New classes of perfectly orderable graphs

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Abstract

This paper generalizes previous works on perfectly orderable graphs by Olariu (Discrete Math. 113 (1992) 143) and by Hoâng et al. (Discrete Math. 102 (1992) 67). Chvátal defined a graph to be perfectly orderable (V. Chvátal, in: C. Berge, V. Chvátal (Eds.), Topics on Perfect Graphs, Annals of Discrete Mathematics, Vol. 21, North-Holland, Amsterdam, 1984, pp. 63–65) if there exists a linear order \(<\) on its set of vertices such that no induced path \(ab\) \(bc\) \(cd\) has both \(a\) \(b\) and \(d\) \(c\). Given a graph \(G\) and a vertex \(v\) in \(G\) such that \(G - v\) is perfectly orderable, we set some conditions on \(v\) for which we deduce that \(G\) is perfectly orderable. Our method allows to construct a new class of such graphs, recognizable in polynomial time, containing quasi-brittle graphs, charming graphs and some other classes of perfectly orderable graphs.

1. Introduction

A natural way to colour the vertices of a graph \(G = (V,E)\) ordered in a sequence \(v_1 < v_2 < \cdots < v_n\) by the set of ‘colours’ \(\{1,2,3,\ldots\}\) consists of scanning the sequence from \(v_1\) to \(v_n\) and assigning to each \(v_j\) the smallest integer assigned to none of its neighbours \(v_i\) such that \(i < j\). This way of colouring the vertices of \(G\) is called the greedy algorithm. Certainly, this way may use a number of colours greater than the chromatic number of \(G\). A graph \(G\) is said to be perfectly orderable [2] if there exists a linear order \(<\) such that no induced \(P_4\) \(abcd\) in \(G\) has both \(a < b\) and \(d < c\) (such a \(P_4\) is called an obstruction in \((G; < ))\). Obviously, such an order, called perfect order, is also a perfect order on every \(V' \subseteq V\), therefore the family of perfectly orderable graphs is hereditary. Recall that C. Berge defined a graph \(G\) to be perfect if the vertices of every induced subgraph \(H\) of \(G\) can be coloured with \(\omega(H)\) colours, where \(\omega(H)\) is the

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maximum clique size in $H$. It can be easily seen that a graph $G$ is perfect if and only if every induced subgraph $H$ contains a stable set intersecting every maximum clique in $H$. Berge and Duchet [1] defined a graph $G$ to be strongly perfect if every induced subgraph $H$ of $G$ contains a stable set that intersects every maximal clique in $H$ (as usual, ‘maximal’ is meant with respect to set-inclusion). Chvátal [2] has shown that perfectly orderable graphs are strongly perfect and that the greedy algorithm applied to a perfect order gives an optimal colouring.

For any integer $k \geq 3$, an induced path (respectively cycle) on $k$ vertices will be denoted by $P_k$ (respectively $C_k$). An induced subgraph of a graph $G$ isomorphic to a $P_k$ (respectively $C_k$) is simply said to be a $P_k$ in $G$ (resp. a $C_k$ in $G$). A vertex $v$ in a graph $G$ is said to be semi-simplicial [6] if $v$ is midpoint of no $P_4$ in $G$. A vertex $v$ in a graph $G$ is said to be charming [8] if $v$ is not end-vertex in a $P_5$ in $G$, is not end-vertex in a $P_5$ in $\tilde{G}$ and does not lie on a $C_5$ in $G$.

A rooted graph is a pair $H=(F,u)$ where $F$ denotes a graph and $u$ a vertex in $F$. Let $\mathcal{F}$ be a family of rooted graphs. A vertex $v$ in a graph $G$ is said to be $\mathcal{F}$-free if there is no induced subgraph $F$ of $G$ such that $H=(F,v)$ is isomorphic to a rooted graph of $\mathcal{F}$. For example, a semi-simplicial vertex is $\mathcal{F}$-free for the family $\mathcal{F} = \{(P_4,u)\}$ where $u$ denotes a midpoint of $P_4$, a charming vertex is $\mathcal{F}$-free for the family $\mathcal{F}$ described in Fig. 1.

Hoàng et al. [8] call charming any graph in which every induced subgraph has a charming vertex, and prove that every charming graph is perfectly orderable. A natural idea is to consider the following related notion: a vertex $v$ in a graph $G$ is said to be nice if $v$ is not internal vertex in a $P_5$ in $G$, is not internal vertex in a $P_5$ in $\tilde{G}$ and does not lie on a $C_5$ in $G$. A nice vertex is $\mathcal{F}$-free for the family $\mathcal{F}$ described in Fig. 2. We call nice any graph in which every induced subgraph has a nice vertex, and we will prove that every nice graph is perfectly orderable (see below). In fact, this result is a corollary of Theorem 3 in Section 5.

An edge $ab$ in a graph $G$ is called a symmetric wing if there exist vertices $c,d,p,q$ such that both $abcd$ and $baqp$ are induced $P_4$'s in $G$. A vertex $v$ in $G$ is said to be special [10] if $v$ is incident with no symmetric wing in $G$ and $\tilde{G}$.

**Lemma 1.** An edge $ab$ in a graph $G$ is a symmetric wing if and only if there is an induced subgraph of $G$ depicted in Fig. 3 such that $ab$ is the edge whose ends are circled.
Proof. Let $c,d,p$ and $q$ such that both $abcd$ and $baqp$ are $P_4$’s in $G$. There are two cases according as $d = p$ or $d \neq p$. If $d = p$ then $cq$ is the only one possible edge. If $d \neq p$ then $cq, cp, dq, dp$ are possible edges. By examining every case, it is a routine matter to obtain the announced result. \qed

Lemma 2. A vertex $v$ in a graph $G$ is special if and only if $v$ is $\mathcal{F}_1$-free, where $\mathcal{F}_1$ is the family of rooted graphs $\{H_1, H_2, \overline{H}_2, H_4, \overline{H}_4\}$ (see Figs. 2 and 4).

Proof. By considering an end-vertex $a$ of a symmetric wing and the rooted graphs $(C_5, a)$, $\overline{(P_5, a)}$ and $(A, a)$ in Fig. 3 and their (rooted) complements, we obtain rooted graphs isomorphic to $H_1, H_2, \overline{H}_2, H_4$ or to $\overline{H}_4$ (see Figs. 2 and 4). The other graphs in Fig. 3, that is $(P_6, a)$, $(C_6, a)$, $(B, a)$, and their complements give rooted graphs containing a rooted graph $(F, a)$ isomorphic either to $H_2$ or to $\overline{H}_2$ depicted in Fig. 2.

Conversely, let $v$ be a vertex belonging to a rooted graph $H$ of $\mathcal{F}_1$. If $H$ belongs to $\{H_1, \overline{H}_2, H_4\}$ (resp. $\{H_2, \overline{H}_4\}$) then $v$ is end-vertex in a symmetric wing in $G$ (resp. $\overline{G}$). \qed
In [10] Olariu defined a graph $G$ to be quasi-brittle if every induced subgraph $H$ of $G$ contains a special vertex, and proved that such a graph is perfectly orderable. Indeed, an attentive examination of his proof shows that he obtained the following more general result:

**Theorem 1.** Let $G$ be a graph and $v$ be a special vertex in $G$. Then, $G$ is perfectly orderable if and only if $G - v$ is perfectly orderable.

Note that such a result was known if $v$ is semi-simplicial [6] or if $v$ is a charming vertex [8].

Our purpose is to generalize these previous results and to obtain new families of perfectly orderable graphs by proving statements of the following type:

**Assertion A.** Let $\mathcal{F}$ be a family of rooted graphs. Let $G$ be a graph and $v$ be a $\mathcal{F}$-free vertex in $G$. Then, $G$ is perfectly orderable if and only if $G - v$ is perfectly orderable.

2. Definitions and notations

For terms not defined here, the reader is referred to [5]. All the graphs in this paper are simple and a graph is denoted $G = (V(G), E(G))$ (or simply $G = (V, E)$ if no confusion exists); its complement $\bar{G}$ is the graph $(V, \bar{E})$ where $\bar{E} = \{xy \mid x \in V, y \in V$ and $xy \notin E\}$. For any subset $A$ of $V$, the subgraph of $G$ (resp. $\bar{G}$) induced by the set $A$ will be denoted by $G[A]$ (resp. $\bar{G}[A]$).

For any vertex $v$ in $V$, the *neighbourhood* $N_G(v)$ of $v$ in $G$ is defined as the set of all vertices adjacent to $v$ in $G$. These vertices are called *neighbours* of $v$. When there is no possibility of confusion, the notation $N(v)$ will also be used to denote the neighbourhood of $v$. The set of vertices in $G$ which are not adjacent to $v$ is denoted by $M(v)$ (this set is equal to $N_{\bar{G}}(v)$, the neighbourhood of $v$ in $\bar{G}$). For any subset $W$ of $V\setminus\{v\}$ such that $W \cap N(v) \neq \emptyset$ and $v$ is not adjacent to at least one vertex in $W$, we will say that $v$ sees $W$ partially. If $W \subseteq N(v)$, we will say that $v$ sees $W$ completely.

If $A \subseteq V$, then $G\setminus A$ will denote the graph induced by $V \setminus A$. The particular case where $A = \{x\}$ will be denoted $G - x$; also if $H$ is a subgraph of $G$, we denote by $H + x$ the subgraph induced in $G$ by $V(H) \cup \{x\}$ (if $x \notin V(H)$).
For any path $P$, the length of $P$ is the number of its edges. If $V(P) = \{v_1, \ldots, v_k\}$ and $E(P) = \{v_i v_{i+1} \mid i \in \{1, \ldots, k-1\}\}$, $P$ is also denoted by $[v_1, \ldots, v_k]$. The vertices $v_1$ and $v_k$ are its end-vertices while any vertex $v_i$, with $1 < i < k$, is said to be an internal vertex. An induced path on $k$ vertices will be denoted by $P_k$. For a $P_4$ with vertices $a, b, c, d$ (in this order on the path), the notation will be slightly changed: it will be denoted $abcd$. The two internal vertices $b$ and $c$ will be called midpoints while the end-vertices $a$ and $d$ will be also referred as endpoints.

Similarly, a chordless cycle on $k$ vertices is denoted by $C_k$ or by $[v_1, \ldots, v_k, v_1]$ if its vertex set is $\{v_1, \ldots, v_k\}$ and its edge set is $\{v_iv_{i+1} \mid 1 \leq i \leq k - 1\} \cup \{v_kv_1\}$.

A set $A \subseteq V$ of vertices is called a module if every vertex in $V \setminus A$ is either adjacent to all vertices in $A$, or none of them. A module in $G$ is also a module in $\tilde{G}$. A module is an homogeneous set if it is not a trivial one, i.e. not an empty set, a singleton or $V$ itself. A graph with more than two vertices is prime if it only has trivial modules.

Let $P_4$ be the smallest prime graph and that any prime graph distinct from $P_4$ has at least five vertices.

Let $G$ and $H$ be two graphs and let $v$ be a vertex in $G$. The $\chi$-join of $H$ and $G$ into $v$ is the graph obtained from the union of $H$ and $G - v$ by adding all the edges $xy$ with $x \in V(H)$ and $y \in N_G(v)$; this graph is denoted by $\chi(H, G; v)$. Since an induced $P_4$ in $\chi(H, G; v)$ not contained in $H$ and not in $G - v$ has exactly one vertex in $H$ and the three other vertices in $G - v$, note that if $v_1 < v_2 < \cdots < v_n$ is a perfect order of $G$ and if $w_1 < w_2 < \cdots < w_p$ is a perfect order of $H$, then $v_1 < \cdots < v_{i-1} < w_1 < v_i < \cdots < v_{i+1} < \cdots < v_n$ is a perfect order of $\chi(H, G; v_i)$. Then we have

**Lemma 3** (Chvátal [3]). Let $G$ and $H$ be two perfectly orderable graphs and let $v$ be a vertex in $G$. Then $\chi(H, G; v)$ is perfectly orderable.

### 3. Pleasant vertices

A vertex $v$ in a graph $G$ is said to be a pleasant vertex if $v$ is $\mathcal{F}_0$-free, with $\mathcal{F}_0 = \{H_1, H_2, \tilde{H}_2\}$ where these rooted graphs are described in Fig. 2. Note that special vertices and nice vertices are pleasant vertices. We call pleasant any graph in which every induced subgraph has a pleasant vertex.

For any vertex $v$ in a graph $G$ let $S_1(v), \ldots, S_q(v)$ be the connected components of $\tilde{G}[N(v)]$, and let $Q_1(v), \ldots, Q_k(v)$ be the connected components of $G[M(v)]$. For the sake of simplicity we will denote these components by $S_1, \ldots, S_q$ and by $Q_1, \ldots, Q_k$.

We give here a structural result that we will often use.

**Proposition 1.** Let $v$ be a pleasant vertex in a graph $G = (V, E)$. For every $i \in \{1, \ldots, q\}$ and every $j \in \{1, \ldots, k\}$ one of the following statement is true:

1. every vertex in $S_i$ is adjacent to every vertex in $Q_j$,
2. every vertex in $S_i$ has no neighbour in $Q_j$,
(3) there exists a non trivial part \( A_i \) of \( S_i \) such that for every vertex \( x \) in \( S_i \) and for every vertex \( y \) in \( Q_j \), \( xy \) belongs to \( E \) if and only if \( x \) belongs to \( A_i \),
(4) there exists a non trivial part \( B_j \) of \( Q_j \) such that for every vertex \( x \) in \( S_i \) and for every vertex \( y \) in \( Q_j \), \( xy \) belongs to \( E \) if and only if \( y \) belongs to \( B_j \).

**Proof.** If \(|S_i| = 1\) or \(|Q_j| = 1\) then the result is clearly true. We suppose that \(|S_i| \geq 2\), \(|Q_j| \geq 2\) and that none of the conditions (1)–(3) is satisfied. Then, there exist \( x \in S_i \), \( a \) and \( b \in Q_j \) such that \( xa \in E \) and \( xb \notin E \). Since \( Q_j \) is connected in \( G \), we can suppose without loss of generality that \( ab \in E \). Since \( S_i \) is connected in \( G \), there is \( y \in S_i \) distinct from \( x \) such that \( xy \notin E \). Since the set of vertices \( \{y,v,x,a,b\} \) induces no cycle \( C_5 \), no \( P_3 \) and no \( \bar{P}_5 \), we have \( ya \in E \) and \( yb \notin E \).

Now we want to prove that any vertex \( h \) in \( Q_j \) is either adjacent to every vertex in \( S_i \) or adjacent to no vertex in \( S_i \). A contrario, suppose that there exist \( h \) in \( Q_j \), \( c \) and \( d \) in \( S_i \) such that \( hc \in E \) and \( hd \notin E \). Since \( S_i \) is connected in \( G \), we may suppose without loss of generality that \( cd \notin E \). Let \( k \) be a neighbour of \( h \) in \( Q_j \). Since the set \( \{k,h,c,v,d\} \) induces no \( C_5 \), no \( P_3 \) and no \( \bar{P}_5 \), we have \( kc \in E \) and \( kd \notin E \). Since \( Q_j \) is connected, \( c \) is adjacent to every vertex in \( Q_j \) (and \( d \) is adjacent to no vertex in \( Q_j \)). But \( b \) belongs to \( Q_j \) and is adjacent to no vertex in \( S_i \), so we obtain a contradiction. Thus, the statement 4) is true. \( \square \)

**Corollary 1.** Let \( v \) be a pleasant vertex in a graph \( G = (V,E) \).

(i) Let \( S_i \) be a component of \( \tilde{G}[N(v)] \). Let \( a \) and \( b \) be two vertices in \( M(v) \) such that \( N(a) \cap S_i \neq \emptyset \), \( N(b) \cap S_i \neq \emptyset \) and \( N(a) \cap S_i \neq N(b) \cap S_i \). Then \( a \) and \( b \) belong to distinct components of \( G[M(v)] \).

(ii) Let \( Q_j \) be a component of \( G[M(v)] \). Let \( a \) and \( b \) be two vertices in \( N(v) \) such that \( N(a) \cap Q_j \neq \emptyset \), \( N(b) \cap Q_j \neq \emptyset \) and \( N(a) \cap Q_j \neq N(b) \cap Q_j \). Then \( a \) and \( b \) belong to distinct components of \( G[N(v)] \).

4. Acyclic orientation and obstructions

A perfect order of \( G \) can be seen as an acyclic orientation of the edges of \( G \) such that every induced \( P_4 \), \( abcd \), is not an obstruction, that is \( abcd \) cannot have simultaneously \((a,b)\) and \((d,c)\) as arcs. Our aim is the following: given a perfect order \( < \) of \( G - v \), how to orient the edges incident with \( v \) and how to modify the orientation of some edges in \( G - v \) so that the oriented graph \( G \) has no circuit and no obstruction. We may define a linear order \( <' \) of \( G \) in the following way: let us partition \( V - v \) into \( L \) and \( R \), and denote \( N(v) \cap L \) by \( LH \), \( M(v) \cap L \) by \( LL \), \( N(v) \cap R \) by \( RH \) and \( M(v) \cap R \) by \( RL \); we do not modify the orientations in \( G[L] \) and \( G[R] \), every edge \( xy \) such that \( x \in L \) and \( y \in R \) is oriented such that \( x <' y \), every edge \( xv \) with \( x \in L \) is oriented such that \( x <' v \) and, at last, every edge \( yv \) such that \( y \in R \) is oriented such that \( v <' y \) (see Fig. 5). Now, our problem is to define the sets \( LH \), \( LL \), \( RH \) and \( RL \) in a suitable way to avoid obstructions.
We note that, to obtain their result, Hoang et al. [8] set $LH = N(v)$, $LL = \emptyset$, $RH = \emptyset$, and $RL = M(v)$.

We consider a pleasant vertex $v$ of the graph $G$, and we suppose $G - v$ perfectly orderable. Let $<$ be a perfect order on $G - v$. Following a similar idea as used by Olariu [10], a component $S_i$ of $\tilde{G}[N(v)]$ is called impure if a vertex $a$ in $M(v)$ and vertices $b, d$ in $S_i$ exist such that $abvd$ is a $P_4$ and $a < b$. A component $S_i$ is called pseudo-pure if there are vertices $a, b$ in $M(v)$ and a vertex $c$ in $S_i$ such that $abcv$ is a $P_4$ and $a < b$. A component which is neither impure nor pseudo-pure is referred to as pure.

**Lemma 4.** No pseudo-pure component is impure.

**Proof.** Assume that $S_i$ is both pseudo-pure and impure: let $a, b, x$ in $M(v)$ and $c, y, z$ in $S_i$ be such that $abcv$ and $xyez$ and $P_4$ with $a < b$ and $x < y$. Since $ac \not\in E$ and $bc \in E$, the vertex $b$ sees $S_i$ completely (by Proposition 1). As $bz \in E$, $xy \in E$ and $xz \not\in E$, the component $Q_j$ containing $x$ is different from the component containing $a$ and $b$ (by Corollary 1). Then $abxy$ is an obstruction in $(G - v; \prec)$, a contradiction.

We can now define the sets $LH$, $RH$, $LL$ and $RL$: $LH$ contains all the pseudo-pure components and, perhaps, some pure components; $RH$ contains all the impure components and the remaining pure components; $LL$ is the union of some components $Q_j$ and $RL$ is the union of the other components of $G[M(v)]$ ($LL$ or $RL$ can be empty). By construction, the oriented graph $(G; <')$ presents no circuit.

**Claim 1.** $(G; <')$ presents no obstruction containing $v$.

**Proof.** If $abcv$ is an obstruction in $(G; <')$, then $a$ and $b$ belong to the same component $Q_j$ and $a < b$. Thus, $c$ belongs to a pseudo-pure component and $v < 'c$, a contradiction. If $abvd$ is an obstruction in $(G; <')$, then the component $S_i$ containing $b$ and $d$ is
contained in $LH$ ($d <' v$) and $a$ is in $LL$ $(a <' b)$. Thus $a < b$ and $S_i$ is impure, a contradiction. □

We are going to observe now the possible obstructions. Let $abcd$ be an obstruction in $(G; <')$. Without loss of generality, we can suppose that $a \in L$, $b \in R$ and $b < a$. Since there is no edge between $LL$ and $RL$, we notice that $a$ or $b$ is a neighbour of the vertex $v$.

**Proposition 2.** If $abcd$ is an obstruction in $(G; <')$, then one of the following situations depicted in Fig. 6 is realized:

- **Situation 1**: $a \in LH$ and $b, c, d \in Q_j \subseteq RL$.
- **Situation 2**: $a \in LL$, $b, c \in RH$ and $d \in RL$.
- **Situation 3**: $a \in LL$ and $b, c, d \in RH$.
- **Situation 4**: $a \in LL$, $b, d \in RH$ and $c \in RL$.
- **Situation 5**: $a, d \in LL$ and $c, b \in RH$.
- **Situation 6**: $a, d \in LL$, $c \in LH$ and $b \in RH$.

**Proof.** Since $d <' c$, we notice that $c \notin L$ while $d \in R$. We distinguish three cases:

- **Case A**: $c, d \in R$;
- **Case B**: $d \in L$ and $c \in R$;
- **Case C**: $c, d \in L$.

**Case A**: $c, d \in R$, then $d < c$. 

![Fig. 6.](image-url)
Case A1: We suppose that \( a,b \in \mathcal{N}(v) \). Let \( S_i \) be the component (pure or pseudo-pure) containing \( a \) and let \( S_j \) be the component (pure or impure) containing \( b \). Since \( ad \notin \mathcal{E} \) and \( bd \notin \mathcal{E} \), \( d \) belongs to \( M(v) \). If \( c \in \mathcal{N}(v) \) then \( c \) belongs to \( S_i \) and this component is impure (because \( d < c \)), a contradiction. Thus, \( c \) and \( d \) belong to \( M(v) \). But now, we consider the \( P_4 \) \( dcbv \) and we see that \( S_j \) is pseudo-pure, which gives also a contradiction. Then, Case A1 does not occur.

Case A2: We suppose that \( a \notin \mathcal{N}(v) \) and \( b \notin \mathcal{N}(v) \). As \( ac, ad \notin \mathcal{E}, b, c \) and \( d \) belong to the same component \( Q_j \). This is the first situation.

Case A3. We suppose that \( a \notin \mathcal{N}(v) \) and \( b \in \mathcal{N}(v) \). As the component containing \( b \) is not pseudo-pure, one of the vertices \( c \) or \( d \) is in \( \mathcal{N}(v) \). We obtain three other possibilities: Situations 2–4.

Case B1: We suppose that \( a \in \mathcal{N}(v) \). Since \( ac, bd \notin \mathcal{E} \) and \( cd \in \mathcal{E} \), \( c \) belongs to \( M(v) \) and \( d \) belongs to \( \mathcal{N}(v) \). Since \( ad, bd \notin \mathcal{E} \), \( a \) and \( d \) are in the same component \( S_i \), \( b \) and \( c \) are in the same component \( Q_j \). By Proposition 1, \( abcd \) cannot be a \( P_4 \); this is a contradiction. Then, Case B1 cannot occur.

Case B2: We suppose that \( a \notin \mathcal{N}(v) \) and \( b \in \mathcal{N}(v) \). Since \( bd \notin \mathcal{E} \) and \( cd \in \mathcal{E} \), we have \( d \notin \mathcal{N}(v) \) and \( c \in \mathcal{N}(v) \). We obtain the fifth situation.

Case C1: We suppose that \( a, b \in \mathcal{N}(v) \). As \( bd \notin \mathcal{E} \), necessarily \( d \notin \mathcal{N}(v) \). If \( c \in \mathcal{N}(v) \), then \( c \) belongs to the component \( S_i \) containing \( a \), and \( S_i \) is impure; this is a contradiction. If \( c \notin \mathcal{N}(v) \), we consider the graph induced by \( \{d,c,b\} \) and we see that the component \( S_j \) containing \( b \) is pseudo-pure. This is also a contradiction; then Case C1 is impossible.

Case C2: We suppose that \( a \notin \mathcal{N}(v) \) and \( b \notin \mathcal{N}(v) \). Necessarily, \( a \) and \( c \) are in the same component \( S_i \). If \( d \in \mathcal{N}(v) \), then \( d \) is in the component \( S_i \) and \( bads \) is a \( P_4 \) where \( b < a \). Then \( S_i \) is impure, which is impossible. If \( d \notin \mathcal{N}(v) \), then \( dcva \) is a \( P_4 \) and \( d < c \). So the component \( S_i \) is impure. Thus, Case C2 is impossible.

Case C3: We suppose that \( a \notin \mathcal{N}(v) \) and \( b \in \mathcal{N}(v) \). If \( c \notin \mathcal{N}(v) \), then \( dcbv \) is a \( P_4 \), where \( d < c \), and the component containing \( b \) is pseudo-pure. So \( c \in \mathcal{N}(v) \), and we obtain the last situation.

From the previous proposition, we deduce immediately the following two results (we shall see later that we can improve Corollary 3).

**Corollary 2.** Let \( v \) be a pleasant vertex of a graph \( G \). If \( v \) is endpoint of no \( P_5 \) then \( G \) is perfectly orderable if and only if \( G - v \) is perfectly orderable.

**Proof.** We choose \( LL = \emptyset \) and \( RL = M(v) \): none of Situations 2–6 appears. Since \( v \) is endpoint of no \( P_5 \), Situation 1 cannot be realized.

**Corollary 3.** Let \( v \) be a pleasant vertex of a graph \( G \). If \( v \) is \( \{H_5, H_6\} \)-free (\( H_5 \) and \( H_6 \) are described in Fig. 7), then \( G \) is perfectly orderable if and only if \( G - v \) is perfectly orderable.
**Fig. 7.**

**Proof.** We choose $LL = M(v)$ and $RL = \emptyset$; the Situations 1, 2, 4 cannot appear. Since $v$ is a vertex $\{H_5, H_6\}$-free, none of the Situations 3, 5, and 6 is realized. □

5. Main results

Let us consider again the following

**Assertion A.** Let $\mathcal{F}$ be a family of rooted graphs. Let $G$ be a graph and $v$ be a $\mathcal{F}$-free vertex in $G$. Then, $G$ is perfectly orderable if and only if $G - v$ is perfectly orderable.

We have solely to prove that if $G - v$ is perfectly orderable then $G$ is perfectly orderable. As a consequence of Lemma 3, we have:

**Claim 2.** To prove Assertion A by induction on the number of vertices, it can be supposed that $G$ is prime.

**Proof.** If every graph in $\mathcal{F}$ has $p$ vertices or more, then this result is true for any graph $G$ such that $|V(G)| < \min(4, p)$. Let us suppose that for any graph $H$ such that $|V(H)| < |V(G)|$, $H$ is perfectly orderable if $w$ is a $\mathcal{F}$-free vertex in $H$ and $H - w$ is perfectly orderable. We suppose that $G$ contains a homogeneous set $M$.

(i) If $v \notin M$, Let $H$ be the graph obtained from $G$ by contracting $M$ in a single vertex $z$. Since $H$ is isomorphic to an induced subgraph of $G$, $v$ is a $\mathcal{F}$-free vertex in $H$. Since $H - v$ is an induced subgraph of $G - v$, $H - v$ is perfectly orderable. By induction, $H$ is perfectly orderable. The subgraph $G[M]$ is an induced subgraph of $G - v$, thus it is a perfectly orderable graph. By Lemma 3, $G = \chi(G[M], H_v)$ is perfectly orderable.

(ii) If $v \in M$ then $G[M] - v$ is an induced subgraph of $G - v$, thus $G[M] - v$ is perfectly orderable. Since $v$ is a $\mathcal{F}$-free vertex in $G[M]$, by induction $G[M]$ is perfectly orderable. The graph $H = G - M + u$ with $u \in M - v$, a subgraph of $G - v$, thus $H$ is perfectly orderable. By Lemma 3, $G = \chi(G[M], H_v)$ is perfectly orderable. □
The following statement generalizes Theorem 1.

**Theorem 2.** Let $G$ be a graph and $v$ be a vertex $\mathcal{F}_2$-free, where $\mathcal{F}_2$ is the family $\{H_1, H_2, \tilde{H}_2, \tilde{H}_4\}$ (see Figs. 2 and 4). Then, $G$ is perfectly orderable if and only if $G - v$ is perfectly orderable.

**Lemma 5.** Let $G$ be a prime graph and $v$ be a $\mathcal{F}_2$-free vertex in $G$. If $S_i$ is a component of $\tilde{G}[N(v)]$ and $u$ is a non-isolated vertex in $G[M(v)]$, then $u$ is adjacent to every vertex of $S_i$ or none.

**Proof.** On the contrary, we suppose that the vertex $u$ sees partially the component $S_i$: there are vertices $a, b$ in $S_i$ such that $au \in E$ and $bu \notin E$. Without loss of generality, we can suppose that $ab \notin E$. As $G$ is prime, the component $Q_j$ containing $u$ is not an homogeneous set, and there are vertices $x$ in $N(v)$ and $y, z$ in $Q_j$ such that $xy \in E$ and $xz \notin E$. We can suppose that $yz \in E$. By Proposition 1, we have $ay \in E$, $az \in E$, $by \notin E$, $bz \notin E$, and by Corollary 1, $x \notin S_i$. Then, the rooted graph $(G[v, a, b, y, z, x], v)$ is isomorphic to $\tilde{H}_4$. 

**Lemma 6.** Let $v$ be a pleasant vertex in a graph $G$, such that $G - v$ is perfectly orderable. Let $< \in$ be a perfect order of $G - v$. $S_i$ be an impure component and $x$ be a vertex in $M(v)$ which sees $S_i$ completely. Then there exists a vertex $y \in S_i$ such that $x < y$.

**Proof.** Let $S_i$ be an impure component. There exist vertices $a \in M(v)$ and $b, y \in S_i$ such that $aby$ is a $P_4$ and $a < b$. If $x \in M(v)$ sees $S_i$ completely then, by Corollary 1, $x$ and $a$ are not in the same component of $G[M(v)]$. Since $abxy$ is not an obstruction in $(G - v; <)$, we have $x < y$.

**Proof of Theorem 2.** By Claim 2, we can suppose that $G$ is prime. We choose $LH$ as the union of all the pure or pseudo-pure components, $RH$ as the union of all the impure components, $RL$ as the set of all the isolated vertices in $G[M(v)]$ and $LL$ as the union of the other components of $G[M(v)]$.

**Claim 3.** None of the Situations 1, 3, 4 appears.

**Proof.** In Situation 1, vertices $b, c, d$ are not isolated in $G[RL]$. In Situations 3 and 4, the vertex $a$ belongs to $M(v)$ and sees partially the impure component containing the vertices $b$ and $d$. By Lemma 5, $a$ is isolated in $G[M(v)]$ and cannot belong to $LL$.

**Claim 4.** Situations 5 and 6 are not realized.

**Proof.** In Situations 5 and 6, $a$ and $d$ are not isolated vertices in $G[M(v)]$. Let $S_i$ (resp. $S_j$) be the impure component containing $b$ (resp. $c$). By Lemma 5, $a$ (resp. $d$)
sees completely $S_i$ (resp. $S_j$). Since $ac \notin E$, $S_i$ and $S_j$ are distinct components. By Lemma 6, in Situation 5, there exist vertices $c', b' \in RH$ such that $dc'b'a$ is an obstruction in $(G - v; <)$; in Situation 6, there exists a vertex $b' \in RH$ such that $dcb'a$ is an obstruction in $(G - v; <)$. □

A careful reading of Olariu’s study about quasi-brittle graphs [10, Proof of Theorem 2, pp. 151–152] allows us to write the following lemma:

**Lemma 7** (Olariu [10]). Let $v$ be a $F_2$-free vertex in a graph $G$, such that $G - v$ is perfectly orderable. There exists no obstruction $abcd$ in $(G; <')$, where $a \in LL$ and $d \in RL$.

Thus, we have

**Claim 5.** Situation 2 does not appear.

This ends the proof of Theorem 2. □

Another way to generalize Theorem 1 is to consider the family \{\(H_1, H_2, \widetilde{H}_2, H_4\). Unfortunately, we are unable to prove the corresponding result (we must add the rooted graph \(H_8\) defined below; see Corollary 6).

**Theorem 3.** Let $G$ be a graph and $v$ be a $F_3$-free vertex, where $F_3$ is the family of rooted graphs \{\(H_1, H_2, \widetilde{H}_2, H_7, H_8\)\} (see Figs. 2 and 8). Then, $G$ is perfectly orderable if and only if $G - v$ is perfectly orderable.

**Proof.** Let $<$ be a perfect order on $G - v$. Since $v$ is a pleasant vertex of $G$, by Lemma 4, no pseudo-pure component is impure. Then, let us consider the linear order $<'$ on $G$ defined in Section 4, such that $LH$ contains no pure component (that is $LH$ is the union of the pseudo-pure components) and $LL$ is empty (that is $RL = M(v)$). The oriented graph $(G; <')$ presents no circuit and, by Claim 1, no obstruction containing $v$. Let $A = abcd$ be an obstruction in $(G; <')$. Since $LL = \emptyset$, by Proposition 2, Situation 1 is the only one possibility. We suppose without loss of generality that $a$ belongs to

![Fig. 8](image-url)
Moreover, a¡b and bx∈E. Since Proposition 1, for every x∈C(A) we have bx∈E, cx∉E and dx∉E. Moreover, a <′ b, b < a and d < c.

Claim 6. There exist s and t in Q(A) such that t < s and for all x in C(A), sx∈E and tx∉E.

Proof. Since C(A)⊆LH, C(A) is a pseudo-pure component. Then, there exist t and s in a component Qj such that t < s, st∈E and for all x in C(A), sx∈E and tx∉E. If Qj ≠ Q(A) then tsab is an obstruction in (G−v; <′), a contradiction. Thus, Qj = Q(A).

Let \( \mathcal{A} \) be the set of obstructions in \((G; <′)\) and let \( \mathcal{C}(\mathcal{A}) \) be the set of vertices x in R such that there exist A = abcd in \( \mathcal{A} \) and an oriented path, \( P[x, b] \), from x to b, in \((G−v; <)\). Note that there is no edge xy such that \( x \in \mathcal{C}(\mathcal{A}) \), \( y \in R\backslash \mathcal{C}(\mathcal{A}) \) and \( y < x \).

Now, set \( L′ = L \cup \mathcal{C}(\mathcal{A}) \) and \( R′ = R\backslash \mathcal{C}(\mathcal{A}) \). We apply to this new partition \( \{L′, R′\} \) of \( V \backslash \{v\} \) the rules given in Section 3 to obtain from \((G−v; <)\) a new oriented graph \((G; <′′)\).

Remark 1. If there exists xy in E such that \( y <′′ x \) and \( y > x \), then \( x \in R′ \) and \( y \in LH = L′H\backslash \mathcal{C}(\mathcal{A}) \).

Note that, by Remark 1, Situations 2–6 do not appear in \((G−v; <′′)\).

Remark 2. By definition of \( \mathcal{C}(\mathcal{A}) \), Situation 1 does not appear in \((G−v; <′′)\).

Clearly, \((G; <′′)\) presents no circuit. Remarks 1 and 2 imply that, if B is an obstruction in \((G; <′′)\) then B contains v.

If \( B = v_{2}v_{3}v_{4} \) then \( v_{2} \in R′H \) (otherwise \( v_{2} <′′ v \)). Clearly, \( v_{3} \) and \( v_{4} \) belong to \( M(v) \) and by Remark 1, \( v_{4} < v_{3} \). But the component \( S_{i} \) containing \( v_{2} \) is pseudo-pure and then \( v_{2} \in LH \subseteq L′H \), a contradiction.

If \( B = v_{1}v_{2}v_{3} \) then \( v_{1} \in L′H \) (otherwise \( v <′′ v_{1} \)). Clearly, \( v_{1} \) and \( v_{2} \) are in the same component \( S_{i} \) and \( v_{3} \in M(v) \). Since \( v_{3} <′′ v_{2} \), by Remark 1, \( v_{3} < v_{2} \). Hence, \( S_{i} \) is an impure component, that is \( v_{1} \notin LH \). Since \( v_{1} \in L′H \), \( v_{1} \notin \mathcal{C}(\mathcal{A}) \).

Thus, there exist A = abcd in \( \mathcal{A} \) and an oriented path in \((G−v; <)\), \( P[v_{1}, b] \), from \( v_{1} \) to \( b \). Since \( C(A) \subseteq LH \), \( C(A) ≠ S_{i} \), \( av_{1} \in E \) and \( av_{2} \in E \). Since \( b < a \), \( v_{1} < a \) (otherwise \( v_{1}P[v_{1}, b]av_{1} \) is a circuit in \((G−v; <)\)). Moreover, \( v_{3}a \in E \), otherwise \( v_{1}av_{2}v_{3} \) would be an obstruction in \((G−v; <)\).

We distinguish two cases:

Case A: \( v_{1}b \in E \). Either \( P[v_{1}, b] = v_{1}b \) or \( v_{1}P[v_{1}, b]b_{1}v_{1} \) is a cycle. In each case \( v_{1} < b \). By Corollary 1, \( v_{3} \notin Q(A) \). Then \( v_{3}b \notin E \). Note that \( v_{2}b \notin E \), otherwise \( v_{1}bv_{2}v_{3} \) would
be an obstruction in \((G - v; <)\). By Proposition 1, \(v_1\) is adjacent to every vertex in \(Q(A)\) and \(v_2\) is adjacent to no vertex of \(Q(A)\). Now, the subgraph of \(G\) induced by \(\{v, a, v_1, v_2, v_3, b, c, d\}\) is isomorphic to the rooted graph \(H_8\) depicted in Fig. 8, a contradiction.

**Case B:** If \(v_1 b \notin E\). By Claim 6 there exist \(s\) and \(t\) in \(Q(A)\) such that \(sa \in E\), \(ta \notin E\) and \(t < s\). Since \(v_1 ast\) is not an obstruction in \((G - v; <)\), \(v_1 s \in E\) or \(v_1 t \in E\). Then, by Proposition 1, \(v_1\) and \(v_2\) have the same neighbourhood in \(Q(A)\). Thus, by Corollary 1, \(v_3 \notin Q(A)\).

**Case B1:** If \(v_1 t \in E\). Clearly \(t \neq b\) and \(v_2 t \in E\). Since \(v_1 tv_2v_3\) cannot be an obstruction, we have necessarily \(t < v_1\). Moreover \(tb \in E\), otherwise \(tv_1ab\) would be an obstruction in \((G - v; <)\). But now, the subgraph of \(G\) induced by \(\{v, a, v_1, v_2, v_3, b, t\}\) is isomorphic to the rooted graph \(H_7\) depicted in Fig. 8, a contradiction.

**Case B2:** If \(v_1 t \notin E\). Then necessarily, \(v_1 s \in E\) and \(v_2 s \in E\). Since \(tsv_2v_3\) is not an obstruction in \((G - v; <)\), \(v_2 t \in E\). Then \(v_1 t \in E\), a contradiction. □

From Theorem 3, we deduce the following three corollaries:

**Corollary 4.** Let \(G\) be a graph and \(v\) be a nice vertex in \(G\) (see Fig. 2). Then, \(G\) is perfectly orderable if and only if \(G - v\) is perfectly orderable.

**Proof.** The rooted graph \(H_3\) is a rooted subgraph of \(H_8\) and \(\tilde{H}_3\) is a rooted subgraph of \(H_7\). □

**Corollary 5.** Let \(G\) be a graph and \(v\) be a \(F_4\)-free vertex, where \(F_4\) is the family of rooted graphs \(\{H_1, H_2, \tilde{H}_2, H_5\}\) (see Figs. 4 and 7). Then, \(G\) is perfectly orderable if and only if \(G - v\) is perfectly orderable.

**Proof.** The rooted graph \(H_5\) is a rooted subgraph of \(H_7\) and \(H_8\). □

**Corollary 6.** Let \(G\) be a graph and \(v\) be a \(F_5\)-free vertex, where \(F_5\) is the family of rooted graphs \(\{H_1, H_2, \tilde{H}_2, H_4, H_8\}\) (see Figs. 2, 4 and 7). Then, \(G\) is perfectly orderable if and only if \(G - v\) is perfectly orderable.

**Proof.** The rooted graph \(H_4\) is a rooted subgraph of \(H_7\). □

**Remark 3.** Since \(H_4\) is a rooted subgraph of \(H_7\) and \(\tilde{H}_4\) is a rooted subgraph of \(H_8\), we note that Theorem 1 is also a corollary of Theorem 3.

6. Conclusion

Chvátal [3] defined a graph \(G\) to be **brittle** if each induced subgraph \(F\) of \(G\) contains a vertex that is not a midpoint of any \(P_4\) (semi-simplicial vertex) or not an endpoint of any \(P_4\). If \(v\) is not a midpoint of any \(P_4\) or not an endpoint of any \(P_4\) and if \(F - v\) is
perfectly orderable then $F$ is perfectly orderable. Thus, every brittle graph is perfectly orderable [3] (see also [7]).

Let $\{G_i\}_{1 \leq i \leq k}$ be a finite set of finite families of rooted graphs such that for any $i \in \{1, \ldots, k\}$ Assertion A is true. For example, $G_1 = \{(P_4, v)\}$ where $v$ denotes a midpoint of $P_4$, $G_2 = \{(P_4, v)\}$ where $v$ denotes an endpoint of $P_4$, $G_3$ is the family depicted in Fig. 1 ($G_3$-free vertices are charming vertices), $G_4$ is the family in Fig. 2 ($G_4$-free vertices are nice vertices), $G_5 = F_2$, $G_6 = F_3$. We define a class $\mathcal{P}(G_1, \ldots, G_k)$ of perfectly orderable graphs in the following way: a graph $G$ belongs to $\mathcal{P}(G_1, \ldots, G_k)$ if for any subgraph $F$ of $G$ there exist a family $G_i$ and a $G_i$-free vertex $v$ in $F$.

According to the previous example, we note that $\mathcal{P}(G_1, G_2)$ is the class of brittle graphs and that $\mathcal{P}(G_1, \ldots, G_6) = \mathcal{P}(G_3, \ldots, G_6)$. The last class contains brittle graphs, charming graphs, nice graphs and quasi-brittle graphs.

Although recognizing general perfectly orderable graphs is NP-complete [9], clearly, there is a polynomial time recognition algorithm for $\mathcal{P}(G_1, \ldots, G_k)$.

One may ask the following question: is it true that if $v$ is a pleasant vertex in a graph $G$ such that $G - v$ is perfectly orderable, then $G$ is perfectly orderable? In [4] we show a graph giving a negative answer to this question.

References

[3] V. Chvátal, Perfect Graph Seminar, McGill University, Montreal, Canada, 1983.