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Indefinite Sturm–Liouville Problems and Half-Range Completeness*

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1. INTRODUCTION

We consider an operator of Sturm–Liouville type

$$Au = -(pu')' + qu \tag{1}$$

on an open interval $I \subseteq \mathbb{R}$, with self-adjoint boundary conditions. It is assumed that A is nonnegative as operator in $H = L^2(I; dx)$. By an indefinite eigenvalue problem for A we mean

$$Au = \lambda hu, \tag{2}$$

where the function h *changes sign* on I. Such problems occur in certain physical models, particularly in transport theory and statistical physics; see [2–9, 11, 13], and references therein. As for the standard eigenvalue problem, it is true under quite general conditions (and not difficult to prove) that there is a set of eigenfunctions $\{u_n\}$ which is complete in an appropriate Hilbert space of functions. This is known as *full-range completeness*. For the physical problems the question of interest is generally that of *half-range completeness*. To formulate this we set

$$I^+ = \{x \in I; h(x) > 0\}, \qquad I^- = \{x \in I; h(x) < 0\}. \tag{3}$$

Given $u \in H$ we set

$$u^\pm = \text{restriction of } u \text{ to } I^\pm. \tag{4}$$

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Let $\{\lambda_n\}$ be the eigenvalues for the solutions $\{u_n\}$ of problem (2). The half-range completeness questions are

Is $\{u_n^+ : \lambda_n > 0 \text{ or } \lambda_n \geq 0\}$ independent and complete in $L^2(I^+, h(x)\, dx)$? (5_+)

Is $\{u_n^- : \lambda_n < 0 \text{ or } \lambda_n \leq 0\}$ independent and complete in $L^2(I^-, -h(x)\, dx)$? (5_-)

We show that the answers are positive in quite general circumstances. When there is a single u_0 with eigenvalue $\lambda_0 = 0$, then u_0^+ must be included if and only if

$$\int_I h(x)\, u_0(x)^2\, dx \geq 0 \tag{6}$$

and similarly for u_0^- with reversal of sign in (6).

The same techniques allow for many possible extensions: equations of higher order, systems, certain partial differential and integrodifferential operators. We hope that the exposition here makes it clear how to proceed in any such case of interest. We may also consider variants of the questions (5_+), (5_-) themselves. In [5] the interval $I = \mathbb{R}$ and $h(x) = x$. For $\alpha \in \mathbb{R}$ we ask whether the restrictions to $\mathbb{R}^+$ of the functions $u_n(x) - \alpha u_n(-x)$, $\lambda_n \geq 0$, are independent and complete, and the answer is positive for $|\alpha| < 1$. The argument is an extension of that given here for $\alpha = 0$.

Our proof of half-range completeness follows a route proposed by the author in [3]. It seems worthwhile to give it in detail here, for several reasons. The question continues to be raised, but not settled, in the physical literature, e.g. [8, 9, 11]. The author considered indefinite Sturm–Liouville problems in [4] but failed to follow with care the method of [3] and reached some incorrect conclusions. Finally, the crucial technical point of the argument is the equivalence of two norms. The proof of equivalence sketched in [3] is so condensed as to be cryptic; it is given in more detail for a particular case in [5] and we establish the general case here. A different proof has since been found by Kaper *et al.* [12].

2. Statement of Results

All functions considered take real values. The regularity assumptions made here can be weakened in various ways; see the discussion at the end of Section 3. We have chosen here to minimize technicalities.

Regularity Assumptions

The functions p and q are continuous on the (open) interval I, and p is positive. The function h is continuous on $I^+ \cup I^-$. Each of I^+ and I^- is a finite union of open intervals. At a point x_0 where h changes sign, there is a constant $a > -\frac{1}{2}$ and a C^1 function g such that $g(x_0) \neq 0$ and such that in a neighborhood of x_0

$$h(x) = \operatorname{sgn}(x - x_0)\, |x - x_0|^a\, g(x). \tag{7}$$

As above, $H = L^2(I; dx)$, with inner product and norm denoted

$$(u, v) = \int_I u(x)\, v(x)\, dx, \qquad |u| = (u, u)^{1/2}. \tag{8}$$

Denoting distribution derivatives by primes, we set

$$D_1 = \{u \in H\colon u' \text{ is measurable and } p(u')^2 \in L^1(I; dx),\ qu^2 \in L^1(I; dx)\}. \tag{9}$$

When u, v belong to D_1 we define the pairing

$$(u, v)_A = \int_I p(x)\, u'(x)\, v'(x)\, dx + \int_I q(x)\, u(x)\, v(x)\, dx. \tag{10}$$

Positivity Assumption

There is a linear subspace $D \subset D_1$, containing the space $C^1_{\mathrm{C}}(I)$ of compactly supported C^1 functions, such that

$$(u, u)_A \geqslant 0 \qquad \text{if} \quad u \in D. \tag{11}$$

Moreover, there is a finite-dimensional subspace $K \subset D$ and a constant C such that

$$(u, u)_A = 0 \qquad \text{if} \quad u \in K, \tag{12}$$

$$|u|^2 \leqslant C(u, u)_A \qquad \text{if} \quad u \in D \quad \text{and} \quad u \perp K. \tag{13}$$

(Here orthogonality is with respect to the inner product in H.)

In particular, if $K = (0)$ the form (11) is positive definite on D and majorizes (8).

Remark. Whenever the usual oscillation arguments apply, K will have dimension $\leqslant 1$. Since the results go over directly to systems and to other generalizations, we shall consider higher dimensionalities as well.

On D we define a norm

$$|u|_1^2=(u,u)_A+(u,u), \qquad u\in D. \tag{14}$$

Let H_A be the completion of D with respect to this norm. In the usual way the inclusion $H_A\subset H$ extends naturally to

$$H_A\subset H\subset H'_A, \tag{15}$$

the inner product (8) extends to a pairing between H_A and its dual H'_A:

$$(u,v), \quad u\in H_A, v\in H'_A \quad \text{or} \quad u\in H'_A, v\in H_A, \tag{16}$$

and H'_A is the completion of H with respect to the norm

$$|u|_{-1}=\sup_{|v|_1=1, v\in H_A} |(u,v)|. \tag{17}$$

We realize the operator A which is formally given by (1) as an operator from H_A to H'_A:

$$(Au,v)=(u,v)_A, \qquad u,v\in H_A; \tag{18}$$

here we have extended (10) to H_A. It follows from the definitions and the positivity assumptions that

$$\begin{gathered}\|A\|\leqslant 1, \qquad A^*=A, \qquad \ker A=K,\\ \operatorname{ran} A=\{v\in H'_A\colon (u,v)=0, \forall u\in K\}.\end{gathered} \tag{19}$$

Compactness Assumption

Multiplication by $h(x)$ defines a continuous mapping from H_A to H and a compact mapping from H_A to H'_A.

Remark. When A has compact resolvent the inclusion $H\subset H'_A$ is compact, so compactness follows from continuity of the map from H_A to H.

DEFINITION. Given $\lambda\in R$ and $u\in H_A$, we say that u is an *eigenvector* with *eigenvalue* λ if

$$Au=\lambda hu. \tag{20}$$

In particular, (20) implies that $-(pu')'+qu=\lambda hu$ in the sense of distributions.

As usual, a set $\{v_n\}$ in a Hilbert space will be called a *basis* if each element of the space is the limit (in norm) of a unique series $\sum a_n v_n$, $a_n\in\mathbb{R}$.

THEOREM 1. *Under the assumptions above, there is a sequence of eigenfunctions* $\{u_n\}$ *with eigenvalues* $\{\lambda_n\}$ *which is a basis for* $L^2(I^+ \cup I^-; |h(x)|\, dx)$. *If* $\dim K = 0$, *then* $\{u_n^+; \lambda_n > 0\}$ *is a basis for* $L^2(I^+; h(x)\, dx)$ *and* $\{u_n^-; \lambda_n < 0\}$ *is a basis for* $L^2(I^-; h(x)\, dx)$.

THEOREM 2. *Under the assumptions above, suppose* $\dim K = 1$ *and* $K = \operatorname{span}\{u_0\}$. *Then* u_0^+ *must be included with* $\{u_n^+; \lambda_n > 0\}$ *to obtain a basis for* $L^2(I^+; h(x)\, dx)$ *if and only if*

$$\int_I h(x)\, u_0(x)^2\, dx \geqslant 0. \tag{21}$$

Similarly, u_0^- *must be included with* $\{u_n^-; \lambda_n < 0\}$ *if and only if the inequality* (21) *is reversed.*

To state the general result, we say that a linearly independent set $\{v_1, ..., v_N\} \subset K$ is *nonnegative* (resp. *nonpositive*) if for every $v \neq 0$ in the span of $\{v_1, ..., v_N\}$ we have $\int hv^2 \geqslant 0$ (resp. $\int hv^2 \leqslant 0$).

THEOREM 3. *Under the assumptions above,* $\{u_n^+; \lambda_n > 0\}$, *together with the restrictions of any maximal nonnegative set in K, gives a basis for* $L^2(I^+; h(x)\, dx)$. *Similarly,* $\{u_n^-; \lambda_n < 0\}$, *together with the restrictions of any maximal nonpositive set in K, gives a basis for* $L^2(I^-; -h(x)\, dx)$.

EXAMPLE 1. In the electron scattering model of Bothe [7], we seek to solve

$$\begin{gathered} x\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] = 0, \qquad t > 0,\ |x| < 1, \\ u(x, 0) = g(x), \qquad 0 < x < 1, \end{gathered} \tag{22}$$

with u bounded as $t \to +\infty$. Separating variables leads [6, 2] to

$$u(x, t) = \alpha + \beta(2x - t) + \sum_{\lambda_n > 0} a_n e^{-\lambda_n t} u_n(x), \tag{23}$$

where $\{u_n\}$, $\{\lambda_n\}$ correspond to

$$Au = -[(1-x^2)\, u']', \qquad h(x) = x, \qquad I = (-1, 1). \tag{24}$$

The Legendre operator A has compact resolvent and multiplication by h is bounded in H, so the assumptions of Theorem 2 hold, with $u_0 \equiv 1$. Equality holds in (21). Therefore the unique solution of (22) which is bounded as $t \to +\infty$ is obtained by choosing $\beta = 0$ and choosing α, a_n so that

$$\alpha + \sum_{\lambda_n > 0} a_n u_n^+ = g. \tag{25}$$

EXAMPLE 2. The one-dimensional linear Fokker–Planck equation for diffusion is

$$x\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(xu) = f, \qquad t > 0, \quad x \in \mathbb{R}; \tag{26}$$

see [13, 8, 5]. Separation of variables and a simple transformation lead to (2) with

$$Au = -u'' + (x^2 - 1)\,u, \qquad h(x) = x, \quad I = \mathbb{R}. \tag{27}$$

Here $H_A = \{u \in H;\ xu \text{ and } u' \text{ are in } H\}$. The Hermite operator A has compact resolvent and multiplication by h clearly maps H_A to H, so it follows that the assumptions of Theorem 2 hold with $u_0(x) = \exp(-\frac{1}{2}x^2)$ and with equality in (21). This leads to the solution of (26) with "purely absorbing" boundary condition [5].

3. THE POSITIVE DEFINITE CASE

With the assumptions and notation of the last section, we assume here that $K = (0)$. Therefore (10) gives a positive definite inner product on H_A with the associated norm equivalent to the norm (14). The operator A is an isomorphism from H_A onto H'_A. Denote $Tu = hu$, so $T\colon H_A \to H'_A$ is compact, and let

$$S = A^{-1}T\colon H_A \to H_A. \tag{28}$$

Thus S is compact. As observed by Hangelbroek [10] for the indefinite problem associated to the neutron transport equation, S is *self-adjoint* for the inner product (10) in H_A,

$$\begin{aligned}(Su, v)_A &= (ASu, v) = (Tu, v) = (u, Tv)\\ &= (Au, A^{-1}Tv) = (u, Sv)_A, \qquad u, v \in H_A.\end{aligned} \tag{29}$$

It follows immediately that S has a set of eigenfunctions $\{u_n\}$ with eigenvalues $\{\lambda_n^{-1}\}$ such that the u_n are complete in the orthogonal complement of the kernel of S in H_A. The equation

$$Su_n = \lambda_n^{-1} u_n \tag{30}$$

is just the equation

$$Au_n = \lambda_n h u_n. \tag{31}$$

Let $P_\pm$ be the positive and negative spectral projections of S; thus $P_\pm$ vanish on $\ker S$ and

$$P_+ u_n = 0, \quad \lambda_n < 0; \qquad P_+ u_n = u_n, \quad \lambda_n > 0; \tag{32}$$

$$P_- u_n = u_n, \quad \lambda_n < 0; \qquad P_- u_n = 0, \quad \lambda_n > 0. \tag{33}$$

Then

$$|S| = S(P_+ - P_-) = (P_+ - P_-)\, S. \tag{34}$$

On H_A we introduce an inner product and seminorm

$$(u, v)_S = (|S|\, u, v)_A, \qquad |u|_S = (u, u)_S^{1/2}. \tag{35}$$

We introduce another inner product and seminorm in H_A by

$$(u, v)_T = (|h|\, u, v), \qquad |u|_T = (u, u)_T^{1/2}. \tag{36}$$

This seminorm is equivalent to that in (35). To prove equivalence we use the following adaptation of a lemma of Baouendi–Grisvard [1].

LEMMA 1. *There are continuous linear maps X, Y: $H_A \to H_A$ such that*

$$Xu = u \qquad \text{on} \quad I^+, \tag{37}$$

$$|h|\, Xu = Y^*(hu). \tag{38}$$

A similar result holds with I^+ replaced by I^-. Here Y^* is the L^2-adjoint.

Proof. Suppose first that $I = \mathbb{R}$ and $I^\pm = \mathbb{R}^\pm$. Choose $\varphi \in C_c^1(R)$ with $\varphi(0) = 1$ and set

$$\begin{aligned} Xu(x) &= u(x), && x \geqslant 0, \\ Xu(x) &= \varphi(x)[\alpha_1 t_1 u(-t_1 x) + \alpha_2 t_2 u(-t_2 x)], && x < 0. \end{aligned} \tag{39}$$

Here t_1, t_2 are distinct positive reals and α_1, α_2 are to be chosen. The necessary and sufficient condition for X to map H_A to itself is that X not introduce a jump at $x = 0$. Thus we need

$$\alpha_1 t_1 + \alpha_2 t_2 = 1. \tag{40}$$

For (38) we must have

$$\begin{aligned} Y^*u(x) &= u(x), && x \geq 0, \\ Y^*u(x) &= \varphi(x)[\alpha_1 t_1\, g_1(x)\, u(-t_1 x) + \alpha_2 t_2\, g_2(x)\, u(-t_2 x)], && x < 0, \end{aligned} \tag{41}$$

where

$$g_j(x) = -h(x)/h(-t_j x), \qquad x < 0. \tag{42}$$

Note that (7) implies that g_j is C^1 up to $x = 0$. Then

$$\begin{aligned} Yu(x) &= 0, && x < 0, \\ Yu(x) &= u(x) + \alpha_1(\varphi g_1 u)(-x/t_1) \\ &\quad + \alpha_2(\varphi g_2 u)(-x/t_2), && x > 0. \end{aligned} \tag{43}$$

To have $Y\colon H_A \to H_A$ the necessary and sufficient condition is

$$\alpha_1\, g_1(0_-) + \alpha_2\, g_2(0_-) = -1. \tag{44}$$

Condition (7) implies that (40), (44) has a unique solution for any distinct positive t_1, t_2.

To pass to the general case, we may use a C^1 partition of unity to decompose $u \in H_A$ as a finite sum such that each summand is supported on an interval in which h changes sign at most once. Then the analogue of the preceding construction may be carried out for each piece. This completes the proof of Lemma 1.

As self-adjoint operator in H, T has spectral projections $Q_\pm$,

$$Q_+ u(x) = 0, \quad h(x) \leq 0; \qquad Q_+ u(x) = u(x), \quad h(x) > 0; \tag{45}$$

$$Q_- u(x) = u(x), \quad h(x) < 0; \qquad Q_- u(x) = 0, \quad h(x) \geq 0. \tag{46}$$

Then $|T|$ is multiplication by $|h(x)|$ and

$$|T| = T(Q_+ - Q_-) = (Q_+ - Q_-)\, T. \tag{47}$$

Let

$$H_T = L^2(I^+ \cup I^-;\, |h(x)|\, dx). \tag{48}$$

Since $C_c^1(I) \subset H_A \subset H$, it is clear that H_T can be considered as the completion of H_A with respect to the seminorm (36). Moreover, $Q_\pm$ extend to complementary orthogonal projections in H_T.

PROPOSITION 1. *The seminorms* $|\ |_S$ *and* $|\ |_T$ *are equivalent on* H_A.

Proof. Considered as an operator in the subspace $P_+(H_A)$, S is positive and has a densely defined self-adjoint inverse B. Given $u \in \operatorname{dom}(B)$, set

$$v(t) = e^{-tB}u, \qquad t \geqslant 0. \tag{49}$$

Then $v(t) \to 0$ in H_A as $t \to +\infty$, exponentially. Moreover, v is continuously differentiable, with

$$Tv'(t) = -TBv(t) = -ASBv(t) = -Av(t), \qquad t \geqslant 0. \tag{50}$$

Using Lemma 1, we have

$$\begin{aligned} |Q_+ u|_T^2 &= \int_{I^+} h(x)\, u(x)^2\, dx \leqslant (|T|\, Xu, Xu) \\ &= -\int_0^\infty \frac{d}{dt}(|T|\, Xv(t), Xv(t))\, dt = -2\int_0^\infty (|T|\, Xv', Xv)\, dt \\ &= -2\int_0^\infty (Tv', YXv)\, dt = 2\int_0^\infty (Av, YXv)\, dt \\ &\leqslant C\int_0^\infty |v|_A^2\, dt = C\int_0^\infty (e^{-2tB}u, u)_A\, dt \\ &= \tfrac{1}{2}C(B^{-1}u, u)_A = \tfrac{1}{2}C(Su, u)_A = \tfrac{1}{2}C\, |u|_S^2. \end{aligned} \tag{51}$$

Similarly, for $u \in \operatorname{dom}(B) \subset P_+(H_S)$ we have

$$|Q_- u|_T^2 \leqslant \tfrac{1}{2}C\, |u|_S^2. \tag{52}$$

Since $\operatorname{dom}(B)$ is dense, (51) and (52) hold on all of $P_+(H_A)$. Working with $\exp(tB)$ on $P_-(H_A)$ gives a similar estimate there. Thus for any $u \in H_A$,

$$\begin{aligned} |u|_T^2 = |P_+ u + P_- u|_T^2 &\leqslant 2\,|P_- u|_T^2 + 2\,|P_- u|_T^2 \\ &\leqslant C(|P_+ u|_S^2 + |P_- u|_S^2) = C\, |u|_S^2. \end{aligned} \tag{53}$$

To obtain the converse inequality we write

$$\begin{aligned} |u|_S^2 &= (Su, P_+ u - P_- u)_A = (Tu, P_+ u - P_- u) \\ &= (|T|(Q_+ - Q_-)\, u, P_+ u - P_- u) \\ &\leqslant |Q_+ u - Q_- u|_T\, |P_+ u - P_- u|_T \\ &= |u|_T\, |P_+ u - P_- u|_T \\ &\leqslant C\, |u|_T\, |P_+ u - P_- u|_S = C\, |u|_T\, |u|_S. \end{aligned} \tag{54}$$

Thus $|u|_S \leqslant C\, |u|_T$.

PROPOSITION 2. *Let H_S and H_T denote the completions of H_A with respect to $|\ |_S$ and $|\ |_T$, respectively. Then*

$$H_S = H_T = L^2(I^+ \cup I^-; |h(x)|\, dx). \tag{55}$$

The projections $P_\pm$ extend to complementary orthogonal projections in H_S with respect to the inner product (35), *while $Q_\pm$ are complementary orthogonal projections in H_T with respect to the inner product* (36). *Finally, $\{u_n, \lambda_n > 0\}$ and $\{u_n; \lambda_n < 0\}$ are orthogonal bases for $P_+(H_S)$ and $P_-(H_S)$, respectively.*

Proof. Proposition 1 implies the first equality in (55) and we have already remarked on the second equality. The remaining statements are obvious from the various definitions.

There are natural identifications

$$Q_\pm(H_T) = L^2(I^\pm; |h(x)|\, dx). \tag{56}$$

Therefore the half-range completeness questions can be phrased as follows.

$$\text{Is } Q_\pm \text{ an isomorphism as mapping from } P_\pm(H_S) \text{ to } Q_\pm(H_T)? \tag{57}$$

The two questions can be combined into the single question:

$$\text{Is } V = Q_+ P_+ + Q_- P_- \text{ an isomorphism from } H_S \text{ onto } H_T? \tag{58}$$

To show that the answer is yes, we note first the four identities for $u, v \in H_A$,

$$(Q_\pm u, P_\pm v)_S = (Q_\pm u, P_\pm v)_T; \tag{59}$$

$$(Q_\pm u, P_\mp v)_S = -(Q_\pm u, P_\mp v)_T. \tag{60}$$

For example,

$$\begin{aligned}(Q_+ u, P_- v)_S &= -(Q_+ u, SP_- v)_A \\ &= -(Q_+ u, TP_- v) = -(TQ_+ u, P_- v) \\ &= -(Q_+ u, P_- v)_T.\end{aligned}$$

Set

$$W = Q_+ P_- + Q_- P_+. \tag{61}$$

From (59), (60) we obtain the identity

$$|Vu|_T^2 = |Wu|_T^2 + |u|_S^2, \qquad u \in H_A. \tag{62}$$

In view of the equivalence of norms, this carries over to $u \in H_S$ and shows that the bounded operator V has closed range and kernel (0). The identities (59), (60), show that as mappings from H_S to H_T, V and W have adjoints

$$V' = P_+ Q_+ + P_- Q_-, \qquad W' = -P_+ Q_- - P_- Q_+. \tag{63}$$

But V' and W' satisfy (62) with S and T interchanged. Thus V' is 1–1 and we have shown that V is an isomorphism.

Remarks. The regularity assumptions can be weakened in various ways without any change in the arguments. For example, we only need

$$p > 0, \qquad p \text{ and } p^{-1} \text{ in } L^\infty_{\text{loc}}(I), \qquad q \in L^2_{\text{loc}}(I).$$

For h we need only the assumptions necessary to make Lemma 1 go through: h need not have any continuity properties except near the points where it changes sign, the number of components of I^+ and I^- may sometimes be infinite, and condition (7) may be changed in various ways. The simplest is to allow

$$h(x) = \pm |x - x_0|^a g_\pm(x), \qquad \pm(x - x_0) > 0$$

near x_0, where $g_\pm$ are C^1 and positive at x_0. Even this assumption gives solvability of the linear equations in the proof of Lemma 2 with almost any choice of $t_1, t_2 > 0$, whereas we only need one choice at each change of sign.

3. The General Case

Here we drop the assumption that the subspace K of the positivity assumption be (0). Note that the positivity assumption still implies that $A = H_A \to H'_A$ has closed range and adjoint A, while $\ker A = K$. Therefore

$$\operatorname{ran} A = K^\perp = \{v \in H'_A : (u, v) = 0, \text{ all } u \in K\}. \tag{64}$$

The strategy is to modify A by a finite rank operator to obtain an invertible operator A_1, in such a way that there is a subspace of finite codimension in which $Au = \lambda Tu$ has the same solutions as $A_1 u = \lambda Tu$.

LEMMA 3. *K is the direct sum of the subspaces K_0, K_1, K_2, where*

$$\begin{gathered} K_0 = K \cap \ker T, \qquad K_1 = \{u \in K : Tu \in K, u \in K_0^\perp\}, \\ K_2 = \{u \in K : Tu \in K^\perp, u \in K_0^\perp\}. \end{gathered} \tag{65}$$

Proof. Let P_1, P_2 denote the orthogonal projections of H onto K and $K^\perp$, respectively. Then $P_j\colon H_A \to H_A$ and has adjoint P_j, so P_jT is defined on H_A. Clearly

$$K \cap \ker(P_1 T) \cap \ker(P_2 T) = K_0,$$
$$K \cap (\ker(P_1 T) + \ker(P_2 T)) = K,$$

and it follows that K is the direct sum of K_0 and $K_j = K_0^\perp \cap K \cap \ker(P_j T)$, $j = 1, 2$, as desired.

Define

$$K_3 = \{u \in H_A \cap K^\perp\colon Au \in T(K_2)\}, \tag{66}$$

and let E_0, E_1, E_2 be the orthogonal projections in H with ranges

$$E_0(H) = K_0, \qquad E_1(H) = T(K_1), \qquad E_2(H) = T(K_3). \tag{67}$$

Set

$$A_1 = A + E_0 + E_1 + E_2\colon H_A \to H'_A. \tag{68}$$

LEMMA 4. *A_1 is an isomorphism of H_A onto H'_A. Moreover*

$$(A_1 u, u) > 0 \qquad \textit{if} \quad u \in H_A, \quad u \neq 0. \tag{69}$$

Proof. To prove (69) we note that A_1 is the sum of 4 nonnegative operators and that A is strictly positive on $K^\perp$ while E_0 is strictly positive on K_0. Therefore it is enough to show that E_j is 1–1 on K_j, $j = 1, 2$. If $0 \neq u \in K_1$, there is $v \in K$ such that $0 \neq (Tu, v) = (u, Tv)$. Now $T(K_0 + K_2) = T(K_2) \subset K^\perp$, so we may assume $v \in K_1$. Then

$$(E_1 u, Tv) = (u, Tv) = (Tu, v) \neq 0.$$

If $0 \neq u \in K_2$, then $0 \neq Tu \in K^\perp = \operatorname{ran} A$, so there is v such that $Av = Tu$; we may take $v \in K^\perp$, so $v \in K_3$. Then

$$(E_2 u, Tv) = (u, Tv) = (Tu, v) = (Av, v) > 0.$$

Thus (69) is valid and $\ker A_1 = 0$. But A is a Fredholm operator with index 0 and A_1 is a perturbation by an operator of finite rank, so A_1 is also Fredholm with index 0. Thus A_1 is an isomorphism onto.

Set

$$(u, v)_A = (A_1 u, v), \qquad |u|_A = (u, u)_A^{1/2}, \quad u, v \in H_A. \tag{70}$$

Then $|\ |_A$ is a norm equivalent to $|\ |_1$. Let

$$S = A_1^{-1}T\colon H_A \to H_A. \tag{71}$$

As before, S is self-adjoint with respect to the inner product (70) and is compact.

LEMMA 5. *The subspaces K_0, K_1, and $K_2 + K_3$ are orthogonal with respect to the inner product* (70) *and invariant for S.*

Proof. Note that K_j and K_k are orthogonal with respect to (70) if and only if $A_1(K_j) \perp K_k$ (orthogonality with respect to the L^2-pairing). Also, $T(K_j) \perp K_k$ if and only if $K_j \perp T(K_k)$. In particular, since $T(K_0) = 0$ we have $K_0 \perp T(K_1 + K_3)$, so $(E_1 + E_2)(K_0) = 0$ and

$$A_1(K_0) = E_0(K_0) = K_0 \perp (K_1 + K_2 + K_3). \tag{72}$$

Next, $K_1 \perp K_0$ and $T(K_1) \perp K_3$, $K_1 \perp T(K_2)$, so $(E_0 + E_2)(K_1) = 0$ and

$$A_1(K_1) = E_1(K_1) = TK_1 \perp (K_0 + K_2 + K_3). \tag{73}$$

Similarly,

$$A_1(K_2) = E_2(K_2) = TK_3 \perp (K_0 + K_1), \tag{74}$$

$$\begin{aligned} A_1(K_3) &= A(K_3) + E_2(K_3) \\ &= T(K_2) + E_2(K_3) \\ &\subset T(K_2) + T(K_3) \perp (K_0 + K_1). \end{aligned} \tag{75}$$

(The fact that there is equality in the second relation in (73)–(75) comes from a count of dimensions.) The relations (72)–(75) give the desired conclusions.

Set

$$\left.\begin{aligned} &H_{A,0} = K_0 + K_1 + K_2 + K_3, \\ &H_{A,1} = \text{orthogonal complement of } H_{A,0} \text{ with respect to } (\ ,\ )_A. \end{aligned}\right. \tag{76}$$

LEMMA 6. *Let $\{u_n\}$ be a complete orthogonal set of eigenvectors of S in $H_{A,1}$ with nonzero eigenvalues $\{\lambda_n^{-1}\}$. Then the $\{u_n\}$ are a basis for the orthogonal complement of the kernel of S in $H_{A,1}$. Moreover*

$$Au_n = \lambda_n Tu_n. \tag{77}$$

Proof. Since $H_{A,0}$ is invariant for the self-adjoint operator S, so is $H_{A,1}$.

Since S is compact, the completeness of the eigenfunctions in the orthogonal complement of the kernel follows. Finally, $u \in H_{A,1}$ implies

$$u \perp A_1(H_{A,0}) \supset T(K_1 + K_3) + K_0,$$

so $(E_0 + E_1 + E_2)\,u = 0$ and thus $Au = A_1 u$. In particular,

$$\lambda_n T u_n = \lambda_n A_1 S u_n = A_1 u_n = A u_n.$$

LEMMA 7. *Suppose $u \in H_A$ and $Au = \lambda T u$ with $\lambda \neq 0$. Then $u \in H_{A,1} + K_0$.*

Proof. Write $u = u_0 + u_1$ with $u_j \in H_{A,j}$ and $u_0 = \sum v_j$ with $v_j \in K_j$. Note that $Au_1 = A_1 u_1$, as above, and $Au_0 = Av_3 \in T(K_2)$. Then

$$0 = A_1^{-1}(A - \lambda T)\,u_0 + A_1^{-1}(A_1 - \lambda T)\,u_1$$

and the summands are in $H_{A,0}$ and $H_{A,1}$, respectively. Therefore each summand vanishes. Then

$$0 = (A - \lambda T)\,u_0 = Av_3 - \lambda T(v_1 + v_2 + v_3)$$

so

$$Av_3 - \lambda T v_2 = \lambda T(v_1 + v_3). \tag{78}$$

Since Av_3 is in $T(K_2)$ and T is 1–1 on $K_1 + K_2 + K_3$, both sides of (78) vanish. From the right-hand side we conclude $v_1 = v_3 = 0$, and then (78) implies $v_2 = 0$. Thus $u = v_0 + u_1$.

Let m_+ (resp. m_-) be the dimension of the positive (resp. negative) subspace of S restricted to $K_1 + K_2 + K_3$.

LEMMA 8. *The cardinality of any maximal non-negative (resp. non-positive) set in K is equal to m^+ (resp. m^-).*

Proof. The cardinality of a maximal nonnegative set is an invariant. Such a set can be obtained by taking a basis for K_2 (where $(Tu, v) \equiv 0$), together with a basis for the positive eigenspace of the self-adjoint operator on $K_1 \subset H$ which corresponds to the nondegenerate symmetric form (Tu, v) on K_1. On the other hand, if $\{v_j\}$ is an orthonormal basis for $T(K_1)$, $v_j = Tu_j$, $u_j \in K_1$, then

$$A_1 u_k = E_1 u_k = \sum (u_k, Tu_j)\, Tu_j,$$

so

$$u_k = \sum (u_k, Tu_j)\, Su_j.$$

In this basis for K_1, $\{(u_j, Tu_k)\}$ is the matrix of S^{-1}, so the dimension of the positive subspace of S on K_1 is the dimension of the positive subspace for the operator corresponding to the form. It remains to show that the dimension of the positive subspace for S in K_2+K_3 is $\dim(K_2)=\frac{1}{2}\dim(K_2+K_3)$. To prove this we write the orthogonal decomposition with respect to the inner product (70) as

$$K_2+K_3=K_2\oplus K_3'.$$

It follows from (74), (75) that the restriction S_1 of S to K_2+K_3 has the form

$$\begin{aligned} S_1 &= S_{23}+S_{32}+S_{33}, \\ S_{23}&: K_3'\to K_2,\ S_{32}=S_{23}^*, \\ S_{33}&: K_3'\to K_3',\ S_{33}=S_{33}^*. \end{aligned} \tag{79}$$

Set

$$S_t=S_{23}+S_{32}+tS_{33}, \qquad 0\leqslant t\leqslant 1. \tag{80}$$

This is a homotopy through invertible self-adjoint operators, so the dimension of the positive subspace is unchanged. Now

$$S_0(K_2)=K_3', \qquad S_0(K_3')=K_2.$$

Therefore S_0^2 and $|S_0|$ preserve K_2 and K_3'. The operator R with

$$R=S_0\,|S_0|^{-1} \quad \text{on } K_2, \qquad R=-S_0\,|S_0|^{-1} \quad \text{on } K_3' \tag{81}$$

is unitary and $RS_0=-S_0R$. Therefore R interchanges the positive and negative subspaces for S_0. Consequently these subspaces have dimension $\frac{1}{2}\dim(K_1+K_2)$, and the proof is complete for m_+; obviously the argument for m_- is the same.

As in the last section we introduce the norms $|\ |_S$ and $|\ |_T$ in H_A and have for the completions

$$H_S=H_T=L^2(I^+\cup I^-;\,|h(x)|\,dx).$$

Let $H_{S,j}$ and $H_{T,j}$ be the completions of $H_{A,j}$, $j=1, 2$. Then it is clear that

$$H_{S,0}=H_{T,0}=K_1+K_2+K_3, \tag{82}$$

$$H_{S,1}=H_{T,1}, \tag{83}$$

and $H_S=H_T$ is the direct sum of these two subspaces. The last result we need for the proof of Theorems 1–3 is

LEMMA 9. *Suppose* $0 \neq u \in H_{S,0}$ *and suppose* $(Tu, u) \geqslant 0$ (*resp.* $(Tu, u) \leqslant 0$). *Then* $Q_+ u$ (*resp.* $Q_- u$) *does not belong to* $Q_+ P_+(H_{S,1})$ (*resp.* $Q_- P_-(H_{S,1})$).

Proof. Suppose u as above and $(Tu, u) \geqslant 0$. Suppose $v \in P_+(H_{S,1}) \subset H_{S,1}$. Then for $w = u - v$ we have

$$\begin{aligned} 2\,|Q_+ w|_T^2 &= (|T|(Q_+ + Q_- + Q_+ - Q_-)\, w, w) \\ &= |w|_T^2 + (Tw, w). \end{aligned} \tag{84}$$

Now

$$0 = (u, v)_S = (u, |S|\, v)_A = (u, Sv)_A = (u, Tv) = (Tu, v), \tag{85}$$

so

$$\begin{aligned} (Tw, w) &= (Tv, v) + (Tu, u) \\ &\geqslant (Tv, v) = (Sv, v)_A = |v|_S^2. \end{aligned} \tag{86}$$

In both (85) and (86) we have used the assumption $v \in P_+(H_{S,1})$. From (84) and (86) we obtain

$$2\,|Q_+(u - v)|_T^2 \geqslant |u - v|_S^2 \geqslant |u|_S^2 > 0. \tag{87}$$

The proof of the other assertion is similar.

Proof of Theorems 1–3. As in the last section, the operator $V = Q_+ P_+ + Q_- P_-$ is an isomorphism from H_S onto $H_T = H_S$. Therefore the positive eigenfunctions for S are taken by Q_+ to a basis for $Q_+(H_T)$. It follows that the $\{Q_+ u_n; \lambda_n > 0\}$, where u_n are again the eigenfunctions in $H_{S,1}$, are a basis for $Q_+ P_+(H_{S,1})$. These functions are exactly the eigenfunctions for $Au = \lambda Tu$ with $\lambda > 0$, $Au \neq 0$. The codimension of $Q_+ P_+(H_{S,1})$ in $Q_+ P_+(H_S)$ is the dimension of the positive subspace of S restricted to $H_{S,0}$, which is the cardinality of a maximal nonnegative set in K. From Lemma 9, if $\{v_1,..., v_N\}$ is such a set, then $\{Q_+ v_j\}$ is independent of $Q_+ P_+(H_{S,1})$. Therefore $\{Q_+ v_j, Q_+ u_n, \lambda_n > 0\}$ are a basis for $Q_+(H_T)$. The same argument applies to $Q_-(H_T)$.

Note added in proof. Hangelbroek [10] first introduced the operator V of (58) in connection with the neutron transport problem and proved its invertibility in that case by a different method. Van der Mee [14] considered similar problems for some bounded semidefinite A and independently obtained the analogue of Lemma 9. Half-range completeness for Example 1, which is conjectured in [11], was already proved (implicitly) in [2].

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