *Let G be a finite group. Then the number of maximal subgroups of index p is a multiple of p if and only if G has no normal subgroups of index p.*

The following is a generalization of [11, Proposition 2.2]. We use isomorphisms instead of automorphisms and our family S_U narrows down the possible subgroups of G' that appear in the expression for $\theta(G/U)$.

Theorem 5.2. *Let G and G' be finite groups and $\theta : B(G) \to B(G')$ a normalized isomorphism. For any subgroup D of G, let D' denote a subgroup of G' such that*

$\theta(x_D) = x_{D'}$. Let U be a soluble subgroup of G. Then U' is soluble, $|U'| = |U|$, $|N_{G'}(U')| = |N_G(U)|$, and $\theta(G/U) = G'/U' + \sum_{T\in S_U} a_T G'/T$ where S_U is the family of soluble subgroups T of G' such that $|T|$ is a proper divisor of $|U|$.

Proof. We shall use induction on the order of the soluble subgroup U. The case $|U| = 1$ is just the fact that θ is normalized. Assume that the result holds for all groups G and G', for all isomorphisms and for all soluble subgroups with order less than $|U|$. The proof is split into several steps:

Step 1. Using the formula

$$x_U = \big[U : D(U)\big]_0 G/U + \sum_{D<U} b_D G/D,$$

a similar formula for $x_{U'}$, and the fact that $\theta(x_U) = x_{U'}$, we get

$$\big[U : D(U)\big]_0 \theta(G/U) = \big[U' : D(U')\big]_0 G'/U' + \sum_{R<U'} c_R G'/R - \sum_{D<U} b_D \theta(G/D).$$

By induction on the proper subgroups D of U, and using the fact that S_D is contained in S_U if $|D|$ divides $|U|$, we conclude that we can write

$$\big[U : D(U)\big]_0 \theta(G/U) = \big[U' : D(U')\big]_0 G'/U' + \sum_{R<U'} f_R G'/R + \sum_{T\in S_U} a_T G'/T,$$

where the sum over S_U absorbs all possible elements from the other sum. In fact, we shall later prove that it absorbed all elements from that sum.

Step 2. The group U' is not in S_U, because if it were, then U' would be soluble of order less than $|U|$, and by induction we would have $|U| = |U'|$ and it cannot be a proper divisor of $|U|$, which is a contradiction. Therefore, the coefficient of G'/U' in the last formula from Step 1 is $[U' : D(U')]_0$, which must be divisible by $[U : D(U)]_0$. We can improve that formula to get

$$\theta(G/U) = tG'/U' + \sum_{R<U'} f_R G'/R + \sum_{T\in S_U} a_T G'/T',$$

where t is $[U' : D(U')]_0$ divided by $[U : D(U)]_0$. Note that this can be applied in any situation where we have an isomorphism and a soluble subgroup of the same order as U.

Step 3. For any prime number p dividing $[U : D(U)]_0$, we shall construct a subgroup M' which is normal of index p in U' such that p does not divide $t\beta(M', U')$. Since U is soluble and non-trivial, note that $[U : D(U)]_0 \neq 1$. Moreover, any prime number p dividing $[U : D(U)]_0$ does not divide t, because the highest power of p dividing $[U' : D(U')]_0$ is p^1. Since $[U : D(U)]_0$ divides $[U' : D(U')]_0$, U' also has normal subgroups of index p.

By Lemma 5.1, the number of subgroups of U' of index p is not divisible by p. Consider the following two families:

$$C_1 = \{T \leqslant U' \mid [U' : T] = p \text{ and there exists } g \in G' \text{ such that } T^g \text{ is normal in } U'\},$$
$$C_2 = \{T \leqslant U' \mid [U' : T] = p \text{ and there does not exist } g \in G' \text{ such that } T^g \text{ is normal in } U'\}.$$

As we have seen, $|C_1| + |C_2|$ is not divisible by p. Note that p divides $|C_2|$, since C_2 is a union of U'-orbits, each of which has size p (because $N_{U'}(T) = T$ for all $T \in C_2$). Therefore $|C_1|$ is not divisible by p. Every element in C_1 is G'-conjugate to a normal subgroup of U' of index p, so that we can write $|C_1|$ as a sum of $\beta(M', U')$ for certain normal subgroups of U' of index p. Since p does not divide $|C_1|$, then there exists a normal subgroup M' of U' of index p such that p does not divide $\beta(M', U')$.

Step 4. Let p and M' be as in Step 3 and fix them for the rest of the proof. We shall prove that $M' \in S_U$, i.e., that M' is soluble and $|M'|$ is a proper divisor of $|U|$. Since M' is a maximal subgroup of U', the only subgroup R of U' containing M' is M' itself. Evaluating $\varphi_{M'}$ on both sides of the formula from Step 2, we get

$$\varphi_M(G/U) = \varphi_{M'}\big(\theta(G/U)\big) = t\varphi_{M'}(G'/U') + f_{M'}\varphi_{M'}(G'/M') + \sum_{T \in S_U} a_T \varphi_{M'}(G'/T),$$

which becomes

$$\varphi_M(G/U) = t\frac{|N_{G'}(M')|}{|U'|}\beta(M', U') + f_{M'}\frac{|N_{G'}(M')|}{|M'|} + \sum_{T \in S_U} a_T \frac{|N_{G'}(M')|}{|T|}\beta(M', T).$$

If $\varphi_M(G/U) \neq 0$, then M could be chosen as a proper subgroup of U, so M would be soluble of smaller order, and by the induction hypothesis M' would be soluble of order $|M|$, which is a proper divisor of $|U|$, so $M' \in S_U$. We may assume then that $\varphi_M(G/U) = 0$ and $M' \notin S_U$. The previous formula becomes

$$0 = t\frac{|N_{G'}(M')|}{|U'|}\beta(M', U') + f_{M'}\frac{|N_{G'}(M')|}{|M'|}.$$

Multiplying by $|U'|$ and dividing by $|N_{G'}(M')|$, we get

$$0 = t\beta(M', U') + f_{M'}p,$$

which contradicts the fact that p does not divide $t\beta(M', U')$. Therefore, $M \in S_U$.

Step 5. By the previous step, M' is soluble, and since M' is normal in U' of index p, it follows that U' is also soluble.

Step 6. We claim that $|U| = |U'|$. If $|U| > |U'|$, since U' is soluble, by the induction hypothesis we would have $|U| = |U'|$, so we may assume that $|U| \leqslant |U'|$. Take the p and M' from Step 3. Once again we evaluate $\varphi_{M'}$ as in Step 4 to get

$$\frac{|N_G(M)|}{|U|}\beta(M,U) = t\frac{|N_{G'}(M')|}{|U'|}\beta(M',U') + \sum_{T\in S_U} a_T \frac{|N_{G'}(M')|}{|T|}\beta(M',T).$$

Recall that the term G'/M' appears inside the sum over S_U. Using the fact that $|N_G(M)| = |N_{G'}(M')|$ (which we know by the induction hypothesis on M'), we can cancel this out from all terms and then multiply by $|U|$ and $|U'|$ to get

$$|U'|\beta(M,U) = |U|t\beta(M',U') + \sum_{T\in S_U} a_T \frac{|U||U'|}{|T|}\beta(M',T).$$

Now divide everything by $|M| = |M'|$ (which is a common divisor of $|U|$ and $|U'|$, since $M' \in S_U$ and $M' < U'$) to obtain:

$$p\beta(M,U) = \frac{|U|}{|M|}t\beta(M',U') + p\sum_{T\in S_U} a_T \frac{|U|}{|T|}\beta(M',T).$$

Note that $|T|$ divides $|U|$, so all the fractions in the previous formula are integers. Since p does not divide $t\beta(M',U')$, we must have that p divides

$$\frac{|U|}{|M|} \leqslant \frac{|U'|}{|M'|} = p,$$

so it follows that $|U| = |U'|$.

Step 7. Since U and U' are both soluble of the same order, we can interchange their roles to conclude that $[U' : D(U')]_0 = [U : D(U)]_0$, that is, $t = 1$ (see the remark at the end of Step 2). Moreover, every proper subgroup R of U' is soluble and its order divides $|U'| = |U|$, so R is in S_U, and we have the desired form for $\theta(G/U)$.

Step 8. We know that the isomorphism θ is such that

$$\frac{|N_G(U)|}{|U|}\big[U : D(U)\big]_0 = \frac{|N_{G'}(U')|}{|U'|}\big[U' : D(U')\big]_0.$$

Since $|U| = |U'|$ and $[U : D(U)]_0 = [U' : D(U')]_0$, we must have that $|N_G(U)| = |N_{G'}(U')|$. $\square$

We can combine our main theorem with the previous one to obtain the following result.

Corollary 5.3. *Let G and G' be finite groups such that their Burnside rings are isomorphic. Then there is a one-to-one correspondence between the conjugacy classes of soluble subgroups of G and G' which preserves order of subgroup and cardinality of the conjugacy class* (*so we can also define a bijection between the families of soluble subgroups of G and G'*).

Proof. By Theorem 4.7, we may assume that there is a normalized isomorphism from $B(G)$ to $B(G')$. By Theorem 5.2, the assignment $U \mapsto U'$ is the desired correspondence. □

Hence groups with isomorphic Burnside rings have "the same soluble subgroups."

Acknowledgments

The authors benefited greatly from conversations with Professor Wolfgang Kimmerle during his stay at the Instituto de Matemáticas, UNAM in Morelia. He provided a lot of useful insight and he pointed out an error in a previous version of Theorem 5.2. Professor Florian Luca also provided us with valuable advice during the preparation of this article.

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