Lateral Connections for Asymptotic Solutions
around Higher Order Turning Points

ANTHONY LEUNG

Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221

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Asymptotic expansions as $\epsilon \to 0^+$ or $x \to \infty$ for fundamental systems of solutions for $\epsilon u''(x) - p(x) u(x) = 0$ were obtained by Evgrafov and Fedoryuk on unbounded canonical domains with neighborhoods deleted around turning points. When $p(x)$ is a polynomial, lateral connection formulas were found by Evgrafov and Fedoryuk, and Leung for two fundamental systems of solutions with known expansions in the interior of two different unbounded overlapping canonical regions with common first or second order turning point at their boundaries. In this paper, lateral connection formulas are found when the turning points are of any higher order. Recent results of uniform simplification by Sibuya around higher order turning point is used.

1. INTRODUCTION

Evgrafov and Fedoryuk [1] found asymptotic series solution for the differential equation

$$\epsilon u''(x) - p(x) u(x) = 0,$$

where $p(x)$ is a polynomial. The expansions are valid as $\epsilon \to 0^+$ or $x \to \infty$ in various unbounded subregions of the $x$-plane around turning points $x = x_0$ where $p(x_0) = 0$. Relationship between different solutions with expansions in different subregions around simple and second order turning points (i.e., first and second order zeros of $p(x)$) are found respectively by Evgrafov and Fedoryuk [1], Wasow [8], and Leung [6].

This paper finds such a relationship—connection formulas—when the polynomial $p(x)$ has a zero of any order. A result more general than given in [6] is obtained by using the recent uniform simplification of Sibuya [7]. Techniques and results of Wasow [8], R. Lee [5], and Hanson and Russell [2] are used to find the "central connection" of the solution of Evgrafov and Fedoryuk with solutions with known expansions around the turning point. These known expansions are obtained from the results of Hsich and Sibuya [4].
Subsequently, "lateral connection" formulas are found. The principal results are Theorem 5.1, Corollary 5.1 and Theorem 6.1.

2. Doubly Asymptotic Solutions in Unbounded Domains

Consider the differential equation

$$\epsilon^2 u''(x) - p(x) u(x) = 0, \quad (2.1)$$

where $p(x)$ is a polynomial with $p^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, (m - 1)$ and $p^{(m)}(x_0) \neq 0$. Here $m$ is a positive integer. A Stokes curve for the equation is a curve on the $x$-plane proceeding from $x_0$, along which

$$\text{Re} \int_{x_0}^x (p(z))^{1/2} \, dz = 0.$$ 

A canonical domain on the $x$-plane is a domain which is bounded by Stokes curves containing no turning points, i.e., zeros of $p(x)$, in its interior, and which is mapped by the function

$$\xi(x) = \int_{x_0}^x (p(z))^{1/2} \, dz \quad (2.2)$$

onto the whole $\xi$-plane cut by a finite number of vertical rays each of which is unbounded. If all these vertical cuts start from the images of some turning points and extend to infinity in the same direction, then the canonical domain is called consistent; otherwise, it is called inconsistent. (Such terminologies were introduced by Evgrafov and Fedoryuk [1], and Wasow [8].)

The purpose of this paper is to find the transition matrix from one fundamental system with known asymptotic expansion on one canonical domain to another such fundamental system on another canonical domain with a common $m$th order zero $x_0$ for $p(x)$ at the boundary.

Let the $(m + 2)$ Stokes curves at $x_0$ be $l_1, \ldots, l_{m+2}$, counting in the counterclockwise direction. For convenience, let $l_1$ be also be denoted by $l_{m+3}$. By making a suitable choice of roots, the transformation

$$t(x) = \left(\frac{m + 2}{2} \xi(x)\right)^{2/(m+2)} \quad (2.3)$$

is uniquely defined near $x = x_0$ and takes the curves $l_1, \ldots, l_{m+2}$ respectively into the $(m + 2)$ rays

$$\arg t = \frac{\pi}{m + 2}, \frac{3\pi}{m + 2}, \frac{5\pi}{m + 2}, \ldots, \frac{2m + 3}{m + 2} \pi.$$
The function $t(x)$ is holomorphic and univalent in a neighborhood of $x = x_0$. It maps subregions of the $(m + 2)$ domains bounded between $l_k, l_{k+1}$, $(k = 1, \ldots, m + 2)$, holomorphically and univalently to simply-connected regions in the $t$-plane bounded by the image rays of the corresponding Stokes curves. Furthermore, the image of these regions and Stokes curves in the $t$-plane consists of the entire plane, except for a finite number of cuts which are analytic curves tending to infinity and starting at the images of turning points other than $x_0$. However, for the domain of definition of $t(x)$, there may be choices of unbounded domains in the $x$-plane, bounded by curves starting at turning points other than $x_0$ and along which $\Re \xi(x) = \text{constant}$. After making a particular choice of $(m + 2)$ open regions: $D_k$ between $l_k, l_{k+1}$, for $k = 1, \ldots, m + 2$, together with the curves $l_1, \ldots, l_{m+2}$ in the $x$-plane for the domain of $t(x)$, the inverse function $x(t)$ would be holomorphic and univalent on the entire $t$-plane, except on the cuts.

In matrix form Eq. (2.1) is equivalent to

$$
\epsilon \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ \phi(x) & 0 \end{bmatrix} Y,
$$

(2.4)

where

$$
Y = \begin{bmatrix} u \\ \epsilon(u/dx) \end{bmatrix}.
$$

By the transformation

$$
Y = \begin{bmatrix} 1 & 0 \\ 0 & dt/dx \end{bmatrix} Y^*,
$$

(2.5)

Eq. (2.4) is transformed into

$$
\epsilon \frac{dY^*}{dt} = \begin{bmatrix} 0 & 1 \\ \tau^m & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 \\ 0 & -\frac{d^2t}{dx^2} \left(\frac{dt}{dx}\right)^2 \end{bmatrix} Y^*
$$

(2.6)

since $(dt/dx)^{-2} \phi(x) = \tau^m$. The coefficient matrix in (2.6) has a simpler leading part than in (2.4). Let

$$
q(t) = (dx/dt)^{1/2}
$$

be an arbitrary but fixed root, holomorphic for $t$ in the image $t(x)$ of the domain formed by our chosen regions $D_k, k = 1, \ldots, m + 2$ together with
the curves \( l_k, k = 1, \ldots, m + 2 \). From now on, let us employ the following
fixed conventions, unless otherwise stated:

\[
\frac{1 + 2(k - 1)}{(m + 2)} \pi \leq \arg t \leq \frac{1 + 2k}{m + 2} \pi \quad \text{for all } t \text{ in } \mathcal{t}(D_k)
\] (2.7)

closure \( k = 1, \ldots, m + 2 \), and

\[
\arg t^\alpha = \alpha \arg t,
\] (2.8)

whenever we take roots in the \( t \)-plane. To avoid confusion, denote

\[
\xi(x) = \frac{2}{m + 2} |t(x)|^{(m+2)/2},
\] (2.9)

which is a fixed choice of \( \xi(x) \), by convention (2.8). For each \( k = 1, \ldots, m + 1 \),
the domain \( \xi(D_k \cup l_{k+1} \cup D_{k+1}) \) is the whole complex plane except for a
finite number of vertical cuts. From \( \xi(D_k \cup l_{k+1} \cup D_{k+1}) \) delete circular
neighborhoods of radius \( \delta_0 \) about the endpoints of the cuts. Starting on the
boundary of each neighborhood draw two vertical half-lines tangent to it,
tending to infinite in the direction of the cut. Delete these open half-strips
of width \( 2\delta_0 \), attached on the deleted neighborhoods. Let \( \Omega_\delta^k \) be the resulting
set after all these neighborhoods and half-strips containing the cuts are
deleted. Denote the preimage of \( \Omega_\delta^k \) in the \( x \)-plane by \( \Omega_\delta^k \). Note that \( \delta_0 \)
is an arbitrary small positive constant.

**Theorem 2.1 (Evgrafov and Fedoryuk).** For each \( k = 1, \ldots, m + 1 \),
there exist two solutions \( u_k \pm(x, \epsilon) \) for (2.1) such that for \( x \in \Omega_\delta^k \), \( 0 < \epsilon < \epsilon_k(\delta_0) \),
(\( \epsilon_k(\delta_0) \) is a constant depending on \( k \) and \( \delta_0 \)), \( u_k \pm(x, \epsilon) \) are expressible as follows:

\[
u_k \pm(x, \epsilon) = \rho(x)^{-1/4} \hat{u_k} \pm(x, \epsilon) \exp\{\pm(1/\epsilon) \hat{\xi}(x)\}.
\] (2.10)

The functions \( \hat{u_k} \pm(x, \epsilon) \) have asymptotic expansions

\[
\hat{u_k} \pm(x, \epsilon) \sim \sum_{r=0}^\infty \hat{u_k}^r(x) \epsilon^r, \quad \hat{u_0} \pm(x) \equiv 1 \quad \text{as} \quad \epsilon \to 0^+,
\] (2.11)

or as \( x \to \infty \) in \( \Omega_\delta^k \) with \( \Re \hat{\xi}(x) \to -\infty \) for \( \hat{u_k}^+(x, \epsilon) \), or as \( x \to \infty \) in \( \Omega_\delta^k \)
with \( \Re \hat{\xi}(x) \to +\infty \) for \( \hat{u_k}^-(x, \epsilon) \).

In a more precise sense, (2.11) means that for each \( k = 1, \ldots, m + 1 \),

\[
\left| \hat{u_k} \pm(x, \epsilon) - \sum_{r=0}^N \hat{u_k}^r(x) \epsilon^r \right| \leq R_{\epsilon,N}^k(x) \epsilon^{N+1}
\] (2.12)
for \( x \in \Omega_{\delta_0}^k, 0 < \epsilon < \epsilon_k(\delta_0) \). The function \( R_{\delta_0}^\pm(x) \) is bounded in compact subsets of the corresponding \( \Omega_{\delta_0}^k \), and is of the order \( O(|x|^{-(h+2)/2}(N+1)} \) uniformly for \( 0 < \epsilon < \epsilon_k(\delta_0) \) as \( x \to \infty \) in \( \Omega_{\delta_0}^x \) in such a manner that \( \text{Re} \, \hat{\xi}(x) \to +\infty \), where \( h \) is the degree of \( p(x) \). The functions \( \hat{u}_k^\pm(x) \) are holomorphic in \( D_k \cup D_{k+1} \) and

\[
\hat{u}_k^\pm(x) = O(|x|^{-(h+2)/2}r)
\]
as \( x \to \infty \) in \( \Omega_{\delta_0}^k \) in such a manner that \( \text{Re} \, \hat{\xi}(x) \to +\infty \). (Each formula above combines two: take the upper or lower sign throughout.) In formula (2.10) we define \( u_k^\pm(x, \epsilon) \) uniquely by specifying

\[
p^{-1/4}(x) = q(t) t^{-m/4}, \quad \frac{\pi}{m + 2} \leq \arg t < \frac{2m + 5}{m + 2} \pi
\]
for

\[
t \in t \left( \bigcup_{k=1}^{m+2} \{ D_k \cup l_k \} \right).
\]
(We shall make this choice for all subsequent formulas in this paper unless otherwise stated.) The asymptotic formulas above may be formally differentiated.

Suppose \( D_k \cup l_{k+1} \cup D_{k+1} \) is a consistent canonical domain, \( 1 \leq k \leq m + 1 \). From \( \hat{\xi}(D_k \cup l_{k+1} \cup D_{k+1}) \) delete circular neighborhoods of radius \( \delta \) about the endpoints of the cuts, as well as sectors of central angle \( \delta \) that have their vertices at the endpoints of the cuts and are bisected by the cuts. The resulting domain in the \( \hat{\xi} \)-plane may be denoted by

\[
\hat{\xi}(D_k \cup l_{k+1} \cup D_{k+1})_0;
\]
the corresponding domains in the \( x \)-plane or \( t \)-plane may be denoted by \( (D_k \cup l_{k+1} \cup D_{k+1})_0 \) or \( t(D_k \cup l_{k+1} \cup D_{k+1})_0 \) respectively. For a consistent canonical domain, Theorem 2.1 can be improved so that the asymptotic relations, as \( x \to \infty \), are valid in \( D_k \) and in \( D_{k+1} \) for both \( u_k^+ \) and \( u_k^- \).

**Corollary 2.1** (Evgrafov and Fedoryuk). For \( 1 \leq k \leq m + 1 \), suppose \( D_k \cup l_{k+1} \cup D_{k+1} \) is a consistent canonical domain. Then the solutions \( u_k^\pm(x, \epsilon) \) for (2.1) of Theorem 2.1 are expressible in the form (2.10) where

\[
\hat{u}_k^\pm(x, \epsilon) \sim \sum_{r=0}^\infty \hat{u}_{kr}^\pm(x) \epsilon^r, \quad \hat{u}_0^\pm(x) \equiv 1 \quad \text{as} \quad \epsilon \to 0^+
\]
or \( x \to \infty \) in \( (D_k \cup l_{k+1} \cup D_{k+1})_0 \) in the sense that

\[
\left| \hat{u}_k^\pm(x, \epsilon) - \sum_{r=0}^N \hat{u}_{kr}^\pm(x) \epsilon^r \right| \leq C_k(N, \delta) (|x|^{-(h+2)/2} \epsilon)^{N+1}
\]

for \( x \in \Omega_{\delta_0}^k, 0 < \epsilon < \epsilon_k(\delta_0) \). The function \( R_{\delta_0}^\pm(x) \) is bounded in compact subsets of the corresponding \( \Omega_{\delta_0}^k \), and is of the order \( O(|x|^{-(h+2)/2}(N+1)} \) uniformly for \( 0 < \epsilon < \epsilon_k(\delta_0) \) as \( x \to \infty \) in \( \Omega_{\delta_0}^x \) in such a manner that \( \text{Re} \, \hat{\xi}(x) \to +\infty \), where \( h \) is the degree of \( p(x) \). The functions \( \hat{u}_k^\pm(x) \) are holomorphic in \( D_k \cup D_{k+1} \) and

\[
\hat{u}_k^\pm(x) = O(|x|^{-(h+2)/2}r)
\]
as \( x \to \infty \) in \( \Omega_{\delta_0}^k \) in such a manner that \( \text{Re} \, \hat{\xi}(x) \to +\infty \). (Each formula above combines two: take the upper or lower sign throughout.) In formula (2.10) we define \( u_k^\pm(x, \epsilon) \) uniquely by specifying

\[
p^{-1/4}(x) = q(t) t^{-m/4}, \quad \frac{\pi}{m + 2} \leq \arg t < \frac{2m + 5}{m + 2} \pi
\]
for

\[
t \in t \left( \bigcup_{k=1}^{m+2} \{ D_k \cup l_k \} \right).
\]
(We shall make this choice for all subsequent formulas in this paper unless otherwise stated.) The asymptotic formulas above may be formally differentiated.

Suppose \( D_k \cup l_{k+1} \cup D_{k+1} \) is a consistent canonical domain, \( 1 \leq k \leq m + 1 \). From \( \hat{\xi}(D_k \cup l_{k+1} \cup D_{k+1}) \) delete circular neighborhoods of radius \( \delta \) about the endpoints of the cuts, as well as sectors of central angle \( \delta \) that have their vertices at the endpoints of the cuts and are bisected by the cuts. The resulting domain in the \( \hat{\xi} \)-plane may be denoted by

\[
\hat{\xi}(D_k \cup l_{k+1} \cup D_{k+1})_0;
\]
the corresponding domains in the \( x \)-plane or \( t \)-plane may be denoted by \( (D_k \cup l_{k+1} \cup D_{k+1})_0 \) or \( t(D_k \cup l_{k+1} \cup D_{k+1})_0 \) respectively. For a consistent canonical domain, Theorem 2.1 can be improved so that the asymptotic relations, as \( x \to \infty \), are valid in \( D_k \) and in \( D_{k+1} \) for both \( u_k^+ \) and \( u_k^- \).

**Corollary 2.1** (Evgrafov and Fedoryuk). For \( 1 \leq k \leq m + 1 \), suppose \( D_k \cup l_{k+1} \cup D_{k+1} \) is a consistent canonical domain. Then the solutions \( u_k^\pm(x, \epsilon) \) for (2.1) of Theorem 2.1 are expressible in the form (2.10) where

\[
\hat{u}_k^\pm(x, \epsilon) \sim \sum_{r=0}^\infty \hat{u}_{kr}^\pm(x) \epsilon^r, \quad \hat{u}_0^\pm(x) \equiv 1 \quad \text{as} \quad \epsilon \to 0^+
\]
or \( x \to \infty \) in \( (D_k \cup l_{k+1} \cup D_{k+1})_0 \) in the sense that

\[
\left| \hat{u}_k^\pm(x, \epsilon) - \sum_{r=0}^N \hat{u}_{kr}^\pm(x) \epsilon^r \right| \leq C_k(N, \delta) (|x|^{-(h+2)/2} \epsilon)^{N+1}
\]
for $x \in (D_k \cup l_{k+1} \cup D_{k+1})_\delta$, $0 < \epsilon < \epsilon_k(\delta)$. Here, $C_k(N, \delta)$ and $\epsilon_k(\delta)$ are constants. The functions $\hat{u}_{k+}^\pm(x)$ are holomorphic in $D_k \cup l_{k+1} \cup D_{k+1}$ and

$$\hat{u}_{kr}^\pm(x) = O(|x|^{-\frac{(m+2)r}{2}})$$

(2.11c)
as $x \to \infty$ in $(D_k \cup l_{k+1} \cup D_{k+1})_\delta$. These asymptotic formulas may be formally differentiated.

Let $1 \leq k \leq m + 1$, and let us not first require that $D_k \cup l_{k+1} \cup D_{k+1}$ be consistent. We then have $u_{k+}^+$ and $u_{k-}^-$ as stated in Theorem 2.1. When $k$ is odd (or even), $u_{k+}^+$ is subdominant in $D_k$ (or $D_{k+1}$), and $u_{k-}^-$ is subdominant in $D_{k+1}$ (or $D_k$). For each odd $k$, we put $(u_{k+}^+, u_{k-}^-)$ in the first row and $(e(du_{k+}^+/(dt)), e(du_{k-}^-/(dt)))$ in the second row to form a fundamental matrix solution for (2.6). For each even $k$, we put $(u_{k-}^-, u_{k+}^+)$ in the first row and $(e(du_{k-}^-/(dt)), e(du_{k+}^+/(dt)))$ in the second row. Expressing everything as functions of $t$, we obtain $(m + 1)$ fundamental matrix solutions for (2.6) of the form

$$Y_k^r(t, \epsilon) = p(x(t))^{-1/4} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Y_k^r(t, \epsilon)$$

$$\times \text{diag} \left( \exp \left\{ \frac{(-1)^{k+1} 2}{(m + 2) \epsilon} t(m+2)/2 \right\}, \exp \left\{ \frac{(-1)^k 2}{(m + 2) \epsilon} t(m+2)/2 \right\} \right)$$

for $t \in t(\Omega_{\delta_0}^k)$, $0 < \epsilon < \epsilon_k(\delta_0)$, for the corresponding $k$. The matrix $Y_k^r(t, \epsilon)$ has the properties

$$\hat{Y}_k^r(t, \epsilon) \sim \sum_{r=0}^{\infty} \hat{Y}_{kr}(t) \epsilon^r, \quad \hat{Y}_{kr}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(2.15)
in the sense that for $0 < \epsilon < \epsilon_k(\delta_0)$, $t \in t(\Omega_{\delta_0}^k)$

$$\left| \hat{Y}_k^r(t, \epsilon) - \sum_{r=0}^{N} \hat{Y}_{kr}(t) \epsilon^r \right| \leq C_{kN}(t) \epsilon^{N+1}.$$  

(2.16)

$C_{kN}(t)$ is a $2 \times 2$ matrix function of $t$, and is bounded in compact subsets of $t(\Omega_{\delta_0}^k)$; the $\hat{Y}_{kr}(t)$ are holomorphic in $t(\Omega_{\delta_0}^k)$. The absolute value sign and inequality relation apply to each matrix element. The first columns of $\hat{Y}_{kr}(t)$ and $C_{kN}(t)$ are of the order $O(|t|^{-((m+2)/2)r})$ and $O(|t|^{-((m+2)/2)(N+1)})$ respectively as $t \to \infty$ in $t(D_k)$ with $\Re \hat{\xi}(x(t)) \to -\infty$ when $k$ is odd, or with $\Re \hat{\xi}(x(t)) \to +\infty$ when $k$ is even, uniformly for $0 < \epsilon < \epsilon_k(\delta_0)$, (for example, $t \to \infty$ at an angle $((1 + 2(k - 1))/(m + 2)) \pi < \arg t < ((1 + 2k)/(m + 2)) \pi$ for the corresponding $k$). The second columns of $\hat{Y}_{kr}(t)$ and $C_{kN}(t)$ are of the order $O(|t|^{-((m+2)/2)r})$ and $O(|t|^{-((m+2)/2)(N+1)})$ respectively as $t \to \infty$ in $t(D_{k+1})$ with $\Re \hat{\xi}(x(t)) \to +\infty$ when $k$ is odd, or with $\Re \hat{\xi}(x(t)) \to -\infty$.
when $k$ is even, uniformly for $0 < \epsilon < \epsilon_k(\delta_0)$. In case $D_k \cup I_{k+1} \cup D_{k+1}$ is a consistent canonical domain, then we have $\hat{Y}_k^\ell(t, \epsilon)$ expressible in the form (2.14), where the matrix $\hat{Y}_k^\ell(t, \epsilon)$ has an asymptotic expansion of the form (2.15) in the sense that for $\epsilon > 0$ sufficiently small, $t \in t(D_k \cup I_{k+1} \cup D_{k+1})_0$,

$$|\hat{Y}_k^\ell(t, \epsilon) - \sum_{r=0}^{N} \hat{Y}_{kr}(t) \epsilon^r| \leq C_k(N, \delta) \left( |t|^{-(m+2)/2} \epsilon \right)^{N+1}.$$ 

Here, the $C_k(N, \delta)$ are constants and $\hat{Y}_{kr}(t)$ are holomorphic in $t(D_k \cup I_{k+1} \cup D_{k+1})$.

We shall now find the connections between the fundamental matrix solutions $\hat{Y}_k^\ell$, $k = 1, \ldots, m + 1$ by first finding their relationship with solutions having known properties in a full neighborhood of the turning point $t = 0$. Such connection techniques was used by Wasow [8] and Leung [6]. To find solutions with known properties in a neighborhood of the turning point, results of Hsieh and Sibuya [4], and Sibuya [7] will be utilized.

3. Uniform Simplification in a Full Neighborhood of the Turning Point

We will transform Equation (2.6) into a equation whose solutions will be expressed in terms of special functions in Section 4. The transformation will be performed in a full neighborhood of $t = 0$, overlapping the regions $\Omega^k_{\delta_0}$ for each $k = 1, \ldots, (m + 1)$, provided $\delta_0 > 0$ is small enough.

**Lemma 3.1.** There exists a holomorphic transformation in a neighborhood of zero in the $t$-plane $Y^* = P(t, \epsilon) Z$ where $P(t, \epsilon)$ has a uniform asymptotic expansion for $|t| \leq t_0$, $t_0$ sufficiently small, that takes differential equation (2.6) into the form

$$\frac{\epsilon}{dt} \frac{dZ}{dt} = \begin{bmatrix} 0 & 1 \\ t^m + \epsilon \sum_{j=0}^{m-2} \epsilon \beta_j(\epsilon) t^j & 0 \end{bmatrix} Z, \quad (3.1)$$

where $\beta_j(\epsilon), j = 1, \ldots, (m - 2)$ are analytic functions of $\epsilon$ for $\epsilon > 0$ sufficiently small and

$$\beta_j(\epsilon) \sim \sum_{i=0}^{\infty} \beta_{ji} \epsilon^i \quad (3.2)$$

as $\epsilon \to 0^+$. The quantities $\beta_{ji}$ are complex constants.
Proof. Direct calculation shows that \((dq/dt) = - \frac{1}{2} g(t) q(t)\), where
\[
g(t) = \frac{d^2q}{dt^2} \left( \frac{dt}{dx} \right)^{-2} = -\frac{d^2x}{dt^2} \left( \frac{dt}{dx} \right)^{-1}.
\]
Thus, if we let
\[
y^* = \begin{bmatrix} q(t) \\ -\frac{1}{2} \epsilon gq \end{bmatrix},
\]
then straightforward computations shows that
\[
\epsilon \frac{dY^{**}}{dt} = \begin{bmatrix} 0 & 1 \\ t^m + \epsilon^2 \left( \frac{dg}{dt} + \frac{g^2}{2} \right) & 0 \end{bmatrix} Y^{**}. \tag{3.4}
\]
Hanson and Russell [2] show that (3.4) can be formally transformed into
\[
\epsilon \frac{dZ}{dt} = \begin{bmatrix} 0 & 1 \\ t^m + \epsilon \sum_{j=0}^{m-2} \epsilon^j \beta_j(\epsilon) t^j & 0 \end{bmatrix} Z \tag{3.5}
\]
where \(\beta_j(\epsilon), j = 1, \ldots, m - 2\) are formal series in \(\epsilon\) with constant coefficients: \(\sum_{i=0}^{\infty} \beta_{ji} \epsilon^i\), by the formal transformation
\[
Y^{**} = M(t, \epsilon) Z \tag{3.6}
\]
\[
M = \begin{bmatrix} q_1(t, \epsilon) & q_2(t, \epsilon) \\ \epsilon \frac{dq_1}{dt} + q_2 \left( t^m + \epsilon \sum_{k=1}^{\infty} s_k(t) \epsilon^k \right) & q_1 + \epsilon \frac{dq_2}{dt} \end{bmatrix} \tag{3.7}
\]
where \(q_1(t, \epsilon), q_2(t, \epsilon)\) are formal series of the forms
\[
q_1(t, \epsilon) = 1 + \sum_{k=1}^{\infty} q_{1k}(t) \epsilon^k,
\]
\[
q_2(t, \epsilon) = \sum_{k=1}^{\infty} q_{2k}(t) \epsilon^k.
\]
The functions \(q_{1k}, q_{2k}\) are holomorphic for all \(k\), for \(|t| \leq \tilde{t}_0, \tilde{t}_0\) sufficiently small; and \(s_k(t), k = 1, 2, \ldots\) are polynomials in \(t\) of degree at most \(m - 2\). They can all be constructed recursively by formulas given in [2]. A recent
result of Sibuya [7], shows that there exists a holomorphic transformation

$$Y^{**} = Q(t, \epsilon) Z$$  \hspace{1cm} (3.8)

on a disk with center $t = 0$ that takes (3.4) into (3.1), with $\beta_j(\epsilon)$ analytic for small $\epsilon > 0$ and satisfying (3.2), while $Q(t, \epsilon) \sim M(t, \epsilon)$ as $\epsilon \to 0^+$, uniformly for $|t| \leq t_0$, $t_0$ sufficiently small. The fact that term of $O(\epsilon)$ in the lower left-hand corner of the coefficient matrix in (3.4) is of order $O(\epsilon^2)$ ensures that there is no $s_0(t)$ in (3.7), and calculation according to procedures in [2] will show that the coefficient of $t^j$, $0 \leq j \leq m - 2$, in the $(2, 1)$th entry for the coefficient matrix in (3.1) are of order $O(\epsilon^2)$.

On combining transformations (3.3) and (3.8) the transformation $Y^* = P(t, \epsilon) Z$ is obtained where

$$P(t, \epsilon) = \begin{bmatrix} q(t) & 0 \\ \frac{1}{2} \epsilon q(t) & q(t) \end{bmatrix} Q(t, \epsilon).$$  \hspace{1cm} (3.9)

It transforms (2.6) into (3.1). Therefore, $P(t, \epsilon)$ has, for $|t| \leq t_0$, the uniform asymptotic expansion

$$P(t, \epsilon) \sim q(t) I + \sum_{r=1}^{\infty} P_r(t) \epsilon^r$$  \hspace{1cm} (3.10)

with $P_r(t)$ holomorphic for each $r$. This completes the proof of Lemma 3.1.

4. Solutions Around the Turning Point, in Terms of Special Functions

Next, we proceed to investigate the solutions of (3.1) in terms of solutions to a particular class of ordinary differential equations with polynomial coefficients, studied by Hsieh and Sibuya [4]. The first element of the vector $Z$ satisfies the equation

$$e^\omega \frac{d^2 x}{dt^2} - \left( t^m + \epsilon \sum_{j=0}^{m-2} \epsilon \beta_j(\epsilon) t^j \right) x = 0. $$  \hspace{1cm} (4.1)

The change of independent variable

$$t = \epsilon^{2/(m+2)} \eta$$  \hspace{1cm} (4.2)

reduces (4.1) to

$$\frac{d^2 \psi}{d\eta^2} (\eta, \epsilon) + \left( \eta^m + \sum_{j=2}^{m} \left[ \epsilon^{4/(m+2)+(3/(m+2))(m-j)} \beta_{m-j}(\epsilon) \eta^{m-j} \right] \right) \psi(\eta, \epsilon) = 0$$  \hspace{1cm} (4.3)
in the sense that solutions \( z(t, \epsilon) \) of (4.1) can be related to solutions \( v(\eta, \epsilon) \) of (4.3) by

\[
z(t, \epsilon) = v(\epsilon^{-2/(m+2)}, \epsilon) = v(\eta, \epsilon). \tag{4.4}
\]

Let

\[
a = a(\epsilon) = (a_1(\epsilon), a_2(\epsilon), \ldots, a_m(\epsilon)),
\]

where \( a_1(\epsilon) = 0, \)

\[
a_j(\epsilon) = \epsilon^{j/(m+2) + (j/(m+2))(m-j)} \cdot \beta_{m-j}(\epsilon), \quad \text{for } j = 2, \ldots, m. \tag{4.5}
\]

[4] shows that Eq. (4.3) has solutions given by

\[
v(\eta, \epsilon) = \mathcal{Y}_{m}(\eta, a(\epsilon)) = \mathcal{Y}_{m}(\omega^{-k} \eta, G^k(\epsilon)), \tag{4.6}
\]

for each \( k = 1, 2, \ldots, m + 2 \), where

\[
\omega = \exp \left\{ i \frac{2}{m+2} \pi \right\}
\]

\[
G^k(a) = (\omega^{-k}a_1, \omega^{-2k}a_2, \ldots, \omega^{-mk}a_m).
\]

The function \( \mathcal{Y}_{m}(s, a) \) is the unique solution of the differential equation

\[
\frac{d^2 y}{ds^2} - P(s) y = 0, \quad P(s) = s^m + a_1 s^{m-1} + \cdots + a_{m-1} s + a_m,
\]

where \( a = (a_1, \ldots, a_m) \) are complex parameters, such that

(i) \( \mathcal{Y}_{m}(s, a) \) is an entire function of \((s, a)\);

(ii) \( \mathcal{Y}_{m}(s, a) \) and \( \mathcal{Y}_{m}^{'}(s, a) \) admit respectively the asymptotic representations

\[
\mathcal{Y}_{m} = s^m \left[ 1 + O(s^{-1/2}) \right] \exp \left[ -E_m(s, a) \right] \tag{4.7}
\]

\[
\mathcal{Y}_{m}^{'} = s^{(m/2)+r_m} \left[ -1 + O(s^{-1/2}) \right] \exp \left[ -E_m(s, a) \right]
\]

uniformly on each compact set in the \((a_1, \ldots, a_m)\)-space as \( s \) tends to infinity in any closed subsector of the open sector: \( |\arg s| < (3/(m + 2)) \pi \). Here

\[
r_m = \begin{cases} \frac{1}{4} m & (m: \text{odd}) \\ \frac{1}{2} m - b_{(m/2)+1}(a) & (m: \text{even}) \end{cases}
\]

\[
E_m(s, a) = \frac{2}{m+2} s^{(m+2)/2} + \sum_{1 \leq h < (m/2)+1} \frac{2}{m+2h} b_h(a) s^{(m+2-2h)/2} \tag{4.9}
\]
and
\[ \left[ 1 + \sum_{j=1}^{m} a_j s^{-j} \right]^{1/2} = 1 + \sum_{h=1}^{\infty} b_h(a) s^{-h}. \] (4.10)

The quantities \( b_h(a) \) are polynomials in \( a_1, \ldots, a_m \); and
\[ s^r = \exp\{r(\log |s| + i \arg s)\} \]
for any constant \( r \).

For the remaining computations we will always use the following rules, unless otherwise stated:
\[ x^r = \exp\{r(\log |x| + i \arg x)\}, \]
\[ \arg(xy) = \arg x + \arg y, \]
for any complex numbers \( x, y, r \);
\[ \arg e = 0, \quad \text{and} \quad \arg \omega = \frac{2}{m+2} \pi. \]

Using (4.5)-(4.10), we have the following asymptotic representations for
\[ \mathcal{Y}_{m,k}(t^{e^{-2/(m+2)}}, a(e)) = \mathcal{Y}_m(\omega^{-k}t^{e^{-2/(m+2)}}, G^k(a(e))) \]
as \( \epsilon \to 0^+ \), for each \( k = 1, \ldots, m + 2 \):
\[ \mathcal{Y}_{m,k}(t^{e^{-2/(m+2)}}, a(e)) \]
\[ = \omega^{(k/4)m2-m/4}e^{m/2(m+2)} \exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2} \epsilon^{-1} \right\} \left[ 1 + O(e^{1/(m+2)}) \right] \]
\[ \frac{d}{dt} \mathcal{Y}_{m,k}(t^{e^{-2/(m+2)}}, a(e)) \]
\[ = \omega^{km/4}\epsilon^{-m/2}e^{m/2} \exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2} \epsilon^{-1} \right\} \]
\[ \times \left[ (-1)^{k+1} + O(e^{1/(m+2)}) \right] \] (4.11)
uniformly for \( \delta_0 \leq |t| \leq \delta_0 \), in any corresponding closed subsector of \( |\arg t - (2k\pi)/(m + 2)| < 3\pi/(m + 2) \). Here \( \delta_0 \) is any arbitrary positive constant less than \( \delta_0 \). We have such a convenient exponential factor because all except the first term of \( E_m(\omega^{-k}t^{e^{-2/(m+2)}}, G^k(a(e))) \) in (4.9) are of the order \( O(e) \) for \( |t| \leq t_0 \), which can be easily shown using (4.5) and (4.10). Furthermore, the functions \( \mathcal{Y}_{m,k} \) satisfy the connection formulas
\[ \mathcal{Y}_{m,k}(t^{e^{-2/(m+2)}}, a(e)) = C(G^k(a(e))) \mathcal{Y}_{m,k+1}(t^{e^{-2/(m+2)}}, a(e)) \]
\[ + \tilde{C}(G^k(a(e))) \mathcal{Y}_{m,k+2}(t^{e^{-2/(m+2)}}, a(e)) \] (4.12)
1 \leq k \leq m, \text{ for } |t| \leq t_0, \text{ where}

\begin{equation}
C(G^k(c)) = \frac{\omega^{1/2} - \omega}{\omega^{1/2} - 1} + O(\epsilon^{4/(m+2)}),
\end{equation}
\begin{equation}
C(G^\kappa(c)) = -\omega + O(\epsilon^{4/(m+2)}),
\end{equation}
as \epsilon \to 0^+ (\text{refer to } [7, \text{ p. 50–54}]).

We are now ready to investigate fundamental matrix solutions of (3.1) for \( \delta_0 \leq |t| \leq t_0 \). Let

\begin{equation}
Z_k(t, \epsilon) = \begin{bmatrix}
\frac{\omega^m/4}{m,k} & \frac{\omega^m/4}{m,k+1} \\
\frac{\partial^m/4}{dt} & \frac{\partial^m/4}{dt}
\end{bmatrix}
\begin{bmatrix}
C(G^k(t, \epsilon), a(\epsilon)) \\
C(G^\kappa(t, \epsilon), a(\epsilon))
\end{bmatrix}
\end{equation}

For \( \delta_0 \leq |t| \leq t_0 \), \( t \) in any closed subsector of

\begin{equation}
\frac{2k-1}{m+2} \pi < \arg t < \frac{2k+3}{m+2} \pi,
\end{equation}

we have the following formula for \( Z_k(t, \epsilon), k = 1, \ldots, (m + 1):\)

\begin{equation}
Z_k(t, \epsilon) = \frac{\omega^{(k+1)/4}}{t^{m/2}(m+1/2)}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\end{equation}

\begin{equation}
X_k(t, \epsilon),
\end{equation}

where

\begin{equation}
X_k(t, \epsilon)
= \begin{bmatrix}
\exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\exp \left\{ (-1)^{k} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\}
\end{bmatrix}
\end{equation}

as \( \epsilon \to 0^+. \) Formula for \( Z_k(t, \epsilon), \) for

\begin{equation}
\frac{2k+1}{m+2} \pi < \arg t < \frac{2k+5}{m+2} \pi,
\end{equation}

can be readily obtained from (4.12) and (4.13):

\begin{equation}
\begin{aligned}
\frac{\omega^m/4}{m,k} & \frac{\omega^m/4}{m,k+1} \\
\frac{\partial^m/4}{dt} & \frac{\partial^m/4}{dt}
\end{aligned}
\begin{bmatrix}
C(G^k(t, \epsilon), a(\epsilon)) \\
C(G^\kappa(t, \epsilon), a(\epsilon))
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\end{equation}

\begin{equation}
X_k(t, \epsilon),
\end{equation}

where

\begin{equation}
X_k(t, \epsilon)
= \begin{bmatrix}
\exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\exp \left\{ (-1)^{k} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\}
\end{bmatrix}
\end{equation}

\begin{equation}
\frac{2k+1}{m+2} \pi < \arg t < \frac{2k+5}{m+2} \pi,
\end{equation}

\begin{equation}
\begin{aligned}
\frac{\omega^m/4}{m,k} & \frac{\omega^m/4}{m,k+1} \\
\frac{\partial^m/4}{dt} & \frac{\partial^m/4}{dt}
\end{aligned}
\begin{bmatrix}
C(G^k(t, \epsilon), a(\epsilon)) \\
C(G^\kappa(t, \epsilon), a(\epsilon))
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\end{equation}

\begin{equation}
X_k(t, \epsilon),
\end{equation}

where

\begin{equation}
X_k(t, \epsilon)
= \begin{bmatrix}
\exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\exp \left\{ (-1)^{k} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\}
\end{bmatrix}
\end{equation}

\begin{equation}
\frac{2k+1}{m+2} \pi < \arg t < \frac{2k+5}{m+2} \pi,
\end{equation}

\begin{equation}
\begin{aligned}
\frac{\omega^m/4}{m,k} & \frac{\omega^m/4}{m,k+1} \\
\frac{\partial^m/4}{dt} & \frac{\partial^m/4}{dt}
\end{aligned}
\begin{bmatrix}
C(G^k(t, \epsilon), a(\epsilon)) \\
C(G^\kappa(t, \epsilon), a(\epsilon))
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\end{equation}

\begin{equation}
X_k(t, \epsilon),
\end{equation}

where

\begin{equation}
X_k(t, \epsilon)
= \begin{bmatrix}
\exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\exp \left\{ (-1)^{k} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\}
\end{bmatrix}
\end{equation}

\begin{equation}
\frac{2k+1}{m+2} \pi < \arg t < \frac{2k+5}{m+2} \pi,
\end{equation}

\begin{equation}
\begin{aligned}
\frac{\omega^m/4}{m,k} & \frac{\omega^m/4}{m,k+1} \\
\frac{\partial^m/4}{dt} & \frac{\partial^m/4}{dt}
\end{aligned}
\begin{bmatrix}
C(G^k(t, \epsilon), a(\epsilon)) \\
C(G^\kappa(t, \epsilon), a(\epsilon))
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) \\
1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\end{equation}

\begin{equation}
X_k(t, \epsilon),
\end{equation}

where

\begin{equation}
X_k(t, \epsilon)
= \begin{bmatrix}
\exp \left\{ (-1)^{k+1} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\exp \left\{ (-1)^{k} \frac{2}{m+2} t^{(m+2)/2}\epsilon^{-1} \right\}
\end{bmatrix}
\end{equation}
as $\epsilon \to 0^+$, $\delta_0 \leq |t| \leq t_0$. Thus in any closed subsector of

$$\frac{2k + 1}{m + 2} \pi < \arg t < \frac{2k + 5}{m + 2} \pi, \quad \delta_0 \leq |t| \leq t_0,$$

$Z_k(t, \epsilon)$ has the representation:

$$Z_k(t, \epsilon) = t^{-m/4} e^{m/2(m+2)} \left( \begin{array}{c} \frac{\omega^{-1} - \omega}{\omega^{-1} - 1} \omega^{((k+2)/4)m} + O(\epsilon^{1/(m+2)}) \\ \omega^{((k+1)/4)m} + O(\epsilon^{1/(m+2)}) \\ t^{m/2}(-1)^k \omega^{-1} - \omega \omega^{((k+2)/4)m} + O(\epsilon^{1/(m+2)}) \\ t^{m/2}(-1)^k \omega^{((k+1)/4)m} + O(\epsilon^{1/(m+2)}) \end{array} \right)$$

$$\cdot \exp \left\{ (-1)^k \frac{2}{m + 2} t^{(m+2)/2} e^{-1} \right\}$$

$$+ \left[ t^{m/2}(-1)^k \omega^{1+((k+3)/4)m} + O(\epsilon^{1/(m+2)}) \right] 0$$

$$\times \exp \left\{ (-1)^{k+1} \frac{2}{m + 2} t^{(m+2)/2} e^{-1} \right\}$$

(4.17)

as $\epsilon \to 0^+$, $k = 1, \ldots, m$. Note that the union of regions of validity for formulas (4.15), $k = 1, \ldots, (m + 1)$, covers an annulus around $t = 0$ except for one ray $\arg t = \pi/(m + 2)$.

5. LATERAL CONNECTION FORMULAS FOR FUNDAMENTAL SYSTEMS WITH ASYMPTOTIC SERIES IN DIFFERENT UNBOUNDED DOMAINS

For each $k = 1, \ldots, m + 1$, consider the region:

$$\delta_0 \leq |t| \leq t_0, \quad \frac{2k - 1}{m + 2} \pi < \arg t < \frac{2k + 3}{m + 2} \pi.$$

It will have nonempty intersection with $t(\Omega^k_{\delta_0})$ if $\delta_0$ is reduced to sufficiently small. We have asymptotic formulas for two fundamental systems $Y_k^\epsilon(t, \epsilon)$ and $P(t, \epsilon)Z_k(t, \epsilon)$ of (2.6) in the intersection region, analyzed respectively from (2.14) to (2.16) and from (4.14) to (4.16). Furthermore, Eq. (4.17) leads to formula for $P(t, \epsilon)Z_k(t, \epsilon)$ for

$$\delta_0 \leq |t| \leq t_n, \quad \frac{2k + 1}{m + 2} \pi < \arg t < \frac{2k + 5}{m + 2} \pi, \quad k = 1, \ldots, m.$$
Thus, for each \( k = 1, \ldots, m \), relationship between \( Y_k^F \) and \( Y_{k+1}^F \) can be calculated by using \( PZ_k \).

**Lemma 5.1 (Central Connections).**

\[
Y_k^F(t, \epsilon) = (PZ_k)(t, \epsilon) \, C_k(\epsilon) \quad \text{for} \quad k = 1, \ldots, m + 1, \quad |t| \leq t_0
\]

and

\[
Y_{k+1}^F(t, \epsilon) = (PZ_k)(t, \epsilon) \, \tilde{C}_k(\epsilon) \quad \text{for} \quad k = 1, \ldots, m, \quad |t| \leq t_0.
\]

Here,

\[
C_k(\epsilon) = \omega^{-(k+1)/4} e^{-m/2(m+2)} \left[ 1 + O(\epsilon^{1/(m+2)}) \right]
\]

and

\[
\tilde{C}_k(\epsilon) = e^{-m/2(m+2)}
\]

\[
\times \begin{bmatrix}
\beta_k(\epsilon) & -\omega^{1-2(k+1)/4} + O(\epsilon^{1/(m+2)}) \\
\omega^{-(k+1)/4} - m/4 + O(\epsilon^{1/(m+2)}) & \omega^{1-1-2(k+1)/4} - m/4 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\]

(5.1)

where \( \alpha_k(\epsilon), \tilde{\alpha}_k(\epsilon) \sim 0 \) as \( \epsilon \to 0^+ \), \( k = 1, \ldots, m + 1 \) and \( \beta_k(\epsilon) \sim 0 \) as \( \epsilon \to 0^+ \), \( k = 1, \ldots, m \).

**Proof.** \( C_k(\epsilon) = Z_k^{-1} P^{-1} Y_k^F \) which can be calculated using formulas (4.15), (3.10) and (2.14) for

\[
\frac{2k - 1}{m + 2} \pi < \text{arg} \, t < \frac{2k + 3}{m + 2} \pi, \quad \delta_0 \leq |t| \leq t_0;
\]

\[
C_k(\epsilon) = \omega^{-(k+1)/4} e^{-m/2(m+2)}
\]

\[
\times \begin{bmatrix}
1 + O(\epsilon^{1/(m+2)}) & \exp \left\{ (-1)^k \frac{4}{m+2} \epsilon^{(m+2)/2} - 1 \right\} O(\epsilon^{1/(m+2)}) \\
\exp \left\{ (-1)^{k+1} \frac{4}{m+2} \epsilon^{(m+2)/2} - 1 \right\} O(\epsilon^{1/(m+2)}) & 1 + O(\epsilon^{1/(m+2)})
\end{bmatrix}
\]

for \( k = 1, \ldots, m + 1 \). Take \( t = t_0 e^{\delta(k+1)\pi i/(m+2)} \), then

\[
\exp \left\{ (-1)^k \frac{4}{m+2} \epsilon^{(m+2)/2} - 1 \right\} \sim 0 \quad \text{as} \quad \epsilon \to 0^+;
\]

while, take \( t = t_0 e^{k\pi i/(m+2)} \), then

\[
\exp \left\{ (-1)^{k+1} \frac{4}{m+2} \epsilon^{(m+2)/2} - 1 \right\} \sim 0 \quad \text{as} \quad \epsilon \to 0^+.
\]
Similarly,
\[
\tilde{C}_k(\epsilon) = Z^{-1}_k P^{-1} Y_{k+1}^F, \quad k = 1, \ldots, m,
\]
can be calculated by using formulas (4.17), (3.10) and (2.14) for
\[
\frac{2k + 1}{m + 2} \pi < \arg t < \frac{2k + 5}{m + 2} \pi, \quad \delta_0 \leq |t| \leq t_0:
\]
\[
\tilde{C}_k(\epsilon) = \begin{pmatrix}
\frac{e^{-m/2(m+2)}}{(-1)^{k+1} 2 t^{m/2} \omega^{1+(k/2)+1} m + O(\epsilon^{1/(m+2)})} \\
+ \exp \left\{ (-1)^{k} \frac{4}{m + 2} t^{(m+2)/2} e^{-1} \right\} O(\epsilon^{1/(m+2)}) \\
+2(-1)^{k+1} t^{m/2} \omega^{1+(k+3)/4} m + O(\epsilon^{1/(m+2)})
\end{pmatrix}
\]
\[
\times \begin{pmatrix}
\exp \left\{ (-1)^{k} \frac{4}{m + 2} t^{(m+2)/2} e^{-1} \right\} O(\epsilon^{1/(m+2)}) \\
2(-1)^{k} t^{m/2} \omega^{1+(k+1)/4} m + O(\epsilon^{1/(m+2)}) \\
+2(-1)^{k+1} t^{m/2} \omega^{1+(k+3)/4} m + O(\epsilon^{1/(m+2)})
\end{pmatrix}
\]
\[
\exp \left\{ (-1)^{k} \frac{4}{m + 2} t^{(m+2)/2} e^{-1} \right\} O(\epsilon^{1/(m+2)}) \\
+2(-1)^{k+1} t^{m/2} \omega^{1-(k+3)/4} m + O(\epsilon^{1/(m+2)})
\]

To evaluate the first row and the lower left term of the above matrix, take \( t = t_0 e^{2(k+1)\pi i/(m+2)} \), then
\[
\exp \left\{ (-1)^{k} \frac{4}{m + 2} t^{(m+2)/2} e^{-1} \right\} \sim 0 \quad \text{as} \quad \epsilon \to 0^+.
\]
To evaluate the lower right term of the above matrix, take \( t = t_0 e^{(2k+3)\pi i/(m+2)} \), then both the absolute values of
\[
\exp \left\{ (-1)^{k} \frac{4}{m + 2} t^{(m+2)/2} e^{-1} \right\} \quad \text{and} \quad \exp \left\{ (-1)^{k+1} \frac{4}{m + 2} t^{(m+2)/2} e^{-1} \right\}
\]
are 1. Thus we have Eq. (5.2) for \( k = 1, \ldots, m \). Observe that all the points at which we evaluate the entries in \( C_k(\epsilon) \) and \( \tilde{C}_k(\epsilon) \) are contained in \( t(\Omega^k_\delta) \), for small enough \( \delta_0 \).

**Theorem 5.1 (Lateral Connections).** For \( k = 1, \ldots, m \),
\[
Y_{k+1}^F(t, \epsilon) = Y_k^F(t, \epsilon) N_k(\epsilon)
\]
where

\[ N_k(\epsilon) = \begin{bmatrix} \eta_k(\epsilon) & 1 + O(\epsilon^{1/(m+2)}) \\ 1 + O(\epsilon^{1/(m+2)}) & \frac{\omega^{-1} - \omega}{\omega^{-1} - 1} \omega^{-1} - (m/4) + O(\epsilon^{1/(m+2)}) \end{bmatrix} \]  

(5.3)

The function \( \eta_k(\epsilon) \sim 0 \) as \( \epsilon \to 0^+ \).

Proof. \( N_k(\epsilon) = C_k^{-1}(\epsilon) \tilde{C}_k(\epsilon) \) which can be computed directly using Lemma 5.1.

**Corollary 5.1.** Suppose \( D_k \cup l_{k+1} \cup D_{k+1} \) is a consistent canonical domain. Then doubly asymptotic series for \( u_{k+1}^- \) and \( u_{k+1}^+ \) as \( \epsilon \to 0^+ \) or \( x \to \infty \) in all of \( D_k \cup l_{k+1} \cup D_{k+1} \) can be obtained from the formulas:

\[ [u_{k+1}^-(x, \epsilon), u_{k+1}^+(x, \epsilon)] = [u_k^+(x, \epsilon), u_k^-(x, \epsilon)] N_k(\epsilon) \]

for \( k = 1, \ldots, m \), and \( k \) odd; while

\[ [u_{k+1}^+(x, \epsilon), u_{k+1}^-(x, \epsilon)] = [u_k^-(x, \epsilon), u_k^+(x, \epsilon)] N_k(\epsilon) \]

for \( k = 1, \ldots, m \), and \( k \) even.

6. Connecting One Full Circle Around the Turning Point

Let \( \tilde{\xi}(x) \) be defined to be the choice of \( \xi(x) \) for \( x \in D_{m+2} \cup l_1 \cup D_1 \cup l_2 \cup D_2 \) such that \( \tilde{\xi}(x) = \xi(x) \) for \( x \in D_{m+2} \) (refer to (2.9)), and \( \tilde{\xi}(x) \) is the continuation of \( \xi(x) \) to \( l_1 \cup D_1 \cup l_2 \cup D_2 \) from \( D_{m+2} \) across \( l_1 \). Let \( \Omega_{\delta_0}^{m+2} = \Omega_{\delta_0}^{m+3} \). In \( \tilde{\xi}(D_{m+2} \cup l_1 \cup D_1) \) delete circular neighborhoods of radius \( \delta_0 \) and half-strips of width \( 2\delta_0 \) containing the cuts, as before. Denote the resulting region by \( \Omega_{\delta_0}^{m+3} \), and its preimage in the \( x \)-plane by \( \Omega_{\delta_0}^{m+2} \). Theorem 2.1 apply for \( k = m + 2 \) and \( m + 3 \), with \( \tilde{\xi}(x) \) everywhere replaced by \( \tilde{\xi}(x) \), and \( p(x)^{-1/4} \) in formula (2.10) defined by (2.13) for \( x \in D_{m+2} \) and by the continuation across \( l_1 \) for \( x \in D_1 \cup l_2 \cup D_2 \). (Analogous results of Corollary 2.1, of course, apply.)

For each \( k = m + 2 \) and \( m + 3 \), put \( u_k^\pm \) and \( \epsilon(du_k^\pm/dt) \) on two rows, according to the rules stated above formula (2.14), to form two fundamental matrix solutions for (2.6), \( k = m + 2, m + 3 \):

\[
Y_k^F(t, \epsilon) = q(t) \epsilon^{-m/4} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{k+1} \epsilon^{m/2} \end{bmatrix} Y_k^F(t, \epsilon) \\
\times \text{diag} \left( \exp \left\{ \frac{(-1)^{k+1} 2}{(m + 2)} \epsilon^{(m+2)/2} \right\}, \exp \left\{ \frac{(-1)^k 2}{(m + 2)} \epsilon^{(m+2)/2} \right\} \right)
\]

(6.1)
\( x(t) \in \Omega_{\delta_0}^k, \ 0 < \epsilon < \epsilon_k(\delta_0). \) Here we use the rules:

\[
\text{arg } t^z = \alpha \text{ arg } t,
\]

\[
\left( 2 - \frac{1}{m+2} \right) \pi \leq \text{arg } t(x) \leq \left( 2 + \frac{1}{m+2} \right) \pi \quad \text{for } x \in D_{m+2} \text{ closure},
\]

\[
\left( 2 + \frac{1}{m+2} \right) \pi \leq \text{arg } t(x) \leq \left( 2 + \frac{3}{m+2} \right) \pi \quad \text{for } x \in D_1 \text{ closure},
\]

\[
\left( 2 + \frac{3}{m+2} \right) \pi \leq \text{arg } t(x) \leq \left( 2 + \frac{5}{m+2} \right) \pi \quad \text{for } x \in D_2 \text{ closure},
\]

and \( Y_k^F(t, \epsilon) \) satisfies the properties described in (2.15) and (2.16) for the corresponding \( k = m+2 \) or \( m+3 \) with \( \tilde{\xi}(x(t)) \) replaced by \( \xi(x(t)) \) and \( D_{m+3}, \text{ } D_{m+4} \) defined respectively as \( D_1, \text{ } D_2 \).

On the other hand, formulas (4.12) and (4.13) are valid even for \( k = m+1, m+2 \) where

\[
\mathcal{Y}_{m,k}(t \epsilon^{-2/(m+2)}, a(\epsilon)) = \mathcal{Y}_m(\omega^{-k} t \epsilon^{-2/(m+2)}, G^k(\epsilon)), \quad k = m+3, m+4
\]

has formulas (4.11) in corresponding closed subsector of

\[
\left| \text{arg } t - \frac{2k\pi}{m+2} \right| < \frac{3}{m+2} \pi,
\]

and roots are taken according to (2.8). (Note that

\[
\mathcal{Y}_{m,k}(t \epsilon^{-2/(m+2)}, a(\epsilon)) = \mathcal{Y}_{m,k}(t \epsilon^{-2/(m+2)}, a(\epsilon))
\]

if \( k = h \text{ mod } m+2 \). Refer to [7].) Define \( Z_{m+2}(t, \epsilon) \) by formula (4.14) with \( k \) replaced by \( m+2 \). It follows that (4.15) and (4.17) are valid respectively in subsector of

\[
\frac{2k-1}{m+2} \pi < \text{arg } t < \frac{2k+3}{m+2} \pi \quad \text{and} \quad \frac{2k+3}{m+2} \pi < \text{arg } t < \frac{2k+5}{m+2} \pi,
\]

\[
|t| \leq t_0, \quad \epsilon \to 0^+, \quad k = m+1, m+2,
\]

where roots are taken according to (2.8).

**Lemma 6.1.**

\[
Y_{m+2}^F(t, \epsilon) = (PZ_{m+2})(t, \epsilon) C_{m+2}(\epsilon), \quad |t| \leq t_0,
\]

where \( C_{m+2}(\epsilon) \) satisfies formula (5.1) with \( h \) replaced by \( m+2 \).

\[
Y_{k+1}^F(t, \epsilon) = (PZ_k)(t, \epsilon) \tilde{C}_k(\epsilon), \quad k = m+1, m+2, \quad |t| \leq t_0,
\]

where \( \tilde{C}_k(\epsilon) \) satisfies formula (5.2) for the corresponding \( k \).
Proof. \( C_{m+2}(\varepsilon) = Z_{m+2}^{-1}P^{-1}Y_{m+2}^{-} \) which can be calculated using (4.15), (3.10) and (6.1). The proof here is exactly the same as the first part of Lemma 5.1, with \( k \) replaced by \( m + 2 \).

\[
\tilde{C}_k(\varepsilon) = Z_{k}^{-1}P^{-1}Y_{k+1}^{-}, \quad k = m + 1, m + 2,
\]
can be calculated by using formulas (4.17), (3.10) and (6.1). The proof here is exactly the same as the second part of Lemma 5.1, with \( k \) replaced by \( m + 1, m + 2 \).

Lemmas 5.1 and 6.1 thus immediately give:

**Theorem 6.1.**

\[
Y_{k+1}^{-}(t, \varepsilon) = Y_{k}^{-}(t, \varepsilon) N_{k}(\varepsilon), \quad k = m + 1, m + 2
\]

where \( N_{k}(\varepsilon) \) satisfies formula (5.3) for the corresponding \( k \).

Analogous results of Corollary 5.1 can be obtained.

**Remarks.** (1) Both \( Y_{1}^{-}(t, \varepsilon) \) and \( Y_{m+2}^{-}(t, \varepsilon) \) are fundamental matrix solutions for (2.6) with first column and second column subdominant respectively in \( t(D_1) \) and \( t(D_2) \). Using Theorems 5.1 and 6.1, we have

\[
Y_{m+2}^{-}(t, \varepsilon) = Y_{1}^{-}(t, \varepsilon) N_{1}(\varepsilon) N_{2}(\varepsilon) \cdots N_{m+2}(\varepsilon).
\]

Formula (5.3) gives the roughest answer for the connection matrix \( N_{1}(\varepsilon) N_{2}(\varepsilon) \cdots N_{m+2}(\varepsilon) \) which serves to give a check on the results. The values for \( N_{1} \cdots N_{m+2} \) are \(-I + O(\varepsilon^{1/8})\) for \( m = 2 \) and \( I + O(\varepsilon^{1/8})\) for \( m = 4 \). (Here \( I \) is the identity matrix.) The leading matrices are respectively \( I \) for \( m = 2 \) and \( m = 4 \) because of the choices of \( \rho(x)^{-1/4} = q(t) t^{-m/4} \) in \( Y_{1}^{-} \) and \( Y_{m+2}^{-} \).

(2) The formula (5.3) differs from the result of Theorem 2.2 in [6]. This is due to arithmetic error in the formula for \( x_{0}(t, \varepsilon) \) above (2.27) in [6], and thus subsequent changes in multiplication should be made there. Such minor corrections in arithmetic will be published and the results in [6], after correction, has been found to match with our present result in this paper.

**References**

1. M. A. Evgrafov and M. V. Fedoryuk, Asymptotic behavior of solutions of the equation \( w''(z) - \rho(z, \lambda) w(z) = 0 \) as \( \lambda \to \infty \) in the complex \( z \)-plane, *Uspehi Mat. Nauk* 21 (1966), 3–50 (in Russian).


