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Axion condensate as a model for dark matter halos

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ABSTRACT

Localized solutions of an axion-like scalar model with a periodic self-interaction are analyzed as a model of dark matter halos. It is shown that such a cold *Bose–Einstein* type condensate can provide a substantial contribution to the observed rotations curves of galaxies, as well provide a *soliton type* interpretation of the dark matter 'bullets' observed via gravitational lensing in merging clusters.

1. Introduction

The dominant nonvisible "dark fraction" of the total energy density or tension of the Universe is known to exist only from its gravitational effects. Since the *dark matter* (DM) part is distributed over rather large distances, its interaction, including a possible self-interaction [44], must be rather weak [6]. The most prominent candidates for such weakly interacting particles (WIMPs) are the (universal) *axion* or the lightest particle in hypothetical supersymmetric extensions of the standard model, as e.g. the neutralino, cf. Ref. [45].

Moreover, heterotic string theory provides a very light universal *axion* which may avoid [11] the strong CP problem in QCD [34]. Given the existence of such almost massless (pseudo-)scalars, it has been speculated that a *coherent non-topological soliton* (NTS) type solution of a nonlinear Klein–Gordon equation can account for the observed halo structure, simulating a non-relativistic *Bose–Einstein condensate* (BEC) of astronomical size [5,13,17,18,28]. Previously, a nonlinear toy model with a Φ^6 potential provided us with *exact* Emden type solutions [10,27], including a flattening of halos due to possible torus-like substructures [32]. An ellipticity $e<1$ was observed via microlensing [15,39]. Since axions are also obeying Bose–Einstein statistics [14], we are going to probe if the instanton-induced periodic axion potential yields reasonable DM halos which can simulate the apparent wiggles in the rotation curve of our Galaxy [42].

2. Effective axion potential

The discovery of instanton solutions in non-Abelian gauge theories topologically classified by a Pontrjagin term of type $\vec{E}\cdot\vec{B}$ has posed a problem in quantum chromodynamics (QCD): The experimental data for the electric dipole moment of the neutron lead to the bound $\theta := \theta_{\rm QCD} + \operatorname{Arg}\operatorname{Det} M < 10^{-10}$ on the effective vacuum angle, after diagonalizing the quark masses. A non-zero θ would imply parity or even CP violation.

The Peccei–Quinn (PQ) solution [34] to the strong CP problem is to introduce a dynamical field, the axion a, as a pseudo-Nambu–Goldstone boson associated with a new global $U(1)_{\rm PQ}$ symmetry, spontaneously broken at a scale f_a. Non-perturbative effects of QCD induce a potential $U(\theta)$ whose minimum at $a := \theta f_a$ cancel θ and thus solve the strong CP problem.

It is characteristic for the axion that it couples derivatively to spinor matter, it couples non-derivatively to two gluons and it has, via the *Primakoff effect*, an effective coupling to photons being used for its detection [3].

Standard QCD is based on the Yang–Mills Lagrangian

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$$\tilde{L}_{\rm YM} := -\frac{1}{2}\,{\rm Tr}\big(G\wedge {}^*G\big) + \frac{\theta}{2}\,dC, \tag{1}$$

amended by a coupling to the topological Pontrjagin term dC proportional to the so-called θ angle.

After integrating out the fermion fields, cf. Ref. [19], the generating functional for QCD including the term $\theta\,dC$ induces an *effective axion potential*

$$\begin{aligned} V(\theta) &= \frac{1}{2}U(\theta) = \Lambda_\theta(1-\cos\theta) \\ &= m_a^2 a^2\left(\frac{1}{2} - \frac{a^2}{12 f_a^2} + \frac{a^4}{360 f_a^4} - \cdots\right) \end{aligned} \tag{2}$$

induced by QCD instantons. In terms of the θ-angle, this potential displays a periodicity with a period of 2π and has a minimum at $\theta = 0$, as required. The resulting curvature of the potential at the minimum can be interpreted as an induced axion mass which for zero temperature is of the order $m_a = \sqrt{\Lambda_\theta}/f_a \simeq (6\times 10^{12}\ \text{GeV})/f_a$ μeV.

If the axion exist in its "invisible" form [19] and its energy scale f_a is not far from 10^{12} GeV, it may constitute a substantial fraction of the dark matter (DM) of the universe [8]. However, searches for the conversion an axion into a single photon via the inverse Primakoff effect have already excluded the $\mu\text{eV}/c^2$ mass range as a possible constituent of the local dark matter halo. On the other hand, in anthropic scenarios [14] or for a torsion-induced quintaxion [30], large $f_a > 10^{13}$ GeV are permitted for small misalignment angle θ_i.

For an ultralight axion of $m_a < 10^{-11}$ eV the absence of gamma rays from the supernova SN 1987A yield the bound of $g_{a\gamma} < 0.3\times 10^{-11}\ \text{GeV}^{-1}$ on the coupling constant of a photon mixing with axion-like particles [43].

3. Anharmonic oscillatory axion density

In the following, let us consider the periodic axion potential (2) with an arbitrary coupling constant Λ_θ. It will get absorbed by a rescaling of the radial coordinate.

Stationary solutions $\theta = P_l(r)\exp(-i\omega\sqrt{\Lambda_\theta}t)Y_m^l(\vartheta,\varphi)$ of the axidilaton [41] or axion-like field obey the nonlinear Klein–Gordon equation

$$\left(\Box + \frac{dV}{d\theta}\right)\theta = 0 \tag{3}$$

in flat spacetime. After averaging out the angular dependence as in Ref. [26], the radial equation reads

$$P'' + \frac{2}{x}P' - \frac{l(l+1)}{x^2}P + \omega^2 P - \sin P = 0, \tag{4}$$

where $' = d/dx$ is the derivative with respect to *dimensionless* radial coordinate $x := \sqrt{\Lambda_\theta}r$. A real axion corresponds to the limiting case $\omega = 0$, cf. [23].

Then the energy density of the axionic soliton is given by

$$\begin{aligned} \rho_\theta &= \frac{1}{2}\left[-\dot{P}^2 + \left(\frac{dP}{dr}\right)^2 + U\right] \\ &= \frac{\Lambda_\theta}{2}\big[\omega^2P^2 + (P')^2 + 2(1-\cos P)\big], \end{aligned} \tag{5}$$

since we adhere to the flat spacetime approximation. Scalar fields in a BEC type configuration, in contradistinction to cold dark matter (CDM), also exert the radial pressure

$$\begin{aligned} p_\theta &= \rho_\theta - U \\ &= \frac{\Lambda_\theta}{2}\big[\omega^2P^2 + (P')^2 - 2(1-\cos P)\big]. \end{aligned} \tag{6}$$

3.1. Localized solutions of the linearized equations

As a first step, let us consider small axion amplitudes $P \ll 1$ such that $\sin P \simeq P$ holds in this approximation. This holds, e.g. for a small initial misalignment $\theta_i \ll 1$. Then, the *nonlinear* differential equation (4) turns into the linear equation

$$x^2P'' + 2xP' + \big[(\omega^2-1)x^2 - l(l+1)\big]P \simeq 0. \tag{7}$$

According to Ref. [22], its solutions are the so-called *spherical Bessel functions*[1]

$$P_l = \sqrt{\frac{\pi}{2\tilde{x}}}\big[A\,J_{l+1/2}(\tilde{x}) + B\,Y_{l+1/2}(\tilde{x})\big], \tag{8}$$

where $\tilde{x} := x\sqrt{\omega^2-1}$ is for $\omega^2 > 1$ a real rescaled radial coordinate. For non-vanishing angular momentum $l \geqslant 1$, these solitons correspond to vortices in a BEC, see also Refs. [25,36]. Since the half-integer Bessel and Neumann functions are known to be elementary functions, we can rewrite our result in terms of parity eigenstates of angular momentum l as follows:

$$P_l = (-1)^l\tilde{x}^l\left(\frac{d}{\tilde{x}\,d\tilde{x}}\right)^l\frac{A\sin\tilde{x} - B\cos\tilde{x}}{\tilde{x}}. \tag{9}$$

For the spherically symmetric case $l = 0$, a regular solution obeying the initial condition $P(0) = \theta_i = \pi/2$ of maximal inicial misalignment[2] reads

$$P_0(x) = \frac{\pi}{2}\,\frac{\sin(x\sqrt{\omega^2-1})}{x\sqrt{\omega^2-1}}. \tag{10}$$

It corresponds to the choice $A = \pi/2$ and $B = 0$, where the latter is a necessary condition for avoiding a singularity at the origin. Such a solution was first considered by Schunck [38] and recently rediscovered in Ref. [25].

In this approximation, it can be regarded as a standing wave of an axion field coherently oscillating in its potential U, cf. Ref. [20]. Since in our approximation $U(\theta) \simeq \Lambda_\theta P^2$ holds, we find for the energy density and pressure

$$\rho_\theta \simeq \frac{\Lambda_\theta}{2}\big[(\omega^2+1)P_l^2 + (\omega^2-1)(dP_l/d\tilde{x})^2\big], \tag{11}$$

$$p_\theta \simeq \frac{\Lambda_\theta}{2}(\omega^2-1)\big[P_l^2 + (dP_l/d\tilde{x})^2\big] = \rho_\theta - \Lambda_\theta P_l^2, \tag{12}$$

respectively.

For a real axion with $\omega = 0$ the pressure will always be negative. Due to the recurrence relations for Bessel functions, the first derivative in Eqs. (11) and (12) may as well be expressed via

$$\frac{d}{d\tilde{x}}P_l = P_{l-1} - \frac{l+1}{\tilde{x}}P_l. \tag{13}$$

3.2. Numerical solutions

In order to see the effect of the nonlinear potential, let us consider a spherically symmetric solution for which the axion is maximally 'misaligned' at the origin, i.e. that satisfies the initial condition $P_0(0) = \pi/2$ and $P_0'(0) = 0$. Such a configuration will be *localized* at the origin, provided the change of the slope remains initially negative, i.e. $P'' < 0$, which, according to (4), holds for $\omega > \sqrt{2/\pi}$. For this initial conditions, we have solved numerically the nonlinear differential equation

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[1] For $l = 0$ and $\omega = 1$, Eq. (7) admits the singular solution $P = A + B/x$ for which the radial pressure (12) vanishes.

[2] The related average $\langle\theta_i\rangle = \pi^2/3$ is considered in Ref. [14].

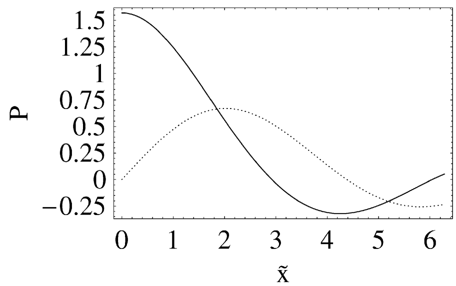

Fig. 1. Distribution of the dimensionless axion field P in a DM halo, initially 'misaligned' by $\theta_0 = \pi/2$, in the spherically symmetric case ($l = 0$) and for $\omega = \sqrt{2}$. Its anharmonic oscillatory form complies with the behavior of the Bessel functions of the first kind. The dotted line represents a 'vortex solution' with $l = 1$.

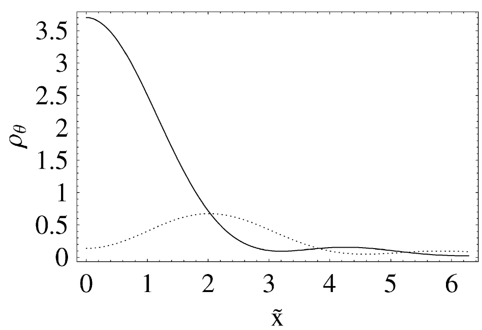

Fig. 2. Decay of the normalized axion density $\rho_\theta/\Lambda_\theta$ in dependence of the rescaled radius $\tilde{x} := x\sqrt{\omega^2 - 1}$ for $\omega = \sqrt{2}$. The spherically symmetric axionic DM halo exhibits a constant core.

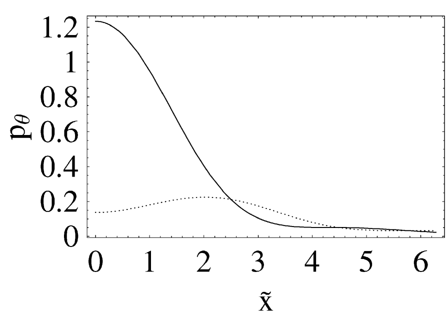

Fig. 3. Normalized radial pressure distribution p_θ/Λ_θ in the DM halo for $\omega = \sqrt{2}$ which always remains positive.

$$\tilde{x}^2 \ddot{P} + 2\tilde{x}\dot{P} + \left[\tilde{x}^2 - l(l+1)\right]P + \frac{\tilde{x}^2}{\omega^2 - 1}(P - \sin P) = 0 \tag{14}$$

with the built-in Nsolve packet of Mathematica 6.0 and plotted the result in Fig. 1. Since now '·' denotes the derivative with respect to $\tilde{x} := x\sqrt{\omega^2 - 1}$, for $\omega^2 > 1$, Eq. (14) is equivalent to (4). At the center, we will apply the same initial conditions as in the linearized case (10). Away from the center, the axion amplitude is oscillating around $\langle\theta\rangle = 0$. As expected, the corresponding axion density ρ_θ and pressure p_θ, drawn in Figs. 2 and 3, turn out to be rather localized at the center. Since we required $\dot{P}(0) = 0$ at the origin, the energy density (5) exhibits an almost *constant core* as observed near the center of DM dominated galaxies. This is in contradistinction to numerical simulations of CDM, for which typically a cusp evolves at the center.

For angular momentum states with $l \geqslant 1$ and $B = 0$, the axion field is initially vanishing, i.e. $P_l(0) = 0$. Nevertheless, formula (13) for the localized solutions of the linearized equation would provide us for $l = 1$ with the initial slope $dP_1/d\tilde{x}|_0 = A(1 - 2/3) = A/3$ for starting the numerical integration close to the origin. The deviations from the analytical expression (9) in Fig. 4 are small.

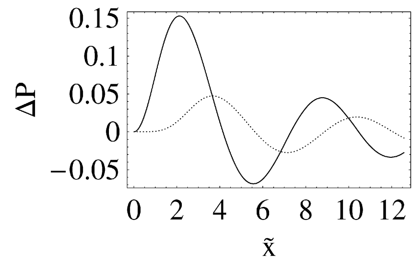

Fig. 4. Residue $\Delta P := P_{\text{num}} - P_{\text{lin}}$ of the numerically generated axion solution and the corresponding exact result of the linearized equations. The same for $l = 1$ (dotted line).

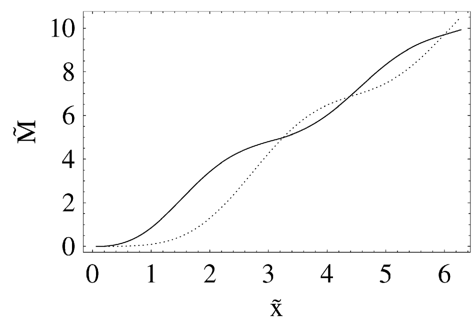

Fig. 5. Normalized Newtonian mass function $\tilde{M} := 2M\sqrt{\omega^2 - 1}/\pi^3\sqrt{\Lambda_\theta}$ oscillating around its linearly rising growth in an axionic DM halo.

4. Rotation curve with wiggles

The circular velocities of stars or HI gas in galaxies are bounded by $v_\varphi/c \leqslant 10^{-3}$ and thus are *non-relativistic*. Nevertheless, we depart from the general relativistic formula [33]

$$v_\varphi^2 := \frac{r}{2}\frac{d\nu}{dr} = \frac{\kappa}{2}\left[\frac{M(r) + 4\pi p_\theta r^3}{4\pi r - \kappa M(r)}\right] \simeq \frac{\kappa}{2}\left[\frac{M(r)}{4\pi r} + p_\theta r^2\right] \tag{15}$$

for the tangential velocity squared, in order to keep track of the effect of the radial pressure. Here $\kappa = 8\pi G$ is the gravitational coupling constant and, for a spherically symmetric metric, $\nu = 2\ln N$ is related to be the lapse function N in general relativity. Even the approximation above for a small mass function goes beyond the usual Kepler law $v_{\varphi,\text{Newt}}^2 \simeq GM(r)/r$ inasmuch as it takes the effect of the radial pressure p_θ into account.

The total mass enclosed in a halo of radius r is given by the familiar Newtonian *mass function*

$$M(r)^l := 4\pi \int_0^r \rho_\theta \tilde{r}^2 \, d\tilde{r} \tag{16}$$

shown in Fig. 5. For a round halo, i.e. for $l = 0$, the solution (10) of the linearized equations yields

$$M(r) \simeq \frac{\pi^3\sqrt{\Lambda_\theta}}{2\sqrt{\omega^2 - 1}}\left[\omega^2 - \frac{\sin(2\tilde{x})}{2\tilde{x}} - \left(\omega^2 - 1\right)\frac{\sin^2\tilde{x}}{\tilde{x}^2}\right]\tilde{x}. \tag{17}$$

Likewise, an l-dependent DM halo profile can be obtained by integrating (11) with the aid of Mathematica and further simplification [22].

Our post-Newtonian approximation reveals that the radial pressure term $p_\theta r^2 \simeq \pi^2[\tilde{x}^2 - \tilde{x}\sin(2\tilde{x}) + \sin^2\tilde{x}]/8\tilde{x}^2$ for $l = 0$ of a scalar 'fluid' contributes to the shape of the configuration as well as to the rotation curve. For a round halo the "Keplerian" term $M(r)/4\pi r \simeq \pi^2[2\omega^2\tilde{x}^2 - \tilde{x}\sin(2\tilde{x}) - 2(\omega^2 - 1)\sin^2\tilde{x}]/16(\omega^2 - 1)\tilde{x}^2$ is partially dominating over the pressure term, with the result that

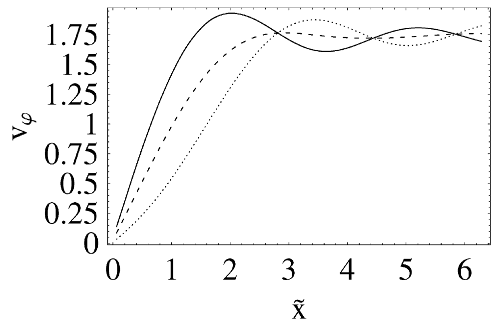

Fig. 6. Rotation curve of a spherically symmetric axionic DM halo normalized by $\sqrt{\kappa/2}$; with the dotted line resulting from the 'vortex solution' with $l = 1$. The dashed line represents the arithmetic mean thereby eliminating in the 'superposed' velocity almost all the wiggles.

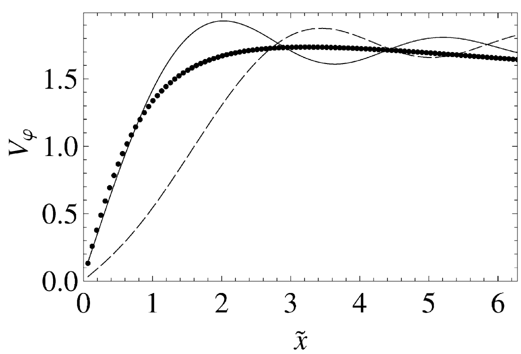

Fig. 7. Axionic rotation curves in comparison to the empirical Burkert fit (fat dots).

$$v_\varphi^2 \simeq \frac{\pi^2(2\omega^2-1)\kappa}{16(\omega^2-1)}\left[1-\frac{\sin(2\tilde{x})}{2\tilde{x}}\right]. \tag{18}$$

At the origin, the tangential velocity is, due to l'Hospital's rule, zero and tends to the constant velocity $v_\infty = \sqrt{(2\omega^2-1)\kappa/(\omega^2-1)} \times \pi/4$ at infinity.

The rotation curves of many spiral galaxies, such as the Milky Way, are also approximately flat, except for some faint irregularities which are usually interpreted as due to the presence of spiral arms. Alternatively, these irregularities have been associated with caustics arising from the CDM model, which in the inner region may form rings [42]. It is gratifying that a BEC of axions weakly self-interacting due to the periodic potential (2) provides a similar halo substructure. When 'superposed' with its 'vortex solution' with angular momentum $l = 1$, a 'destructive interferences' partially eliminates the wiggles, cf. Fig. 6 and an asymptotically almost flat rotation curve remains. Likewise, for the nonlinear Emden model to exhibit torus-like rings [32], a superposition of a spherically symmetric configuration with angular momentum states is necessary; the additional bonus [10] is a scaling relation $\rho_0 \propto 1/r_c$ between the central density and the core radius r_c.

As a preliminary test, let us compared our results with the empirical formula

$$v_{\varphi B}^2 = \frac{v_0^2}{2x}\left\{\ln\left[(1+x)^2\left(1+x^2\right)\right] - 2\arctan(x)\right\}, \tag{19}$$

originally proposed by Burkert in 1995 for DM dominated dwarf galaxies [7]. Nevertheless, it appears to be a rather *universal* profile [37] fitting also well the whole spiral luminosity sequence as well as low surface brightness (LSB) galaxies. After a maximum at $x = 3.3$ in dimensionless units $x = r/r_c$, it amounts to a *logarithmic modification* $v^2 \to \ln x/x$ of the Kepler law at spatial infinity. The concordance with the axionic velocity curve in Fig. 7 is rather good, in particular close to the center. At large radii, the Burkert fit approaches the 'superposed' profile, except for small wiggles occurring also in the observed rotation curve of a dwarf galaxy [12].

Asymptotically, the deviation increases due to the almost linearly rising mass (17) of the axionic halo. Then, modifications of the rotation curve due to *self-gravity* are mandatory, cf. Ref. [27] for related attempts in a general-relativistic setting.

5. Discussion

For a BEC below its critical temperature

$$T_c = \frac{2\pi\hbar^2}{mk_B}\left[\frac{N}{\xi(3/2)V}\right]^{2/3}, \tag{20}$$

where $\xi(z)$ is the Riemann zeta function, the $N = nV$ scalar particles in a volume V are in a *coherent state* and thus do not collide. Since the axion is very weakly interacting, the reheating phase after initial inflation cannot destroy coherence [2]. Thus an axion-like DM halo can be in a non-relativistic BEC state even until now.

It is interesting to note that both, the critical temperature T_c as well as the Kaup limit [31]

$$M_{\mathrm{Kaup}} \leqslant \frac{2}{\pi}\frac{M_{\mathrm{Planck}}^2}{m} \tag{21}$$

for the stable branch of spherical localized boson configurations are *inversely* proportional to the constituent mass m. Thus an *almost vanishing* mass is mandatory in order to extrapolate a BEC to a galactic scale.[3]

Consequently, in order that a BEC can account for DM halos, the Compton wave length of DM particles has to be of the order of the size of a galaxy, i.e. $\lambda_{\mathrm{Compton}} = h/mc \sim 10$ kpc. This naive estimate would lead to an ultra light mass of $m \simeq 10^{-26}$ eV which is twenty orders of magnitude below the usual mass range $m_a \simeq$ μeV of axions. If we identify the average radius $\langle r\rangle = \int_0^\infty \rho_\theta r\,dr / \int_0^\infty \rho_\theta\,dr \propto \Lambda_\theta^{-1/2} = 1/m_a f_a$ of our axion condensate with the size of a galaxy, we obtain a much higher constituent mass, depending on the axion coupling f_a.

Recently, the so-called *'bullet cluster'* (1E0657-558) provided indirect evidence [9] for the existence of DM, see also Ref. [24] for a dynamical simulation of the baryonic and DM components. In fact, the dynamics of galaxies in the Coma cluster provided the first clue for Zwicky in postulating 'missing matter'. Clusters provide an excellent laboratory for constraining DM parameters such as their lifetime [35]. During a collision or merging of galaxy clusters, the DM halos are displaced from the stars and apparently pass each other in a soliton-like way. There will be no annihilation, if the halos are described by two coherent states of nonlinearly interacting scalar fields [4,21]. Although the observations in the bullet cluster [9] can be given a rather conventional interpretation [24], axionic BECs offer the possibility of lump-like DM structures [1], similarly to those of boson stars [41].

On the other hand, the real part of a scalar field can be regarded as a conformal mode of the spacetime metric which, via a Legendre transformation, induces a nonlinear modification of general relativity, exhibiting an intriguing bifurcating into several nearly Einsteinian branches [29,40]. Ultimately, this may indicate that soliton models of DM halos are equivalent to the purely gravitational *geons* [16] of Wheeler.

[3] For a small critical temperature $T_c \simeq 3$ K, the resulting equivalent bound $M_{\mathrm{Kaup}} \leqslant 4k_B T_c[\xi(3/2)/n]^{2/3} M_{\mathrm{Planck}}^2/h^2$ would require a tiny number density n, independent of the constituent mass m.

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