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On the diagram of 132-avoiding permutations

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Abstract

The diagram of a 132-avoiding permutation can easily be characterized: it is simply the diagram of a partition. Based on this fact, we present a new bijection between 132-avoiding and 321-avoiding permutations. We will show that this bijection translates the correspondences between these permutations and Dyck paths given by Krattenthaler and by Billey–Jockusch–Stanley, respectively, to each other. Moreover, the diagram approach yields simple proofs for some enumerative results concerning forbidden patterns in 132-avoiding permutations.

1. Introduction

Let $\mathcal{S}_n$ denote the symmetric group on $\{1, \ldots, n\}$. Given a permutation $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$ and a permutation $\tau = \tau_1 \cdots \tau_k \in \mathcal{S}_k$, we say that π *contains the pattern* τ if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the elements $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ are in the same relative order as $\tau_1\tau_2 \cdots \tau_k$. Otherwise, π *avoids the pattern* τ, or alternatively, π is τ*-avoiding*. For any finite set $\{\tau_1, \ldots, \tau_s\}$, we write $\mathcal{S}_n(\tau_1, \ldots, \tau_s)$ to denote the set of permutations in $\mathcal{S}_n$ which avoid each of the patterns $\tau_1, \ldots, \tau_s$.

It is an often quoted fact that $|\mathcal{S}_n(\tau)|$ is equal to the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ for each pattern $\tau \in \mathcal{S}_3$. Because of obvious symmetry arguments, from an enumerative viewpoint there are only two distinct cases to consider, $\tau \in \{123, 321\}$ and $\tau \in \{132, 213, 231, 312\}$. Several authors established bijections between permutations avoiding a pattern of each of these classes. The first one was given by Simion and Schmidt [18]; West described in [19] a construction using trees; and recently, Krattenthaler [10] connected 123-avoiding and 132-avoiding permutations via Dyck paths.

In Section 2, we present a simple bijection between $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$ basing on another interesting combinatorial object, the diagrams. Our correspondence has the

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advantage that the excedances of a permutation in $\mathcal{S}_n(321)$ are precisely the descents of its image in $\mathcal{S}_n(132)$.

An *excedance of* π is an integer $i \in \{1, \ldots, n-1\}$ such that $\pi_i > i$. Here the element π_i is called an *excedance letter for* π. Given a permutation π, we denote the set of excedances of π by $\mathsf{E}(\pi)$ and the number $|\mathsf{E}(\pi)|$ by $\mathsf{exc}(\pi)$. An integer $i \in \{1, \ldots, n-1\}$ for which $\pi_i > \pi_{i+1}$ is called a *descent of* π. If i is a descent, we say that π_{i+1} is a *descent bottom for* π. The set of descents of π is denoted by $\mathsf{D}(\pi)$, its cardinality is denoted by $\mathsf{des}(\pi)$, as usual.

There are several one-to-one correspondences between restricted permutations and lattice paths, in particular, Dyck paths. A *Dyck path* is a path in the (x, y)-plane from the origin to $(2n, 0)$ with steps $[1, 1]$ (called up-steps) and $[1, -1]$ (called down-steps) that never falls below the x-axis.

For 321-avoiding permutations, such a bijection was given by Billey et al. [1]; for 132-avoiding permutations, Krattenthaler proposed a correspondence to Dyck paths in [10]. In Section 3, we will show that the Dyck path obtained for any $\pi \in \mathcal{S}_n(321)$ by the first mentioned correspondence coincides with the Dyck path associating by Krattenthaler's correspondence with the image (with respect to our bijection) $\sigma \in \mathcal{S}_n(132)$ of π.

Moreover, it will turn out that the diagram of a 132-avoiding permutation is closed related to the corresponding Dyck path.

In Section 4, the diagram approach will be used to obtain some enumerative results concerning the restriction of 132-avoiding permutations by additional patterns. These results are already known (see [12]) but we will give bijective proofs for them.

The paper ends with a note on how to obtain the number of occurrences of the pattern 132 in an arbitrary permutation via the diagram.

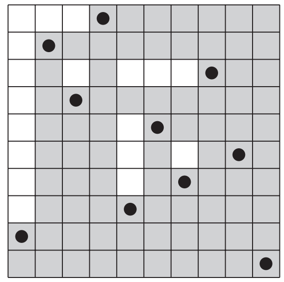
2. A bijection between 132-avoiding and 321-avoiding permutations

Let $\mathcal{Y}_n = \{(\lambda_1, \ldots, \lambda_{n-1}) : 0 \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \cdots \leq \lambda_1 \leq n-1, \lambda_i \leq n-i \text{ for all } i\}$ be the set of partitions whose Young diagram fits in the shape $(n-1, n-2, \ldots, 1)$. (We will identify a partition with its Young diagram and vice versa.) In [16] respectively [17], we have already described a bijection between $\mathcal{Y}_n$ and a class of pattern-avoiding permutations, namely $\mathcal{S}_n(321)$. For 132-avoiding permutations, a simple one-to-one correspondence to partitions with restricted diagram can be given, as well.

The key object in our derivation is the *diagram of a permutation*. (For an introduction see [11, Chapter 1].) Given a permutation $\pi \in \mathcal{S}_n$, we obtain its diagram $D(\pi)$ as follows. Let π be represented by an $n \times n$-array with a dot in each of the squares (i, π_i). (The other cells are white.) Shadow all squares due south or due east of some dot and the dotted cell itself. The diagram $D(\pi)$ is defined as the region left unshaded after this procedure.

Recovering a permutation from its diagram is trivial: row by row, put a dot in the leftmost shaded square such that there is exactly one dot in each column.

Example 2.1. The diagram of $\pi = 4\,2\,8\,3\,6\,9\,7\,5\,1\,10 \in \mathcal{S}_{10}$ contains the white squares of

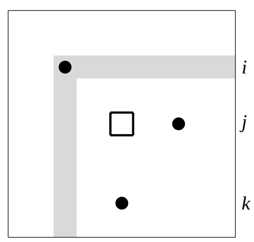


By the construction, each of the connected components of $D(\pi)$ is a Young diagram. Their corners are defined to be the elements of the *essential set* $\mathcal{E}(\pi)$ of the permutation π. In [9], Fulton introduced this set which together with a rank function was used as a tool for algebraic treatment of Schubert polynomials. In [7], Eriksson and Linusson characterized the essential sets that can arise from arbitrary permutations, as well as those coming from certain classes of permutations.

It is very easy to characterize the diagrams of 132-avoiding permutations.

Theorem 2.2. *Let $\pi \in \mathcal{S}_n$ be a permutation not equal to the identity. Then π is* 132-*avoiding if and only if its diagram consists of only one component and* $(1, 1) \in D(\pi)$.

Proof. If there are indices $i < j < k$ such that $\pi_i < \pi_k < \pi_j$, then the square (j, π_k) belongs to $D(\pi)$, but it is not connected with $(1, 1)$:



Clearly, the existence of such a square is also sufficient for π containing the pattern 132.

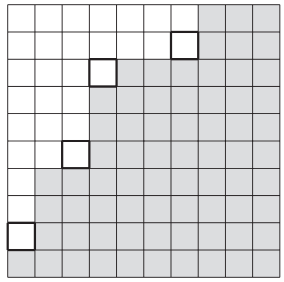
Note that the square $(1, 1)$ must be an element of $D(\pi)$ for any 132-avoiding permutation $\pi \neq \text{id}$, otherwise we would have $\pi_1 = 1$ and hence $\pi_i = i$ for all $i = 1, \ldots, n$. $\square$

Thus the diagram $D(\pi)$ of a permutation $\pi \in \mathcal{S}_n(132)$ is the graphical representation of a partition. By construction, $D(\pi)$ is just the diagram of an element of $\mathcal{Y}_n$: the square $(i, j(i))$ belongs to $D(\pi)$ if and only if no index $k \leq i$ satisfies $\pi_k \leq j$. Thus we have $j(i) \leq n - i$.

In [17, Remark 3.6], a simple one-to-one correspondence between $\mathcal{S}_n(321)$ and $\mathcal{Y}_n$ was given. It is characteristic for 321-avoiding permutations that the subwords consisting of the excedance letters and the non-excedance letters, respectively, are increasing. Therefore such a permutation is uniquely determined by its excedances and excedance letters. Consequently, the map which takes $\pi \in \mathcal{S}_n(321)$ with excedances $i_1, \ldots, i_e$ to the Young diagram with corners $(i_k, n + 1 - \pi_{i_k})$, for $k = 1, \ldots, e$, is bijective.

Composing both bijections, that one from $\mathcal{S}_n(321)$ to $\mathcal{Y}_n$, and that one from $\mathcal{Y}_n$ to $\mathcal{S}_n(132)$, yields a bijection between 321-avoiding and 132-avoiding permutations which is denoted by Φ in the following.

Example 2.3. For the permutation $\pi = 1\ 4\ 7\ 2\ 3\ 8\ 5\ 6\ 10\ 9 \in \mathcal{S}_{10}(321)$ we have $\mathsf{E}(\pi) = \{2, 3, 6, 9\}$. Hence it corresponds to the 132-avoiding permutation having the diagram



that is, $\Phi(\pi) = 8\,9\,5\,4\,6\,7\,2\,3\,10\,1 \in \mathcal{S}_{10}(132)$.

It is an essential property of Φ that it respects classical permutation statistics.

As observed by Fulton in [9], every row of a permutation diagram containing a white corner (that is, an element of the essential set) corresponds to a descent. Thus we have $\mathsf{des}(\Phi(\pi)) = \mathsf{exc}(\pi)$ for all $\pi \in \mathcal{S}_n(321)$. But there is more to it than that: the excedance set of π and the descent set of $\Phi(\pi)$ have not only the same number of elements; the sets are even identical.

Proposition 2.4. *We have* $\mathsf{E}(\pi) = \mathsf{D}(\Phi(\pi))$ *for all* $\pi \in \mathcal{S}_n(321)$.

Proof. Any excedance i of π corresponds to a corner $(i, n + 1 - \pi_i)$ of $D(\Phi(\pi))$. Obviously, by constructing $\Phi(\pi)$ from its diagram we obtain a descent of $\Phi(\pi)$ at the position i. $\square$

Remarks 2.5. (a) As mentioned above, every 321-avoiding permutation is completely determined by its excedances and excedance letters. Our bijection shows that it is sufficient for fixing a 132-avoiding permutation to know the descents, the descent bottoms, and the first letter. Let $i_1 < \cdots < i_e$ be the excedances of $\pi \in \mathcal{S}_n(321)$, and let $\sigma = \Phi(\pi)$. Then we have

$$\sigma_1 = n + 2 - \pi_{i_1}, \qquad \sigma_{i_k+1} = n + 2 - \pi_{i_{k+1}}, \qquad \sigma_{i_e+1} = 1$$

where $k = 1, \ldots, e - 1$. It is clear from the construction that these elements are precisely the left-to-right minima of σ. (A *left-to-right minimum* of a permutation σ is an element σ_i which is smaller than all elements to its left, i.e., $\sigma_i < \sigma_j$ for every $j < i$.) Based on this, we can determine the permutation σ since it avoids 132.

For example, let $\pi = 1\ \underline{4}\ \underline{7}\ 2\ 3\ \underline{8}\ 5\ 6\ \underline{10}\ 9 \in \mathcal{S}_{10}(321)$ again. (The underlined positions are just the excedances of π.) As described above, we obtain the left-to-right minima of $\Phi(\pi)$ and their positions

$$8 * 5\,4 * * 2 * * 1,$$

and hence, by putting the remaining elements $a = 3, 6, 7, 9, 10$ on the first possible position following $a - 1$, the permutation $\Phi(\pi) = 8\ 9\ 5\ 4\ 6\ 7\ 2\ 3\ 10\ 1$.

(b) In [17, Corollary 3.7], we prove that the number of excedances is Narayana distributed over $\mathcal{S}_n(321)$. Using the correspondence between $\mathcal{S}_n(321)$ and $\mathcal{Y}_n$, this fact can be derived from a result given in [6] concerning the distribution of Dyck paths according to the number of valleys. Therefore, there are $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ permutations in $\mathcal{S}_n$ having k descents and avoiding the pattern 132.

3. Correspondences to Dyck paths

Both for 321-avoiding and for 132-avoiding permutations, one-to-one correspondences to lattice paths were given by several authors. In [1, p. 361], Billey, Jockusch, and Stanley established a bijection Ψ_{BJS} between 321-avoiding permutations on $\{1, \ldots, n\}$ and Dyck paths of length $2n$. Recently in [10, Section 2], Krattenthaler exhibited a Dyck path correspondence Ψ_K for 132-avoiding permutations. Our bijection Φ translates these constructions into each other.

Theorem 3.1. *Let $\pi \in \mathcal{S}_n(321)$. Then we have $\Psi_{BJS}(\pi) = \Psi_K(\Phi(\pi))$.*

Proof. Let $\pi \in \mathcal{S}_n(321)$ have the excedances $i_1 < \cdots < i_e$, and let $\sigma = \Phi(\pi)$. The bijection Ψ_{BJS} constructs the Dyck path corresponding to π as follows:

(1) Let $a_k = \pi_{i_k} - 1$ for $k = 1, \ldots, e$ and $a_0 = 0$, $a_{e+1} = n$. Furthermore, let $b_k = i_k$ for $k = 1, \ldots, e$ and $b_0 = 0$, $b_{e+1} = n$.
(2) Generate the Dyck path (starting at the origin) by adjoining $a_k - a_{k-1}$ up-steps and $b_k - b_{k-1}$ down-steps, for $k = 1, \ldots, e + 1$.

As shown in the preceding section, the elements

$$\begin{aligned} c_1 &:= \sigma_1 = n + 2 - \pi_{i_1}, \\ c_{k+1} &:= \sigma_{i_k+1} = n + 2 - \pi_{i_{k+1}} \qquad \text{for } k = 1, \ldots, e - 1, \\ c_{e+1} &:= \sigma_{i_e+1} = 1 \end{aligned}$$

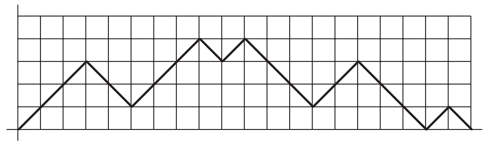
are the left-to-right minima of σ. With the convention $c_0 = n + 1$ we have $c_{k-1} - c_k = a_k - a_{k-1}$ for all $k = 1, \ldots, e + 1$. For the number d_k of the positions between the kth and (including) the $(k + 1)$st left-to-right minimum we obtain $d_k = b_k - b_{k-1}$ for $k = 1, \ldots, e + 1$. (Let $n + 1$ be the position of the imaginary $(e + 2)$nd minimum, so $d_{e+1} = n - b_e$.)

Hence the translation of Ψ_{BJS} by Φ constructs the Dyck path corresponding to $\sigma \in \mathcal{S}_n(132)$ as follows:

(1) Let $c_1 > \cdots > c_{e+1}$ be the left-to-right minima of σ. Furthermore, let d_k be one plus the number of the letters in σ between c_k and c_{k+1}, for $k = 1, \ldots, e + 1$. Initialize $c_0 = n + 1$.
(2) Generate the Dyck path (starting at the origin) by adjoining $c_{k-1} - c_k$ up-steps and d_k down-steps, for $k = 1, \ldots, e + 1$.

But this is precisely the description of Ψ_K proposed in [10]. $\square$

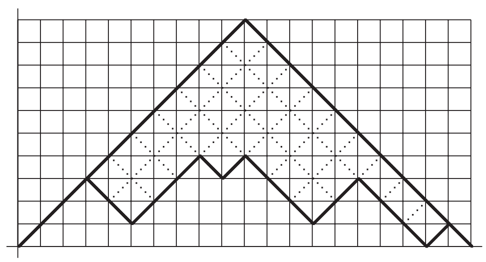
Example 3.2. Let $\pi = 1\ 4\ 7\ 2\ 3\ 8\ 5\ 6\ 10\ 9 \in \mathcal{S}_{10}(321)$, and let $\sigma = \Phi(\pi) = 8\ 9\ 5\ 4\ 6\ 7\ 2\ 3\ 10\ 1$. Billey–Jockusch–Stanley's bijection takes π to the Dyck path



which is exactly the path corresponding to σ by Krattenthaler's bijection.

It is obvious that the Dyck path $\Psi_K(\pi)$ and the diagram of a 132-avoiding permutation π are closely related to each other. Given a permutation $\pi \in \mathcal{S}_n(132)$, its diagram $D(\pi)$ is just the region bordered by the lines between the lattice points $(0, 0)$ and (n, n) and between (n, n) and $(2n, 0)$, respectively, and the path $\Psi_K(\pi)$. (The northwest-to-southeast diagonals correspond to the diagram columns.)

Example 3.3. For $\pi = 8\ 9\ 5\ 4\ 6\ 7\ 2\ 3\ 10\ 1 \in \mathcal{S}_{10}(132)$ we obtain:



Remarks 3.4. (a) In [8, p. 7], Fulmek gave a graphical construction of Krattenthaler's bijection in terms of permutation graphs. For 312-avoiding permutations, he pictures the construction as follows. Represent $\pi \in \mathcal{S}_n(312)$ as an $n \times n$-array with a dot in the square $(n + 1 - \pi_i, i)$. Consider all squares which lie below and to the right of some dot representing a left-to-right maximum, that is, of some dot having no dots to its northwest. (A *left-to-right maximum* is an element which exceeds all the elements to its left.) Define the path to be the upper boundary of the union of these squares. This yields $\Psi_K(\sigma)$ where σ is the permutation obtained from π by replacing π_i with $n + 1 - \pi_i$. (Note that $\sigma \in \mathcal{S}_n(132)$.)

(b) In [3], Brändén et al. studied the number e_k of increasing subsequences of length $k + 1$ in 132-avoiding permutations. By means of Krattenthaler's correspondence, the statistics e_k were translated into Dyck path characteristics. In particular, the sum of heights in $\Psi_K(\pi)$ equals $e_0(\pi) + 2e_1(\pi)$ where $\pi \in \mathcal{S}_n(132)$. (Here the *height* w_i of the ith path step is defined to be the ordinate of the starting point.) This fact becomes immediately clear from the relation between the path and diagram. For any permutation in $\mathcal{S}_n$, the square number of its diagram is equal to the number of its inversions (see [11, p. 9]). Consequently, we have $w_1 + \cdots + w_{2n} = n^2 - 2\mathsf{inv}(\pi)$ for all $\pi \in \mathcal{S}_n(132)$.

(c) In the same paper, the distribution of right-to-left maxima over $\mathcal{S}_n(132)$ was determined. (An element is called a *right-to-left maximum* if it is larger than all elements to its right.) Therefore, the number of permutations in $\mathcal{S}_n(132)$ with k such maxima equals the *ballot number*

$$b(n-1, n-k) = \frac{k}{2n-k}\binom{2n-k}{n}.$$

For their bijective proof, the authors of [3] used that Ψ_K translates any right-to-left maximum of $\pi \in \mathcal{S}_n(132)$ into a return of the associated Dyck path. (A *return* of a Dyck path is a down-step landing on the x-axis.) The number of returns of Dyck paths is known to have a distribution given by $b(n-1, n-k)$; see [5].

By the construction of Ψ_{BJS}, a return, except the last (down-) step, appears if and only if i is an excedance of $\pi \in \mathcal{S}_n(321)$ with $\pi_i = i+1$. The very last step of $\Psi_{BJS}(\pi)$ is a return by definition. Thus $b(n-1, n-k)$ counts the number of 321-avoiding permutations π having $k-1$ elements $\pi_i = i+1$.

Furthermore, there are $b(n-1, n-k)$ Young diagrams fitting in $(n-1, n-2, \ldots, 1)$ with $k-1$ corners in the diagonal $i + j = n$. The condition $i + j = n$ for a corner (i, j) of the diagram $D(\pi)$ also appears in the following section in context with the avoidance of the pattern 213 in a 132-avoiding permutation π.

4. Forbidden patterns in 132-avoiding permutations

Now we will use the correspondence between $\mathcal{S}_n(132)$ and $\mathcal{Y}_n$ for the enumeration of multiple restrictions on permutations. The results concerning the Wilf-equivalence of several pairs $\{132, \tau\}$ where $\tau \in \mathcal{S}_k$ are already known, see [4, 10, 13, 14]. (We say that $\{132, \tau_1\}$ and $\{132, \tau_2\}$ are *Wilf-equivalent* if $|\mathcal{S}_n(132, \tau_1)| = |\mathcal{S}_n(132, \tau_2)|$ for all n.) While the proofs given in these papers are analytical we present bijective ones.

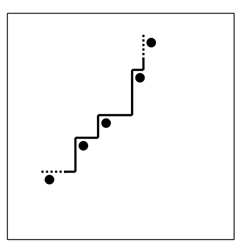
Theorem 4.1. *Let $\pi \in \mathcal{S}_n(132)$ be a permutation not equal to the identity, and $k \geq 3$. Then*

(a) *π avoids $k(k-1)\cdots 1$ if and only if $D(\pi)$ has at most $k-2$ corners. In particular, then we have $\mathsf{des}(\pi) \leq k-2$.*

(b) *π avoids $12\cdots k$ if and only if $D(\pi)$ contains the diagram $(n+1-k, n-k, \ldots, 1)$.*

(c) *π avoids $213\cdots k$ if and only if every corner (i, j) of $D(\pi)$ satisfies $i + j \geq n+3-k$.*

Proof. (a)



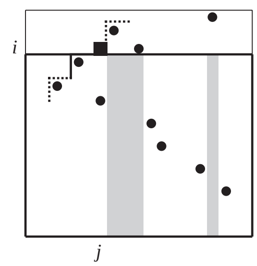
Obviously, π contains a decreasing subsequence of length k if the diagram of π has at least $k-1$ corners.

On the other hand, if there are at most $k-2$ corners in $D(\pi)$, we have $\mathsf{des}(\pi) \leq k-2$ and hence $\pi \in \mathcal{S}_n(k\cdots 1)$. (Note that each corner of $D(\pi)$ corresponds to a descent in π.)

(b) If the diagram $(n+1-k, n-k, \ldots, 1)$ fits in $D(\pi)$ then we have $\pi_i \geq n+3-k-i$ for all i. Hence any increasing subsequence of π is of length at most $k-1$: if $\pi_i = n+3-k-i$ then $i-1$ elements from $i+k-3$ many ones in $\{n+4-k-i, n+5-k-i, \ldots, n\}$ appear in $\pi_1\cdots\pi_{i-1}$.

Conversely, let i be the smallest integer with $i+\pi_i < n+3-k$. Furthermore, choose j such that $\pi_j = n+3-k-i$ (by definition of i, we have $j > i$), and let $\pi_{i_1} < \cdots < \pi_{i_{k-2}}$ be the elements of $\{n+4-k-i, n+5-k-i, \ldots, n\}$ which are not equal to $\pi_1, \ldots, \pi_{i-1}$. Note that $j < i_1 < \cdots < i_{k-2}$ since π is 132-avoiding. Thus $\pi_i\pi_j\pi_{i_1}\cdots\pi_{i_{k-2}}$ is an increasing sequence.

(c)



Let (i, j) be the top corner of $D(\pi)$ for which $i + j < n + 3 - k$. By removing the rows $1, \ldots, i$ and the columns $\pi_1, \ldots, \pi_i$, we obtain the diagram of a permutation $\sigma \in \mathcal{S}_{n-i}(132)$ whose letters are in the same relative order as $\pi_{i+1}\cdots\pi_n$ where $\sigma_1 = \pi_{i+1} \leq j < \pi_i$. As discussed in the proof of part (b), the element σ_1 is the first one of an increasing sequence of length $(n-i)-\sigma_1+1$ in σ. (Since $\sigma_1 \leq j < n+3-k-i$ the index $l = 1$ is the smallest one with $l+\sigma_l < (n-i)+3-k$.) Clearly, the first $j+1-\sigma_1$ terms are restricted by j. Thus there is an increasing sequence of length $n-(i+j) > k-3$ in σ whose all elements are larger than π_i. Note that the elements $j+1, j+2, \ldots, \pi_i - 1$ appear in $\pi_1\cdots\pi_{i-1}$.

To prove the converse, suppose that every corner of $D(\pi)$ satisfies the condition given above. Then we have $\pi_i + i > n+3-k$ for all $i \in \mathsf{D}(\pi)$. Hence for each descent i of π there exist at most $k-3$ elements π_j with $j > i$ and $\pi_j > \pi_i$. Since π is 132-avoiding these elements form an increasing sequence. Thus there is no pattern $2134\cdots k$ in π. $\square$

Remarks 4.2. (a) From the statement of (a), it follows that the maximum length of a decreasing subsequence of $\pi \in \mathcal{S}_n(132)$ is equal to the number of corners of $D(\pi)$ plus one, or in terms of permutation statistics, $\mathsf{des}(\pi)+1$. It is well known that the length of the longest decreasing sequence can easily be obtained for any permutation in $\mathcal{S}_n$ via the Robinson–Schensted correspondence: it is just the number of rows of one of the tableaux P and Q corresponding to $\pi \in \mathcal{S}_n$.

It is clear from the construction that every left-to-right-minimum of π appears in the first column of P. Any other entry of this column must be the largest element of a

132-subsequence in π. Hence in case of 132-avoiding permutations the elements of the first column of P are precisely the left-to-right-minima. As observed in Section 2, these minima are just the descent bottoms and the first letter of π. Thus for $\pi \in \mathcal{S}_n(132)$ the tableau P has $\mathsf{des}(\pi) + 1$ rows.

(b) By part (b), the length of a longest increasing subsequence of $\pi \in \mathcal{S}_n(132)$ equals the maximum value of $n + 1 - i - \lambda_i$ with $1 \le i \le n - 1$ where $\lambda_i \ge 0$ is the length of the ith row of $D(\pi)$. This also follows from [11, p. 9] according to which λ_i is equal to the ith component of the *code of* π, that is, the number of integers $j > i$ satisfying $\pi_j < \pi_i$. Thus $n - i - \lambda_i$ counts the number of elements on the right of π_i which are larger than π_i. (Since π contains no pattern 132 these elements appear in increasing order.)

Corollary 4.3. $|\mathcal{S}_n(132, k(k-1)\cdots 1)| = \frac{1}{n}\sum_{i=1}^{k-1}\binom{n}{i}\binom{n}{i-1}$ *for all n and $k \ge 3$. In particular, there are* $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ *permutations in* $\mathcal{S}_n(132)$ *whose longest decreasing subsequence has length exactly k.*

Proof. As mentioned in Remarks 2.5(b), the number of partitions in $\mathcal{Y}_n$ whose diagram has exactly i corners is equal to the Narayana number $N(n, i+1) = \frac{1}{n}\binom{n}{i}\binom{n}{i+1}$. Thus there are $\sum_{i=0}^{k-2} N(n, i+1)$ diagrams with at most $k - 2$ corners. $\square$

The following result also follows from a special case of [14, Theorem 2.6].

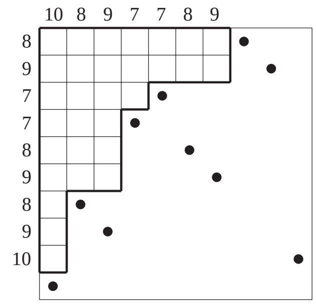
Corollary 4.4. $|\mathcal{S}_n(132, 12\cdots k)| = |\mathcal{S}_n(132, 213\cdots k)|$ *for all n and $k \ge 3$.*

Proof. There is a simple bijection between the restricted diagrams which contain $(n+1-k, n-k, \ldots, 1)$ and those ones whose all corners satisfy the condition $i + j \ge n + 3 - k$. (Note that the empty diagram associated with the identity in $\mathcal{S}_n$ belongs to the latter ones.) For each corner (i, j) of the diagram $(n+1-k, n-k, \ldots, 1)$ we have $i + j = n+2-k$. Thus every diagram containing $(n + 1 - k, n - k, \ldots, 1)$ is uniquely determined by its corners outside this shape (which are precisely the corners with $i + j \ge n + 3 - k$). Given such a diagram D, the corresponding diagram D' is defined to be that one whose corners are the corners of D which are not contained in $(n + 1 - k, n - k, \ldots, 1)$. Conversely, for any diagram D' whose all corners satisfy $i + j \ge n+3-k$ we construct the corresponding diagram D as the union of D' and $(n + 1 - k, n - k, \ldots, 1)$. $\square$

Theorem 4.1 deals with patterns whose existence in a 132-avoiding permutation can be checked without effort. The characterization of the avoidance of the patterns considered now is more technical.

Given a permutation $\pi \in \mathcal{S}_n(132)$, let $\lambda_1, \ldots, \lambda_l$ be the positive parts of the partition with diagram $D(\pi)$. Let $a_i = n - (i + \lambda_i)$ for $i = 1, \ldots, l$ and $b_i = n - (i + \lambda'_i)$ for $i = 1, \ldots, \lambda_1$ where λ' denotes the conjugate of λ. Furthermore, for $i = 1, \ldots, l$, let h_i be the length of the longest increasing sequence in $b_{\lambda_i} b_{\lambda_i - 1} \cdots b_1$ whose first element is b_{λ_i}. We call the number h_i the *height of* a_i. In particular, a_i and a_j are of the same height if $\lambda_i = \lambda_j$.

For example, the permutation $\pi = 8\ 9\ 5\ 4\ 6\ 7\ 2\ 3\ 10\ 1 \in \mathcal{S}_{10}(132)$ generates the diagram of $\lambda = (7, 7, 4, 3, 3, 3, 1, 1, 1)$:



The numbers $i + \lambda_i$ and $i + \lambda'_i$ are given on the left-hand side and on the top of the diagram, respectively.

So we obtain $a(\pi) = (2, 1, 3, 3, 2, 1, 2, 1, 0)$ and $h(\pi) = (3, 3, 1, 2, 2, 2, 1, 1, 1)$. (Note that we have $b(\pi) = (0, 2, 1, 3, 3, 2, 1)$.)

Theorem 4.5. *Let $\pi \in \mathcal{S}_n(132)$ be a permutation, and let $a(\pi)$, $h(\pi)$ be as above. Then π avoids the pattern $s(s+1)\cdots k12\cdots(s-1)$, where $2 \le s \le k$ and $k \ge 3$, if and only if the longest decreasing subsequence of $a(\pi)$ whose final term is an element of height $\ge s-1$ is of length at most $k-s$.*

Proof. Let i_1 be an integer such that $a_{i_1} \ge a_i$ for all $i < i_1$. Hence a decreasing sequence whose first element is a_{i_1} cannot be extended to the left. Then $\lambda_{i_1} < \lambda_{i_1-1}$ where $\lambda_0 := n$. (Note that the condition $\lambda_i < \lambda_{i-1}$ is equivalent to $a_{i-1} \le a_i$.) As shown in Section 2, the element π_{i_1} is a left-to-right minimum of π. Thus any increasing subsequence in π which starts with π_{i_1} is left maximal. In particular, we have $\pi_{i_1} = \lambda_{i_1} + 1$.

Now let a_j be an element of $a(\pi)$ with $a_j < a_{i_1}$, $j > i_1$, and $a_{i_1+1}, \ldots, a_{j-1} > a_j$. Since $j - i > \lambda_i - \lambda_j$ for $i = i_1, i_1+1, \ldots, j-1$, all the elements $\lambda_j + 1, \lambda_j + 2, \ldots, \lambda_{i_1}$ occur in $\pi_{i_1+1}\pi_{i_1+2}\cdots\pi_{j-1}$. Hence $\pi_{i_1} < \pi_j$.

Consequently, if $a_{i_1} > a_{i_2} > \cdots > a_{i_r}$ with $i_1 < i_2 < \cdots < i_r$ is a sequence such that there is no integer i with $i_l < i < i_{l+1}$ and $a_{i_l} > a_i \ge a_{i_{l+1}}$ for any l, then $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_r}$ is an increasing subsequence of π which is maximal with respect to the property that π_{i_1} and π_{i_r} are its first and last elements, respectively. Note that the relations $\pi_{i_1} < \pi_{i_l}$ where $2 \le l \le r$ imply that $\pi_{i_1}, \ldots, \pi_{i_r}$ is increasing since π avoids the pattern 132.

It is clear from the definition that h_i is the maximal length of an increasing sequence formed from dots southwest of the dot (i, π_i), beginning with the top dot southwest of (i, π_i). Thus, if a_{i_r} is an element of height $\ge s-1$ then there exist at least $s-1$ integers $i_r < j_1 < j_2 < \cdots < j_{s-1}$ with $\pi_{j_1} < \cdots < \pi_{j_{s-1}} < \pi_{i_r}$. Since π is 132-avoiding, we even have $\pi_{j_1} < \cdots < \pi_{j_{s-1}} < \pi_{i_1}$. Choosing i_1 and i_r minimal and maximal, respectively, proves the assertion. $\square$

For any permutation $\pi \in \mathcal{S}_n(132)$, denote by $l_s(\pi)$ the largest integer l such that π contains the pattern $s(s+1)\cdots l12\cdots(s-1)$ where $s \ge 2$.

For example, if $\pi = 8\,9\,5\,4\,6\,7\,2\,3\,10\,1$ we have $l_2(\pi) = l_3(\pi) = l_4(\pi) = 5$.

By the theorem, $l_s(\pi)$ is equal to $s-1$ plus the maximum length of a decreasing sequence in $a(\pi)$ whose smallest element is of height at least $s-1$.

Clearly, the sequence $L(\pi) := (l_2(\pi) - 1, l_3(\pi) - 2, \ldots)$ is a partition, that is, $l_s(\pi) + 1 \geq l_{s+1}(\pi)$ for all s. Since π avoids the pattern $s(s+1)\cdots k12\cdots(s-1)$ if and only if no pattern $(k+2-s)(k+3-s)\cdots k12\cdots(k+1-s)$ occurs in π^{-1}, the partition $L(\pi^{-1})$ is the conjugate of $L(\pi)$. (Obviously, for any permutation $\pi \in \mathcal{S}_n$ the diagram of the inverse π^{-1} is just the transpose of $D(\pi)$. Hence the set $\mathcal{S}_n(132)$ is closed under inversion.)

Remark 4.6. Using the relation between the diagram $D(\pi)$ and the Dyck path $\Psi_K(\pi)$, it is easy to see that the number a_i is just the height at which the ith down-step of $\Psi_K(\pi)$ ends. (Here we only consider the down-steps before the last up-step.) The numbers b_i needed for the construction of $h(\pi)$ are (in reverse order) the starting heights of the up-steps after the first down-step. Denoting the ith down-step in $\Psi_K(\pi)$ by d_i, the integer $h_i(\pi)$ is precisely the difference between the maximum height of a peak to the right of d_i and the height of the first valley following d_i. Hence in case $s = 2$, the theorem yields the second part of [10, Lemma Φ].

We shall prove now that the number of permutations in $\mathcal{S}_n$ which avoid both 132 and the pattern $s(s+1)\cdots k12\cdots(s-1)$ where $k \geq 3$ and $1 \leq s \leq k$ does *not* depend on s.

Proposition 4.7. *Let $\pi \in \mathcal{S}_n(132)$, and let l be the maximum length of an increasing subsequence of π. Then π corresponds in a one-to-one fashion to a permutation $\sigma \in \mathcal{S}_n(132)$ with $l_2(\sigma) = l$.*

Proof. Let λ and μ be the partitions whose diagrams coincide with $D(\pi)$ and $D(\sigma)$, respectively. Given $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathcal{Y}_n$, we define the sequence $\hat{\mu}$ by

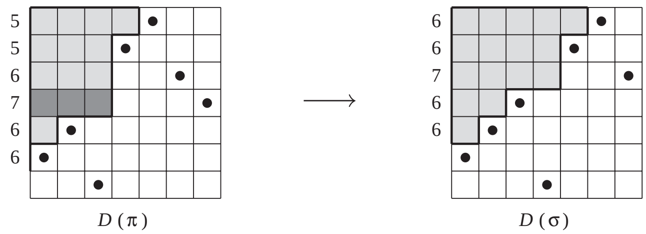
$$\hat{\mu}_i = \begin{cases} \lambda_i + 1 & \text{if } \lambda_i + i < n \\ 0 & \text{if } \lambda_i + i = n \end{cases}$$

for $i = 1, \ldots, n-1$, and obtain the partition μ by sorting $\hat{\mu}$. (Delete all parts $\hat{\mu}_i = 0$ with nonzero $\hat{\mu}_{i+1}$ and add the corresponding number of zeros at the end of the sequence.) It is obvious that $\mu \in \mathcal{Y}_n$, and it is easy to see that the map $\lambda \mapsto \mu$ is injective, and thus a bijection on $\mathcal{Y}_n$. To recover λ from μ, first set $\hat{\lambda}_i = \mu_i - 1$ for all positive μ_i. Then for each $\mu_i = 0$, let j be the largest integer for which $\hat{\lambda}_j + j \geq n - 1$, and redefine $\hat{\lambda}$ to be the sequence obtained by inserting $n - 1 - j$ between $\hat{\lambda}_j$ and $\hat{\lambda}_{j+1}$. If there is no such integer j, then prepend $n - 1$ to $\hat{\lambda}$. The partition resulting from this procedure equals λ.

Consider now the sequence $\bar{a}(\pi) = (n - i - \lambda_i)_{i=1,\ldots,n-1}$. (For the statement of Theorem 4.5 it suffices to consider the reduced sequence $a(\pi)$ which is obtained by omitting the final terms $\bar{a}_i = n - i$.) By Remarks 4.2(b), we have $l = \max \bar{a}_i + 1$ where $1 \leq i \leq n - 1$. Let j_1 be an integer satisfying $\bar{a}_{j_1} = l - 1$. As mentioned in Remark 4.6, the ith down-step of $\Psi_K(\pi)$ lands on the level $\bar{a}_i(\pi)$. In particular, l is equal to the maximum height of a peak of $\Psi_K(\pi)$. Therefore, it is obvious that there exist some integers $j_1 < j_2 < \cdots < j_{l-1} \leq n - 1$ with $\bar{a}_{j_i} = l - i$ for $i = 1, \ldots, l - 1$. Clearly, the indices $j_1, \ldots, j_{l-1}$ can be chosen in a way that $\bar{a}_j(\pi) > 0$ for all $j_1 \leq j \leq j_{l-1}$.

By the construction, the elements of the sequence $a(\sigma)$ correspond to the nonzeros of $\bar{a}(\pi)$. More exactly, we have $a_i(\sigma) = \alpha_i - 1 + \beta_i$ where α_i denotes the ith positive element in $\bar{a}(\pi)$, and β_i counts the number of zeros to the left of α_i. Thus the elements corresponding to $\bar{a}_{j_1}, \bar{a}_{j_2}, \ldots, \bar{a}_{j_{l-1}}$ form a decreasing sequence of maximal length in $a(\sigma)$. Consequently, we have $l_2(\sigma) = (l-1)+1$. (Note that the height of any element a_i is always positive due to the definition.) □

Example 4.8. Consider the permutation $\pi = 5\ 4\ 6\ 7\ 2\ 1\ 3 \in \mathcal{S}_7(132)$. Its longest increasing subsequence is of length 3. As described in the proof, we obtain the diagram of the corresponding permutation $\sigma = 6\ 5\ 7\ 3\ 2\ 1\ 4 \in \mathcal{S}_7(132)$ from $D(\pi)$:



The length of the longest decreasing subsequence on the positive terms of $\bar{a}(\pi) = (2, 2, 1, 0, 1, 1)$ coincides with the length of the longest decreasing sequence in $a(\sigma) = (1, 1, 0, 1, 1)$; it equals 2. Thus $l_2(\sigma) = 3$.

The following result can be derived from the corresponding generating functions which were given for the first time by Chow and West ([4, Theorem 3.1]). Several different analytical proofs appeared recently in [10, Theorems 2 and 6] and [13, Theorem 3.1].

Corollary 4.9. $|\mathcal{S}_n(132, 12\cdots k)| = |\mathcal{S}_n(132, 23\cdots k1)| = |\mathcal{S}_n(132, k12\cdots(k-1))|$ *for all* n *and* $k \geq 3$.

Proof. The first identity is an immediate consequence of the preceding proposition. For the second one use that π avoids $23\cdots k1$ if and only if π^{-1} contains no pattern $k12\cdots(k-1)$. □

Now we will generalize the result from Proposition 4.7 for any $s \geq 2$.

Proposition 4.10. *Let* $\pi \in \mathcal{S}_n(132)$, *and let* $l \geq s-1$ *be the maximum length of an increasing subsequence of* π. *Then* π *corresponds in a one-to-one fashion to a permutation* $\sigma \in \mathcal{S}_n(132)$ *with* $l_s(\sigma) = l$.

Proof. For $l = s-1$, Corollary 4.9 yields the correspondence between π and a permutation σ avoiding $s12\cdots(s-1)$, that is, satisfying $l_s(\sigma) = s-1$. Thus we may assume that $l \geq s$.

The reasoning is similar to that done for Proposition 4.7 (we preserve the notation) but the analysis of the bijection $\lambda \mapsto \mu$ requires more technical effort. (Following the proof we give a detailed example for illustrating.)

Given $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathcal{Y}_n$, we define the sequence $\hat{\mu}$ by

$$\hat{\mu}_i = \begin{cases} \lambda_i + s - 1 & \text{if } \lambda_i + i < n + 2 - s \\ 0 & \text{if } \lambda_i + i = n + 2 - s \\ 1 & \text{if } \lambda_i + i = n + 3 - s \\ \vdots & \qquad \vdots \\ s - 2 & \text{if } \lambda_i + i = n \end{cases}$$

for $i = 1, \ldots, n+1-s$, and $\hat{\mu}_i = \lambda_i$ otherwise, and obtain the partition μ from $\hat{\mu}$ by the following procedure:

(1) For $i = 1, \ldots, n+1-s$, if $\hat{\mu}_i \geq s-1$ then increase $\hat{\mu}_i$ by the number of elements $\hat{\mu}_j$ satisfying $1 \leq j < i$ and $0 < \hat{\mu}_j < s-1$.

(2) For $i = 1, \ldots, n+1-s$, if $\hat{\mu}_i < s-1$ then increase $\hat{\mu}_i$ by the number of elements $\hat{\mu}_j$ satisfying $i < j \leq n+2-s$ and $0 < \hat{\mu}_j < s-1$.

For the sake of clearness we denote the sequence obtained now by $\check{\mu}$.

(3) Arrange the elements of $\check{\mu}$ in decreasing order.

First we show that the map $\lambda \mapsto \mu$ is really a bijection on $\mathcal{Y}_n$.

Why $\mu \in \mathcal{Y}_n$? Let $i_1 < i_2 < \cdots < i_r \leq n+1-s$ be the indices for which $\lambda_i + i < n+2-s$, that is, $\hat{\mu}_i \geq s-1$, and $j_1 < j_2 < \cdots < j_{n-1-r}$ the remaining ones. (Because of the assumption $l \geq s$ we have $r \geq 1$.) Furthermore, let $c(a, b)$ denote the number of elements $\hat{\mu}_k \in \{1, 2, \ldots, s-2\}$ with $a < k < b$.

It is not difficult to see that both $\check{\mu}_{i_k} \geq \check{\mu}_{i_{k+1}}$ and $\check{\mu}_{j_k} \geq \check{\mu}_{j_{k+1}}$ for some k, and $\check{\mu}_{i_r} > \check{\mu}_{j_1}$. To verify the first relation, note that $\lambda_{i_k} + i_k = n+1-s$ and $\hat{\mu}_{i_k+1} = 0$ if i_k and i_{k+1} are non-consecutive integers. Thus we only have to consider the case $i_{k+1} - i_k > 2$ in which we obtain

$$\begin{aligned}
\check{\mu}_{i_{k+1}} &= \lambda_{i_{k+1}} + s - 1 + c(0, i_k) + c(i_k, i_{k+1}) \\
&\leq \lambda_{i_{k+1}} + s - 1 + c(0, i_k) + i_{k+1} - i_k - 2 \\
&\leq n + 1 - s + s - 1 + c(0, i_k) - i_k - 2 \\
&= \lambda_{i_k} + s - 1 + c(0, i_k) - 2 = \check{\mu}_{i_k} - 2
\end{aligned}$$

since $\lambda_{i_{k+1}} + i_{k+1} \leq n+1-s$. For the second relation, we may assume that j_k and j_{k+1} are consecutive. Otherwise we have $\hat{\mu}_{j_{k+1}} = 0$ as a result of $\hat{\mu}_{j_{k+1}-1} \geq s-1$, and there is nothing to show. So, let $j_{k+1} = j_k + 1 \leq n+1-s$. By the definition, we have $\hat{\mu}_j = \lambda_j + j - (n+2-s)$ and therefore $\hat{\mu}_{j_{k+1}} \leq \hat{\mu}_{j_k} + 1$. In case of equality, it follows that $c(j_k, n+3-s) = c(j_k+1, n+3-s) + 1$ and hence $\check{\mu}_{j_k} = \check{\mu}_{j_{k+1}}$. If $\hat{\mu}_{j_{k+1}} \leq \hat{\mu}_{j_k}$ we have $\check{\mu}_{j_k} \geq \check{\mu}_{j_{k+1}}$ since $c(j_k, n+3-s) \geq c(j_k+1, n+3-s)$. Furthermore, the map is defined in a way that $\check{\mu}_j = \hat{\mu}_j = \lambda_j \leq s-2$ for $n+2-s \leq j \leq n-1$. If $j_k = n+1-s$ we have $\lambda_{n+1-s} \geq 1$ (otherwise we would have $\hat{\mu}_{j_k} = s-1$) which yields $\check{\mu}_{j_k} = \lambda_{n+1-s} - 1 + c(n+1-s, n+3-s) \geq \lambda_{n+2-s}$. To show the relation $\check{\mu}_{i_r} > \check{\mu}_{j_1}$, note that $c(i_r, n+3-s) \leq \lambda_{i_r}$ since $\lambda_{i_r} + i_r = n+1-s$ and $\hat{\mu}_{i_r+1} = 0$. Thus the assertion follows immediately for $i_r = j_1 - 1$ (which is the only possible case satisfying $i_r < j_1$). If $j_1 < i_r$ we have

$$\begin{aligned}
\check{\mu}_{j_1} &= \hat{\mu}_{j_1} + c(j_1, i_r) + c(i_r, n+3-s) \leq s - 2 + c(j_1, i_r) + \lambda_{i_r} \\
&= \hat{\mu}_{i_r} + c(j_1, i_r) - 1 = \check{\mu}_{i_r} - c(0, j_1+1) - 1.
\end{aligned}$$

Consequently, $\mu_k = \check{\mu}_{i_k}$ for $1 \leq k \leq r$ and $\mu_k = \check{\mu}_{j_{k-r}}$ for $r+1 \leq k \leq n-1$. In particular, the partitions λ and μ coincide in the final $s-2$ parts. Thereby it becomes obvious why $\mu \in \mathcal{Y}_n$. On the one hand, we have $\mu_k + k \leq n$ for all $k \leq r$ since $\hat{\mu}_{i_k}$ is increased at most by the number of indices $j < i_k$ with $\hat{\mu}_j \leq s-2$. On the other hand, we obtain

$$\begin{aligned}\mu_k + k &= \check{\mu}_{j_{k-r}} + j_{k-r} + |\{i > j_{k-r} : \hat{\mu}_i \geq s-1\}| \\ &\leq \check{\mu}_{j_{k-r}} + j_{k-r} + n + 1 - s - j_{k-r} - c(j_{k-r}, n+2-s) \\ &= \lambda_{j_{k-r}} + j_{k-r} - (n+2-s) + c(j_{k-r}, n+3-s) \\ &\quad + n + 1 - s - c(j_{k-r}, n+2-s) \\ &= \lambda_{j_{k-r}} + j_{k-r} - 1 + c(j_{k-r}, n+3-s) - c(j_{k-r}, n+2-s) \\ &\leq \lambda_{j_{k-r}} + j_{k-r} \leq n\end{aligned}$$

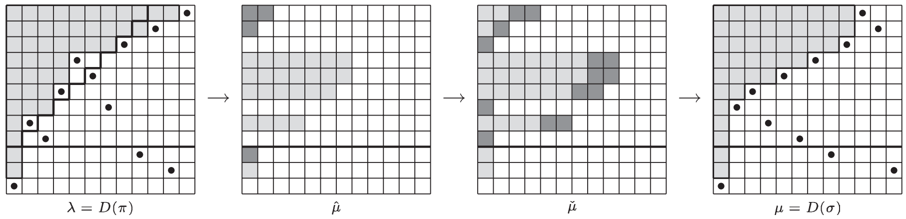
for $k = r+1, \ldots, n+1-s$.

How does the inverse map work? The partition λ can be recovered from μ by the following procedure. Here, let t be the greatest integer i for which $h_i(\sigma) \geq s-1$.

(1) Let $\hat{\lambda}_i = \mu_i - (s-1)$ for $i = 1, \ldots, t$ and $\hat{\lambda}_i = \mu_i$ for $i = n+2-s, \ldots, n-1$.
(2) For $i = n+1-s, n-s, \ldots, t+1$, let $\hat{\lambda}_i = \mu_i - c$ where c denotes the number of the positive elements $\hat{\lambda}_j$ satisfying $i < j \leq n+2-s$.
(3) Initialize $\check{\lambda} = \hat{\lambda}$. For $i = t+1, \ldots, n+1-s$, increase $\check{\lambda}_i$ by $n+2-s-i$. Furthermore, let $c_i = 1$ if $\hat{\lambda}_i$ is a positive integer and $c_i = 0$ otherwise, and set $j = i$. While $j > 1$ and $\check{\lambda}_{j-1} < \check{\lambda}_j$, replace $\check{\lambda}_{j-1}$ with $\check{\lambda}_j + 1$ and $\check{\lambda}_j$ with $\check{\lambda}_{j-1} - c_i$, and decrease j by 1. This routine yields just the partition λ.

Finally, why we have $l_s(\sigma) = l$? To answer this, we compare the sequences $\bar{a}(\pi)$ and $\bar{a}(\sigma)$ again. By the previous discussion, the r elements exceeding $s-2$ in $\bar{a}(\pi)$ correspond to the first r elements of $\bar{a}(\sigma)$. More exactly, we have $\bar{a}_k(\sigma) = \bar{a}_{i_k}(\pi) - (s-1) + \beta_k$, for $k = 1, \ldots, r$ where β_k counts the number of elements $\bar{a}_j(\pi) = s-2$ satisfying $j < i_k$. The remaining elements of $\bar{a}(\sigma)$ can be determined by $\bar{a}_{r+k}(\sigma) = \bar{a}_{j_k}(\pi) + \beta_k$, for $k = 1, \ldots, n-1-r$ where β_k counts now the number of elements $\bar{a}_j(\pi) = s-2$ satisfying $j_k < j \leq n+2-s$. The relevant sequence $a(\sigma)$ is obtained by omitting the terms $\bar{a}_k = n-k$. (Note that $a(\sigma)$ contains at least the first r terms since μ_r is always positive.) In contrast to the case $s = 2$, now we have to take the heights of the elements into consideration. The element $a_r(\sigma)$ is of height at least $s-1$. By Remark 4.6, we obtain $h_r(\sigma)$ as difference between the maximum height of a peak to the right of the rth down-step of $\Psi_K(\sigma)$ and the height of the first valley following this step. As shown above, we have $\mu_r > \mu_{r+1}$. Thus the valley in question is of height $n - r - \mu_r = \bar{a}_r(\sigma)$. The height of the peak equals $\max \bar{a}_i(\sigma) + 1$ where $r < i \leq n-1$. From the above derivation follows the existence of some i for which $\bar{a}_i(\sigma) - \bar{a}_r(\sigma) \geq s-2$. In addition, the rth element is the last one in $a(\sigma)$ whose height is greater than $s-2$. By the construction, we have $\mu_k - \mu_{k+1} \leq s-2$ for $k > r$. Thus the original definition of the height of an element yields immediately $h_k(\sigma) \leq s-2$ for all $k > r$. Consequently, the integer t which appears in the description of the inverse map is precisely r. By reasoning similar as applied in the case $s = 2$, we obtain $l_s(\sigma) = l$. (The length of the longest decreasing subsequence of $(\bar{a}_{i_1}(\pi), \bar{a}_{i_2}(\pi), \ldots, \bar{a}_{i_r}(\pi))$ equals $l + 1 - s$ and coincides with the length of the longest decreasing subsequence of $(a_1(\sigma), a_2(\sigma), \ldots, a_r(\sigma))$.) $\square$

Example 4.11. Consider the permutation $\pi = 12\ 10\ 8\ 5\ 6\ 4\ 7\ 2\ 3\ 9\ 11\ 1 \in \mathcal{S}_{12}(132)$. Its longest increasing subsequence contains five elements. We determine the corresponding permutation σ for $s = 4$. Starting with $D(\pi)$, we obtain the diagram of σ as follows:



For $i \leq 9$, the terms of the sequence $\hat{\mu}$ depend on the differences $\lambda_i + i - 10$. In case of a negative difference, we increase λ_i by 3, otherwise $\hat{\mu}_i$ is just defined to be the difference. (In the array, the first ones are the light grey parts.) The final two terms of λ and $\hat{\mu}$ coincide by the definition. The procedure makes changes to $\hat{\mu}_1, \ldots, \hat{\mu}_9$ as follows. Step one increases each light grey part by the number of (positive) dark grey parts above it (in this example, at each case by two) while step two increases each term $\hat{\mu}_i$ being at most 2 by the number of (positive) dark grey parts below it. (Note that $\hat{\mu}_{10}$ is counted in if it is positive.) In this way, we obtain $\check{\mu}$, and by sorting the parts in decreasing order, finally, the partition μ.

This construction associates the terms of $\bar{a}(\pi) = (0, 1, 2, \underline{4}, \underline{3}, \underline{3}, 2, \underline{3}, 2, 1, 0)$ which are greater than 2 with the elements of $a(\sigma) = (\underline{2}, \underline{1}, \underline{1}, \underline{2}, 3, 4, 4, 3, 2, 1, 0)$ being of height at least 3 or having an element of height at least 3 to their right. (We have $h(\sigma) = (4, 4, 4, 3, 2, 1, 1, 1, 1, 1, 1)$.) The length of the longest decreasing subsequence is the same for these two (underlined) subsequences; it equals 2. By Theorem 4.5, we have $l_4(\sigma) = 3 + 2 = 5$.

To complete the picture, we illustrate the working of the inverse map, applied to the partition $\mu = (9, 9, 8, 6, 4, 2, 1, 1, 1, 1, 1)$. The maximum index of an element $\bar{a}_i(\sigma)$ of height at least 3 is $t = 4$. The first two procedure steps generate the sequence $\hat{\lambda}$:

$$\begin{aligned}
(6, 6, 5, 3 \mid *, *, *, *, * \mid 1, 1) &\to (6, 6, 5, 3 \mid *, *, *, *, 0 \mid 1, 1)\\
&\to (6, 6, 5, 3 \mid *, *, *, 0, 0 \mid 1, 1)\\
&\to (6, 6, 5, 3 \mid *, *, 0, 0, 0 \mid 1, 1)\\
&\to (6, 6, 5, 3 \mid 1, 0, 0, 0 \mid 1, 1)\\
&\to (6, 6, 5, 3 \mid 2, 1, 0, 0, 0 \mid 1, 1).
\end{aligned}$$

(The bars mark the intervals $[1, t]$, $[t + 1, n + 1 - s]$, and $[n + 2 - s, n - 1]$.) Now for $i = 5, \ldots, 9$, increase $\hat{\lambda}_i$ by $10 - i$, and exchange the term with the previous one while as the first i elements of the sequence are in decreasing order. Any element putting to the left is increased by 1, any element putting to the right is decreased by 1 if $\hat{\lambda}_i$ is a non-zero.

$$\begin{aligned}
&i = 5 : (c_5 = 1)\\
&(6, 6, 5, 3, 7, 1, 0, 0, 0, 1, 1) \to (6, 6, 5, 8, 2, 1, 0, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (6, 6, 9, 4, 2, 1, 0, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (6, 10, 5, 4, 2, 1, 0, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 5, 5, 4, 2, 1, 0, 0, 0, 1, 1)
\end{aligned}$$

$$
\begin{aligned}
&i = 6 : (c_6 = 1)\\
&(11, 5, 5, 4, 2, 5, 0, 0, 0, 1, 1) \to (11, 5, 5, 4, 6, 1, 0, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 5, 5, 7, 3, 1, 0, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 5, 8, 4, 3, 1, 0, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 9, 4, 4, 3, 1, 0, 0, 0, 1, 1)\\
&i = 7 : (c_7 = 0)\\
&(11, 9, 4, 4, 3, 1, 3, 0, 0, 1, 1) \to (11, 9, 4, 4, 3, 4, 1, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 9, 4, 4, 5, 3, 1, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 9, 4, 6, 4, 3, 1, 0, 0, 1, 1)\\
&\qquad\qquad\qquad\qquad\qquad\quad \to (11, 9, 7, 4, 4, 3, 1, 0, 0, 1, 1)\\
&i = 8 : (c_8 = 0)\\
&(11, 9, 7, 4, 4, 3, 1, 2, 0, 1, 1) \to (11, 9, 7, 4, 4, 3, 3, 1, 0, 1, 1)\\
&i = 9 : (c_9 = 0)\\
&(11, 9, 7, 4, 4, 3, 3, 1, 1, 1, 1)
\end{aligned}
$$

In this way we obtain the partition $(11, 9, 7, 4, 4, 3, 3, 1, 1, 1, 1)$ which is just equal to λ.

Proposition 4.10 yields immediately the following result that was proved in an analytical way by Mansour and Vainshtein [14, Theorem 2.4].

Corollary 4.12. $|\mathcal{S}_n(132, 12\cdots k)| = |\mathcal{S}_n(132, s(s+1)\cdots k12\cdots(s-1))|$ *for all* n *and* $k \geq 3$ *and* $2 \leq s \leq k$.

5. A final note

As shown in Section 2, the permutation diagram indicates whether or not the permutation contains the pattern 132. If so, we even obtain the exact number of occurrences.

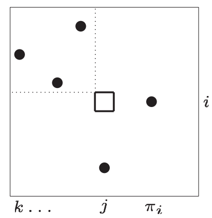
In [9], Fulton defined the following rank function on the essential set. Given a corner (i, j) of the diagram $D(\pi)$, i.e. $(i, j) \in \mathcal{E}(\pi)$, its *rank* is defined to be the number of dots northwest of it and is denoted by $\rho(i, j)$.

It is clear from the construction that the number of dots in the northwest is the same for all diagram squares which are connected. Hence we can extend the rank function on $D(\pi)$. The information about the number of sequences of type 132 contained in a permutation is encoded by the ranks of its diagram squares.

Theorem 5.1. *Let* $\pi \in \mathcal{S}_n$ *be a permutation, and let* $D(\pi)$ *be its diagram. Then the number of occurrences of the pattern* 132 *in* π *is equal to*

$$\sum_{(i,j)\in D(\pi)} \rho(i, j).$$

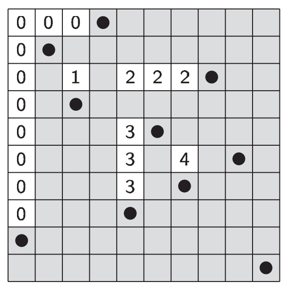
Proof. It is easy to see that each square (i, j) of $D(\pi)$ corresponds to exactly $\rho(i, j)$ subsequences of type 132 in π, namely the sequences $k\ \pi_i\ j$ where k ranges over all column indices of dots northwest of (i, j):



$\square$

Remark 5.2. As mentioned above, we have $|D(\pi)| = \mathsf{inv}(\pi)$ for all $\pi \in \mathcal{S}_n$. Hence the non-weighted sum $\sum_{(i,j)\in D(\pi)} 1$ counts the number of occurrences of the pattern 21 in π.

Example 5.3. The ranked diagram of $\pi = 4\ 2\ 8\ 3\ 6\ 9\ 7\ 5\ 1\ 10 \in \mathcal{S}_{10}$ is



Thus π contains 20 subsequences of type 132 and 18 inversions.

The first enumerative result concerning permutations that contain a given positive number r of occurrences of the pattern 132 was given by Bóna [2]. He showed that there are $\binom{2n-3}{n-3}$ permutations in $\mathcal{S}_n$ which contain 132 exactly once. (By Theorem 5.1, these permutations are characterized to be such ones having exactly one diagram square of rank 1 and only rank 0 squares otherwise.) In [15], Mansour and Vainshtein determined the generating function for the number of permutations in $\mathcal{S}_n$ having exactly r occurrences of pattern 132 for all $r \geq 0$.

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