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Rank 2 stable vector bundles on Fano 3-folds of index 2

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Abstract

Let X be a Fano 3-fold of the first kind with index 2. In this paper, we characterize the chern classes of rank 2 stable vector bundles on X and we find a bound for the least twist of a rank 2 reflexive sheaf on X which has a global section.

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0. Introduction

In the study of vector bundles on non-singular projective 3-folds X, two important questions are to characterize the Chern classes of rank 2 stable vector bundles on X and to find in terms of the Chern classes the least twist of the vector bundle which has a global section. In the case of rank 2 stable vector bundles on $\mathbf{P}^3$ the first problem was solved by Atiyah and Rees in [1], and the second problem was completely solved by Hartshorne in [7]. We will devote this work to the case of rank 2 stable vector bundles on Fano 3-folds X of the first kind with index 2, namely, the anticanonical bundle $-K_X$ is ample on X, $-K_X = 2H$ and $Pic(X) \cong \mathbf{Z}H$. By Shokurov's Theorem [9] one knows that there exits a smooth member H_0 in the linear system $|H|$, which is a Del Pezzo surface. Furthermore, $1 \leq d = H^3 \leq 7$ and if $3 \leq d \leq 7$ then H is very ample.

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In Section 1, we recall some basic facts that we will need in the sequel. In Section 2, first using Serre's correspondence we characterize the Chern classes of rank 2 stable vector bundles on a Fano 3-fold X of index 2 (see Proposition 2.1), then we find a bound for the least twist of a rank 2 reflexive sheaf on X which has a global section (Theorem 2.2). Our proof goes by induction on c_1 using the standard construction of Gruson–Peskine (see [5]). In Section 3, given a rank 2 stable reflexive sheaf E on X with Chern classes c_i we bound c_3 in terms of c_1 and c_2. Finally, in Section 4, we extend Theorem 2.2 to arbitrary 3-folds V with Neron–Severi group $NS(V) \cong \mathbf{Z}$ (Theorem 4.1). As before we proceed by induction on c_1; one difficulty is to prove the initial cases and the other difficulty is to mantain adequate control on the Chern classes when we apply the standard construction of Gruson–Peskine, so that the induction will work.

Notation. Throughout this work $\mathbf{k}$ will be an algebraically closed field of characteristic 0. For a coherent sheaf E on a projective variety X we will often write H^iE (resp. h^iE) for $H^i(X,E)$ (resp. $dim_{\mathbf{k}}H^i(X,E)$). The dual of E is written $E^v := \mathscr{H}om(E, \mathcal{O}_X)$ and we will say that E is reflexive if and only if the natural map $E \to E^{vv}$ is an isomorphism. We refer to [6] for general facts on reflexive sheaves.

1. Preliminaries

In this section we will recall some basic facts needed later on. Let (X,H) be a Fano 3-fold of the first kind with index 2, namely, the anticanonical bundle $-K_X$ is ample on X, $-K_X = 2H$ and $Pic(X) \cong \mathbf{Z}H$. Furthermore, $H^4(X,\mathbf{Z})$ is generated by a line L and $H^6(X,\mathbf{Z})$ by a point P. Hence, we may identify Chern classes with integers in the usual way, i.e. identifying the Chern classes with the coefficients at H, L and P of the total Chern class. In this notation, $HL = 1$ and $H^2 = dL$.

For a rank 2 reflexive sheaf E on (X,H) with Chern classes c_i the following formulae can be checked in a standard way:

(1) $c_1E(t) = c_1 + 2t$,

(2) $c_2E(t) = c_2 + c_1td + t^2d$,

(3) $c_3E(t) = c_3 = h^0(\mathscr{E}xt^1(E, \mathcal{O}_X)) \geq 0$,

(4) $\chi(E) = (c_3/2) + (c_1+2)(c_1(c_1+1)d - 3c_2 + 6)/6$. In particular, if $c_1 = -1$ then c_2 is even.

1.1. Definition. (a) Let E be a rank 2 reflexive sheaf on (X,H) we say that E is normalized if and only if $c_1E = 0$ or -1.

Note that using a suitable twist we can normalize any rank 2 reflexive sheaf on (X,H).

(b) Let E be a rank 2 reflexive sheaf on (X,H) we say that E is stable (resp. semistable) if and only if for any coherent subsheaf F of E of rank 1, the inequality $c_1F < (c_1E/2)$ (resp. $c_1F \leq (c_1E/2)$) holds.

The following well known lemmas will be very useful:

1.2. Lemma. Let E be a normalized rank 2 reflexive sheaf on (X,H). Then, E is stable (resp. semistable) if and only if $H^0E = 0$ (resp. $H^0E(-1) = 0$)

1.3. Lemma. Let E be a rank r semistable vector bundle on (X,H). Then, $2rc_2E - (r-1)(c_1E)^2d \geq 0$. Furthermore, if E is a rank 2 stable vector bundle on (X,H) then $4c_2E - (c_1E)^2d > 0$.

Proof. See [8]. □

2. Chern classes and global sections of rank 2 stable vector bundles on Fano 3-folds

As before (X,H) will be a Fano 3-fold of the first kind with index 2, namely, the anticanonical bundle $-K_X$ is ample on X, $-K_X = 2H$ and $Pic(X) \cong \mathbf{Z}H$.

2.1. Proposition. *Given integers $c_1, c_2 \in \mathbf{Z}$ there exits a normalized rank 2 stable vector bundle E on (X,H) with Chern classes c_1, c_2 if and only if $c_1 = 0$ and $c_2 \geq 2$ or $c_1 = -1$ and $c_2 \geq 2$ even.*

Proof. Using Riemann–Roch theorem and Lemma 1.3 we easily see that these conditions are necessary unless $c_1 = 0$ and $c_2 = 1$.

Claim. *There is no rank 2 stable vector bundle E on (X,H) with Chern classes $c_1 = 0$ and $c_2 = 1$.*

Proof. Let E be a rank 2 vector bundle on (X,H) with Chern classes $c_1 = 0$ and $c_2 = 1$, it is enough to see that $H^0E \neq 0$. Assume $H^0E = 0$. Since $0 < \chi(E) \leq h^2E$ we can take a non-trivial extension $0 \neq e \in Ext^1(E, \mathcal{O}_X(-2)) = (H^2E)^*$:

$$e : 0 \to \mathcal{O}_X(-2) \to F \to E \to 0$$

where F is a rank 3 reflexive sheaf on (X,H) with Chern classes $(c_1F, c_2F) = (-2, 1)$. Since $(c_1F)^2d - 3c_2F > 0$, F is not stable. If there exists a non-zero section $\mathcal{O}_X(n) \to F$ with $n \geq -2/3 > -1$ then it induces a non-zero section of E. Otherwise, there is a non-zero map $F \xrightarrow{\varphi} \mathcal{O}_X(n)$ with $n < -2/3$

$$\begin{array}{ccccccccc} e : 0 & \longrightarrow & \mathcal{O}_X(-2) & \xrightarrow{\sigma} & F & \longrightarrow & E & \longrightarrow & 0 \\ & & & \searrow^{\varphi\sigma} & \downarrow{\scriptstyle\varphi} & & & & \\ & & & & \mathcal{O}_X(n) & & & & \end{array}$$

Either $\varphi\sigma : \mathcal{O}_X(-2) \to \mathcal{O}_X(n)$ is zero and φ lifts to a map $\psi : E \to \mathcal{O}_X(n)$ giving us $H^0E \neq 0$ or $\varphi\sigma : \mathcal{O}_X(-2) \to \mathcal{O}_X(n)$ is non-zero. In this last case, since $e \in Ext^1(E, \mathcal{O}_X(-2))$ is a non-trivial extension, we have $n \geq -1$ which together with $n < -2/3$ gives us $n = -1$. Set $F' := Ker(F \to \mathcal{O}_X(-1))$. F' is a rank 2 reflexive sheaf on (X,H) with Chern classes $c_1F' = c_1F + 1 = -1$ and $c_2F' \leq c_2E - d \leq 0$. Hence, F' is non-stable and we get $H^0F' \neq 0$ and so $H^0E \neq 0$ which proves our claim. □

Conversely, let us see that the stated conditions are sufficient. We analyze separately the two possible cases:

Case 1: Assume $c_1 = -1$. Taking the disjoint union of $1 \leq s$ conics $C_1, \dots, C_s$ and a non-zero extension $0 \neq \rho \in Ext^1(I_C(1), \mathcal{O}_X)$, where $C := \bigcup_{i=1}^s C_i$, we construct, via Serre's correspondence, a rank 2 stable vector bundle E on (X,H):

$$0 \to \mathcal{O}_X(-1) \to E \to I_C \to 0$$

with Chern classes $(-1, 2s)$.

Case 2: Assume $c_1 = 0$. Taking the disjoint union of $2 \leq s$ lines $L_1, \dots, L_s$ and a non-zero extension $0 \neq \rho \in Ext^1(I_C(2), \mathcal{O}_X)$, where $C := \bigcup_{i=1}^s L_i$, we construct a rank 2 stable vector bundle E on (X,H):

$$0 \to \mathcal{O}_X(-1) \to E \to I_C(1) \to 0$$

with Chern classes $(0, s)$. □

Our next goal is to find the least twist of a rank 2 reflexive sheaf E on (X,H) which has a global section.

2.2. Theorem. *Let E be a rank 2 reflexive sheaf on (X,H) with Chern classes c_i. If $c_1 \geq -1$ and $c_1(c_1+1)d - 3c_2 + 6 > 0$, then $H^0E \neq 0$.*

Proof. We will use induction on c_1. Let us check the initial cases. If $-1 \leq c_1 \leq 0$ then the hypothesis $c_1(c_1+1)d - 3c_2 + 6 > 0$ implies $c_2 \leq 1$ and hence E is not stable (Proposition 2.1) and $H^0E \neq 0$.

Assume $c_1 > 0$ and $H^0E = 0$. Since $c_1(c_1+1)d - 3c_2 + 6 > 0$ implies $h^2E \geq \chi(E) > 0$, there exists a non-trivial extension $0 \neq e \in Ext^1(E, \mathcal{O}_X(-2)) = DH^2E$:

$$e : 0 \to \mathcal{O}_X(-2) \to F \to E \to 0$$

where F is a rank 3 reflexive sheaf on (X,H) with Chern clases $(c_1F, c_2F) = (c_1 - 2, c_2 - 2c_1d)$. It is easy to check that $(c_1F)^2d - 3c_2F > 0$ unless $c_1E = 1$ and $d = 1$. (In this last case the hypothesis $c_1(c_1+1)d - 3c_2 + 6 > 0$ gives us $c_2 \leq 2$ and hence either E is not stable and $H^0E \neq 0$ or E is stable and the result follows from [2].) Hence F is not stable. If there exists a non-zero section $\mathcal{O}_X(n) \to F$ with $n \geq (c_1 - 2)/3 > -1$, then it induces a non-zero section of E. Otherwise, there is a non-zero map $F \xrightarrow{\varphi} \mathcal{O}_X(n)$

with $n < (c_1 - 2)/3$

$$\begin{array}{ccccccccc} e: 0 & \longrightarrow & \mathcal{O}_X(-2) & \xrightarrow{\sigma} & F & \longrightarrow & E & \longrightarrow & 0 \\ & & & \searrow^{\varphi\sigma} & \downarrow{\scriptstyle\varphi} & & & & \\ & & & & \mathcal{O}_X(n) & & & & \end{array}$$

Either $\varphi\sigma : \mathcal{O}_X(-2) \to \mathcal{O}_X(n)$ is zero and φ lifts to a map $\psi : E \to \mathcal{O}_X(n)$ giving us $H^0E \neq 0$ or $\varphi\sigma : \mathcal{O}_X(-2) \to \mathcal{O}_X(n)$ is non-zero. In this last case, since $e \in Ext^1(E, \mathcal{O}_X(-2))$ is a non-trivial extension, we have $n \geq -1$. Set $F' := Ker(F \to \mathcal{O}_X(-n))$. F' is a rank 2 reflexive sheaf on (X, H) with Chern classes $c_1F' = c_1F - n = c_1 - 2 - n < c_1$ $(n \geq -1)$ and $c_2F' \leq c_2F - ndc_1F = c_2 + (n+2)(n - c_1)d$. It is easy to see that $c_1F' \geq -1$ and $c_1F'(c_1F' + 1)d - 3c_2F' + 6 > 0$. By induction, we have $H^0F' \neq 0$ and $H^0E \neq 0$ which proves what we want. □

Remark 2.2.1. We wonder if this result is the best that we can get. Note that it will be enough to see that for all $c_1 = 0$ (resp. $c_1 = -1$) and $c_2 \geq 2$ (resp. $c_2 \geq 2$ even) there exists a rank 2 vector bundle on (X, H) with Chern classes c_1, c_2 and natural cohomology.

Remark 2.2.2. From Theorem 2.2 it follows that if E is a rank 2 stable vector bundle on (X, H) with Chern classes (0,2) then $H^0E(1) \neq 0$. This fact will be very useful for proving that the moduli scheme $M(0, 2)$ studied by Szurek and Wisniewski [10] is irreducible.

3. Bounds on c_3

Let E be a normalized rank 2 reflexive sheaf on a Fano 3-fold (X, H) of index 2, with Chern classes c_i. The goal of this section is to find an upper bound on c_3 in terms of c_1 and c_2. First of all we need to generalize some restriction theorems for reflexive sheaves on $\mathbf{P}^n$ to restriction theorems for reflexive sheaves on (X, H).

3.1. Lemma (Glueing lemma). *Let E be a reflexive sheaf on (X, H). Suppose that for a general Del Pezzo surface $S \in |H|$ and for a general elliptic curve $C = SH$ we have $h^0E_S = h^0E_C = p$ and $h^0E_C(-1) = 0$. Then, $h^0E = p$.*

Proof. It is analogous to the case of reflexive sheaves on $\mathbf{P}^n$ and we leave it to the reader. □

3.2. Proposition. *Let E be a normalized rank 2 stable reflexive sheaf on (X, H) with Chern classes c_i. Then for a general $S \in |H|$ we have $H^0E_S = 0$.*

(Remark. Note that E_S is not necessarily stable.)

Proof. Suppose $H^0E_S \neq 0$ and let m be the least integer such that $H^0E_S(m) \neq 0$. We take a non-zero section $\sigma \in H^0E_S(m)$ and we write the corresponding exact sequence:

$$0 \to \mathcal{O}_S(W) \to E_S(m) \to I_{Z,S}(c_1 + 2m - W) \to 0$$

where $c_1 = c_1(E)$ and W is a divisor on X. If $W = 0$ we argue as in [3, Theorem 1.3]; otherwise from Flenner's estimation of the slope of the members of the Harder–Narasimhan filtration we get $deg(W) = 1$ ([4, Theorem 1.4]). Let $C = SH$ be a general elliptic curve. Restricting the last exact sequence to C we get

$$0 \to \mathcal{O}_C(P) \to E_C(m) \to I_{Z',C}(c_1 + 2m - P) \to 0$$

where $P = C \cap W$ is a point on C and $Z' = Z \cap C$. From this exact sequence we conclude $h^0E_C(m) = 1$ and $h^0E_C(m-1) = 0$ and from the exact cohomology sequence associated to the exact sequence:

$$0 \to E_S(-1) \to E_S \to E_C \to 0$$

we get $h^0E_S(m) = 1$. Now we can apply Lemma 3.1 and we have $h^0E(m) = 1$ which contradicts the stability of E.

3.3. Proposition. *Let E be a normalized rank 2 stable reflexive sheaf on (X,H) with Chern classes c_i. Then the following hold:*
(i) *If $c_1 = -1$, then $c_3 \leq c_2^2 - 3c_2 + 2$,*
(ii) *If $c_1 = 0$, then $c_3 \leq c_2^2 - 4c_2 + 4$.*

Proof. By stability we have $(c_3/2) + (c_1+2)((c_1(c_1+1)d - 3c_2 + 6)/6) = \chi(E) \leq h^2E$ From the cohomology exact sequence:

$$\cdots \to H^1E_S(1) \to H^2E \to H^2E(1) \to \cdots \qquad S \in |H|$$

we get $h^2E \leq h^1E_S(1) + h^2E(1)$. Thus repeating the process we obtain $h^2E \leq \sum_{t\geq 1} h^1E_S(t)$.

Claim. *For all $t \geq 0$, $h^1E_S(t+1) < h^1E_S(t)$ provided $h^1E_S(t) \neq 0$.*

Let us bound h^1E_S. By Proposition 3.2 and by Serre duality $h^0E_S = h^2E_S = 0$. Hence, by Riemann–Roch, $h^1E_S = -\chi(E_S) = c_2 - 2$ which gives us $\sum_{t\geq 1} h^1E_S(t) \leq (c_2-3) + (c_2-4) + \cdots + 1 = \binom{c_2-2}{2}$. Now from $(c_3/2) + (c_1+2)((c_1(c_1+1)d - 3c_2 + 6)/6) \leq h^2E \leq \binom{c_2-2}{2}$ we easily get $c_3 \leq c_2^2 - 3c_2 + 2$ if $c_1 = -1$ and $c_1 = 0 c_3 \leq c_2^2 - 4c_2 + 4$ if $c_1 = 0$. □

4. Generalization

The aim of this section is the proof of Theorem 4.1 which gives a partial extension of Theorem 2.2 to a much wider class of 3-folds (containing all Fano 3-folds with $b_2 = 1$ and all complete intersection, even if of general type).

4.1. Theorem (*char* $\mathbf{k} = 0$). Fix integers e, d with $d > 0$. Then there are functions $f(d,e) \in \mathbf{Z}$ and $\gamma(d,e) \in \mathbf{Z}$ with the following property: let V be a projective 3-fold with Neron–Severi group $NS(V) \cong \mathbf{Z}$; Call $[nH]$ any positive generator of $NS(V)$ and assume $H^3 = d$ and K_V numerically equivalent to eH. Then, for every integers $t \geq f(d,e)$, $c_1 \geq \gamma(d,e)$ and every rank 2 reflexive sheaf E on V with $c_1(E)$ numerically equivalent to $c_1 H$ and $\chi(E) > 0$, there is a line bundle $L \in Pic(V)$ numerically equivalent to tH and with $H^0(E \otimes L) \neq 0$.

The proof of Theorem 4.1 will give us a rather explicit description of the function $\gamma(d,e)$ (which in the case considered in Theorem 2.2 is just the constant -1); this function comes from Riemann–Roch. Furthermore, the proof of Theorem 4.1 will show that "in principle" the function $f(d,e)$ is computable; if one is interested in a particular 3-fold V. The proof of Theorem 4.1 will give us many information on $f(d,e)$; this function will bother only for finitely many cases and each of this can be hopefully solved using the ideas in the proof, for an explicit example, see Proposition 2.1. The proof of Theorem 4.1 is just a modification of the proof of Theorem 2.2 and we will give only the modifications needed. First of all, when a line bundle, $\mathcal{O}(n)$, appeared, use the words "a line bundle, L_n, numerically equivalent to nH"

Proof (*Sketch*). It is elementary and very well known that $K_V(mH)$ has a section for, say, every $m \geq 4$. Therefore, using Riemann–Roch Theorem and Kodaira Vanishing Theorem, $h^0(L) \neq 0$ for every $L \in Pic(V)$ which is numerically equivalent to tH with $t \geq e+4$. Hence, for proving Theorem 4.1 we may assume that $H^0(E) = 0$. Hence by Riemann–Roch Theorem we have $h^2 E \neq 0$ and we can take a non-trivial extension:

$$0 \to K_V \to F \to E \to 0.$$

Now, by the form of $\chi(E)$, we may choose $\gamma(d,e)$ such that $c_1(F)^2 - 3c_2(F) > 0$ and continue with the inductive procedure of Theorem 2.2 lowering c_1 until we arrive at a case with $c_1 < \gamma(d,e)$ (for instance, $\gamma(d,-1) = (24/d) - 2$). Call $m(d,e)$ the computable integer such that if $c_1 > m(d,e)$, then in one step we do not drop the c_1 below $\gamma(d,e)$. The finitely many cases with $\gamma(d,e) < c_1 < m(d,e)$ can be handled in the following way. By Flenner's Theorem ([4, Theorem 1.2]), we have an explicit upper bound for the degree of unstability of the restriction of E to a general curve complete intersection e.g. of 2 surfaces in $|(e+5)H|$. Using this bound we find an integer $f(d,e)$ such that for every $t \geq f(d,e)$ and every $L \in Pic(V)$ with L numerically equivalent to tH, we have $h^0(F' \otimes L) \neq 0$ for every F' with $\chi(F') > 0$ and $\gamma(d,e) < c_1(F') < m(d,e)$.

Now we claim that the map $F' \to E$ appearing in the proof of Theorem 2.2 is an inclusion of sheaves. If the claim fails the map $K_V \to \mathcal{O}(n)$ would be identically zero; this leads (as in the proof of Theorem 2.2) to a contradiction.

By the claim if $h^0(F' \otimes L) \neq 0$, then $h^0(E \otimes L) \neq 0$ and the inductive proof of Theorem 2.2 works to give Theorem 4.1. □

Remark 4.2. By the first part of the proof just given it follows that in the statement of Theorem 4.1 we may assume $\gamma(d,e) = -1$ if instead of assuming $\chi(E) > 0$ we say that there is a computable $\delta(d,e)$ and we assume $\chi(E) > \delta(d,e)$.

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