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Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds

Michael T. Anderson[1]

Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794-3651, USA

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Abstract

This paper studies several aspects of asymptotically hyperbolic (AH) Einstein metrics, mostly on 4-manifolds. We prove boundary regularity (at infinity) for such metrics and establish uniqueness under natural conditions on the boundary data. By examination of explicit black hole metrics, it is shown that neither uniqueness nor finiteness holds in general for AH Einstein metrics with a prescribed conformal infinity. We then describe natural conditions which are sufficient to ensure finiteness.

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0. Introduction

In this paper, we study several aspects of asymptotically hyperbolic (AH) Einstein metrics on an open 4-manifold M with compact boundary ∂M. These metrics are complete Einstein metrics g on M, normalized so that

$$Ric_g = -3g, \tag{0.1}$$

E-mail address: anderson@math.sunysb.edu.
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which are conformally compact in the sense of Penrose, in that there exists a defining function ρ for ∂M in M, such that the conformally equivalent metric

$$\bar{g} = \rho^2 g \tag{0.2}$$

extends to a Riemannian metric on $\bar{M}$. Recall that a defining function ρ for ∂M is essentially just a coordinate function for ∂M in M; thus ρ is a smooth, typically C^∞, function on $\bar{M} = M \cup \partial M$ such that $\rho > 0$ on M, $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M.

Defining functions are unique only up to multiplication by positive functions on $\bar{M}$. Hence only the conformal class $[\bar{g}]$ is uniquely determined by g, as is the conformal class $[\gamma]$ of the induced metric $\gamma = \bar{g}_{\partial M}$ on ∂M. The class $[\gamma]$ is called the *conformal infinity* of g and a choice $\gamma \in [\gamma]$ will be called a *boundary metric*. The metric g is $L^{k,p}$ or $C^{m,\alpha}$ conformally compact if there exists a defining function such that the metric $\bar{g}$ in (0.2) has a $L^{k,p}$ or $C^{m,\alpha}$ extension to $\bar{M}$; here $L^{k,p}$ is the Sobolev space of k weak derivatives in L^p and $C^{m,\alpha}$ is the usual Hölder space. It is easy to see, cf. Section 1, that a conformally compact Einstein metric has curvature decaying to -1 at an exponential rate, so that such manifolds are AH.

Regarding the existence of such metrics, Graham and Lee [19] have proved that any metric γ near the standard metric γ_o on S^{n-1} in a sufficiently smooth topology may be filled in with an AH Einstein metric g on the n-ball B^n having prescribed boundary metric γ. Further, such AH Einstein metrics have a conformal compactification with a certain degree of smoothness. More precisely, they prove that there is an open neighborhood $\mathcal{U}_{\gamma_o}$ of γ_o in the space of $C^{m,\alpha}$ metrics $\mathcal{M}^{m,\alpha}(S^{n-1})$ on S^{n-1}, for any $m \geqslant 2$, such that any metric $\gamma \in \mathcal{U}_{\gamma_o}$ is the boundary metric of an AH Einstein metric g on the n-ball B^n, i.e. $\bar{g}|_{\partial M} = \gamma$. Further, for $m \leqslant n-1$, the metric g is $C^{n-2,\alpha}$ conformally compact for $n > 4$ and $C^{1,\alpha}$ compact, for $n = 4$.

Recently, Biquard [7] has extended this result to boundary metrics γ in an open neighborhood $\mathcal{U}_{\gamma_o} \subset \mathcal{M}^{m,\alpha}(\partial M)$ of the boundary metric γ_o of an arbitrary non-degenerate AH Einstein manifold (M, g). Here non-degenerate means that there are no non-trivial L^2 infinitesimal AH Einstein deformations of (M, g). Biquard's method can be shown to give a C^2 conformal compactification for $n \geqslant 4$, (by choosing $\delta = 2$ in the notation of [7]).

The first purpose of this paper is to study the boundary regularity of conformal compactifications $\bar{g}$ of AHE metrics g, in dimension 4. Namely, given an AH Einstein metric (M, g) which, in some compactification $\bar{g}$ as in (0.2), has a $C^{m,\alpha}$ boundary metric γ, is there a $C^{m,\alpha}$ conformal compactification of g? This issue of boundary regularity was first raised in [16], and has been an open problem for some time. In fact, when $n = \dim M$ is odd, it was discovered in [16] that boundary regularity in general breaks down at the order $m = n-1$, in that there are log terms in the asymptotic expansion of $\bar{g}$ near ∂M at this order, cf. (0.4). These log terms play an important role in the AdS/CFT correspondence relating string theory and conformal field theory, cf. [25,38].

When $n = \dim M$ is even, there are no log terms in the expansion, and it is possible that a C^∞ boundary metric has a C^∞ compactification. We resolve this issue in dimension 4.

Theorem 0.1. *Let M be a 4-manifold and g an AH Einstein metric on M. Suppose g has an $L^{2,p}$ conformal compactification, for some $p > 4$, in which the boundary metric γ is $C^{m,\alpha}$, $m \geqslant 3$, $\alpha > 0$. Then (M, g) has a $C^{m,\alpha}$ conformal compactification with the same boundary metric. This result holds also if $m = \infty$ or $m = \omega$.*

The reason for considering the (weak) condition of $L^{2,p}$ compactification is that the result of Biquard above naturally gives the existence of AH Einstein metrics with such compactifications. Theorem 0.1 is proved in Section 2, cf. Theorem 2.4 and Corollary 2.5.

A second purpose of the paper is to study the uniqueness question, i.e. to what extent an AH Einstein metric (M, g) is uniquely determined by its conformal infinity $(\partial M, [\gamma])$, or by data of (M, g) at infinity. Again, this issue, raised in [16], has been open for some time, cf. also [8].

We summarize some of the results on non-uniqueness here, and refer to Section 4 for detailed statements and constructions. First, it turns out that it has been known to physicists working in Euclidean quantum gravity for a rather long time that uniqueness fails in general. For example, there are distinct, i.e. non-isometric, AH Einstein metrics on $M = S^2 \times \mathbb{R}^2$, which have the same conformal infinity on $\partial M = S^2 \times S^1$. These metrics are from the family of AdS-Schwarzschild metrics, and have been analyzed in detail in a remarkable paper of Hawking and Page [24]. Further there are AH Einstein manifolds (M, g) of distinct topological type, with the same conformal infinity, so that the conformal infinity $(\partial M, \gamma)$ does not determine the topological type of M.

In fact, even in dimension 3, where Einstein metrics are hyperbolic, i.e. of constant curvature, there are numerous examples of non-uniqueness. These examples come from the Thurston theory of Dehn surgery, or the well-known process of "opening cusps", cf. [20,37]. This construction gives infinitely many isometrically distinct hyperbolic 3-manifolds, all with a given conformal infinity.

Of course such Thurston–Dehn surgery is a special feature of hyperbolic 3-manifolds, and does not generalize to hyperbolic n-manifolds, $n \geqslant 4$. However, we will see in Section 4 that such Dehn surgery constructions do generalize in the context of AH Einstein metrics. As in dimension 3, this gives rise, in special situations, to an infinite sequence distinct AH Einstein metrics on a fixed 4-manifold, with a fixed conformal infinity, cf. Proposition 4.4 and Remark 4.5. These metrics again come from 'AdS black hole' metrics, with a toral (T^2) event horizon and have been extensively examined in the AdS/CFT correspondence. In particular, we see that even the finiteness issue, in addition to the uniqueness issue, fails in general.

In sum, there are numerous counterexamples to uniqueness of AH Einstein metrics with a given conformal infinity. We give at least a brief overview of this situation in Section 4.

Thus, it is of interest to understand under what conditions one can uniquely characterize an AH Einstein metric. To do this, recall the Fefferman–Graham expansion of a conformal compactification. Thus, let $\bar{g}$ be a compactification as in (0.2) where the defining function t has the property that $t(x) = dist_{\bar{g}}(x, \partial M)$, in a collar neighborhood U of ∂M. It is easy to see that, given a boundary metric γ in the conformal infinity of (M, g), there is a unique such defining function $t = t_{\gamma}$ having γ as its boundary metric, (cf. Section 1). The Gauss lemma then implies that the metric $\bar{g}$ splits in U as

$$\bar{g} = dt^2 + g_t, \tag{0.3}$$

where g_t is a curve of metrics on ∂M. It follows from Theorem 0.1, cf. Corollary 2.5, that if the boundary metric $\gamma \in C^{m+1,\alpha}$, then the curve g_t is at least a $C^{m,\alpha}$ curve of metrics in t. Following [16], we may then expand g_t as a Taylor series in t as

$$g_t = g_{(0)} + tg_{(1)} + t^2 g_{(2)} + t^3 g_{(3)} + \cdots + t^m g_{(m)} + O(t^{m+\alpha}). \tag{0.4}$$

One has $g_{(0)} = \gamma, g_{(1)} = 0$, while $g_{(2)}$ is determined locally from the geometry of the boundary metric γ. On the other hand, the terms $g_{(j)}$ for $j \geqslant 3$ are not locally determined by the boundary γ in general, cf. Section 3 for further discussion.

Theorem 0.2. *Suppose* $\dim M = 4$ *and the boundary metric* $\gamma \in C^{7,\alpha}$. *Then the data* $(\gamma, g_{(3)})$ *on* ∂M *uniquely determine an AH Einstein metric up to local isometry*, *i.e. if* g^1 *and* g^2 *are two AH Einstein metrics on manifolds* M^1 *and* M^2, *with* $\partial M^1 = \partial M^2 = \partial M$ *such that*, *w.r.t. geodesic compactifications as in* (0.3),

$$\gamma^1 = \gamma^2 \quad \text{and} \quad g^1_{(3)} = g^2_{(3)}, \tag{0.5}$$

then g^1 *and* g^2 *are locally isometric and the manifolds* M^1 *and* M^2 *are commensurable*, *i.e. they have diffeomorphic universal covers.*

It follows that M^1 is diffeomorphic to M^2 and g^1 is isometric to g^2 if $\pi_1(M^1) \cong \pi_1(M^2)$ and the actions of $\pi_1(M^i)$ on the universal cover $\tilde{M}$ of M^i are conjugate in the isometry group of $\tilde{M}$. Theorem 0.2 is proved in Section 3.

The third purpose of the paper is to analyze the finiteness issue, i.e. when a given boundary metric $(\partial M, \gamma)$ bounds finitely or infinitely many AH Einstein 4-manifolds (M_i, g_i). The main result is, roughly speaking, that under reasonably natural conditions only conformally flat boundary data can bound infinitely many AH Einstein metrics. Since the exact statement is somewhat technical, we refer to Theorem 5.3 for details. We also mention here Proposition 5.1, which gives a very simple proof of the result of Witten–Yau [39] on the connectedness of ∂M when ∂M has a component of positive scalar curvature.

This paper does not address the existence question for AH Einstein metrics, i.e. given a conformal class $[\gamma]$ of metrics on ∂M, does there exist an AH Einstein metric on M which has $[\gamma]$ as its conformal infinity. This issue will be discussed in a sequel [4] to this paper; the results obtained here, however, will play an important role in the sequel.

As indicated above, there is a very extensive and active recent physics literature on AH Einstein metrics in relation to the AdS/CFT correspondence and this work is a strong influence on this paper. We refer to [14,23,31,33,38] for some relevant perspectives.

1. Conformally compact Einstein metrics

In this section we discuss some background material for conformally compact Einstein metrics. Although the results of this section hold in arbitrary dimensions, we carry out the computations only in dimension 4, since this is the most relevant case for the paper; cf. also Remark 1.5 for the higher-dimensional case. Thus, we assume that M is an open, connected, oriented 4-manifold, with non-empty and compact boundary. It is not assumed that ∂M is connected.

Let ρ be a defining function for ∂M and as in (0.2), set

$$\bar{g} = \rho^2 g. \tag{1.1}$$

Unless ρ is restricted by the geometry of g, without loss of generality we may, and do, assume that ρ is C^∞ on $\bar{M}$, so that the metric $\bar{g}$ is C^∞ in the interior of M. The metric $\bar{g}$ is a $L^{k,p}$ or $C^{m,\alpha}$ compactification if it extends to a $L^{k,p}$ or $C^{m,\alpha}$ metric on $\bar{M}$. Thus, there are coordinate charts for a neighborhood of ∂M in M such that the local components of $\bar{g}$ in these charts are $L^{k,p}$ or $C^{m,\alpha}$ functions of the coordinates. We will usually assume that

$$k \geqslant 2, \quad p > 4 \quad \text{or} \quad m + \alpha > 1. \tag{1.2}$$

Sobolev embedding, in dimension 4, implies that for $p>4$, $L^{2,p} \subset C^{1,\alpha}$, $\alpha = 1 - \frac{4}{p} > 0$, while for $p>2$, $L^{2,p} \subset C^{\alpha}$, $\alpha = 2 - \frac{4}{p}$. It is well-known that metrics have optimal regularity in harmonic coordinates. Such coordinates exist on $(\bar{M}, \bar{g})$ if $\bar{g}$ is a $L^{k,p}$ metric with $k \geqslant 2, p > 2$; further the components of $\bar{g}$ are in $L^{k,p}$ w.r.t. such harmonic coordinates, cf. [6, Chapter 5E].

If $\bar{g}$ is a $C^{m,\alpha}$ compactification, then the boundary metric $\gamma = \bar{g}|_{\partial M}$ on ∂M is also $C^{m,\alpha}$, while if $\bar{g}$ is $L^{k,p}$, then the boundary metric γ is $L^{k-\frac{1}{p},p}$, cf. [1, 7.56]. In particular, by Sobolev embedding, a $L^{2,p}$ compactification has a $C^{1,\alpha}$ boundary metric when $p>4$. Of course the converse of these statements may not hold in general. The degree of smoothness of the boundary metric γ does not, a priori, imply any degree of smoothness of the compactification.

As noted in Section 0, defining functions ρ on M are not unique, but differ by multiplication by positive functions. Conversely, given any positive smooth function ϕ on M and any defining function ρ, then $\phi \cdot \rho$ is a defining function. Hence only the conformal class of the boundary metric $\gamma = \bar{g}_{\partial M}$ is uniquely determined by (M, g).

The curvatures of the Einstein metric g and a compactification $\bar{g}$ are related by the following formulas in dimension 4:

$$\bar{K}_{ab} = \frac{K_{ab} + |\bar{\nabla}\rho|^2}{\rho^2} - \frac{1}{\rho}\{\bar{D}^2\rho(\bar{e}_a, \bar{e}_a) + \bar{D}^2\rho(\bar{e}_b, \bar{e}_b)\}. \tag{1.3}$$

$$\bar{R}ic = -2\frac{\bar{D}^2\rho}{\rho} + \left(3\rho^{-2}(|\bar{\nabla}\rho|^2 - 1) - \frac{\bar{\Delta}\rho}{\rho}\right)\bar{g}, \tag{1.4}$$

$$\bar{s} = -6\frac{\bar{\Delta}\rho}{\rho} + 12\rho^{-2}(|\bar{\nabla}\rho|^2 - 1). \tag{1.5}$$

The terminology here is the following: D^2 is the Hessian, ∇ is the gradient, $\Delta = tr\, D^2$ the Laplacian, while K_{ab} denotes sectional curvature and $\{e_a\}$ an orthonormal basis. (See [6, Chapter 1J] for example for formulas for conformal changes of metric). All barred quantities are w.r.t. the $\bar{g}$ metric. Eq. (1.4) is equivalent to the Einstein equations (0.1). Similar formulas hold in all dimensions. We also let

$$r = \log\left(\frac{2}{\rho}\right), \quad \rho = 2e^{-r}. \tag{1.6}$$

Since ρ is smooth on $\bar{M}$, it is essentially immediate from (1.4) that if $\bar{g}$ is a $L^{k,p}$ compactification, satisfying (1.2), then $|\bar{\nabla}\rho| \equiv 1$ at ∂M, and hence by (1.3), the sectional curvatures of g tend to -1 at infinity in (M, g). Hence any such conformally compact Einstein manifold is asymptotically hyperbolic (AH). Further at ∂M, we have $\bar{D}^2\rho = A$, where A is the second fundamental form of ∂M in $(\bar{M}, \bar{g})$; A is C^α on ∂M, again by (1.2). Eq. (1.4) further implies that ∂M is *umbilic*, i.e. $A = \lambda \cdot \gamma$, for some function λ on ∂M.

From formulas (1.4) and (1.5), it is clear that defining functions which satisfy

$$|\bar{\nabla}\rho| \equiv 1 \tag{1.7}$$

in a collar neighborhood U of ∂M in M are especially natural. Such defining functions will be called *geodesic* defining functions, (although they are called special defining functions in [18]). A brief computation shows that, (in general),

$$|\bar{\nabla}\rho| = |\nabla r|, \tag{1.8}$$

where the norm on the left is w.r.t. $\bar{g}$ and that on the right w.r.t. g. Thus condition (1.7) is an intrinsic property of (M, g) and r. The function r is a (signed) distance function on (M, g) and the integral curves of ∇r are geodesics in (M, g). Similarly ρ is the distance function from ∂M w.r.t. $\bar{g}$. Geodesic defining functions are geometric, in that they depend on the geometry of $(M, \bar{g})$ or (M, g), and so their smoothness depends on the metric $\bar{g}$. Thus, such functions will not be C^∞ unless the compactification $\bar{g}$ is; if $\bar{g}$ is $C^{m,\alpha}$, then ρ is $C^{m+1,\alpha}$ off the cutlocus of ∂M in $(\bar{M}, \bar{g})$.

Suppose there is a compactification $\tilde{g} = \rho^2 g$ of (M, g) which is at least C^2, (actually $C^{1,1}$ suffices), with boundary metric γ. Then it is easy to see that there is a unique geodesic defining function $t = t(\gamma)$ for (M, g) such that the compactification

$$\bar{g} = t^2 \cdot g \tag{1.9}$$

has boundary metric γ, cf. [19, Lemma 5.2]. Briefly, write $t = u \cdot \rho$ where u is a positive function on $\bar{M}$ with $u \equiv 1$ on ∂M so that the boundary metric of (1.9) is indeed γ. Then the equation that $|\bar{\nabla} t|_{\bar{g}} = 1$, i.e. t is a geodesic defining function, is equivalent to

$$2(\tilde{\nabla}\rho)(\log u) + \rho|\tilde{\nabla} \log u|^2_{\tilde{g}} = \rho^{-1}(1 - |\tilde{\nabla}\rho|^2_{\tilde{g}}). \tag{1.10}$$

This is a non-characteristic first order PDE, with C^2 (or $C^{1,1}$) coefficients, on the right-hand side in C^1 (or Lipschitz). Hence, the Cauchy problem has a unique solution with $u \equiv 1$ on ∂M in a collar neighborhood U of ∂M. Observe, however, that $\bar{g}$ may not be as smooth as $\tilde{g}$; if $\tilde{g} \in C^{m,\alpha}$, $\alpha \geqslant 0$, then we have only $u \in C^{m-1,\alpha}$ and so $\bar{g} \in C^{m-1,\alpha}$

The Gauss lemma implies that the metrics $\bar{g}$ and g split in U, as in (0.3):

$$\bar{g} = dt^2 + g_t \quad \text{and} \quad g = dr^2 + g_r \tag{1.11}$$

with $g_0 = \gamma$ and $g_r = t^{-2} g_t$. The 1-parameter family g_t is a $C^{m,\alpha}$ smooth curve of metrics on ∂M if $\bar{g}$ is a $C^{m,\alpha}$ compactification. Observe also that the second fundamental form $\bar{A}$ of the level sets of t is given by $\bar{A} = \bar{D}^2 t$ and so in particular by (1.4)

$$\bar{A} = 0 \quad \text{at } \partial M, \tag{1.12}$$

i.e. ∂M is totally geodesic in M. In sum, if $\tilde{g}$ is a $C^{m,\alpha}$ compactification with $m \geqslant 2$ and $\alpha \geqslant 0$, then there is a unique compactification $\bar{g}$, at least $C^{m-1,\alpha}$, by a geodesic defining function inducing the boundary metric γ of $\tilde{g}$ on ∂M. Such a compactification will be called the *geodesic compactification* associated with γ.

If $\tilde{g}$ is only $L^{2,p}$ or $C^{1,\alpha}$, then the discussion above does not hold; although geodesics normal to ∂M do exist, they are not necessarily uniquely defined, and so one does not obtain a splitting (1.11) valid up to the boundary ∂M. Nevertheless, the discussion above does hold "to first order" at ∂M, in that there exists (another) $L^{2,p}$ compactification $\hat{g}$ which satisfies (1.12).

Lemma 1.1. *Let $\tilde{g}$ be a $L^{2,p}$ compactification of g, $p > 4$, with boundary metric γ and $L^{3,p}$ defining function ρ. Then there exists another (possibly equal) $L^{2,p}$ compactification $\hat{g}$ of g, with the same boundary metric γ, such that*

$$\hat{A} = 0 \quad \text{at } \partial M.$$

Proof. Let ϕ be a $L^{2,p}$ positive function on $\bar{M}$, with $\phi \equiv 1$ on ∂M and set $\hat{g} = \phi^2 \tilde{g}$. Then $\hat{g} \in L^{2,p}$, the boundary metric of $\hat{g}$ is γ, and the second fundamental forms $\hat{A}$ and $\tilde{A}$ of ∂M w.r.t $\hat{g}$ and $\tilde{g}$ are related by

$$\hat{A} = \tilde{A} + \langle \tilde{\nabla} \log \phi, \tilde{\nabla} \rho \rangle \cdot \gamma.$$

Let $\tilde{A}$ be the second fundamental form of the ρ-level sets w.r.t. $\tilde{g}$ and set $\lambda = tr_{\tilde{g}} \tilde{A}/3$, so that $\lambda \in L^{1,p}$. As noted above, $\tilde{A} = \lambda \gamma$ and $\tilde{\nabla} \rho = N$, the $\tilde{g}$ unit normal, both at ∂M. Choosing ϕ to satisfy $\langle \tilde{\nabla} \log \phi, \tilde{\nabla} \rho \rangle = N(\log \phi) = -\lambda$ at ∂M then gives the result.

Observe that if $\tilde{g} \in L^{k,p}$ or $C^{m,\alpha}$ then ϕ may also be chosen to be in $L^{k,p}$ or $C^{m,\alpha}$. $\square$

The next result shows that the Ricci curvature $\bar{R}ic$ at ∂M of a C^2 geodesic compactification $\bar{g}$ is determined by the intrinsic C^2 geometry of the boundary metric γ.

Lemma 1.2. *Let $\bar{g}$ be a C^2 geodesic compactification of an AH Einstein 4-manifold (M, g), with C^2 boundary metric γ. Then at ∂M,*

$$\bar{s} = 6\bar{R}ic(N, N) = \frac{3}{2} s_\gamma, \tag{1.13}$$

where N is the unit normal to ∂M w.r.t. $\bar{g}$. If X is tangent to ∂M, then

$$\bar{R}ic(N, X) = 0, \tag{1.14}$$

while if T denotes the projection onto $T(\partial M)$, then

$$(\bar{R}ic)^T = 2Ric_\gamma - \frac{1}{4} s_\gamma \cdot \gamma. \tag{1.15}$$

Proof. Equality (1.14) follows immediately from (1.4), while the first equality in (1.13) follows from (1.4) and (1.5). For the rest, let t be the geodesic defining function, let $\bar{e}_i$, $i = 1, 2, 3$, be an o.n. basis at any point for a level set $S(t)$ of t, for t near 0, and let $N = \bar{\nabla} t$ be the unit normal field to $S(t)$. By definition, the curvature $\bar{K}_{Ni} = \bar{R}(\bar{e}_i, N, N, \bar{e}_i) = \langle \bar{\nabla}_{\bar{e}_i} \bar{\nabla}_N N, \bar{e}_i \rangle - \langle \bar{\nabla}_N \bar{\nabla}_{\bar{e}_i} N, \bar{e}_i \rangle - \langle \bar{\nabla}_{[\bar{e}_i, N]} N, \bar{e}_i \rangle$. Using the fact that N is tangent to geodesics and N is a gradient, together with the fact that $\bar{D}^2 t = A = 0$ at ∂M by (1.12), gives

$$\bar{K}_{Ni} = -\lim_{t \to 0} t^{-1} \bar{D}^2 t(\bar{e}_i, \bar{e}_i),$$

at ∂M. From (1.4) and (1.5), it then follows that

$$\bar{K}_{Ni} = \frac{1}{2} \bar{R}ic(\bar{e}_i, \bar{e}_i) - \frac{1}{12} \bar{s}, \tag{1.16}$$

again on ∂M. On the other hand, the Gauss–Codazzi equations and (1.12) again imply

$$\bar{R}ic(\bar{e}_i, \bar{e}_i) = Ric_\gamma(\bar{e}_i, \bar{e}_i) + \bar{K}_{Ni},$$

and so substituting in (1.16) gives $\frac{1}{2}\bar{R}ic(\bar{e}_i, \bar{e}_i) = Ric_\gamma(\bar{e}_i, \bar{e}_i) - \frac{1}{12}\bar{s}$. Thus (1.15) follows from the second equality in (1.13).

Finally, to prove this second equality, Gauss–Codazzi and (1.12) again give $\frac{1}{2}s_\gamma = \sum_{i<j\leqslant 3} \bar{K}_{ij}$. But by definition,

$$\frac{1}{2}\bar{s} = \sum_{i\leqslant 3} \bar{K}_{Ni} + \sum_{i<j\leqslant 3} \bar{K}_{ij}.$$

The first term is just $\bar{R}ic(N, N)$, and so $\frac{1}{2}\bar{s} = \bar{R}ic(N, N) + \frac{1}{2}s_\gamma$. From (2.6), this gives $\frac{1}{2}\bar{s} = \frac{1}{6}\bar{s} + \frac{1}{2}s_\gamma$, which gives (1.13). $\square$

We will also need the analogue of Lemma 1.2 when the compactification is not geodesic.

Lemma 1.3. *Let $\bar{g}$ be a geodesic compactification and let $\tilde{g} = \phi^2\bar{g}$ be another compactification with the same boundary metric γ, so that $\phi \equiv 1$ on ∂M. Suppose that $\tilde{A} = 0$ on ∂M, cf. Lemma* 1.1. *Then the Ricci curvature $\tilde{R}ic$ of $\tilde{g}$ at ∂M is determined by the Ricci curvature of the boundary metric γ and the scalar curvature $\tilde{s}$ of $\tilde{g}$ at ∂M. In fact, at ∂M,*

$$\tilde{R}ic = \bar{R}ic + \tfrac{1}{6}(\tilde{s} - \bar{s})(\tilde{g} + 2v \cdot v), \tag{1.17}$$

where $\bar{R}ic$ and $\bar{s}$ are as in (1.13)–(1.15) *and v is the unit conormal at ∂M. If $\tilde{A} \neq 0$ at ∂M, then* (1.17) *holds modulo terms of the form $\tilde{A}^2$.*

Proof. By standard formulas for conformal changes of the metric, cf. [6, 1.159], the Ricci curvatures of $\tilde{g}$ and $\bar{g}$ are related by

$$\tilde{R}ic = \bar{R}ic - 2\phi^{-1}\bar{D}^2\phi + 4(d\log\phi)^2 - \phi^{-1}\bar{\Delta}\phi \cdot \bar{g} - |d\log\phi|^2 \cdot \bar{g}. \tag{1.18}$$

At ∂M, $\phi = 1$ and $d\phi = 0$, since $\tilde{A} = 0$. Hence

$$\tilde{R}ic = \bar{R}ic - 2\bar{D}^2\phi - \bar{\Delta}\phi \cdot \bar{g}.$$

In an orthonormal frame $e_1, \ldots, e_4$, $e_4 = N$, $\bar{D}^2\phi(e_i, e_j) = 0$ at ∂M unless $e_i = e_j = N$, i.e. $\bar{D}^2\phi = [NN(\phi)]v \cdot v$. Hence $\bar{D}^2\phi(N, N) = \bar{\Delta}\phi$, and it follows that

$$\tilde{R}ic = \bar{R}ic - (\bar{\Delta}\phi)(\tilde{g} + 2v \cdot v)$$

at ∂M. Taking the trace of this equation gives $\tilde{s} = \bar{s} - 6\bar{\Delta}\phi$, which implies (1.17). If $\tilde{A} \neq 0$, then the same arguments show that (1.18) gives (1.17) modulo $\tilde{A}^2$ terms. □

Next we have an interesting estimate for the scalar curvature $\bar{s}$ of geodesic compactifications $\bar{g}$.

Proposition 1.4. *Let $\bar{g}$ be a C^2 geodesic compactification with boundary metric γ and scalar curvature $\bar{s}$. Then off the cut locus of t in $(M, \bar{g})$,*

$$\bar{s}' \equiv \langle \bar{\nabla}\bar{s}, \bar{\nabla}t \rangle = 6t^{-1}|\bar{D}^2 t|^2 \geqslant 0. \tag{1.19}$$

In particular, $\bar{s}$ is uniformly bounded below in this region by its boundary value on ∂M, and thus bounded below by the C^2 geometry of γ.

Proof. The computations below are w.r.t. the $\bar{g}$ metric, but we drop the bar from the notation. The flow lines of ∇t are geodesics, and hence, a standard result in Riemannian geometry (cf. [35] for example) implies that the following Ricatti equation holds in U:

$$H' + |A|^2 + Ric(\nabla t, \nabla t) = 0; \tag{1.20}$$

here $H = tr\, A = \Delta t$ and $H' = \langle \nabla H, \nabla t \rangle$. By (1.4), we have $Ric(\nabla t, \nabla t) = -2t^{-1}(D^2 t)(\nabla t, \nabla t) - t^{-1}\Delta t$, and since $|\nabla t| = 1$, $(D^2 t)(\nabla t, \nabla t) = 0$. Hence dividing (1.20) by t gives

$$t^{-1}(\Delta t)' + t^{-1}|D^2 t|^2 - t^{-2}\Delta t = 0. \tag{1.21}$$

But $t^{-1}(\Delta t)' = (t^{-1}\Delta t)' + t^{-2}\Delta t$, and so (1.21) becomes

$$(t^{-1}\Delta t)' + t^{-1}|D^2 t|^2 = 0.$$

Eq. (1.19) then follows from (1.5). □

Remark 1.5. All of the discussion in this section holds in any dimension, with obvious modifications for Sobolev embedding and constants depending on dimension. For instance, in dimension n, with 3 in (0.1) replaced by $n-1$, (1.13) holds with $\frac{3n-4}{2(n-1)(n-2)}$ in place of $\frac{3}{2}$, while the factor of 6 in (1.13) and (1.19) should be replaced by $2(n-1)$.

2. Boundary regularity

In this section we study the boundary regularity of AH Einstein metrics on 4-manifolds and establish Theorem 0.1.

As is well-known [6, Chapter 5], Einstein metrics satisfy an elliptic system of equations in harmonic coordinates, and so one obtains higher-order (C^∞ or C^ω) regularity of such metrics from a local L^p bound on the curvature. With regard to boundary regularity, the boundary of an AH Einstein metric occurs at infinity. If one works in local coordinates for ∂M, the system of Einstein equations becomes degenerate at ∂M, and thus difficult to deal with for regularity issues.

It is a special feature of dimension 4 that Einstein metrics (M, g) also satisfy a conformally invariant equation, namely the Bach equation

$$\delta d\left(Ric - \frac{s}{6} g\right) + \mathring{W}(Ric) = 0, \tag{2.1}$$

cf. [6, (4.77)]. This is the Euler–Lagrange equation for $\mathscr{W}$, the square of the L^2 norm of the Weyl curvature W. Here Ric and g are viewed as 1-forms with values in TM, $d = d^\nabla$ is the exterior derivative $\Lambda^1 \to \Lambda^2$ defined in terms of the metric g and $\mathring{W}$ is the action of the Weyl tensor on symmetric bilinear forms.

Of course Einstein metrics satisfy (2.1) in any dimension, but expression (2.1) is conformally invariant only in dimension 4. Hence (2.1) holds for any conformal compactification $(M, \bar{g})$ of (M, g). Observe that (2.1) is a fourth-order system of equations in the metric g, as opposed to the second-order system of Einstein equations. Note also that, being conformally invariant, Eq. (2.1) is trace-free, i.e. its trace vanishes identically.

We first prove boundary regularity of an $L^{2,p}$ compactification, $p > 4$, given suitable control on the scalar curvature of the compactification. As mentioned in the introduction, we consider compactifications which are $L^{2,p}$, since this level of regularity exists for the AH Einstein metrics constructed by Biquard [7].

Proposition 2.1. *Let (M, g) be an AH Einstein 4-manifold which admits an $L^{2,p}$ conformal compactification $\bar{g}$, with boundary metric γ, for some $p > 4$, so that*

$$|R_{\bar{g}}|_{L^p} \leqslant \Lambda < \infty, \tag{2.2}$$

where R is the curvature tensor.

Let $k \geqslant 1$ and $q \geqslant 2$. If $\gamma \in L^{k+2,q}(\partial M)$, the scalar curvature $\bar{s} \in L^{k,q}(\bar{M})$, with $\bar{s}|_{\partial M} \in L^{k,q}(\partial M)$, then the metric $\bar{g}$ is in $L^{k+2,q}(\bar{M})$. (This last assumption may be realized by assuming $\bar{s} \in L^{k+\frac{1}{q},q}$, cf. [1, 7.56]).

The same result holds with respect to the Hölder $C^{m,\alpha}$ spaces, i.e. if $\gamma \in C^{m+2,\alpha}(\partial M)$ and $\bar{s} \in C^{m,\alpha}(\bar{M})$, then $\bar{g} \in C^{m+2,\alpha}(\bar{M})$. If $\bar{s}$ and γ are C^ω, i.e. real-analytic, then so is $\bar{g}$.

Further, if, with respect to a fixed harmonic coordinate system for $\bar{M}$,

$$||\gamma||_{L^{k+2,q}(\partial M)} \leqslant C \quad \text{and} \quad ||\bar{s}||_{L^{k,q}(\bar{M})} + ||\bar{s}||_{L^{k,q}(\partial M)} \leqslant C \tag{2.3}$$

then

$$||\bar{g}||_{L^{k+2,q}(\bar{M})} \leqslant C_1, \tag{2.4}$$

where C_1 depends only on C, Λ, an upper diameter bound and a lower bound for the geodesic ball volume ratio $vol_{\bar{g}}B_x(s)/s^4$, $x \in M$, $s \leqslant 1$, on $(M, \bar{g})$. The analogous estimate holds with respect to the $C^{m,\alpha}$ Hölder norms.

Proof. The idea of the proof is to apply boundary regularity results for elliptic systems, in connection with the Bach system (2.1). However, the leading operator δd in (2.1) is not elliptic. This can be rectified by considering the operator $\delta d + 2\delta^*\delta$. In fact a standard Weitzenbock formula gives

$$\delta d + 2\delta^*\delta = 2D^*D + \mathscr{R},$$

where D^*D is the rough Laplacian, and $\mathscr{R}$ is a curvature term, cf. [5, p. 288] or [6, (4.71)]. (The exact form of $\mathscr{R}$ is of no importance here). The Bianchi identity $\delta Ric = -\frac{1}{2}ds$, gives $2\delta^*\delta Ric = -D^2 s$, while a straightforward computation, cf. again [5], shows $\delta d(sg) = -2\Delta s \cdot g + 2D^2 s$. Hence (2.1) can be rewritten as

$$2D^*DRic = -\frac{2}{3}D^2 s - \frac{1}{3}\Delta s \cdot g + \mathscr{R}_1, \tag{2.5}$$

where $\mathscr{R}_1 = -\mathscr{R}(Ric) - \mathring{W}(Ric)$ is a term quadratic in curvature. Here and in the following, all metric quantities are w.r.t. $\bar{g}$ but the overbar is omitted from notation.

The assumptions on s and γ give control on the right-hand side of (2.5), while the left-hand side of (2.5) is essentially an elliptic fourth-order system in the metric g. In principle, the result then follows from general elliptic boundary regularity theory, but there are a fair number of details to address. To begin, as noted in Section 1, one may choose local harmonic coordinates for $\bar{M}$ in which the components of g, (i.e. $\bar{g}$), are $L^{2,p}$. With respect to such coordinates, the components of the Ricci curvature are

$$-Ric_{ab} = \frac{1}{2}\Delta g_{ab} + [Q_1(g, \partial g)]_{ab}, \tag{2.6}$$

where Q_1 is of lower order; Q_1 involves only quadratic expressions in g, g^{-1} and ∂g. Similarly, the rough Laplacian D^*D is also the function Laplacian to leading order, in that, in harmonic coordinates

$$-(D^*DRic)_{ab} = \Delta(Ric_{ab}) + [Q_3(g, \partial^j g)]_{ab}, \tag{2.7}$$

where Q_3 involves only derivatives of g up to order 3; we refer to [2, p. 234] for instance for the exact calculations. Hence, in local harmonic coordinates, the left-hand side of (2.5) has the form of the bi-Laplacian of the metric, $\Delta\Delta(g_{ab})$, to leading order. Further, in such coordinates, the Laplacian has the form

$$\Delta = g^{kl}\partial_k\partial_l, \tag{2.8}$$

and so involves the metric only to zeroth order.

The metric g is a weak $L^{2,p}$ solution of Eq. (2.5), i.e. (2.5) holds when it is paired with any $L_o^{2,p'}$ test form h, $(p^{-1} + (p')^{-1} = 1)$ and integration by parts is performed twice; here $L^{2,p'}_o$ is the closure of the space of smooth functions of compact support in M in the $L^{2,p'}$ norm.

Given this setup, to control the boundary behavior it is necessary to make a specific choice of harmonic coordinates. Thus, given the boundary metric $\gamma \in L^{k+2,q}(\partial M)$, (or $C^{m+2,\alpha}(\partial M)$), choose local harmonic coordinates u_a, $a = 1,2,3$ for ∂M w.r.t. γ. The coordinates u_a are in $L^{k+3,q}(\partial M)$, (or $C^{m+3,\alpha}(\partial M)$), and $\gamma_{ab} = \gamma(\partial_a, \partial_b) \in L^{k+2,q}(\partial M)$, $(C^{m+2,\alpha}(\partial M))$; here $\partial_a = \partial/\partial u_a$. Next, the coordinates u_a may be extended to local harmonic coordinate functions for M by solving a local Dirichlet problem: $\Delta_g u_a = 0$, with $u_a|_{\partial M}$ the given function u_a on ∂M. Similarly, choose a local "harmonic defining function" u_4, i.e. $\Delta_g u_4 = 0$, with $u_4|_{\partial M} = 0$. The metric $g = g_{ab} = g(\partial_a, \partial_b)$ has optimal regularity in these coordinates.

Clearly $g_{ab}|_{\partial M} = \gamma_{ab}$, for $a, b \leqslant 3$. However, g_{4a} at ∂M is not apriori determined by the boundary metric γ. The following lemma shows that the components g_{4a} satisfy Neumann boundary conditions on ∂M.

Lemma 2.2. *Let $N = \nabla u_4/|\nabla u_4|$ be the unit normal at ∂M, $\nabla u_4 = g^{4a}\partial_a$. Then the components g^{4a} satisfy the following Neumann boundary condition at ∂M:*

$$N(g^{44}) = -6\lambda (g^{44})^{3/2} \quad \text{and}$$

$$N(g^{4a}) = -\frac{1}{2}(g^{44})^{-1/2} g^{ab}\partial_b g^{44} - 3\lambda\sqrt{g^{44}}g^{4a}, \quad a<4 \text{ on } \partial M, \tag{2.9}$$

where λ is given by $A = \lambda \cdot g$ at ∂M.

Proof. Since $\Delta u_a = 0$ at ∂M, one has, at ∂M,

$$0 = \langle \nabla_{e_i}\nabla u_a, e_i \rangle + \langle \nabla_N \nabla u_\alpha, N \rangle,$$

where e_i is an orthonormal basis for ∂M at a given point. Write $\nabla u_a = (\nabla u_a)^T + (\nabla u_a)^N$. Then $\langle \nabla_{e_i}(\nabla u_a)^T, e_i \rangle = 0$, since u_a is harmonic on ∂M, $a \leqslant 3$, and $u_4 \equiv 0$ on ∂M. Further, $\langle \nabla_{e_i}(\nabla u_a)^N, e_i \rangle = A(e_i, e_i)\langle \nabla u_a, N \rangle = Hg^{4a} = 3\lambda g^{4a}$. This then gives

$$\langle \nabla_N \nabla u_a, N \rangle = -3\lambda g^{4a} \quad \text{at } \partial M. \tag{2.10}$$

Let $a = 4$. Since $\nabla u_4 = g^{4a}\partial_a$, $|\nabla u_4|^2 = g^{44}$, so $|\nabla u_4| = \sqrt{g^{44}}$. The term on the left-hand side in (2.10), with $a = 4$ is then

$$\frac{1}{|\nabla u_4|^2}\langle \nabla_{\nabla u_4}\nabla u_4, \nabla u_4 \rangle = \frac{1}{2}(g^{44})^{-1}\langle \nabla u_4, \nabla g^{44} \rangle = \frac{1}{2}(g^{44})^{-1/2}N(g^{44}),$$

which, with (2.10), gives the first equation in (2.9).

For the second equation, the left-hand side of (2.10) may be written as

$$\frac{1}{|\nabla u_4|^2}\langle \nabla_{\nabla u_4}\nabla u_a, \nabla u_4\rangle = \frac{1}{|\nabla u_4|}N\langle \nabla u_a, \nabla u_4\rangle - \frac{1}{|\nabla u_4|}\langle \nabla u_a, \nabla_N \nabla u_4\rangle.$$

For the first term, compute

$$\langle \nabla u_a, \nabla u_4\rangle = \langle g^{ac}\partial_c, g^{4b}\partial_b\rangle = g^{ac}g^{4b}g_{bc} = g^{4a},$$

so that the first term is $\frac{1}{|\nabla u_4|}N(g^{4a})$.

For the second term, using the gradient property, this is

$$\begin{aligned}\frac{1}{|\nabla u_4|}\langle N, \nabla_{\nabla u_a}\nabla u_4\rangle &= \frac{1}{2|\nabla u_4|^2}\langle \nabla u_a, \nabla g^{44}\rangle = \frac{1}{2|\nabla u_4|^2}\left\langle g^{ab}\partial_b, g^{cd}\frac{\partial g^{44}}{\partial u_d}\partial_c\right\rangle \\ &= \frac{1}{2|\nabla u_4|^2}g^{ad}\partial_d g^{44}.\end{aligned}$$

Combining these estimates gives the second equation in (2.9). $\square$

We are now in position to deal with the proof of Proposition 2.1 itself. We begin with the lowest regularity situation, and so suppose $k=1$ and $2 \leqslant q \leqslant p$. One has R and Ric in $L^p(M)$, and by assumption $s \in L^{1,q}(M)$ and $s|_{\partial M} \in L^{1,q}(\partial M)$. Hence, (2.5) and (2.7) give

$$\Delta(Ric_{ab}) \in L^{-1,q} + L^{p/2} + L^{-1,p}, \tag{2.11}$$

on M, where the right-hand side of (2.11) denotes the sum of three terms, each in the respective spaces, cf. also [2, p. 234]. (The $L^{-1,q}$ term comes from the s terms on the right-hand side in (2.5), the $L^{p/2}$ term comes from the curvature terms in (2.5), while the $L^{-1,p}$ term comes from the Q_3 term in (2.7)). The coefficients of the Laplacian in (2.11) are in $L^{2,p}$. We refer to [30] for treatment of Sobolev spaces with negative exponents, and recall that the dual space of $L^{-1,q}$ is $L_o^{1,q'}$.

A straightforward application of Sobolev embedding then shows that $L^{-1,p} \subset L^{-1,q}$ and $L^{p/2} \subset L^{-1,q}$ and so we may view the Laplacian in (2.11) as a mapping $\Delta : L^{1,q} \to L^{-1,q}$. Now elliptic boundary regularity theory for a Laplacian as in (2.8), cf. [30, Chapter 2.7; 32, Theorem 5.5.5′] shows that $Ric_g \in L^{1,q}$ provided the curvature Ric_g is $L^{1,q}$ at the boundary ∂M. By Lemma 1.3, the Ricci curvature of g at ∂M is determined algebraically by that of intrinsic metric γ, s and A at ∂M. By Lemma 1.1, (and passing to a new compactification equally as smooth as the original), one may assume that $A=0$ at ∂M; alternately, it will be remarked below that the case $A \neq 0$ may be handled by the same arguments. It then follows from the assumptions on Ric_γ and $s|_{\partial M}$ in Proposition 2.1 that $Ric_g|_{\partial M} \in L^{1,q}(\partial M)$ and hence $Ric_g \in L^{1,q}(M)$.

We now basically repeat this argument with (2.6) in place of (2.7). Thus, the left-hand side of (2.6) is now in $L^{1,q}$, with Laplacian of form (2.8) with $L^{2,p}$ coefficients.

The lower-order term $[Q_1]_{ab}$ is in $L^{1,p} \subset L^{1,q}$. For $i,j \leqslant 3$, since $g_{ij} = \gamma_{ij}$ at ∂M, it follows from elliptic boundary regularity that $g_{ij} \in L^{3,q}$, since the boundary metric $\gamma \in L^{3,q}(\partial M)$ by assumption. For the normal terms, g^{4a}, suppose as above, (w.l.o.g.), that $A = 0$ at ∂M and work first with g^{44}. This satisfies the Neumann condition (2.9), where the coefficients of the vector field N are in $L^{2,p}(M)$, and so in $C^{1,\alpha}(\partial M)$. Since g^{44} also satisfies an equation of form (2.6), with right-hand side of the form $Ric^{44} \in L^{1,q}(M)$, it follows from elliptic boundary regularity that $g^{44} \in L^{3,q}(M)$, cf. [32, Theorem 6.3.7] or [30, Chapter 2.7.3] for instance. For the terms g^{4a}, $a < 4$, since now $\partial g^{44} \in L^{2,q}(M)$, the same arguments as above using the Neumann condition (2.9) on g^{4a} give $g^{4a} \in L^{3,q}(M)$.

Thus we have $g_{ij} \in L^{3,q}(M)$, $i,j \leqslant 3$ and $g^{4a} \in L^{3,q}(M)$, $a \leqslant 4$. From this, it is an exercise in linear algebra to see that $g_{ab} \in L^{3,q}(M)$, $a,b \leqslant 4$; my thanks to Rafe Mazzeo for suggesting the argument below. Thus $g^{44} = (\det g_{ab})^{-1} A_{44}$, where A_{44} is the $(4,4)$ cofactor is the matrix g_{ab}. Since A_{44} only involves g_{ij}, $i,j \leqslant 3$, the regularity above gives $A_{44} \in L^{3,q}(M)$, and hence $\det g_{ab} \in L^{3,q}(M)$, since $g^{44} > 0$. The same reasoning on g^{4a} then gives $A_{4a} \in L^{3,q}(M)$, $a \leqslant 3$. Now each determinant A_{4a} may be expanded, (along its last row or column), to obtain a linear form in the variables g_{4i}, $i \leqslant 3$, with coefficients given by 2×2 determinants. Thus, one has a linear system of three equations in the three variables g_{4i} with coefficients given by 2×2 determinants. These 2×2 determinants are cofactors in the 3×3 matrix g_{ij}, $i,j \leqslant 3$. The determinant of the matrix associated to this 3×3 linear system is then easily calculated to be just $-(\det g_{ij})^2$. Since $\det g_{ij}|_{\partial M} = \det \gamma_{ij} > 0$, this linear system in invertible near ∂M. Hence, the variables g_{4i} are rational expressions in A_{4a}, g_{kl}, $k,l \leqslant 3$ and $(\det g_{kl})^{-1}$, each of which is in $L^{3,q}(M)$. It follows that $g \in L^{3,q}(M)$; this completes the proof in case $k = 1$ and $q \leqslant p$.

If $k = 1$ and $q > p$, then the work above gives $g \in L^{3,p}$ and by Sobolev embedding, $L^{3,p} \in L^{2,r}$, for any $r < \infty$, since $p > 4$. Choosing r sufficiently large so that $q \leqslant r$, the work above with r in place of p gives $g \in L^{3,q}$, as required. This completes the proof in case $k = 1$.

The proof for a given $k \geqslant 2$ follows exactly the same 2-step procedure, using induction from the regularity obtained at order $k - 1$.

If $A \neq 0$ at ∂M, the same arguments are valid, with an extra bootstrap or iteration, since A is of lower order. Thus, for instance, working with (2.11), since $A^2 \in L^{1,p}(M)$, $Ric_g|_{\partial M} \in L^{1,q} + L^{1-1/p,p}$ and so elliptic regularity gives $Ric_g \in L^{1-1/p,p}(M)$, (assuming $L^{1,q} \subset L^{1-1/p,p}$). In turn, this leads as before to $g \in L^{3-1/p,p}(M)$, which gives $A^2 \in L^{2-1/p,p}(M)$. Using this new estimate for A^2 and iterating this process again leads to the same result as before.

The proof of regularity in the case of Hölder spaces is also essentially the same although a little easier. Thus suppose $m = 1$, so that $\gamma \in C^{3,\alpha}$ and $s \in C^{1,\alpha} \subset L^{1,p}$, for all $p < \infty$. It follows that $D^2 s \in L^{-1,p}$ and so the work on Sobolev space regularity above implies $g \in L^{3,p}$, for any $p < \infty$. The metric g is thus an $L^{3,p}$ weak solution of (2.5) with $Ric|_{\partial M} \in C^{1,\alpha}$. Elliptic boundary regularity theory applied to (2.7), cf. [17, Chapter 8], again implies that $Ric_g \in C^{1,\alpha}$ on M and the same argument applied to (2.6) gives $g \in C^{3,\alpha}$. The proof when $m \geqslant 2$ follows in the same way, using the Schauder elliptic estimates. This then completes the proof in all cases.

Regarding estimate (2.4), bound (2.2), together with bounds on the diameter and volume ratios of geodesic balls imply uniform $L^{2,p}$ bounds on the metric $\bar{g}$ in harmonic coordinates, as well as an upper bound on the number of such coordinate charts, cf. [2,12]. Thus, bound (2.4) follows from the fact that the elliptic regularity results above are effective, i.e. all the regularity statements are accompanied by estimates.

Finally, Eq. (2.5) has real-analytic coefficients in the metric $\bar{g}$, and smooth solutions of such equations with real-analytic boundary values are real-analytic, cf. [32, Chapter 6.7]. This gives the statement on real analyticity. $\square$

Remark 2.3. The method of proof above, and in particular Lemma 2.2, can also be used to prove other regularity results; for example regularity up to the boundary for metrics whose Ricci curvature is controlled up to the boundary.

Proposition 2.1 shows that the smoothness of the compactification $\bar{g}$ is determined by that of its scalar curvature $\bar{s}$, and that of the boundary metric γ on ∂M. Since (2.1) is trace-free, one cannot improve this result eliminating the dependence on the scalar curvature $\bar{s}$. An improvement can be obtained only by choice of a suitable representative of the conformal class, i.e. a suitable choice of gauge. From formulas (1.4) and (1.5), a natural choice of gauge near ∂M is that given by a geodesic defining function. However, it seems to be difficult to prove higher-order regularity directly in this gauge. The Yamabe, i.e. constant scalar curvature, gauge appears to be much better in this respect.

This leads to the following result, which is essentially Theorem 0.1.

Theorem 2.4. *Let (M, g) be an AH Einstein 4-manifold which admits a $L^{2,p}$ conformal compactification $\bar{g} = \rho^2 g$, $p > 4$. If, for a given $m \geqslant 3$ and $\alpha \in (0,1)$, the boundary metric $\gamma = \bar{g}|_{\partial M}$ is $C^{m,\alpha}$ smooth, then there is another (possibly equal) conformal compactification $\tilde{g}$ of g, with $\tilde{g}|_{\partial M} = \gamma$, such that $\tilde{g}$ is $C^{m,\alpha}$ smooth. If γ is real-analytic, then there is a real-analytic compactification $\tilde{g}$.*

Further, estimate (2.4) holds for $\tilde{g}$ without any dependence on the scalar curvature $\tilde{s}$.

Proof. Suppose $\gamma \in C^{m,\alpha}$, $m \geqslant 3$, and that $\bar{g}$ is an $L^{2,p}$ compactification of g. Let $\tilde{g}$ be a constant scalar curvature metric conformal to $\bar{g}$ on M with $\tilde{g}|_{\partial M} = \gamma$. Thus, for $\tilde{g} = u^2 \cdot \bar{g}$, the function $u > 0$ is a solution to the Dirichlet problem for the Yamabe equation

$$u^3 \mu = -6\bar{\Delta} u + \bar{s} u, \tag{2.12}$$

on M, with $u \equiv 1$ on ∂M and $\tilde{s} = \mu = \text{const}$. It is simplest to choose $\mu = -1$. Then standard methods in elliptic PDE give an $L^{2,p}$ solution to this Dirichlet problem, just as in the negative scalar curvature case of the Yamabe problem on compact manifolds. We refer to [29, Theorem 1.1] and references therein, (cf. also [15, Remark]), for a proof, at least when $\bar{g} \in C^{2,\alpha}$. The same proof holds for $\bar{g} \in L^{2,p}$, $p > 4$; alternately, the compactness result of [22] implies that if $\bar{g}_j \in C^{2,\alpha}(M)$ converges to $\bar{g}$ in $L^{2,p}(M)$, then the Yamabe metrics $\tilde{g}_j$ also converge in $L^{2,p}(M)$ to an $L^{2,p}$ Yamabe metric $\tilde{g} \in [\bar{g}]$, with $u \in L^{2,p}(M)$.

Since the Bach equation (2.1) is conformally invariant, the metric $\tilde{g}$ is an $L^{2,p}$ weak solution of Eq. (2.5). Hence, since $\tilde{s}$ is constant, Proposition 2.1 implies that $\tilde{g}$ is as smooth as the boundary metric γ, which completes the proof.

Estimate (2.4) follows as in the proof of Proposition 2.1, since the scalar curvature $\tilde{s}$ is a priori controlled. $\square$

Theorem 2.4 gives the optimal regularity near ∂M of a conformal compactification in terms of the regularity of the intrinsic metric γ on ∂M, assuming there is a $L^{2,p}$ conformal compactification of g, for some $p > 4$.

We note also the following, essentially immediate, corollary.

Corollary 2.5. *Let g be an AHE metric on M, which admits a $L^{2,p}$ conformal compactification, for some $p > 4$, for which the boundary metric γ is in $C^{m,\alpha}(\partial M)$, for some $m \geqslant 3$.*

Then the geodesic compactification $\bar{g}$ associated to γ is at least a $C^{m-1,\alpha}$ compactification.

Proof. Let $\tilde{g}$ be the $C^{m,\alpha}$ (Yamabe) compactification of g, given by Theorem 2.4. Then the geodesic compactification $\bar{g}$ and the Yamabe compactification $\tilde{g}$ are related by a conformal factor u satisfying (1.10). The coefficients and right-hand side of the first-order system (1.10) are in $C^{m,\alpha}$ and $C^{m-1,\alpha}$, respectively, and so the solution u with $u \equiv 1$ on ∂M is a $C^{m-1,\alpha}$ function on $\bar{M}$. Hence $\bar{g}$ is also $C^{m-1,\alpha}$ on $\bar{M}$. $\square$

Remark 2.6. Theorem 2.4 does not hold in odd dimensions $n \geqslant 5$. Namely by the result of Graham–Lee [19], there are C^∞ metrics γ on S^{n-1} which are boundary metrics of $C^{n-2,\alpha}$ compactifications $\bar{g}$ of AHE metrics g on the n-ball B^n. However, for generic γ, such compactifications $\bar{g}$ have a non-zero $t^{n-1} \log t$ term in the asymptotic expansion (0.4) of $\bar{g}$ near ∂M, cf. [18] for example. Hence, such metrics are at best only $C^{n-1,\alpha}$ smooth. It is unknown if Theorem 2.4 holds in even dimensions $n > 4$.

The geodesic compactification $\bar{g}$ is $C^{m-1,\alpha}$ only off the cutlocus $\bar{C}$ of ∂M in $(\bar{M}, \bar{g})$; at the cutlocus $\bar{C}$, the metric $\bar{g}$ becomes singular, (although of course the Einstein metric g is smooth). However, any smooth approximation to the geodesic defining function t gives a smoothing to the compactification $\bar{g}$.

We conclude this section with the following application of Corollary 2.5. First, let $\{g_i^*\}$ be a sequence of $L^{2,p}$ compactifications, $p > 4$, of AH Einstein metrics (M, g_i)

with $C^{m,\alpha}$ boundary metrics γ_i, $m \geqslant 1$. Suppose (2.2) holds uniformly for $\{g_i^*\}$, and that the bounds on the diameter and volume ratios for geodesic balls of g_i^* hold uniformly. Then a standard application of the L^p Cheeger–Gromov compactness theorem, cf. [2,12] and references therein, implies that $\{g_i^*\}$ is precompact, in that there is a subsequence converging in the weak $L^{2,p}$ and $C^{1,\alpha}$, $\alpha < 1 - \frac{4}{p}$, topologies to an $L^{2,p}$ limit metric g_∞^* on $\bar{M}$.

The following result shows that this convergence can be strengthened, given suitable control on the boundary metrics.

Proposition 2.7. *For $\{g_i^*\}$ as above, suppose the boundary metrics γ_i are bounded in the $C^{m,\alpha}$ topology on ∂M, for some $m \geqslant 3$. Then, for the subsequence above, the $C^{m-1,\alpha}$ geodesic compactifications $\bar{g}_i$ determined by γ_i converge, away from their cutlocus and in the $C^{m-1,\alpha'}$ topology, to the $C^{m-1,\alpha}$ limit $\bar{g}_\infty$, for any $\alpha' < \alpha$. Further, the distance of the cutlocus of each $\bar{g}_i$ to ∂M is uniformly bounded below.*

Proof. Corollary 2.5 and the associated (Hölder) bound (2.4) imply a uniform bound on $\{\bar{g}_i\}$ in the $C^{m-1,\alpha}$ topology on M, away from the cutlocus. Since $m \geqslant 3$, the curvature of $\bar{g}_i$ is thus uniformly bounded, which, by standard Riemannian geometry, implies a uniform lower bound on the distance of the cutlocus of $\bar{g}_i$ to ∂M. Given such a uniform bound on $\{\bar{g}_i\}$, it is then again standard that one has $C^{m-1,\alpha'}$ convergence to the $C^{m-1,\alpha}$ limit $\bar{g}_\infty$, for any $\alpha < \alpha'$; this is essentially the Arzela–Ascoli theorem in harmonic coordinates, cf. [2,12]. $\square$

Using the $C^{1,\alpha}$ compactness, the geodesic compactifications $\bar{g}_i$ may be smoothed near the cutlocus to obtain $C^{m-1,\alpha'}$ convergence of the smoothed metrics on all of $\bar{M}$.

3. Uniqueness

In this section, we prove the uniqueness theorem, Theorem 0.2. Let g be an AH Einstein metric on a 4-manifold M, with $C^{m+1,\alpha}$ boundary metric γ, $m \geqslant 3$. By Corollary 2.5, we may assume that the geodesic compactification $\bar{g}$ associated with γ is a $C^{m,\alpha}$ compactification, so that $\bar{g}$ has a Fefferman–Graham expansion

$$g_t = g_{(0)} + t^2 g_{(2)} + t^3 g_{(3)} + \cdots + t^m g_{(m)} + O(t^{m+\alpha}). \tag{3.1}$$

The coefficients are defined by

$$g_{(j)} = \frac{1}{j!} (\mathscr{L}^{(j)}_{\bar{\nabla} t} \bar{g})|_{\partial M}, \tag{3.2}$$

where $\mathscr{L}^{(j)}$ is the j-fold Lie derivative. Observe that although expression (3.2) gives symmetric bilinear forms on $TM|_{\partial M}$, the vector $\bar{\nabla} t \in Ker\, g_{(j)}$ for all j and so $g_{(j)}$ is uniquely determined by its restriction to $T(\partial M)$. Hence we view $g_{(j)}$ as bilinear forms on ∂M.

The term $g_{(0)} = \gamma$, while the term $g_{(1)}$, equal to the second fundamental form of ∂M in $(M, \bar{g})$, vanishes. Using formulas (1.4) and (1.5) and (1.12)–(1.15), the term $g_{(2)}$ is given by

$$g_{(2)} = -\frac{1}{2}\Big(Ric_\gamma - \frac{s_\gamma}{4}\gamma\Big), \tag{3.3}$$

while the $g_{(3)}$ term satisfies

$$tr_\gamma g_{(3)} = 0, \quad \delta_\gamma g_{(3)} = 0, \tag{3.4}$$

i.e. $g_{(3)}$ is transverse traceless. However, beyond relations (3.3) and (3.4), the Einstein equations at ∂M do not determine the coefficients $g_{(j)}, j \geqslant 3$. In particular, the term $g_{(3)}$ is not a priori determined by the Einstein equations, for a given choice of boundary metric. These results follow from the work of Fefferman–Graham [16], cf. also [18,25]. Related results hold in higher dimensions, given suitable boundary regularity, up to the order $g_{(n-1)}$.

Remark 3.1. The term $g_{(3)}$ has the following interpretation from the AdS/CFT correspondence. First, the expansion (3.1) easily leads to an expansion for the volume of the geodesic 'spheres' $S(r) = \{x \in M\colon\ t(x) = 2e^{-r}\}$, of the form

$$vol\, S(r) = v_{(0)} e^{3r} + v_{(2)} e^{r} + O(e^{-\alpha r}), \tag{3.5}$$

cf. again [18] for instance. The coefficients $v_{(0)}$ and $v_{(2)}$ in (3.5) depend on the compactification $\bar{g}$, and so are not invariantly attached to (M, g). (The term $v_{(3)}$ vanishes by (3.4)). Let $B(r) = \{x \in M\colon\ t(x) \geqslant 2e^{-r}\}$ be the associated geodesic 'ball'. Then integrating (3.5) over r gives

$$vol\, B(r) = \frac{1}{3} v_{(0)} e^{3r} + v_{(2)} e^{r} + V + o(1). \tag{3.6}$$

Now general reasoning from the AdS/CFT correspondence, cf. [38], leads to the conclusion that the constant term V in (3.6) is in fact independent of the compactification $\bar{g}$ and depends only on (M, g).

The term V is the renormalized volume (or action, up to a multiplicative constant) of the AH Einstein metric (M, g). In fact, V may be computed invariantly in terms of the L^2 norm of the Weyl curvature W of (M, g) as

$$\frac{1}{8\pi^2} \int_M |W|^2\, dV = \chi(M) - \frac{3}{4\pi^2} V. \tag{3.7}$$

cf. [3]. Note in particular that (3.7) implies $V \leqslant \frac{4\pi^2}{3}\chi(M)$.

Let dV be the differential of V, acting on infinitesimal AH Einstein deformations h of a given AH Einstein metric (M, g), so that $g_s = g + sh$ is an AH Einstein metric to first-order in s. Let $h_{(0)}$ be the induced first-order variation of the boundary metric γ_s at γ. Then dV is given by

$$dV_g(h) = -\frac{1}{4}\int_{\partial M} \langle g_{(3)}, h_{(0)} \rangle \, dV_\gamma, \tag{3.8}$$

where the inner product and volume form are w.r.t. γ, cf. again [3]. Although (3.8) implies that dV is determined by the behavior at ∂M, dV is not intrinsically determined by the boundary metric γ; it depends on the global AH Einstein filling (M, g). A formula similar to (3.8) also holds in dimensions $n \geqslant 4$, when $g_{(3)}$ is replaced by $g_{(n-1)}$ and lower-order terms, cf. [3], and also [14,33].

For convenience, we restate Theorem 0.2 as follows. Define two manifolds M^1 and M^2 to be commensurable if M^1 and M^2 have covering spaces $\bar{M}^1, \bar{M}^2$ which are diffeomorphic. This is equivalent to the statement that the universal covers are diffeomorphic.

Theorem 3.2. *Let (M, g) be an AH Einstein 4-manifold with $C^{7,\alpha}$ boundary metric. Then the data $(\gamma, g_{(3)})$ on ∂M uniquely determine (M, g) up to local isometry, i.e. if g^1 and g^2 are two such AH Einstein metrics on manifolds M^1 and M^2, with $\partial M^1 = \partial M^2 = \partial M$ such that, w.r.t. geodesic compactifications,*

$$\gamma^1 = \gamma^2 \quad \text{and} \quad g^1_{(3)} = g^2_{(3)}, \tag{3.9}$$

then g^1 and g^2 are locally isometric and the manifolds M^1 and M^2 are commensurable.

The proof will be carried out in several steps below. The main issue is to prove that g^1 and g^2 are isometric within a collar neighborhood U of ∂M in M; given this it is straightforward to prove that g^1 and g^2 are then everywhere locally isometric. The basic idea to establish uniqueness within U is to set up a suitable Cauchy problem for a conformal compactification within U, and then prove uniqueness of solutions to the Cauchy problem.

Step 1: Let g^1 and g^2 be AH Einstein metrics on M satisfying (3.9). By Corollary 2.5, the geodesic compactifications $\bar{g}^1$ and $\bar{g}^2$ of g^1 and g^2 are $C^{6,\alpha}$ compactifications. The discussion preceding Theorem 3.2 implies that the first four terms $g_{(j)}$, $0 \leqslant j \leqslant 3$, of the Taylor expansion (3.1) for g^1 and g^2 agree, i.e.

$$g^1_{(j)} = g^2_{(j)}, \quad j \leqslant 3. \tag{3.10}$$

However, the geodesic defining functions t^i for g^i are not necessarily the same. We rectify this by means of a suitable diffeomorphism. Namely, a geodesic compactification (1.11) gives rise to a natural identification

$$I = I_{\bar{g}} : U \rightarrow I \times \partial M, x \rightarrow (t(x), \sigma_x(0)),$$

where σ_x is the unique $\bar{g}$ geodesic starting in ∂M through x. For distinct AH Einstein metrics $g = g^1$ and g^2 with the same boundary metric γ as above, the resulting identifications are distinct, although of course equivalent. Namely, in a possibly smaller collar neighborhood also called U, the diffeomorphism

$$\phi : U \to U, \quad \phi(I_{\bar{g}^2}^{-1}(t^2(x), \sigma_x^2(0))) = I_{\bar{g}^1}^{-1}(t^1(x), \sigma_x^1(0))$$

has the effect that $\phi^* t^2 = t^1 : U \to \mathbb{R}$. Observe that $\phi \in C^{6,\alpha}$; this is because the vector fields $\nabla_{\bar{g}^1} t^1$ and $\nabla_{\bar{g}^2} t^2$ are $C^{6,\alpha}$ and so have $C^{6,\alpha}$ flows. The map ϕ is a composition of these two flow maps.

Thus, given the fixed metric $\bar{g} = \bar{g}^1$, we pull back the metric $\bar{g}^2$ to the metric

$$\bar{\bar{g}}^2 \equiv \phi^* \bar{g}^2.$$

The $C^{5,\alpha}$ metric $\bar{\bar{g}}^2$ is of course isometric to $\bar{g}^2$ in U, has geodesic defining function t^1, and hence a splitting w.r.t. t^1. Further, the metrics $\bar{\bar{g}}^2$ and $\bar{g}^1$ have the same boundary metric γ.

Since $\phi = id$ on ∂M, the terms $\bar{g}_j^2$ and $\bar{\bar{g}}_j^2$ are equal for $j \leqslant 3$; this can be seen directly from expressions (3.3) and (3.8), cf. also [14] and references therein. It follows then that the Taylor expansions of $\bar{g}^1$ and $\bar{\bar{g}}^2$ w.r.t. t^1 agree up to order 3. In the following, we will always assume that $\bar{g}^2$ is pulled back in this way to make it comparable to a given $\bar{g}$.

Step 2: As discussed in Section 2, Einstein metrics g and their compactifications $\bar{g}$ are solutions of the conformally invariant Bach equation (2.5) in dimension 4,

$$2D^*DRic + \frac{2}{3}D^2 s + \frac{1}{3}\Delta s \cdot g + \mathscr{R}_1 = 0. \tag{3.11}$$

Here and below, we will usually drop the overbar from the notation.

Because of the conformal as well as diffeomorphism invariance of (3.11), one must choose suitable gauges, i.e. representatives of the conformal and diffeomorphism actions, in order to prove any uniqueness. For the conformal gauge, we choose the geodesic compactification, while for the diffeomorphism gauge, we use harmonic coordinates.

The AH Einstein metric (M, g) has a $C^{5,\alpha}$ geodesic compactification $\bar{g}$ with boundary metric γ, and with $C^{6,\alpha}$ geodesic defining function t. By (1.5), the scalar curvature $s = \bar{s}$ is given by $s = -6\frac{\Delta t}{t}$, and in local harmonic charts for a neighborhood U of ∂M one thus has

$$s = -6g^{ij}t^{-1}\partial_i\partial_j t.$$

It follows that the terms $D^2 s$ and Δs in (3.11) involve at most the second derivatives of the metric tensor $\bar{g}_{ij}$, with coefficients that are at least $C^{1,\alpha}$. Thus, (3.11) may be rewritten as

$$D^*DRic + Q_2(x, g, \partial_j g) = 0, \tag{3.12}$$

where Q_2 involves g and its derivatives only up to order 2, with all coefficients at least $C^{1,\alpha}$. As in (2.6) and (2.7), one may then rewrite system (3.11) in a harmonic coordinate atlas $\mathscr{A}$ covering ∂M as

$$\Delta\Delta g + Q_3(x, g, \partial_j g) = 0, \tag{3.13}$$

where Q_3 involves derivatives of g only up to order 3.

By Step 1, if g^1 and g^2 are two distinct metrics satisfying (3.9), we may assume that the associated geodesic defining functions are the same.

Step 3: In this step, we set up and prove uniqueness for the Cauchy problem for system (3.13).

From the work in Steps 1 and 2, to each AH Einstein metric g with boundary metric γ, we have associated a geodesic compactifiction $\bar{g}$ defined in a collar neighborhood U of ∂M, with fixed defining function t. In a harmonic atlas for $\bar{U}$, the metric $\bar{g}$ satisfies system (3.13). This is a four order, non-linear, elliptic system in the metric $\bar{g}$. Further, in these local coordinates, the Cauchy data on ∂M takes the form

$$\partial_t^{(j)}(g_{ij}) = g_{(j)}, \quad 0 \leqslant 3 \leqslant j, \quad \text{on } \partial M. \tag{3.14}$$

Clearly, ∂M is non-characteristic for the Cauchy problem (3.13) and (3.14). As explained at the beginning of Section 3, data (3.14) are determined by data (3.9).

We now claim that this coordinate Cauchy problem has a unique solution in a possibly smaller neighborhood $U' \subset U$ of ∂M. Given this for the moment, if g^2 is another AH Einstein metric with boundary metric γ, then by construction in Steps 1 and 2, the geodesic compactification $\bar{g}^2$ is also a solution (in local harmonic coordinate charts) to the Cauchy problem (3.13) with the same boundary Cauchy data (3.14). Hence, uniqueness to the coordinate Cauchy problem implies that the metrics g^1 and g^2 are isometric in U'.

With regard to uniqueness of the coordinate Cauchy problem, first note that the symbol (or characteristic polynomial) of Δ is

$$\sigma(\Delta) = |\xi|^2 = g^{ij}\xi_i\xi_j : T^*(M) \to \mathbb{R},$$

where g^{ij} is the metric induced on the cotangent bundle. Here and below, all computations are w.r.t. the compactification $\bar{g}$, but we omit the overbar from the notation. Hence, the leading order symbol of the Bach equation in form (3.13) is

$$\sigma(B) = \sigma(\Delta\Delta) = |\xi|^4. \tag{3.15}$$

This symbol has of course no real characteristics. However, it does have double complex characteristics. Namely, for $\xi \in T^*(\partial M)$ with $|\xi| = 1$, the roots of the characteristic form of the leading term $\Delta\Delta$,

$$p(x, \xi, \tau) = \sigma(\xi + \tau\, dt) = 0,$$

are given by

$$\tau = \pm i,$$

independent of $x \in U$ and ξ in the unit sphere bundle within $T^*(\partial M)$. Thus, the operator $\Delta\Delta$ has constant, pure imaginary, double characteristics.

The proof of uniqueness for this coordinate Cauchy problem now essentially follows from the Calderón uniqueness theorem, cf. [10] and especially [11, Theorem 11]. A clear exposition of this result is also given by Nirenberg [34, Sections 6 and 7], (however only in the case of linear equations with C^∞ coefficients); cf. also [36]. We describe below how to reduce the uniqueness problem to the class of problems solved in [11, Theorem 11].

The main issue is to reduce the non-linear Cauchy problem to a linear one. Note that while the non-linearity in the lower-order term Q_3 in (3.13) is complicated, the non-linearity in the leading order term just comes from the fact that the "unknown" g enters in the Laplacian Δ, of the form (2.8).

We do this following the elegant method of [10, Section 5]. Thus, system (3.13) is a non-linear system of 10 fourth-order PDE's in 10 unknowns $g = g_{ij} = g_{ji}$ on (a domain in) $(\mathbb{R}^4)^+$. Let u denote a variable vector in $\mathbb{R}^{10}$, (so that u corresponds to the metric g), and $\{u_\alpha\}$ the collection of all partial derivatives of u of order $\leqslant 4$. System (3.13) may then be formally expressed as

$$F(x, u, u_\alpha) = 0, \tag{3.16}$$

where $F : (\mathbb{R}^4)^+ \times \mathbb{R}^{10} \times \mathbb{R}^{64} \to \mathbb{R}^{10}$ is $C^{1,\alpha}$ smooth. Write

$$F(x, u, u_\alpha) = \Delta_u \Delta_u u + F_3(x, u, u_\alpha), \tag{3.17}$$

where F_3 corresponds to the term Q_3 in (3.13), so that F_3 has order 3.

Now suppose u and v are two solutions of (3.16), corresponding to metrics $g = g^1$ and g^2, with say v fixed. Then one has

$$\begin{aligned} 0 &= \Delta_u \Delta_u u - \Delta_v \Delta_v v + F_3(x, u, u_\alpha) - F_3(x, v, v_\alpha) \\ &= \Delta_u \Delta_u (u - v) + H(x, u, u_\alpha) - H(x, v, v_\alpha), \end{aligned}$$

where

$$H(x, u, u_\alpha) = \Delta_u \Delta_u v + F_3(x, u, u_\alpha).$$

Note that in terms of the metrics $u = g = g_1, v = g_2$, (subscripted here for convenience),

$$\Delta_u \Delta_u v = g_1^{ij} g_1^{kl} \partial_{ijkl} g_2.$$

Now as in [10, Section 5] the mean value theorem applied to H, (with x, v fixed and u varying), gives

$$\begin{aligned} H(x,u,u_\alpha) &- H(x,v,v_\alpha) \\ &= (u-v)\int_0^1 H_u[x, v+(u-v)s, v_\alpha + (u_\alpha - v_\alpha)s]\,ds \\ &\quad + \sum_\alpha (u_\alpha - v_\alpha)\int_0^1 H_\alpha[x, v+(u-v)s, v_\alpha + (u_\alpha - v_\alpha)s]\,ds. \end{aligned} \tag{3.18}$$

Substitute the solutions $u = u(x)$ and $v = v(x)$ in all terms inside the integrals in (3.18), so that the integrals then become coefficient functions in x. Hence, (3.18) becomes a third-order linear system in the unknown $u - v$. It follows that one has a solution $u - v$ of the linear fourth-order system

$$\Delta_u\Delta_u(u-v) + H(x,(u-v),(u_\alpha - v_\alpha)) = 0. \tag{3.19}$$

The leading order symbol of (3.19) is given by (3.15), and $u - v$ has 0 Cauchy data on ∂M. Hence, if the leading coefficients in (3.19) are in $C^{1,\beta}$, $\beta > 0$, and the lower-order coefficients are bounded and measurable, then [11, Theorem 11] implies that $u = v$ in a neighborhood U of ∂M. We have assumed that $u = g^1$, $v = g^2 \in C^{5,\alpha}$. This implies that the lower-order coefficients are at least in C^α, while the leading order coefficient is in $C^{5,\alpha}$.

This completes the proof of uniqueness within a collar neighborhood U. The last step is to extend this to the filling manifolds M^1 and M^2 of g^1 and g^2.

Step 4: Suppose g^1 and g^2 are two AHE metrics on manifolds M^1 and M^2 which agree, up to diffeomorphism on a collar neighborhood U of $\partial M = \partial M^i$, i.e. there is a diffeomorphism $\phi : U \to U$, $\phi = id$ on ∂M, such that

$$\phi^* g^2 = g^1. \tag{3.20}$$

We claim that g^1 and g^2 are locally isometric, i.e. for all $x^1 \in M^1$ there exists $x^2 \in M^2$ together with small open balls $V^i \in M^i$, $x^i \in V^i$ and a diffeomorphism $\psi : V^1 \to V^2$, such that $\psi^* g^2 = g^1$ on V^1. To see this, let K^i be a domain with compact closure in M^i such that $\partial K^i \subset U$, so that $M^i = K^i \cup U$. For each g^i, we may cover K^i by a finite collection of charts which are harmonic w.r.t. g^i, i.e. let $\mathscr{A}^i$ be a finite harmonic atlas for (a thickening of) K^i w.r.t. g^i. By (3.20), without loss of generality we may assume that the charts in $\mathscr{A}^1$ restricted to $U \cap K^1$ are ϕ-pullbacks of charts in $\mathscr{A}^2$ restricted to $U \cap K^2$.

Now it is well-known that in harmonic charts an Einstein metric is real-analytic and hence satisfies unique continuation. Thus, given the expression for the local components of g^1 in one local harmonic chart of $\mathscr{A}^1$, the expression for g^1 in all of the other finitely many harmonic charts of $\mathscr{A}^1$ is uniquely determined, by analytic continuation along paths. The same holds w.r.t. g^2.

Thus, given $x^1 \in K^1$, let σ^1 be an analytic path in K^1 joining x^1 to a point $x^o \in U \cap K^1$. Using the identification $\phi : U \to U$ near ∂M and analyticity, σ^1 gives rise to a unique path σ^2 in K^2, ending at a point $x^2 \in K^2$. Since g^2 and g^1 are isometric in U, analytic continuation along σ^1 and σ^2 implies that g^2 are g^1 are locally isometric near x^2 and x^1. Of course, the local isometry may depend on the homotopy class of the path σ^1. An alternate, but essentially similar argument is to show that the set of points where g^2 and g^1 are locally isometric is both open and closed, cf. also [27, Chapter 6.6].

Finally, since g^1 and g^2 are locally isometric, it follows that they are isometric in the universal covers of each M^i, and hence the manifolds M^1 and M^2 are commensurable. □

Remark 3.3. (i) We point out that the uniqueness, within a collar neighborhood, of self-dual AH Einstein metrics with real-analytic compactifications, has been proved by LeBrun [28], using twistor methods.

(ii) The proof of Theorem 3.2 strongly uses the fact that $dim\, M = 4$, since the conformally invariant Bach equation can be used in that situation. It is unknown if an analogous result holds in n-dimensions, i.e. whether the coefficients $(g_{(0)}, g_{(n-1)})$ uniquely determine an AH Einstein metric up to local isometry. Without working in the compactified setting, this would require a uniqueness result for the Cauchy problem for a highly degenerate elliptic system.

(iii) It follows of course from Theorem 3.2 that all the higher-order terms $g_{(j)}$ in the Fefferman–Graham expansion (3.1) are uniquely determined by the pair $(\gamma, g_{(3)})$. This also follows directly from an obvious analysis of the Bach equation at the boundary ∂M.

(iv) The hypothesis $\gamma \in C^{7,\alpha}$ is needed only for technical reasons arising from the proof. In the sequel paper [4], methods will be developed allowing one to use approximation arguments, so that the hypothesis $\gamma \in C^{7,\alpha}$ can be relaxed to $\gamma \in C^{3,\alpha}$.

Theorem 3.2 implies that the isometry type of (M, g) is determined by $(\gamma, g_{(3)})$ and the action of $\pi_1(M)$ on the universal cover, i.e. the representation of $\pi_1(M)$ as a subgroup of the isometry group $Isom(\tilde{M})$ of $\tilde{M}$. The examples constructed in Section 4.4 are locally isometric, non-isometric metrics on a fixed manifold, with a fixed $(\gamma, g_{(3)})$, but varying representation of $\pi_1(M)$.

4. Non-uniqueness

In this section, we examine in detail several classes of examples which show that in general an AH Einstein metric is not uniquely determined by its conformal infinity. These examples will also illustrate the sharpness of the uniqueness result, Theorem 3.2. These classes of examples are AdS black hole metrics and are discussed in some detail in the literature on the AdS/CFT correspondence. cf. [23,24,33,38], and also [6, (9.118)]. The black hole topologies may be arbitrary surfaces, i.e. S^2, T^2 or Σ_g, where Σ_g is any oriented surface of genus $g \geqslant 2$. The most interesting cases, (for the present purposes), are those of S^2 and T^2, which we treat first and last.

4.1. We begin with a discussion of the AdS-Schwarzschild metric, following [23]. On the manifold $M = \mathbb{R}^2 \times S^2$, consider the metric

$$g_m = g_m^{(+1)} = V^{-1}\, dr^2 + V\, d\theta^2 + r^2 g_{S^2(1)}, \tag{4.1}$$

where $g_{S^2(1)}$ is the standard metric of curvature $+1$ on S^2 and $V = V_m(r)$ is the function

$$V = 1 + r^2 - \frac{2m}{r}. \tag{4.2}$$

The mass m is any positive number, $m > 0$; if $m < 0$, metric (4.1) has a singularity at $r = 0$ and so it is no longer complete. The parameter r runs over the interval $[r_+, \infty)$, where r_+ is the largest root of the equation $V(r) = 0$. The locus $\Sigma = \{r = r_+\}$ in M is thus a totally geodesic round 2-sphere, of radius r_+. The circular parameter θ runs over an interval $[0, \beta]$ of length β. Smoothness of the metric g_m at Σ requires that

$$\lim_{r \to r^+} V^{1/2} \frac{d(V^{1/2})}{dr} \beta = 2\pi;$$

otherwise, the metric has a cone singularity along and normal to Σ. It follows easily from this and (4.2) that g_m is smooth everywhere exactly when

$$\beta = \frac{4\pi r_+}{1 + 3r_+^2}. \tag{4.3}$$

Observe also that the radius r_+ increases monotonically from 0 to ∞ as the mass parameter m increases from 0 to ∞.

If one sets $m = 0$ in (4.2) and $\beta = \infty$, then metric (4.1) is the hyperbolic metric $H^4(-1)$ on the 4-ball B^4, (decomposed along equidistants from $H^3(-1) \subset H^4(-1)$). This can be seen by the change of coordinates $r = \sinh s$. Here of course the sphere Σ has collapsed to a point. However, the metrics g_m do not converge (globally) to the hyperbolic metric as $m \to 0$, due to the restriction on β in (4.3). As $m \to 0$, $r_+ \to 0$, and so $\beta \to 0$. Nevertheless, for r large, the term $2m/r$ in (4.2) is small and so the local geometry of the metric g_m, for any $m > 0$, approximates hyperbolic geometry. In fact, it is easily verified that the metrics g_m are conformally compact, with conformal infinity given by the conformal class of the product metric $S^1(\beta) \times S^2(1)$.

The 1-parameter family of metrics g_m are Einstein metrics satisfying (0.1), and are isometrically distinct, i.e. g_{m_1} is not isometric to g_{m_2} for $m_1 \neq m_2$. The parameter β in (4.3) however does not increase monotonically with m or r_+. In fact, β has a maximal value β_o,

$$\beta_o = 2\pi/\sqrt{3},$$

achieved at $r_+ = 1/\sqrt{3}$. As $m \to 0$, or $m \to \infty$, $\beta \to 0$. In particular, for any $m_1 \neq 2/(3)^{3/2}$, there is an $m_2 \neq m_1$ such that the AH Einstein metrics g_{m_1} and g_{m_2} on $S^2 \times \mathbb{R}^2$ are not isometric but have the same conformal infinity. This is the first example of non-uniqueness.

As indicated above, as $\beta \to 0$, these metrics degenerate, as does the conformal structure of the boundary metric. Observe also that since $\beta \leqslant \beta_o$, the boundary metrics $S^1(\beta) \times S^2(1)$ for $\beta > \beta_o$ are not achieved in this family.

Remark 4.1. (i) There is another AH Einstein metric with conformal infinity $S^1(\beta) \times S^2(1)$. Namely let γ be a geodesic in the hyperbolic space $H^4(-1)$ and let $(M, g) = (H^4(-1)/\mathbb{Z}, g_{-1})$, where the $\mathbb{Z}$ action is generated by translation of length β along γ. This hyperbolic metric also has conformal infinity given by $S^1(\beta) \times S^2(1)$. Note that the topological type here, $\mathbb{R}^3 \times S^1$, is distinct from that of the Schwarzschild family.

In this situation, all values of the length β may be realized as boundary metrics. Further, if one replaces the (pure hyperbolic) translation along γ by a loxodromic translation, i.e. translation along γ together with a rotation in the orthogonal $H^3(-1)$, then the resulting conformal structure at infinity is a bent product $S^1(\beta) \times_\alpha S^2(1)$, where the angle α between the factors corresponds to the twist rotation.

(ii) There are a number of other explicit examples of S^2 black hole AdS metrics; for example the AdS Taub-Bolt metrics on non-trivial line bundles over S^2, cf. [23] and references therein.

4.2. Next, consider the class of AdS black hole metrics on surfaces $\Sigma = \Sigma_g$, of genus $\geqslant 2$. As above, on the manifold $M = \mathbb{R}^2 \times \Sigma$, consider the metric

$$g_m = g_m^{(-1)} = V^{-1}\, dr^2 + V\, d\theta^2 + r^2 g_\Sigma, \tag{4.4}$$

where g_Σ is a hyperbolic metric on Σ, i.e. any point in the moduli space of Riemann surfaces. Now $V = V(r)$ is given by

$$V = -1 + r^2 - \frac{2m}{r} \tag{4.5}$$

with $r \geqslant r_+$, the largest root of $V(r) = 0$. As before, the locus $\Sigma = \{r = r_+\}$ in M is totally geodesic and isometric to (Σ, g_Σ), and smoothness at the horizon requires $\theta \in [0, \beta)$ with

$$\beta = \frac{4\pi r_+}{-1 + 3r_+^2}. \tag{4.6}$$

The metrics g_m are AH Einstein metrics on M, with conformal infinity $S^1(\beta) \times \Sigma$, and are non-isometric for distinct values of m.

In contrast to the case of S^2, the function β here is monotone decreasing as m or r_+ increases, so that β is a single valued function of m or r_+. Further, the metric g_m is well-defined whenever $r_+ > 1/\sqrt{3}$, which is equivalent to

$$m > m_o = -3^{-3/2}.$$

Hence the mass parameter may assume (some) negative values.

When $m = 0$, so that $r_+ = 1$, the metric g_o is the hyperbolic metric on $\mathbb{R}^2 \times \Sigma$, and β has the value 2π. When $m \to \infty$, $\beta \to 0$, while when $m \to m_o$, $\beta \to \infty$. Thus, at these extremes, both the metrics and the conformal infinity degenerate. In particular, we see that this family does not provide examples of non-uniqueness.

4.3. Before proceeding to discuss T^2 AdS black hole metrics, in this subsection we review the well-known theory of Dehn surgery on hyperbolic 3-manifolds. This review mainly motivates the construction to follow in dimension 4 in Section 4.4, but also shows that uniqueness fails even in the category of conformally compact hyperbolic 3-manifolds.

Let (T^2, g_o) be a torus with a fixed flat metric g_o, representing a fixed point in the moduli space of flat structures on T^2. Let σ be a given simple closed geodesic in T^2, with length $L = L(\sigma)$.

Next, let γ be a complete geodesic in $H^3(-1)$ and let $T(R)$ be the R-tubular neighborhood about γ in $H^3(-1)$. The metric on $T(R)$ then has the form

$$g_{-1} = dr^2 + \sinh^2 r \, d\theta^2 + \cosh^2 r \, ds^2,$$

where s is the parameter for γ and $\theta \in [0, 2\pi]$. The boundary $\partial T(R)$ is a flat cylinder $S^1 \times \mathbb{R}$, with metric

$$\tilde{g}_o = \sinh^2 R \, d\theta^2 + \cosh^2 R \, ds^2. \tag{4.7}$$

Now choose R so that

$$2\pi \sinh R = L(\sigma).$$

There is then a unique free $\mathbb{Z}$-action on the cylinder $\partial T(R)$ such that the quotient $S^1 \times_{\mathbb{Z}} \mathbb{R}$ with the induced metric is the given flat torus (T^2, g_o) and such that the meridian circle $S^1 = \partial D^2$ of length $2\pi \sinh R$ in the cylinder is mapped to σ.

This action extends to an isometric action on $T(R)$ and so produces a hyperbolic metric g_{-1} on the solid torus $D^2 \times S^1$, with boundary isometric to (T^2, g_o), and with the geodesic σ in T^2 bounding the disc D^2 in $D^2 \times S^1$. This metric is the tube of radius R about the core closed geodesic γ. Observe that the length of the core geodesic γ, of distance R to $\partial T(R)$, is on the order of $O(\sinh^{-1} R) \ll 1$, for R large.

It is clear that this hyperbolic metric extends to a complete hyperbolic metric on $D^2 \times S^1$ with smooth conformal infinity. Since (T^2, g_o) is the metric $\tilde{g}_o$ on $S^1 \times \mathbb{R}$

divided out by the $\mathbb{Z}$ action, the conformal infinity is given by the conformal class $(T^2, [g_\infty])$ where

$$g_\infty = (e^{2R}\, d\theta^2 + e^{2R}\, ds^2)/\mathbb{Z}. \tag{4.8}$$

The classes $[g_\infty]$ and $[g_o]$ do not agree, (although $[g_\infty] \to [g_o]$ on any sequence where $L(\sigma) \to \infty$). However, the construction above can easily be modified so that the conformal infinity is fixed instead of fixing the conformal structure g_o on $\partial T(R)$. Namely, for any fixed R, write $s' = s'(R) = \frac{\cosh R}{\sinh R} s$, so that in these new coordinates, the metric $\tilde{g}_o$ in (4.7) has the form

$$\tilde{g}_o = \sinh^2 R (d\theta^2 + (ds')^2). \tag{4.9}$$

Now divide $D^2(R) \times \mathbb{R}$ and $\partial D^2(R) \times \mathbb{R}$ by the same $\mathbb{Z}$ action as before, but with respect to the parameters (θ, s') in place of (θ, s). This gives a complete hyperbolic metric on $D^2 \times S^1$ with prescribed conformal infinity (T^2, g_o), for any choice of closed geodesic $\sigma \subset (T^2, g_o)$.

Summarizing, the discussion above proves:

Proposition 4.2. *For any given flat structure g_o on the torus T^2, and for any given simple closed geodesic σ in (T^2, g_o), there is a unique complete hyperbolic metric g_{-1} on the solid torus $D^2 \times S^1$, with (T^2, g_o) as conformal infinity.*

As σ varies over the class of simple closed geodesics on T^2, the resulting hyperbolic metrics, although of course locally isometric, are not isometric since for instance the lengths of the core geodesics are distinct; compare with the discussion at the end of Section 3. In particular, there are infinitely many distinct hyperbolic 3-manifolds, all diffeomorphic to $D^2 \times S^1$, whose conformal infinity is an arbitrary but fixed (T^2, g_o).

As $L = L(\sigma) \to \infty$, the length of the core geodesic γ tends to 0. Any sequence of such metrics thus converges to the complete (rank 2) hyperbolic cusp

$$g_C = dr^2 + e^{2r} g_o, \tag{4.10}$$

on $\mathbb{R} \times T^2$. This process is the formation of a cusp, or "opening a cusp", cf. [20,37].

Remark 4.3. The process described above of opening a cusp may also be reversed. Thus, given a complete hyperbolic cusp as in (4.10), the Dehn surgery process above closes this cusp by filling in with a hyperbolic solid torus, keeping the conformal structure at infinity fixed. As discussed above, this can be done in infinitely many non-isometric ways.

More generally, let (M^3, g_{-1}) be any complete conformally compact hyperbolic 3-manifold with cusps, so that the ε-thick part of M^3 is conformally compact while the ε-thin part consists of a finite number of cusps (4.10), cf. [37, Chapter 5]. Then the Jorgensen–Thurston theory implies that one can close the cusps by hyperbolic manifolds, at least for all sufficiently short core geodesics, and with at most a small perturbation of the structure of conformal infinity.

In contrast to the situation with solid tori, these manifolds obtained by performing Dehn surgery on the cusps of (M^3, g_{-1}) are generally not diffeomorphic. For a fixed diffeomorphism type, typically only finitely many such hyperbolic manifolds have a fixed conformal infinity.

4.4. The construction for hyperbolic 3-manifolds above is special to dimension 3, and cannot be carried out for hyperbolic manifolds in dimensions $\geqslant 4$. However, we show it can be carried out for AH Einstein metrics in dimension 4, (or greater).

Thus, consider the following T^2 AdS black hole metrics; we first discuss these on the universal cover $\mathbb{R}^2 \times \mathbb{R}^2$, and then descend to the quotient $\mathbb{R}^2 \times T^2$. As before, let

$$g_m = g_m^{(0)} = V^{-1}\, dr^2 + V\, d\theta^2 + r^2(ds_1^2 + ds_2^2), \tag{4.11}$$

where

$$V = V(r) = r^2 - \frac{2m}{r}. \tag{4.12}$$

As previously, we require $r \geqslant r_+ = (2m)^{1/3} > 0$, where r_+ is the (unique) root of the equation $V(r) = 0$, while $s_1, s_2 \in \mathbb{R}$. The metric is smooth provided θ runs over the parameter interval $[0, \beta]$, where $\beta = \beta_m$ is given by

$$\beta = \frac{4\pi}{3r_+}, \tag{4.13}$$

In contrast to the situation with genus $g \neq 1$ black holes, on the space $\mathbb{R}^2 \times \mathbb{R}^2$, the metrics g_m are in fact all isometric; the change of parameters (i.e. diffeomorphism) given by $r = m^{1/3}s$, $\theta = m^{1/3}\psi$ and $s_i = m^{1/3}t_i, i = 1, 2$, gives an isometry between g_m and g_1. Thus, in the following, we set $m = 1$.

Now we essentially repeat the construction in Section 4.3 on these metrics. Thus, fix an arbitrary flat structure g_o on T^2, and fix an arbitrary simple closed geodesic σ in (T^2, g_o). Let $L = L(\sigma)$ be the length of σ in (T^2, g_o). Consider first the three-dimensional metric

$$g_m' = V^{-1}\, dr^2 + V\, d\theta^2 + r^2\, ds_1^2, \tag{4.14}$$

on $D^2 \times \mathbb{R}$, for V as in (4.12) with $m = 1$. Choose R so that

$$V(R)^{1/2} \cdot \beta = L.$$

Thus, at the boundary $\partial(D^2(R) \times \mathbb{R})$, the metric is the flat metric

$$V(R)\, d\theta^2 + R^2\, ds_1^2 \tag{4.15}$$

on the cylinder $S^1 \times \mathbb{R}$. The group of Euclidean isometries acts on this space, and just as before, given (T^2, g_o) there is a unique isometric $\mathbb{Z}$-action on $S^1 \times \mathbb{R}$ such that the $\mathbb{Z}$-quotient metric of (4.15) is (T^2, g_o) for which the meridian θ circle bounding the disc is taken to σ.

This isometric $\mathbb{Z}$-action on the boundary extends to an isometric $\mathbb{Z}$-action on the interior $D^2(R) \times \mathbb{R}$ and the quotient is a solid torus $D^2(R) \times S^1$, with $\partial D^2(R) = \sigma$. The core geodesic, of distance R to the boundary, has length of order $O(e^{-R})$. Further and as before, the metric extends to a complete metric on $D^2 \times S^1$.

In the same way as described in (4.8) and (4.9), one may alter this construction slightly to produce such complete, conformally compact metrics with the conformal infinity (T^2, g_o) prescribed, in place of prescribing the geometry at distance R.

Finally, return to the 4-metric (4.11) and choose an arbitrary, but fixed, range for the parameter s_2, so that $s_2 \in [0, \beta_2]$.

To sum up, the analysis above proves:

Proposition 4.4. *Given any flat structure (T^2, g_o) on the torus, and any simple closed geodesic σ in (T^2, g_o), there is a complete AH Einstein metric g on the 4-manifold $\mathbb{R}^2 \times T^2$, whose conformal infinity is the flat product $(T^2, g_o) \times S^1(\beta_2)$, for any given $\beta_2 > 0$. These metrics on $\mathbb{R}^2 \times T^2$ are all locally isometric, but the isometry type of a metric in this family is uniquely determined by the data (T^2, g_o), β_2 and σ.*

Hence, one has an infinite family of AH Einstein metrics with a given conformal infinity. If σ_i is a sequence of geodesics with $L(\sigma_i) \to \infty$, the corresponding metrics g_i converge to the complete hyperbolic cusp metric

$$g_C = dr^2 + e^{2r} g_{T^3}, \tag{4.16}$$

on $\mathbb{R} \times T^3$, where $(T^3, g_{T^3}) = (T^2, g_o) \times S^1(\beta_2)$. Here the convergence is based at points x_i for which $t_i(x_i) = 1$ for instance, where t_i is the geodesic defining function. Thus, one sees that the regions where the metrics g_i differ a definite amount from a hyperbolic metric are being pushed further and further down the cusp (this corresponds to letting $m \to 0$). We also point out that a brief computation shows that the $g_{(3)}$ term for any of these metrics satisfies $g_{(3)} = 0$; compare with Theorem 3.2.

These examples illustrate that one may, at least in certain situations, open cusps in the class of AH Einstein metrics. Similar but more general examples may be obtained by performing Dehn surgery on a closed geodesic in T^3 in place of $T^2 \times S^1$.

Remark 4.5. In analogy to Remark 4.3, it is an interesting open question whether this process can be reversed in general. Thus, given a complete hyperbolic 4-manifold M, with smooth conformal infinity ∂M, and with a finite number of cusps, does there exist a sequence of AH Einstein manifolds (M_i, g_i), without cusps, such that (M_i, g_i) converges to (M, g), and such that the conformal infinity is either fixed, or converges to that of (M, g)? Again in analogy to Remark 4.3, it is to be expected that this requires M_i to range over an infinite collection of topological types in general.

Remark 4.6. All of the discussion in Sections 4.1, 4.2 and 4.4 above generalizes in a straightforward way to dimensions $n > 4$. Thus, one replaces the surfaces Σ_g, $g \geqslant 0$, of constant curvature ± 1, 0, by $(n-2)$-dimensional compact Einstein manifolds Σ^{n-2} of Ricci curvature $\pm(n-3)$, 0. The function V becomes $V = c + r^2 - \frac{2m}{r^{n-3}}$, with $c = \pm 1$, 0, as before.

For the purposes of the next section for fixed boundary data (T^2, g_o, β_2), consider the behavior of the geodesic compactifications $\bar{g}_i$ on $T^2 \times \mathbb{R}^2$, for g_i as above with $L(\sigma_i) \to \infty$. First, the geodesic compactification $\bar{g}$ of the $\mathbb{R} \times T^3$ hyperbolic cusp metric (4.16) has the form

$$\bar{g} = dt^2 + g_{T^3}, \tag{4.17}$$

i.e. $\bar{g}$ is the flat product metric on $\mathbb{R}^+ \times T^3$. Here of course $t = 2e^{-r}$, and the boundary ∂M occurs at $t = 0$. Note that the "compactification" $\bar{g}$ is not compact, due to the cusp. As $i \to \infty$, the (true) compactifications $\bar{g}_i$ converge to $\bar{g}$, uniformly on compact subsets, based at a point say on $\partial M = \{0\} \times T^3$. In particular, (and this is the main point), we have

$$diam_{\bar{g}_i} M \to \infty, \quad \text{as } i \to \infty \tag{4.18}$$

on $M = \mathbb{R}^2 \times T^2$.

5. Cusp formation and hyperbolic manifolds

Proposition 4.4 shows that one may close a complete hyperbolic cusp $\mathbb{R} \times T^3$ in the class of AH Einstein metrics on the 4-manifold $\mathbb{R}^2 \times T^2$ with a fixed conformal infinity. This implies in particular that the space of AH Einstein metrics on a fixed manifold M with a fixed conformal infinity is not, in general, compact; there are sequences (M, g_i) of AH Einstein metrics which do not converge to an AH Einstein metric on the same space.

In this section, we prove a type of converse of this statement, namely that under reasonable convergence conditions, one can open cusps for AH Einstein metrics only when the resulting limit is a complete hyperbolic 4-manifold. More generally, divergent sequences of AH Einstein metrics with controlled conformal infinity can only limit on complete hyperbolic 4-manifolds with at least one cusp. The exact statement is given in Theorem 5.3.

If $(M, \bar{g})$ is a geodesic compactification of an AH Einstein manifold (M, g) with geodesic defining function t, define its width $Wid_{\bar{g}}(M)$ by

$$Wid_{\bar{g}}(M) = sup\{t(x) \colon x \in M\}. \tag{5.1}$$

Thus, $Wid_{\bar{g}}(M)$ is the length of the longest minimizing $\bar{g}$ geodesic starting at ∂M and orthogonal to ∂M. Note that $Wid_{\bar{g}}(M)$ depends on the choice of geodesic compactification, i.e. the choice of the boundary metric. Two different choices of the boundary metric will give rise to different widths, although they can be estimated in terms of each other by the conformal factor relating the boundary metrics. Observe that the compactifications $\bar{g}_i$ of the AH Einstein metrics g_i discussed following Remark 4.6 satisfy

$$Wid_{\bar{g}_i}(M) \to \infty \quad \text{as } i \to \infty, \tag{5.2}$$

corresponding to (4.18).

As an introduction to the technique, we first show that cusps, (or new ends in general), cannot form when the conformal infinity has positive scalar curvature.

Proposition 5.1. *Let (M, g) be an AH Einstein 4-manifold, with boundary metric γ. Suppose that there is a component of ∂M on which the scalar curvature s_γ of γ satisfies*

$$s_\gamma \geqslant s_o > 0 \tag{5.3}$$

for some constant s_o. Then ∂M is connected and if $\bar{g}$ is the geodesic compactification associated to γ, then

$$Wid_{\bar{g}} M \leqslant D = \sqrt{3}\pi/\sqrt{s_o}. \tag{5.4}$$

Proof. Let $\partial_o M$ be a component of ∂M satisfying (5.3). For $t_1 > 0$ sufficiently small, let $S_0(t_1) = \{x \in M\colon dist_{\bar{g}}(x, \partial_o M) = t_1\}$, so that $S_0(t_1)$ is connected and smooth. We may view $S_0(t_1) \subset (M, g)$, so that the function r as in (1.6) has the value $r_1 = \log(\frac{2}{t_1})$ on $S_0(t_1)$.

We construct now a partial defining function t_o, i.e. a defining function for the component $\partial_o M$ in the obvious way. Thus, set

$$t_o = 2e^{-r_o},$$

where $r_o(x) = sgndist(x, S_0(t_1)) + r_1$, and *sgndist* is the signed distance function on (M, g) to $S_0(t_1)$, i.e. $sgndist(x) = \pm dist_g(x, S_0(t_1))$ according to whether $t(x) < t_1$ or $t(x) > t_1$.

Note that if $\partial M = \partial_o M$, then $t = t_o$ is a (full) defining function for the boundary. Otherwise however, $t \neq t_o$ and the function t_o compactifies only the end of (M, g) corresponding to $\partial_o M$, in that $\bar{g}_o = t_o^2 g$ is compact only on this boundary component. The other boundary components of $(M, \bar{g}_o)$ are all of infinite $\bar{g}_o$-distance to $\partial_o M$.

In either case, it then suffices to show that the maximal length L of a (minimizing) t_o-geodesic $\sigma(t_o)$ of $\bar{g}_o$ satisfies (5.4). This will imply that ∂M is connected, since ∂M disconnected implies $L = \infty$, giving a contradiction. When ∂M is connected, $\bar{g}_o = \bar{g}$, $L = Wid_{\bar{g}}(M)$, and so (5.4) also follows.

By (1.19), we have $\bar{s}'_o = 6t_o^{-1}|\bar{D}^2 t_o|^2 \geqslant 0$ along σ, so that $\bar{s}_o(\sigma(t_o))$ is monotone increasing along σ. Further, (1.4) and (1.5) and (1.13) imply that $Ric_{\bar{g}_o}(N, N) =$

$\frac{1}{6}\bar{s}_o \geqslant \frac{1}{4}s_\gamma$, where $N = \bar{\nabla}t_o$ is the unit tangent vector to σ. Hence, along $\sigma(t_o)$, one has

$$Ric_{\bar{g}_o}(N,N) \geqslant \frac{1}{4}s_o.$$

Now a standard result (Rauch comparison theorem) in Riemannian comparison geometry, cf. [35], implies that $\sigma(t_o)$ must have a focal point at distance $D \leqslant \sqrt{3}\pi/\sqrt{s_o}$, which gives (5.4).

An alternate, even more elementary argument is as follows. Eq. (1.19) together with the obvious estimate $|\bar{D}^2 t_o|^2 \geqslant \frac{1}{3}(\bar{\Delta}t_o)^2$ and (1.5) imply $\bar{s}_o' \geqslant \frac{1}{18}t_o\bar{s}_o^2$. Dividing by $\bar{s}_o^2$ and integrating gives (5.4) with the slightly weaker estimate $D = 6/\sqrt{s_o}$. $\square$

Remark 5.2. Proposition 5.1 holds in all dimensions, with the same proof, cf. also Remark 1.5. As such, it gives a simple new proof of the connectedness result of Witten–Yau [39]. More generally, suppose (5.3) is replaced by the weaker condition that $s_\gamma \geqslant 0$ and ∂M is not connected. Then there is an infinite t_o-geodesic σ of $\bar{g}_o$ joining $\partial_o M$ with a distinct boundary component of M. The argument of Proposition 5.1 implies that $\bar{s}_o \equiv 0$ along σ, and hence, via (1.19), $D^2 t_o \equiv 0$ along σ. By (1.5), this means that $\bar{g}_o$ is Ricci-flat and has a parallel vector field ∇t_o along σ, and so the metric $\bar{g}_o$ has an infinitesimal splitting as a product of $\mathbb{R}$ with a Ricci-flat metric γ_{o}. We will see later in Lemma 5.5 that this infinitesimal splitting may be globalized to a full splitting, using arguments as in the Cheeger–Gromoll splitting theorem. It follows that either ∂M is connected or (M,g) is a complete cusp of the form

$$g = dr^2 + e^{2r}\gamma_{\mathrm{o}},$$

where $(\partial_o M, \gamma_{\mathrm{o}})$ is a compact Ricci-flat manifold. This result has been proved by Cai–Galloway [9] using different although related methods. Of course, under the weaker bound $s_\gamma \geqslant 0$, even if ∂M is connected one no longer has the effective bound (5.4).

We now begin the analysis of the formation of cusps. More generally, we study the behavior of sequences $\{g_i\}$ of AH Einstein metrics on 4-manifolds which have controlled conformal infinities, but which diverge in the sense that $\{g_i\}$ does not converge to an AH Einstein metric on the same manifold.

Thus, let (M_i, g_i) be a sequence of AH Einstein 4-manifolds, with a fixed boundary $\partial M_i = \partial M$. Suppose that the conformal infinities $[\gamma_i]$ of g_i are $C^{m+1,\alpha}$, $m \geqslant 2$, and converge to a limit, so that there are representative metrics $\gamma_i \in [\gamma_i]$ such that $\gamma_i \to \gamma$ in $C^{m+1,\alpha}(\partial M)$. Let $\bar{g}_i$ be the associated $C^{m,\alpha}$ geodesic compactifications, with t_i the associated $C^{m+1,\alpha}$ geodesic defining functions. We assume the following:

Convergence condition: The compactifications $(M_i, \bar{g}_i)$ converge in the $C^{m,\alpha}$ topology, for some $m \geqslant 2$, and uniformly on *compact subsets*, to a limit metric $(N, \bar{g})$, with boundary metric γ.

This convergence condition should be understood in light of Proposition 2.7. In particular, the metrics $\bar{g}_i$ and $\bar{g}$ are smoothed near their cutloci to obtain $C^{m,\alpha}$ convergence across the cutlocus. It turns out that this convergence condition is not a strong assumption at all, but this will only be completely clear in the sequel paper [4].

Since they are distance functions to ∂M, the defining functions t_i then also converge to the limit $C^{m+1,\alpha}$ geodesic defining function t for $\bar{g}$. Given base points $x_i \in t_i^{-1}(t_o) \subset M_i$, for some fixed t_o with $0 < t_o < Wid_{\bar{g}_i} M$, it follows that the AH Einstein manifolds (M_i, g_i, x_i) converge, uniformly on compact subsets, to a limit complete Einstein manifold (N, g, x_∞), $x_\infty = \lim x_i$, with "compactification" $\bar{g} = t^2 \cdot g$. The convergence is in the pointed Gromov–Hausdorff topology based at x_i, cf. [21, Chapter 3], and also in the C^∞ topology, since C^2 convergence of Einstein metrics implies C^∞ convergence, by elliptic regularity.

Now if the width $Wid_{\bar{g}_i}(M_i)$ of the manifolds $(M_i, \bar{g}_i)$ is uniformly bounded above, it follows by a standard application of the Cheeger–Gromov compactness theorem, cf. [2,12], that, in a subsequence, the manifolds M_i are all diffeomorphic to a fixed manifold M, $M = N$, and $\bar{g}_i \to \bar{g}$ in the $C^{m,\alpha}$ topology on M. (The curvature, volume and diameter of $(M_i, \bar{g}_i)$ are all uniformly bounded.) Hence, in a subsequence, g_i is a sequence of AH Einstein metrics on M, converging to a limit AH Einstein metric g on M, for which the boundary metrics $\gamma_i \to \gamma$; compare again with Proposition 2.7. In other words, the sequence (M_i, g_i) is not divergent in this situation.

On the other hand, if

$$Wid_{\bar{g}_i}(M_i) \to \infty,$$

then any limit complete Einstein manifold (N, g), (again in a subsequence), has a non-empty collection of "new" ends, whose boundary $\partial_\infty N$ is at infinite $\bar{g}$-distance to $\partial N = \partial M$. In particular, although for any fixed $T < \infty$, the domains $U_i(T) = t_i^{-1}[0, T] \subset M_i$ are diffeomorphic to $U(T) = t^{-1}[0, T] \subset N$, (for T a regular value and i sufficiently large), the full manifold N is not diffeomorphic to any M_i. The discussion concerning and following Proposition 4.4 exhibits examples where the infinite end of N is a cusp, although this of course does not follow automatically in general.

To state the main result on the structure of (N, g) below, we need the following two definitions. First, let $\Omega = \Omega(1) = t^{-1}[1, \infty) \subset N$ and let $E \subset \Omega$ be any end of Ω; (note that E is distinct from an end of N corresponding to a boundary component of ∂M). Let $S_E(t) = S(t) \cap E$, where $S(t)$ is the t-level set of the geodesic defining function t and define

$$T_o(E) = sup\{t : inf_{S_E(t)} \bar{s} < 0\}. \tag{5.5}$$

If $\bar{s} \geqslant 0$ in E, set $T_o(E) = 0$. Recall again from (1.19) that $\bar{s}$ is non-decreasing along t-geodesics in $(\Omega, \bar{g})$.

Next, define an end $E \subset \Omega$ as above to be *weakly hyperbolic* if

$$|K+1|(x) \to 0, \quad \text{as } t(x) \to \infty \quad \text{in } E, \tag{5.6}$$

where K denotes the sectional curvature of (E, g) at any plane in $T_x E$.

Recall also the definition of a conformally compact hyperbolic manifold with cusps, as in Remark 4.3 (but in dimension 4 instead of 3). We then have the following partial characterization of the limits (N, g) obtained above; this result may be considered as a converse to the results of Section 4.4, cf. Proposition 4.4.

Theorem 5.3. *Let (M_i, g_i) be a sequence of AH Einstein 4-manifolds, $\partial M_i = \partial M$, which satisfy the convergence condition. Suppose that the Euler characteristics $\chi(M_i)$ satisfy $\chi(M_i) \leqslant \chi_o$, for some $\chi_o < \infty$, that*

$$Wid_{\bar{g}_i}(M_i) \to \infty, \tag{5.7}$$

and that either one of the following two conditions hold:

(i) *There is an end $E \subset \Omega \subset N$ such that $T_o(E) < \infty$.*
(ii) *(Ω, g) has a weakly hyperbolic end.*

Then the limit (N, g) is a complete conformally compact hyperbolic manifold with cusps, with conformal infinity $[\gamma]$ on ∂M. In particular, $(\partial M, [\gamma])$ is a conformally flat 3-manifold.

Understandably, the proof is rather long and so is broken into several steps.

Step I: First, one needs to control the global size of the AH Einstein manifolds (M_i, g_i) away from ∂M.

Lemma 5.4. *Under the assumptions of Theorem 5.3, let $\Omega_i = \{x \in (M_i, \bar{g}_i)$: $t_i(x) \geqslant 1\} = \{x \in (M_i, g_i)$: $r_i(x) \leqslant \log 2\}$, where t_i is the geodesic defining function and r_i is as in (1.6). Then there is a constant $V^o < \infty$ such that, for all i,*

$$vol_{g_i} \Omega_i \leqslant V^o. \tag{5.8}$$

Proof. For any (M_i, g_i), the geodesic 'spheres' $S(t) = S_{g_i}(t)$, i.e. the t level sets of the functions t_i, have the asymptotic expansion (3.5):

$$vol_{g_i} S(t) = v_{(0)} t^{-3} + v_{(2)} t^{-1} + o(t). \tag{5.9}$$

Now by the convergence condition, the geometry of $(M_i, \bar{g}_i)$ between $t = 0$ and 1 is uniformly controlled in $C^{m,\alpha}$, and so converges smoothly to that of the limit $(N, \bar{g})$ in this region. So do the defining functions $t_i \to t$, and the coefficients $v_{(0)}, v_{(2)}$. Thus, expansion (5.9) is uniform on $S(t)$, in that the lower-order term $o(t)$ is small for t small, independent of i. Hence, for t_o small but fixed, by integrating (5.9) over the region $t \geqslant t_o$, we obtain, for i sufficiently large,

$$vol_{g_i} B(t_o) \leqslant \tfrac{1}{3} v_{(0)} t_o^{-3} + v_{(2)} t_o^{-1} + V + 1,$$

where $V = V_i$ is the renormalized volume of (M_i, g_i), cf. (3.6), and $B(t_o) = t_i^{-1}([t_o, \infty)) \subset (M_i, g_i)$.

Now by (3.7), the upper bound on $\chi(M_i)$ gives a uniform upper bound on V. This gives a uniform upper bound on $vol_{g_i} B(t_o)$ and hence (5.8) follows. $\square$

Summarizing, we have the following description of the structure of the limit (N, g, x_∞) of (M_i, g_i, x_i). In the collar neighborhood $U_i = M_i \setminus \Omega_i$ where $t_i \leqslant 1$, the convergence condition implies that the compactifications $\bar{g}_i$ converge smoothly to the compactification $\bar{g} = t^2 \cdot g$ of the limit. In particular, each U_i is diffeomorphic to a collar neighborhood U of ∂M. By Lemma 5.4, the complementary domains Ω_i have uniformly bounded volume, and hence the limit region $\Omega \subset N$ also has finite volume. Further, by (5.7), $Wid_{\bar{g}} \Omega = \infty$, so that $\Omega \subset N$ has "new" ends, formed from the limiting behavior of (M_i, g_i).

Each end E of Ω is 'cusp-like' in that it has finite volume, and so $vol B_x(1) \to 0$, as $x \to \infty$ in E. The proof is now split into two cases, according to hypotheses (i) or (ii).

Step II: The following result proves Theorem 5.3 in case (i) holds.

Lemma 5.5. *Suppose that, for some end* $E \subset \Omega$,

$$T_o(E) < \infty. \tag{5.10}$$

Then (N, g) *is a complete conformally compact hyperbolic manifold with at least one cusp.*

Proof. Given (5.10), the monotonicity of $\bar{s}$ implies that there is a subend $E' \subset E$ on which $\bar{s} \geqslant 0$ and hence there is a t_o such that $\bar{s} \geqslant 0$ on $\Omega_{t_o} = \{x \in E\colon\ t(x) \geqslant t_o\}$. By (1.5), this means that $\bar{H} = \bar{\Delta} t \leqslant 0$ on Ω_{t_o}, where $\bar{H}$ is the mean curvature of the level set $S(t)$, i.e. $S_E(t)$, in the direction $\bar{\nabla} t$. Since $Wid_{\bar{g}} E = \infty$, the discussion in Remark 5.2 shows there is an infinitesimal splitting of $(\Omega_{t_o}, \bar{g})$ along a t-geodesic ray σ in Ω_{t_o}.

To globalize this splitting, consider the domain Ω_{t_o} with respect to the Einstein metric g. By standard formulas for conformal change, cf. also [2, (1.18)], one has

$$H = 3 - t\bar{H},$$

where H is the mean curvature of the Lipschitz hypersurface $S(t)$ w.r.t. the outward normal ∇r, for r and t related as in (1.6). Since $\bar{H} \leqslant 0$ on $S(t)$, $t \geqslant t_o$, the mean curvature of $(\partial \Omega_{t_o}, g)$ satisfies

$$H \geqslant 3.$$

As in the proof of the Cheeger–Gromoll splitting theorem, cf. [6, Chapter 6G], this estimate also holds in the sense of distributions or support functions at the cut points of t on $S(t_o)$ where $S(t_o)$ is not smooth. Since we also have $Ric_g = -3g$, the modification of the Cheeger–Gromoll splitting theorem by Kasue [26], cf. also [13], implies that (Ω_{t_o}, g) splits globally as a warped product, i.e. as a hyperbolic cusp metric

$$g_C = dr^2 + e^{2r} g_{T^3}, \tag{5.11}$$

on $\mathbb{R} \times T^3$, where g_{T^3} is a flat metric on the 3-torus T^3. It follows that the full complete manifold (N, g) is hyperbolic, since Einstein metrics are analytic. □

Step III: In this step, we prove Theorem 5.3 in case (ii) holds, so that there is an end E which is weakly hyperbolic as in (5.6). The next result specifies the geometry of such an end more precisely, using Lemma 5.4.

Lemma 5.6. *A weakly hyperbolic end (E, g) of (N, g) is topologically $\mathbb{R}^+ \times T^3$ and the metric asymptotically approaches a hyperbolic cusp metric g_C, as in* (5.11), *uniformly on compact sets as $t \to \infty$. More precisely, for any $\varepsilon > 0$ and $T < \infty$, there is a $T_o = T_o(\varepsilon, T)$ such that if $t(y) \geqslant T_o$, then the geodesic annulus*

$$A_y(T) = \{x \in E : t(x) \in (T^{-1}t(y), Tt(y))\},$$

diffeomorphic to $I \times T^3$, is ε-collapsed, in that $diam_g T^3_s < \varepsilon$, where $T^3_s = t^{-1}(s)$.

Moreover, there exist finite covering spaces $\bar{A}_y(T)$ of $A_y(T)$, unwrapping the collapse of the T^3 factors, such that the metric g is of the form

$$g|_{\bar{A}_y(T)} = g_C + \kappa_y, \tag{5.12}$$

where the perturbation κ_y satisfies $||\kappa_y|| < \varepsilon$ in the C^k topology on $\bar{A}_y(T)$, for any given $k < \infty$.

Proof. The weakly hyperbolic end (E, g) has uniformly bounded curvature, with curvature approaching -1 as $t \to \infty$. Further, Lemma 5.4 implies that (E, g) has finite volume, so that the volumes of unit balls $B_y(1)$ tends uniformly to 0 as $t(y) \to \infty$. This means that the manifolds (E, g, y), based at points y, are collapsing with bounded curvature as y tends to ∞ in E. Hence, the annuli $A_y(T)$ have an F-structure, cf. [12], formed essentially by the collection of short geodesic loops in $A_y(T)$, for $t(y)$ large.

When the curvature is highly pinched about -1, the structure of such collapse is described by the Margulis lemma, cf. [21,37]. Thus, as in the statement of the lemma, there are (in fact abelian) covering spaces of the annuli $A_y(T)$ unwrapping the collapse; the choice of such covering spaces is not unique, but they may be chosen so that the injectivity radius and diameter of the fibers of the F-structure are on the order of 1. (The degree of the covering of course depends on the degree ε of the collapse). In such covering spaces $\bar{A}_y(T)$, the curvature of the metric is uniformly close to -1, while the diameter and volume of this region is uniformly bounded, away from 0 and ∞, for $t(y)$ sufficiently large. The Cheeger–Gromov compactness theorem then implies that the metric g is uniformly close to a hyperbolic metric on $\bar{A}_y(T)$. Further, the covering transformations are uniformly close to hyperbolic isometries. In any limit as $t(y) \to \infty$, the covering group is hence $\mathbb{Z}^3$, the orbits of the F-structure are flat 3-tori, and the limit metric is the hyperbolic cusp metric g_C in (5.11). Thus, if $t(y)$ is sufficiently large, the metric g on $\bar{A}_y(T)$ is uniformly close to a hyperbolic cusp metric. Since the metric g is Einstein, the metrics $\bar{A}_y(T)$ are close to g_C in the C^∞ topology.

Finally, since all geodesic annuli $A_y(T)$ are topologically $I \times T^3$, it follows easily that the end E is topologically $\mathbb{R}^+ \times T^3$. $\square$

Lemma 5.6 describes the structure of the end E in the region where $t \gg 1$. Note that one may let $T \to \infty$, (sufficiently slowly), as $T_o \to \infty$ in Lemma 5.6. In particular, if y_k is any divergent sequence in E, i.e. $t(y_k) \to \infty$ in (E, g), then the based sequence (E, g, y_k) has subsequences converging, after unwrapping the collapse as above, to the complete hyperbolic cusp metric g_C. The limit parameter $r = r_\infty$ in (5.11) is then given by

$$r_\infty = \lim_{k \to \infty} (r - r(y_k))$$

with $r = \log(\frac{2}{t})$ as in (1.6). Thus, $r_\infty(y) = 0$, where $y = \lim y_k$ is the limit of the base points y_k. (This is of course analogous to the classical construction of Busemann functions.)

The asymptotic behavior $t \gg 1$ of the 'compactification' $\bar{g} = t^2 \cdot g$ of (E, g) has a similar description. Thus let t_k be the geodesic defining function associated with the function $r - r(y_k)$, so that

$$t_k = \frac{1}{t(y_k)} \cdot t. \tag{5.13}$$

Thus t_k renormalizes t at y_k, in that t_k is a geodesic defining function with $t_k(y_k) = 1$. The metrics $\bar{g}_k = t_k^2 \cdot g = (t(y_k))^{-2} \cdot \bar{g}$, when based at y_k, and unwrapped by passing to covers of T^3 as above, have a subsequence converging to the flat product metric

$$g_F = dt^2 + g_{T^3}, \tag{5.14}$$

on $F = \mathbb{R}^+ \times T^3$; this follows for example from formulas (1.3)–(1.5) and (1.19), compare with (4.17). Here, the limit parameter t is given by $t \equiv t_\infty = \lim_{k \to \infty} t_k$, associated to r_∞ as above. Of course for $y = \lim y_k$ as above, $t(y) = t_\infty(y) = 1$, so that $y \in F$ has distance 1 to $\partial F = \{0\} \times T^3$.

This discussion holds for *any* divergent sequence $\{y_k\}$ in E. Note, however, that we do not assert that the flat structure on T^3 is independent of the sequence $\{y_k\}$. A priori it is possible that different sequences may give rise to flat limits (5.14) with distinct flat structures on T^3, although if y_k and y'_k are distinct sequences with

$t(y_k)/t(y_k')$ bounded away from 0 and infinity, then the limit metrics are the same, (i.e. isometric). This possibility of the non-uniqueness of the 'tangent cones at infinity', does not play any role, however, in the remainder of the proof.

It is worth emphasizing again that, for any divergent sequence $\{y_k\}$, $t(y_k) \to \infty$ in (E, g), the sequence of metrics $\bar{g}_k$ as $k \to \infty$ describes the normalized asymptotic behavior in regions about y_k of the *fixed* metric $(E, \bar{g})$, in that the metrics $\{\bar{g}_k\}$ are just rescalings and unwrappings of $\bar{g}$ based at y_k.

An end $E \subset \Omega$ having the structure described in Lemma 5.6 will be called an *asymptotically hyperbolic cusp*. Theorem 5.3 is now an immediate consequence of the following rigidity result.

Proposition 5.7. *Let (N, g) be an AH Einstein 4-manifold, with at least one asymptotically hyperbolic cusp E. Then (N, g) is hyperbolic.*

Proof. As described above, for *any* divergent sequence y_k in E, the Riemannian manifolds (E, g, y_k) converge, in a subsequence and uniformly on compact subsets, to a complete hyperbolic cusp after unwrapping the collapse. The limit parameter $r = r_\infty$ is normalized by $r(y) = 0$, where y is the limit base point. Thus, $\{y_k\}$ determines a sequence of Einstein perturbations of the hyperbolic cusp metric (5.11). If (E, g) itself is not hyperbolic, then the based metrics (E, g, y_k) are not hyperbolic, so that the sequence of perturbations is non-trivial. We will prove that this assumption leads to a contradiction.

For computation, it is convenient, (although not necessary), to work with the compactification $\bar{g}$. Thus, as described above, the compactifications $\bar{g}_k = t_k^2 \cdot g$ based at y_k converge, in a subsequence, to a flat product metric g_F (5.14) on $\mathbb{R}^+ \times T^3$, again after unwrapping the collapse. The convergence of $\bar{g}_k$ to g_F is smooth and uniform on compact subsets of $\mathbb{R}^+ \times T^3$, but is not smooth at the boundary $\{0\} \times T^3$.

Now view the metrics $\bar{g}_k$ as perturbations of the limit flat metric g_F. Note that the metrics $\bar{g}_k$ are all Bach-flat, i.e. satisfy the Bach equation (2.5). If any $\bar{g}_k$ is flat on some open set $U \subset E$ containing some y_k, then g is locally conformally flat in U. Since g, being Einstein, is analytic, it is then everywhere locally conformally flat and hence (N, g) is hyperbolic, i.e. the result follows in this case. Thus, we may and do assume that $\bar{g}_k$ is not flat on any open set, for all k.

To understand the behavior of $\bar{g}_k$ near the flat limit g_F, consider the linearization. Thus write

$$\bar{g}_k = g_F + s_k h_k, \tag{5.15}$$

where $s_k \to 0$ and h_k is a sequence of symmetric bilinear forms with $s_k h_k \to 0$ smoothly on compact subsets. As above, it is understood here and below that the metrics $\bar{g}_k$ are lifted to covering spaces unwrapping the collapse. The parameter s_k is chosen to measure the local size of the curvature at the base point y_k in that

$$s_k = \left(\int_{B_{y_k}(\frac{1}{2})} |R_{\bar{g}_k}|^2 \, dV \right)^{1/2}. \tag{5.16}$$

Since $\bar{g}_k$ is not flat anywhere, $s_k > 0$, for all k. The convergence $\bar{g}_k \to g_F$, (in a subsequence), is smooth, and so the forms h_k are locally bounded, away from $\{0\} \times T^3$, and converge smoothly to a limit symmetric bilinear form h on $\mathbb{R}^+ \times T^3$, with $||h|| \sim 1$ at the base point $y = \lim y_k$. Further, since the convergence of $\bar{g}_k$ to g_F requires unwrapping to larger and larger covers, the limit form h is invariant under the T^3 action on $\mathbb{R}^+ \times T^3$.

The limit h is not uniquely defined, since one may alter the convergence $\bar{g}_k \to g_F$ by diffeomorphisms converging to the identity; this corresponds to changing h to $h + \delta^* X$, for some vector field X. To normalize, h may be chosen so that

$$\beta_{g_F}(h) = 0, \tag{5.17}$$

where β_{g_F} is the Bianchi operator of $g_F, \beta_{g_F}(h) = \delta h + \frac{1}{2} d\, tr\, h$, where the divergence and trace are w.r.t g_F, cf. also [7].

Now the form h is a solution of the linearized Bach equations at the flat metric g_F and the deviation of $\bar{g}_k$ from g_F is measured, to first order, by the size of the linearization h, in that

$$\bar{g}_k = g_F + s_k h + o(s_k), \tag{5.18}$$

where $o(s_k) \ll s_k$ on any given compact subset of $\mathbb{R}^+ \times T^3$. In particular, the curvature $R_{\bar{g}_k}$ on the annuli $A_{y_k}(T)$ satisfies

$$|R_{\bar{g}_k}| \sim s_k |\partial^2 h + Q_1(h)|, \tag{5.19}$$

where $Q_1(h)$ involves only h and its first derivative. Note that $|R_{\bar{g}_k}| \sim s_k$ on the L^2-average in $B_{y_k}(\frac{1}{2})$, by (5.16). The following lemma gives the structure of any such linearization h which arises from an Einstein perturbation of a hyperbolic cusp metric, as above.

Lemma 5.8. *Any T^3 invariant symmetric bilinear form h on $\mathbb{R}^+ \times T^3$ constructed as above and satisfying* (5.17) *is given by*

$$h = C^{(0)} + C^{(1)} t + C^{(2)} t^2 + C^{(3)} t^3 + C^{(4)} t^4, \tag{5.20}$$

where the coefficients $C^{(i)}$ are constant, i.e. parallel forms, on $\mathbb{R}^+ \times T^3$, and $t = t_\infty$ is the parameter on $\mathbb{R}^+$, as in (5.14).

Proof. Since the limit is flat, it is easily seen from (2.5) that the linearized Bach equation is

$$2D^* D(Ric'(h)) = -\frac{2}{3} D^2 s' - \frac{1}{3}(\Delta s') g_F, \tag{5.21}$$

where $Ric'(h) = \frac{d}{ds} Ric(g_F + sh)$ is the linearization of the Ricci curvature at the flat metric g_F and similarly $s' = s'(h)$ is the linearization of the scalar curvature, in the direction h. From standard formulas, cf. [6, Chapter 1K], the normalization (5.17) at the flat metric gives

$$Ric'(h) = \frac{1}{2}D^*Dh \quad \text{and} \quad s'(h) = -\frac{1}{2}\Delta\, tr\, h. \tag{5.22}$$

Hence, (5.21) becomes

$$(D^*D)^2 h = \frac{1}{3}D^2(\Delta\, tr\, h) + \frac{1}{6}(\Delta\Delta\, tr\, h)g_F. \tag{5.23}$$

The task now is to determine the T^3 invariant solutions of (5.23). To do this, let e_i be an orthonormal framing for the flat metric g_F, with $e_1 = \nabla t$, and e_i, $i = 2, 3, 4$ tangent to the T^3 factor and let θ_i be the corresponding coframing. Thus

$$h = \sum h_{ij}\theta_i \cdot \theta_j,$$

where $h_{ij} = h_{ij}(t)$, since h is T^3 invariant. It is straightforward to compute that the Bianchi normalization (5.17) gives the equations

$$\partial_t h(e_1, e_1) = \frac{1}{2}\partial_t\, tr\, h, \quad \partial_t h(e_1, e_i) = 0, \quad i \geqslant 2. \tag{5.24}$$

Next, recall from (1.5) that the scalar curvature of a geodesic compactification is given by $s = -6\frac{\Delta t}{t}$; (as usual we drop the overbars). Hence, $s' = 6t^{-2}(\Delta t)t' - 6t^{-1}(\Delta')(t) - 6t^{-1}\Delta(t')$, where t' is the linearization of t in the direction h. The first term here vanishes, since $\Delta t = 0$ on g_F. For the second term, from [6, Chapter 1K], $(\Delta')(t) = -\langle D^2 t, h\rangle + \langle dt, \beta(h)\rangle = 0$, by (5.17) and the fact that $D^2 t = 0$ on g_F. Thus,

$$s' = -6t^{-1}\Delta(t') = -6t^{-1}\partial_t\partial_t(t'); \tag{5.25}$$

here the second equality follows from the fact that t' is only a function of t, since h is. To compute $\partial_t(t')$, let $g_s = g_F + sh$ and let t_s be distance functions w.r.t. g_s converging to the distance function $t = t_\infty$ on (F, g_F). (For example for $s = s_k$ as in (5.18), $t_s = t_{s_k} = t_k$ is given as in (5.13)). We have $t_s = t + st' + o(s)$ and $|\nabla_{\bar{g}_s} t_s|^2 = 1$, i.e. $\bar{g}_s^{ij}\partial_i t_s \partial_j t_s = 1$. Taking the derivative w.r.t. s then gives, at g_F,

$$2\partial_t(t') = 2\langle \nabla t', \nabla t\rangle = h(\nabla t, \nabla t) = h(e_1, e_1).$$

Combining this with (5.24) and (5.25) results in

$$s' = -\frac{3}{2}t^{-1}\partial_t\, tr\, h, \tag{5.26}$$

which, combined with (5.22) gives $\partial_t\partial_t\, tr\, h = 3t^{-1}\partial_t\, tr\, h$. Hence, $\partial_t\, tr\, h = c_o t^3$ and so

$$tr\, h = \frac{c_o}{4}t^4 + c_1, \tag{5.27}$$

for some constants c_o, c_1. Thus (5.23) reduces to the fourth-order equation

$$h_{ij}^{(iv)} = 2c_o\delta_{1i}\delta_{1j} + c_o\delta_{ij},$$

which implies the result. $\square$

The polynomials of order 1, i.e. the constant and linear forms in t in (5.20), give rise to trivial, i.e. flat deformations of g_F, and so do not contribute to the curvature in (5.19) or (5.22). Since by construction, i.e. by the choice of s_k in (5.16), h is non-trivial, h contains polynomials of degree at least 2. Hence

$$\nabla^2 h = \nabla_T \nabla_T h = 2C^{(2)} + 6C^{(3)}t + 12C^{(4)}t^2. \tag{5.28}$$

Observe that in the context of the perturbation $\bar{g}_k$ in (5.15), (5.19ff) implies that $C^{(2)}$ is uniformly bounded away from 0 and ∞.

Now return to the geometry of the metric (E, g) or $(E, \bar{g})$, (with the original defining function t, in place of $t = t_\infty$ above). Observe that the results above hold for *any* divergent sequence of base points $\{y_k\}$ in $(E, \bar{g})$, i.e. $t(y_k) \to \infty$. This means that for any $y \in E$ with $t(y)$ sufficiently large, the rescaled metrics $\bar{g}_y = t_y^2 \cdot g$, $t_y = t/t(y)$, based at y, are always of form (5.18) on large annuli, with $h = h_y$ of form (5.20). As noted following (5.13), observe also that $\bar{g}_y = t(y)^{-2}\bar{g}$, i.e. $\bar{g}_y$ is a constant (not conformal) rescaling of $\bar{g}$.

Now on the one hand, by the weakly hyperbolic assumption (5.6), all the rescaled metrics $\bar{g}_y$ tend to the flat metric as $t(y) \to \infty$ (i.e. the parameter s_k, now s_y, in (5.16) tends to 0). On the other hand, (5.28) and (5.19) show that, at any given y, the curvature of $\bar{g}_y$, although of necessity small, at least remains bounded away from 0 on $(A_y(T), \bar{g}_y)$, for $T \gg 1$ and t_y large. In other words, for any $y' \in (A_y(T), \bar{g}_y)$ with $t(y') \gg t(y)$, one has

$$|R_{\bar{g}_y}|(y') \geqslant c_o |R_{\bar{g}_y}|(y),$$

where c_o is a fixed numerical constant. For any such y' since $\bar{g}_{y'} = (t(y)/t(y'))^2 \bar{g}_y$, and $t(y)/t(y') \ll 1$, it follows that

$$|R_{\bar{g}_{y'}}|(y') = (t(y')/t(y))^2 |R_{\bar{g}_y}|(y') \gg |R_{\bar{g}_y}|(y)|. \tag{5.29}$$

The estimate (5.29) implies, for instance by iteration, that the curvature of $\bar{g}_y$ cannot decrease to 0 as $t(y) \to \infty$ in E. This is of course a contradiction.

This contradiction implies that there are no non-trivial T^3-invariant Bach-flat deformations of the flat metric arising in this way, and hence no non-trivial deformations of the hyperbolic cusp metric among Einstein metrics. As explained at the beginning of the proof, this contradiction proves Proposition 5.7, which thus also completes the proof of Theorem 5.3. $\square$

It is an interesting open question whether Theorem 5.3 remains valid without one of the hypotheses (i) or (ii).

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