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On the Multivariate Normal Hazard

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It is well known that the hazard rate of a univariate normal distribution is increasing. In this paper, we prove that the hazard gradient, in the case of general multivariate normal distribution, is increasing in the sense of Johnson and Kotz.

1. INTRODUCTION

It is well known that the hazard rate of a univariate normal distribution is increasing. Johnson and Kotz (1975) presented a vector definition of multivariate hazard rates and associated definitions of increasing and decreasing multivariate distributions. They showed that in a bivariate normal case, the bivariate hazard is increasing provided that the correlation coefficient is positive.

In this paper, we first show that in the bivariate normal case, the hazard rate is increasing without the condition on the correlation coefficient imposed by Johnson and Kotz. The result is then extended to the trivariate normal case and finally to the general m dimensional multivariate case. Before proceeding further, we present some definitions and results which are used in establishing the results.

1. Definition 1. The joint multivariate hazard rate of m jointly absolutely continuous random variables $X_1, X_2, ..., X_m$ is defined as the vector

$$h(\mathbf{x}) = \left(-\left(\frac{\partial}{\partial x_1}\right) \cdots -\left(\frac{\partial}{\partial x_m}\right)\right) \ln G(\mathbf{x}) = -\text{grad} \ln G(\mathbf{x})$$

where $G(\mathbf{x}) = P(X_i > x_i,\ i = 1, 2, ..., m)$ is the joint survival function. For convenience, we shall write $-(\partial/\partial x_j) \ln G(x) = h_j(\mathbf{x})$. Note that $h_j(\mathbf{x})$ is the

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jth component of the hazard gradient and is a function of $x_1, x_2, ..., x_m$. Also note that the vector-valued function h uniquely determines the probability distribution; see Marshall (1975).

2. DEFINITION 2. If for all values of $\mathbf{x}$, all components of $h(\mathbf{x})$ are increasing (decreasing) functions of the corresponding variable, i.e., $h_j(\mathbf{x})$ is an increasing (decreasing) function of x_j for $j = 1, 2, ..., m$, then the distribution is called a multivariate IHR (DHR).

3. Noting that

$$G(\mathbf{x}) = P(X_1 > x_1 | X_2 > x_2, ..., X_m > x_m)\, P(X_2 > x_2, ..., X_m > x_m),$$

this gives

$$h_1(\mathbf{x}) = -\frac{\partial}{\partial x_1} \ln P(X_1 > x_1 \mid X_2 > x_2, ..., X_m > x_m),$$

i.e., $h_1(\mathbf{x})$ is the hazard rate of the conditional distribution of X_1 given $X_2 > x_2, ..., X_m > x_m$.

4. The corresponding conditional pdf of X_1 is given by

$$\begin{aligned} f_1(\mathbf{x}) &= h_1(\mathbf{x})\, P(X_1 > x_1 \mid X_2 > x_2, ..., X_m > x_m) \\ &= -\frac{\partial}{\partial x_1} P(X_1 > x_1 \mid X_2 > x_2, ..., X_m > x_m) \\ &= \frac{-\frac{\partial}{\partial x_1} P(X_1 > x_1, X_2 > x_2, ..., X_m > x_m)}{P(X_2 > x_2, ..., X_m > x_m)}. \end{aligned}$$

Likewise the hazard rate and the conditional pdf of $X_2, ..., X_m$ can be obtained.

5. We now present the following result due to Glaser (1980) which will be used to establish the results.

LEMMA. *Let X be a positive random variable with pdf $f(x)$. Assume that $f(x)$ is continuous and twice differentiable on its support. Define a function $\eta(t) = -f'(t)/f(t)$. Then the hazard rate of X is increasing if $\eta'(t) > 0$ for all t.*

Remark. The above result is true even if the suport of X is the whole real line.

6. The following result will be needed: Let $\psi(u_1, u_2, ..., u_m)$ be an integrable function and let $a_1(x), ..., a_m(x)$ be differentiable, then

$$\frac{d}{dx}\int_{a_m(x)}\cdots\int_{a_1(x)} \psi(u_1, u_2, ..., u_m)\, du_1\, du_2 \cdots du_m$$

$$= -\sum_{j=1}^{m} a_j'(x)\int_{a_m(x)}\cdots\int_{a_{j+1}(x)}\int_{a_{j-1}(x)}\cdots\int_{a_1(x)} \psi(u_1, u_2, ..., u_{j-1}, a_j(x), u_{j+1}, ..., u_m)\, du^j,$$

where $du^j = du_1 \cdots du_{j-1}\, du_{j+1} \cdots du_m$.

7. The following result on the conditional distribution will be needed. Let X be $N_m(\mu, \Sigma)$. Partition X, μ, and Σ as below:

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

where X_1 and μ_1 are $k \times 1$ and Σ_{11} is $k \times k$. Let Σ_{22}^{-} be a generalized inverse of Σ_{22}, i.e., a matrix satisfying

$$\Sigma_{22}\Sigma_{22}^{-}\Sigma_{22} = \Sigma_{22}.$$

Then the conditional distribution of X_2 given X_1 is $N_{m-k}(\mu_2 + \Sigma_{21}\Sigma_{11}^{-}(X_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-}\Sigma_{12})$; see Muirhead (1982). In our case Σ is nonsingular and hence the generalized inverse is the same as the regular inverse.

2. TWO DIMENSIONAL CASE

Suppose X_1, X_2 have a joint standard bivariate normal distribution with correlation coefficient ρ (this can be assumed without loss of generality).

It can be verified that in this case

$$h_1(\mathbf{x}) = \frac{\left\{1 - \Phi\left(\frac{x_2 - \rho x_1}{\sqrt{1-\rho^2}}\right)\right\}\phi(x_1)}{\bar{F}_\rho(x_1, x_2)}, \tag{2.1}$$

where $\bar{F}_\rho(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$, $\phi(x)$ and $\Phi(x)$ are the pdf and the cumulative distribution function of a standard normal, respectively.

The corresponding pdf of X_1 is given by

$$f_1(\mathbf{x}) = \frac{\left[1 - \Phi\left(\frac{x_2 - \rho x_1}{\sqrt{1-\rho^2}}\right)\right]\phi(x_1)}{P(X_2 > x_2)}.$$

This gives

$$\eta(x_1) = -\frac{\partial}{\partial x_1} \ln f_1(\mathbf{x})$$

$$= -\left[-x_1 + \frac{\rho}{\sqrt{1-\rho^2}} \frac{\phi((x_2-\rho x_1)/\sqrt{1-\rho^2})}{1-\Phi((x_2-\rho x_1)/\sqrt{1-\rho^2})} \right]$$

$$= -\left[-x_1 + \frac{\rho}{\sqrt{1-\rho^2}} r\left(\frac{x_2-\rho x_1}{\sqrt{1-\rho^2}}\right)\right],$$

where $r(t)$ is the hazard rate of a standard normal at the point t.

This gives

$$\eta'(x_1) = 1 + \frac{\rho^2}{(1-\rho^2)} r'\left(\frac{x_2-\rho x_1}{\sqrt{1-\rho^2}}\right) > 0,$$

since the hazard rate of a univariate normal distribution is increasing at every point. Thus $h_1(\mathbf{x})$ is an increasing function of x_1, and hence the hazard gradient of a bivariate normal distribution is increasing for all values of ρ.

3. THREE DIMENSIONAL CASE

Suppose $X = (X_1, X_2, X_3)$ has a trivariate standard normal distribution with correlation matrix R given by

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

Then

$$f_1(\mathbf{x}) = \frac{-\frac{\partial}{\partial x_1} P(X_1 > x_1, X_2 > x_2, X_3 > x_3)}{P(X_2 > x_2, X_3 > x_3)}. \tag{3.1}$$

The numerator of (3.1) is

$$\frac{1}{(2\pi)^{3/2}\,|R|^{1/2}} \int_{x_3}^{\infty} \int_{x_2}^{\infty} I_2 \, dz_2 \, dz_3, \tag{3.2}$$

where

$$I_2 = \exp\left\{-\frac{1}{2\,|R|}\left(\begin{matrix} C_{11}x_1^2 + C_{22}z_2^2 + C_{33}z_3^2 + 2C_{23}z_2z_3 \\ +2C_{12}x_1z_2 + 2C_{13}x_1z_3 \end{matrix}\right)\right\},$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

is the matrix of the co-factors of R. It can be seen that (3.2) is of the form

$$\frac{1}{(2\pi)^{3/2}\,|R|^{1/2}}$$

$$\times \int_{x_3}^{\infty}\int_{x_2}^{\infty} \exp\left\{-\frac{1}{2\,|R|}\left(\begin{matrix} |R|\,x_1^2 + (z_2 - \xi_2(x_1))^2\,C_{22} + (z_3 - \xi_3(x_1))^2\,C_{33} \\ +2C_{23}(z_2 - \xi_2(x_1))(z_3 - \xi_3(x_1)) \end{matrix}\right)\right\} dz_2\,dz_3$$

$$= \phi(x_1)\int_{x_3}^{\infty}\int_{x_2}^{\infty} \frac{1}{2\pi\,|R|^{1/2}}$$

$$\times \exp\left\{-\frac{1}{2\,|R|}\left(\begin{matrix} C_{22}(z_2 - \xi_2(x_1))^2 + C_{33}(z_3 - \xi_3(x_1))^2 \\ +2C_{23}(z_2 - \xi_2(x_1))(z_3 - \xi_3(x_1)) \end{matrix}\right)\right\} dz_2\,dz_3. \qquad (3.3)$$

Note that the integrand in the above expression is the conditional pdf of $\binom{X_2}{X_3}$ given $X_1 = x_1$ with mean $\binom{\xi_2(x_1)}{\xi_3(x_1)} = \binom{\rho_{12}x_1}{\rho_{13}x_1}$ and covariance matrix $= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, where Σ_{11}, Σ_{12}, Σ_{22}, and Σ_{21} are the partitioning matrices of R as shown below:

$$R = \left[\begin{array}{c|cc} 1 & \rho_{12} & \rho_{13} \\ \hline \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{array}\right] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

By the transformation

$$\mathbf{v} = \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \dfrac{z_2 - \xi_2(x_1)}{\sqrt{1-\rho_{12}^2}} \\ \dfrac{z_3 - \xi_3(x_1)}{\sqrt{1-\rho_{13}^2}} \end{bmatrix} = D_2 \begin{bmatrix} z_2 - \xi_2(x_1) \\ z_3 - \xi_3(x_1) \end{bmatrix},$$

where $D_2 = \text{diag}(1/\sqrt{1-\rho_{12}^2},\ 1/\sqrt{1-\rho_{13}^2})$, (3.3) will reduce to

$$\phi(x_1) \int_{a_3(x_1)}^{\infty} \int_{a_2(x_1)}^{\infty} \frac{1}{2\pi\, |K_2|^{1/2}}\, e^{-(1/2)\, y' K_2^{-1} y}\, dy = \phi(x_1)\, \bar{F}_{K_2}(a_2(x_1), a_3(x_1)), \quad (3.4)$$

where $\bar{F}_{K_2}(a_2(x_1), a_3(x_1))$ is the survival function of a standard bivariate normal with correlation matrix K_2 at the point

$$(a_2(x_1), a_3(x_1)), \qquad K_2 = \begin{bmatrix} 1 & k_{12} \\ k_{12} & 1 \end{bmatrix} = D_2(\Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12})\, D_2',$$

$$a_2(x_1) = \frac{x_2 - \xi_2(x_1)}{\sqrt{1-\rho_{12}^2}}, \qquad a_3(x_1) = \frac{x_3 - \xi_3(x_1)}{\sqrt{1-\rho_{13}^2}}.$$

Now

$$\begin{aligned} \eta(x_1) &= -\frac{\partial}{\partial x_1} \ln f_1(\mathbf{x}) \\ &= -\frac{\partial}{\partial x_1} \ln[\phi(x_1)\, \bar{F}_{K_2}(a_2(x_1), a_3(x_1))] \\ &= -\frac{\partial}{\partial x_1} [\ln \phi(x_1) + \ln \bar{F}_{K_2}(a_2(x_1), a_3(x_1))] \\ &= \left[x_1 - \frac{\partial}{\partial x_1} \ln \bar{F}_{K_2}(u_1, u_2) \right], \end{aligned}$$

where $u_1 = a_2(x_1)$, $u_2 = a_3(x_1)$,

$$\begin{aligned} &= x_1 - \frac{\partial}{\partial u_1} \ln \bar{F}_{K_2}(u_1, u_2)\, \frac{\partial u_1}{\partial x_1} - \frac{\partial}{\partial u_2} \ln \bar{F}_{K_2}(u_1, u_2)\, \frac{\partial u_2}{\partial x_1} \\ &= x_1 + h_1(u_1, u_2)\, \frac{\partial u_1}{\partial x_1} + h_2(u_1, u_2)\, \frac{\partial u_2}{\partial x_1}, \end{aligned}$$

where $h_i(u_1, u_2)$, $i = 1, 2$, is the ith component of the hazard gradient at (u_1, u_2) given by

$$h_1(u_1, u_2) = \frac{\phi(u_1) \displaystyle\int_{u_2}^{\infty} \frac{1}{\sqrt{2\pi}\, |K_2|^{1/2}} \exp\left\{ -\frac{1}{2} \left(\frac{w - k_{12}(u_1)}{\sqrt{1-k_{12}^2}} \right)^2 \right\} dw}{\bar{F}_{K_2}(u_1, u_2)} \quad (3.5)$$

and

$$h_2(u_1, u_2) = \frac{\phi(u_2) \int_{u_1}^{\infty} \frac{1}{\sqrt{2\pi}\,|K_2|^{1/2}} \exp\left\{-\frac{1}{2}\left(\frac{w - k_{12}(u_2)}{\sqrt{1-k_{12}^2}}\right)^2\right\} dw}{\bar{F}_{K_2}(u_1, u_2)}. \tag{3.6}$$

Hence

$$\begin{aligned}
\eta'(x_1) &= \frac{\partial}{\partial x_1}\eta(x_1) \\
&= 1 + \left[\frac{\partial}{\partial u_1} h_1(u_1, u_2)\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial u_2} h_1(u_1, u_2)\frac{\partial u_2}{\partial x_1}\right]\frac{\partial u_1}{\partial x_1} \\
&\quad + \left[\frac{\partial}{\partial u_1} h_2(u_1, u_2)\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial u_2} h_2(u_1, u_2)\frac{\partial u_2}{\partial x_1}\right]\frac{\partial u_2}{\partial x_1}.
\end{aligned} \tag{3.7}$$

Note that

$$\frac{\partial}{\partial u_2} h_1(u_1,u_2)\frac{\partial u_2}{\partial x_1} = \frac{\partial}{\partial u_1} h_1(u_1, u_2)\frac{\partial u_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} = \frac{\partial}{\partial u_1} h_1(u_1, u_2)\frac{\partial u_1}{\partial x_1}.$$

Similarly,

$$\frac{\partial}{\partial u_1} h_2(u_1, u_2)\frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial u_2} h_2(u_1, u_2)\frac{\partial u_2}{\partial x_1}.$$

Hence (3.7) becomes

$$\begin{aligned}
\eta'(x_1) &= 1 + 2\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 \frac{\partial}{\partial u_1} h_1(u_1, u_2) + \left(\frac{\partial u_2}{\partial x_1}\right)^2 \frac{\partial}{\partial u_2} h_2(u_1, u_2)\right] \\
&= 1 + 2\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 \left(\frac{\partial u_2}{\partial x_1}\right)^2\right]\begin{bmatrix} \frac{\partial}{\partial u_1} h_1(u_1, u_2) \\ \frac{\partial}{\partial u_2} h_2(u_1, u_2) \end{bmatrix} > 0,
\end{aligned}$$

since the hazard gradient for dimension 2 has shown to be increasing.

4. GENERAL CASE

Let us assume that the result is true for dimension $(m-1)$.

Suppose $X=(X_1, X_2, ..., X_m)'$ have a multivariate standard normal density with correlation matrix R given as follows:

$$R=[\rho_{ij}]_{m\times m}, \qquad \rho_{ij}=1 \qquad \text{if} \quad i=j.$$

Then, as before,

$$f_1(\mathbf{x})=\frac{-\dfrac{\partial}{\partial x_1} P(X_1>x_1, X_2>x_2, ..., X_m>x_m)}{P(X_2>x_2, X_3>x_3, ..., X_m>x_m)}, \tag{4.1}$$

where

$$\begin{aligned}
&-\frac{\partial}{\partial x_1} P(X_1>x_1, X_2>x_2, ..., X_m>x_m)\\
&\quad=\phi(x_1)\int_{a_m(x_1)}^{\infty}\int_{a_{m-1}(x_1)}\cdots\int_{a_2(x_1)}^{\infty}\frac{1}{(2\pi)^{(m-1)/2}\,|K_{m-1}|^{1/2}}\,e^{-(1/2)\mathbf{v}^1K_{m-1}^{-1}\mathbf{v}}\,d\mathbf{v}\\
&\quad=\phi(x_1)\,\bar{F}_{K_{m-1}}(a_2(x_1), a_3(x_1), ..., a_m(x_1)),
\end{aligned} \tag{4.2}$$

where $\bar{F}_{K_{m-1}}(a_2(x_1), a_3(x_1), ..., a_m(x_1))$ is the survival function of a standard $(m-1)$ normal variate with correlation matrix

$$K_{m-1}=D_{m-1}(\Sigma_{m-1,\,m-1}-\Sigma_{m-1,1}\Sigma_{11}^{-1}\Sigma_{1,\,m-1})\,D'_{m-1},$$

$$D_{m-1}=\operatorname{diag}\left(\frac{1}{\sqrt{1-\rho_{12}^2}}, \frac{1}{\sqrt{1-\rho_{13}^2}}\cdots\frac{1}{\sqrt{1-\rho_{1m}^2}}\right).$$

Also, $\Sigma_{m-1,\,m-1}$, $\Sigma_{m-1,1}$, Σ_{11}, and Σ_{1m-1} are the partitions of R defined as below:

$$R=\left[\begin{array}{c|cccc}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1m}\\
\hline
\rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2m}\\
\rho_{13} & \rho_{23} & 1 & \cdots & \rho_{3m}\\
\vdots & \vdots & \vdots & & \vdots\\
\rho_{1m} & \rho_{2m} & \rho_{3m} & \cdots & 1
\end{array}\right]=\begin{bmatrix}\Sigma_{11} & \Sigma_{1,\,m-1}\\ \Sigma_{m-1,1} & \Sigma_{m-1,\,m-1}\end{bmatrix},$$

and

$$a_i(x_1)=\frac{x_i-\rho_{1i}x_1}{\sqrt{1-\rho_{1i}^2}}, \qquad i=2, 3, ..., m.$$

It may be verified that K_{m-1} is the matrix of partial correlations with $X_1 = x_1$.

Now

$$\eta(x_1) = -\frac{\partial}{\partial x_1} \ln f_1(\mathbf{x})$$

$$= -\frac{\partial}{\partial x_1} \ln[\phi(x_1)\, \bar{F}_{K_{m-1}}(a_2(x_1), a_3(x_1), ..., a_m(x_1))]$$

$$= \frac{\partial}{\partial x_1} [\ln \phi(x_1) + \ln \bar{F}_{K_{m-1}}(a_2(x_1), a_3(x_1), ..., a_m(x_1))]$$

$$= x_1 - \sum_{i=1}^{m-1} \frac{\partial}{\partial u_i} \ln \bar{F}_{K_{m-1}}(u_1, u_2, ..., u_{m-1}) \frac{\partial u_i}{\partial x_1}, \tag{4.3}$$

where $u_i = a_{i+1}(x_1)$ $i = 1, 2, ..., m-1$,

$$= x_1 + \sum_{i=1}^{m-1} h_i(u_1, u_2, ..., u_{m-1}) \frac{\partial u_i}{\partial x_1};$$

note that $h_i(u_1, u_2, ..., u_{m-1})$, $i = 1, 2, ..., m-1$, is the ith component of the hazard gradient at $(u_1, u_2, ..., u_{m-1})$ given by

$$h_i(u_1, u_2, ..., u_{m-1}) = \frac{\phi(u_i) \int_{u_{m-1}}^{\infty} \int_{u_{m-2}}^{\infty} \cdots \int_{u_{i+1}}^{\infty} \int_{u_{i-1}}^{\infty} \cdots \int_{u_1}^{\infty} g_{m-2}(\mathbf{w})\, dw^i}{\bar{F}_{K_{m-1}}(u_1, u_2, ..., u_{m-1})}, \tag{4.4}$$

with $dw^i = (dw_1, dw_2, ..., dw_{i-1}, dw_{i+1}, ..., dw_{m-1})$ and $g_{m-2}(\mathbf{w})$ the $(m-2)$ variate normal density. It may be observed that (4.4) is the analog of (3.5) and (3.6) in three dimension. Hence

$$\eta'(x_1) = \frac{\partial}{\partial x_1} \eta(x_1) = 1 + \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \left[\frac{\partial}{\partial u_j} h_i(u_1, u_2, ..., u_{m-1}) \frac{\partial u_j}{\partial x_1} \right] \frac{\partial u_i}{\partial x_1}.$$

Therefore,

$$\eta'(x_1) = 1 + \sum_{i=1}^{m-1} \left[\left(\frac{\partial u_i}{\partial x_1} \right)^2 \frac{\partial}{\partial u_i} h_i(u_1, u_2, ..., u_{m-1}) + \sum_{j \neq i, j=1}^{m-1} \frac{\partial}{\partial u_j} h_i(u_1, u_2, ..., u_{m-1}) \frac{\partial u_j}{\partial x_1} \frac{\partial u_i}{\partial x_1} \right]. \tag{4.5}$$

Note that

$$\frac{\partial}{\partial u_j} h_i(u_1, u_2, ..., u_{m-1}) \frac{\partial u_j}{\partial x_1} = \frac{\partial}{\partial u_i} h_i(u_1, u_2, ..., u_{m-1}) \frac{\partial u_i}{\partial u_j} \frac{\partial u_j}{\partial x_1}$$

$$= \frac{\partial}{\partial u_i} h_i(u_1, u_2, ..., u_{m-1}) \frac{\partial u_i}{\partial x_1},$$

$j \neq i$ and $j = 1, 2, ..., m-1$.

Hence (4.5) becomes

$$\eta'(x_1) = 1 + (m-1) \sum_{i=1}^{m-1} \left(\frac{\partial u_i}{\partial x_1}\right)^2 \frac{\partial}{\partial u_i} h_i(u_1, u_2, ..., u_{m-1})$$

$$= 1 + (m-1) \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 \left(\frac{\partial u_2}{\partial x_1}\right)^2 \cdots \left(\frac{\partial u_{m-1}}{\partial x_1}\right)^2 \right]$$

$$\times \begin{bmatrix} \frac{\partial}{\partial u_1} h_1(u_1, u_2, ..., u_{m-1}) \\ \frac{\partial}{\partial u_2} h_2(u_1, u_2, ..., u_{m-1}) \\ \vdots \\ \frac{\partial}{\partial u_{m-1}} h_{m-1}(u_1, u_2, ..., u_{m-1}) \end{bmatrix} > 0,$$

since the hazard gradient for dimension $m-1$ is assumed to be increasing. Thus the general result has been established by induction.

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