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Approximate analysis of two-mass–spring systems and buckling of a column

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ABSTRACT

Max–Min Approach (MMA) is applied to obtain an approximate solution of three practical cases in terms of a nonlinear oscillation system. After finding maximal and minimal solution thresholds of a nonlinear problem, an approximate solution of the nonlinear equation can be easily achieved using He Chengtian's interpolation. Numerical results indicate the effectiveness of the proposed method both in respect of the whole range of involved parameters as well as the excellent agreement with the approximate frequencies and periodic solutions with the exact ones. It is predicted that MMA can be found widely applicable in engineering.

1. Introduction

The study of nonlinear oscillators involves today a variety of research fields, such as vibrations, multi-body systems, structural dynamics and transportation [1–25]. An improved He's Energy Balance method for solving nonlinear oscillatory differential equation using a new trial function was presented by Sfahani et al. [18]. The problem considered represents the governing equations of the nonlinear, large amplitude free vibrations of a slender cantilever beam with a rotationally flexible root and carrying a lumped mass at an intermediate position along its span [18]. Ganji et al., [21] analyzed the Jamming Transition Problem (JTP) via Lorentz system. Authors modeled jamming transition in traffic flow as a nonlinear damped oscillator. Differential Transformation Method (DTM) was utilized for solving the nonlinear problem and the obtained analytical results were compared with those obtained by the fourth-order Runge–Kutta Method (RK4) as a numerical method [21].

Ibsen et al., [22] developed a new method called Max–Min method for deriving an accurate approximate analytical solution to Duffing oscillators. They compared the obtained results with the Homotopy Analysis Method (HAM), Energy Balance and numerical solution. The studies conducted before, show the significant importance of nonlinear oscillation systems in deep understanding of the motions of nonlinear single (SDOF) and two degrees of freedom (TDOF) oscillation systems. Single degree of freedom system representing a column is a system whose motion is defined just by a single independent co-ordinate or function in terms of time. Additionally, some dynamic systems that require two impendent co-ordinates, to describe the motion are considered as two degrees of freedom systems (TDOF). The TDOF oscillation systems are mainly modeled with two coupled non-homogeneous ordinary differential equations.

Jefferys [26] reported the results of an investigation of the dynamical systems with two degrees of freedom, with particular emphasis on cases where genuine differences from the integrable cases are apparent due to phenomena such as small divisors.

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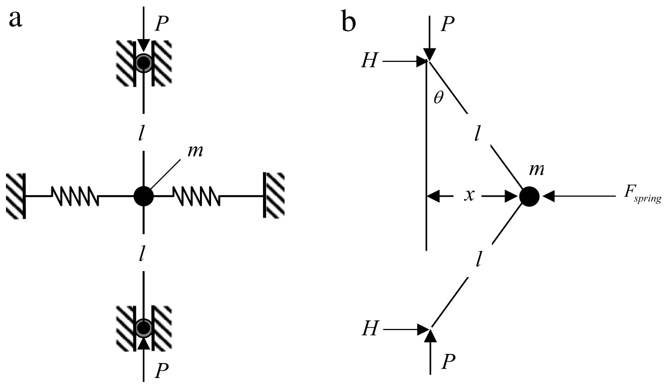

Fig. 1. Model for the bulking of a column [34].

Megahed and Abd El-Razik [27] presented the dynamic modeling and simulation of a proposed modified design of variable inertia vibration absorbers for the vibration control of single degree of freedom systems. They used Lagrange formulation to obtain its dynamic model in an analytical form.

Periodic motion of single and two degrees of freedom systems has been analyzed through many works. The existence of an infinite number of periodic motions of arbitrarily long periods is established for spring–mass systems [28]. The concept was to reduce the associated Hamiltonian to a normal form under the assumption which a certain determinant is non-zero. A two degrees of freedom vibratory system with a clearance and repeated impacts is considered by Luo et al. [29]. Perfectly plastic impact case is utilized in their work to study the dynamics of the system. Additionally, dynamics of the plastic impact oscillator describing free flight and sticking solutions of two masses of the system, supplemented by transition conditions at the instants of impacts are analyzed by Luo et al. [30].

Many researchers have addressed the nonlinear vibration of two-mass–spring systems with linear and nonlinear stiffness, both analytically and numerically. Kachapi et al., [31] developed an analytical approach based on introducing the transformation of two nonlinear differential equations for a two-mass system using proper intermediate variables into a single one and then the displacement of the system was obtained directly from the linear second-order differential equation using a first-order variational approach.

The presented work by Abrarova [32] is devoted to the existence, stability and bifurcation of the steady motion for a system composed of two bodies connected by an elastic torsional spring. The forward rectilinear motion of two rigid bodies along a horizontal plane is given by Chernousko [33] as well. Chernousko [33] constructed a periodic motion in which the system moves along a straight line in his research.

2. The models of nonlinear oscillation systems

In this section, a practical case of nonlinear oscillation system of SDOF and two cases of TDOF systems are analyzed.

2.1. Case 1: model of a bulking column

In this section, we consider a column as shown in Fig. 1. The mass m moves in the horizontal direction only. Using this model representing a column, we demonstrate how one can study its static stability by determining the nature of the singular point at $u = 0$ of the dynamic equations.

Neglecting the weight of springs and columns, shows that the governing equation for the motion of m is [34]:

$$m\ddot{u} + \left(k_1 - \frac{2P}{l}\right)u + \left(k_3 - \frac{2P}{l^3}\right)u^3 + \cdots = 0, \tag{1}$$

where $u(0) = A$, $\dot{u}(0) = 0$. The spring force is given by:

$$F_{spring} = k_1 u + k_3 u^3 + \cdots. \tag{2}$$

Case 2. *Two-mass system with three springs.*

Two-mass system with three springs is modeled in Fig. 2. In this figure, two equal masses m are linked with the fixed supports using spring k_1. The connection between two masses makes a compact item which is a spring with nonlinear properties. The linear coefficient of spring elasticity is k_2 and the cubic nonlinearity is k_3, thus, the system shows two degrees of freedom. The generalized co-ordinates are x and y.

The mathematical model of the system is presented herein [11]:

$$\ddot{v} + \left[\frac{k_1 + 2\,k_2}{m}\right]v + \left[\frac{2\,k_3}{m}\right]v^3 = 0, \qquad v(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{v}(0) = 0. \tag{3}$$

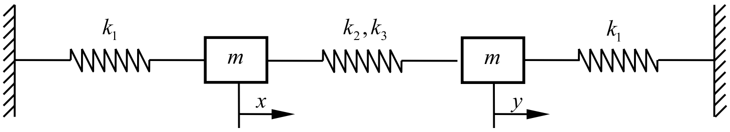

Fig. 2. The two-mass system with three springs [11].

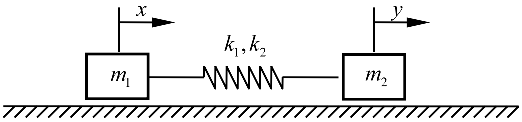

Fig. 3. The two-mass system with spring [9].

Note that the case of $k_3 > 0$ corresponds to a hardening spring while $k_3 < 0$ indicates a softening one.

Case 3. *Two-mass system with a connection spring.*

Similarly, the system with one spring is modeled through Fig. 3. Two masses, m_1 and m_2, are linked with a spring which linear coefficient of rigidity is k_1 while the nonlinear one is k_3. The system has two degrees of freedom.

The generalized co-ordinates of the system are x and y. The motion of the system is described by [9]:

$$\ddot{v} + \left[\frac{k_1\,(m_1+m_2)}{m_1 m_2}\right] v + \left[\frac{k_2\,(m_1+m_2)}{m_1 m_2}\right] v^3 = 0, \qquad v(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{v}(0) = 0. \tag{4}$$

As mentioned above, these models can be transformed to a cubic nonlinear differential equation in general form with different values α and β. The general form of cubic nonlinear differential is described as follows:

$$\ddot{v} + \alpha\, v + \beta\, v^3 = 0, \qquad v(0) = A, \qquad \dot{v}(0) = 0. \tag{5}$$

3. Basic idea of Max–Min approach

Let us consider the general nonlinear oscillators as follows:

$$\ddot{u} + N(u, \dot{u}, \ddot{u}, t) = 0, \qquad u(0) = A, \qquad \ddot{u}(0) = 0 \tag{6}$$

where $N(u, \dot{u}, \ddot{u}, t)$ is a function with nonlinear term. Due to the fact that MMA requires neither a small parameter nor a linear term in a differential equation, Eq. (6) can be approximately solved using MMA. The basic concept of this method is come from an ancient history book written by He Chengtian [35]. He actually uses the following inequality:

If

$$\frac{a}{b} < x < \frac{d}{c} \tag{7}$$

where a, b, c and d are real numbers, then

$$\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c}. \tag{8}$$

And x is approximated by

$$x = \frac{ma+nd}{mb+nc} \tag{9}$$

where m and n are weighting factors. Briefly, we can rewrite Eq. (6) in the following form:

$$\ddot{u} + \zeta(u, \dot{u}, \ddot{u}, t).u = 0 \tag{10}$$

where $\zeta(u, \dot{u}, \ddot{u}, t)$ is $N(u, \dot{u}, \ddot{u}, t)/u$. We can identify the frequency value as the following form:

$$\frac{a}{b} < \omega^2 = \frac{ma+nd}{mb+nc} < \frac{d}{c} \tag{11}$$

where a, b, c, d are real numbers and m, n are weighting factors. So, we have

$$\ddot{u} + \omega^2.u = \ddot{u} + N(u, \dot{u}, \ddot{u}, t) + \Omega(u, \dot{u}, \ddot{u}, t). \tag{12}$$

And

$$\Omega(u, \dot{u}, \ddot{u}, t) = 0. \tag{13}$$

Substituting $A\cos\omega t$ as initial assumption into Eq. (13), ω can be obtained.

4. Analysis of mechanical models applying MMA

In this section, the proposed method, MMA, is applied in order to solve and analyze three nonlinear cases presented in the Section 2. For this reason, the Cubic nonlinear equation (5) is considered as the following steps. Initially, Eq. (5) can be rewritten:

$$\ddot{v} + (\alpha + \beta v^2).v = 0. \tag{14}$$

If we choose the trial-function in the form $v(t) = A.\cos(\omega t)$, where ω is the frequency. By using the trial-function, the maximal and minimal values of $\alpha + \beta\, v^2$ are $\alpha + \beta\, A^2$ and A, respectively. So we can obtain:

$$\frac{\alpha}{1} \prec \omega^2 = \alpha + \beta v^2 \prec \frac{\alpha + \beta A^2}{1}. \tag{15}$$

According to He Chengtian's interpolation, we set

$$\omega^2 = \frac{m.\alpha + n.(\alpha + \beta A^2)}{m+n} = \alpha + k.\beta.A^2 \tag{16}$$

where m and n are weighting factors, $k = n/n + m$. So the frequency can be approximated as:

$$\omega = \sqrt{\alpha + k.\beta.A^2}. \tag{17}$$

Then Eq. (14), can be rewritten in the following form:

$$\begin{aligned} &\ddot{v} + v.(\alpha - k.\beta.A^2) = \ddot{v} + \alpha.v + \beta.v^3 + \Omega \\ &\Omega = k.v.\beta.A^2 - \beta.v^3. \end{aligned} \tag{18}$$

Substituting the trial function into Ω, and using Fourier expansion series, it is obvious that:

$$\begin{aligned} A^3\beta.\cos(\omega t)[k - \cos^2(\omega t)] &= \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\omega t] \\ &= b_1\cos(\omega t) + \sum_{n=1}^{\infty} b_{2n+1}\cos[(2n+1)\omega t] \\ &\approx b_1\cos(\omega t). \end{aligned} \tag{19}$$

For avoiding secular term we set $b_1 = 0$:

$$b_1 = \frac{4A^3\beta.}{\pi}\int_0^{\pi/2}[(k - \cos^2\varphi)\cos^2\varphi \mathrm{d}\varphi] = \frac{A^3\beta(4k-3)}{4} = 0. \tag{20}$$

From Eq. (20), the value of k is:

$$k = \frac{3}{4} \tag{21}$$

Substituting Eq. (21) into Eq. (17), yields:

$$\omega_{\text{MMA}} = \sqrt{\alpha + \frac{3A^2\beta}{4}}. \tag{22}$$

Consequently, we obtain the following period:

$$T_{\text{MMA}} = \frac{2\pi}{\omega_{\text{MMA}}}. \tag{23}$$

5. Results and discussions

The analytical results in terms of frequency values as well as outcomes associated with period of Eq. (5) are considered for different values of α and β. Subtitling $\alpha = (k_1 + 2P/l)/m$ and $\beta = (k_3 + 2P/l^3)/m$ in Eqs. (22) and (23) yields the following results for bucking of a column as a nonlinear SDOF system presented in Section 2:

$$\omega_1 = \frac{1}{2}\sqrt{\frac{4k_1l^3 - 8Pl^2 + 3k_3A^2l^3 - 6PA^2}{m\,l^3}}, \tag{24}$$

$$T_1 = \frac{4\pi\sqrt{m\,l^3}}{\sqrt{4k_1l^3 - 8Pl^2 + 3k_3A^2l^3 - 6PA^2}}. \tag{25}$$

Table 1
Comparison of approximate and exact periods for case 1.

Constant parameters						Approximate solutions	Exact solution	Error percentage
m	l	p	k_1	k_3	A	T_{MMA}	T_e	$\lvert T-T_{ex}\rvert/T_{ex}$ (%)
1	1	1	10	5	1	1.96254	1.96451	0.101
5	1.5	5	5	6	3	3.23743	3.32368	2.664
10	10	10	10	50	10	0.32418	0.33143	2.210
50	25	40	30	100	20	0.25640	0.26208	2.216
70	20	−30	50	100	10	0.60486	0.61809	2.187
100	50	150	70	20	100	0.16221	0.16580	2.218
500	150	220	120	500	0.5	9.67637	9.71672	0.417
1000	500	1000	500	500	1	6.73241	6.75871	0.391

Table 2
Comparison of approximate and "Exact" frequencies for case 2.

Constant parameters						Approximate solutions	Exact solution	Error percentage
m	k_1	k_2	k_3	X_0	Y_0	ω_{MMA}	ω_{ex}	$\lvert\omega-\omega_{ex}\rvert/\omega_{ex}$ (%)
1	1	1	1	5	1	5.1962	5.1078	1.73
2	1	3	5	8	10	4.3012	4.2406	1.43
5	10	20	30	−10	10	60.08328	58.7856	2.21
10	50	70	90	20	−40	220.4972	215.7113	2.22
10	25	20	0.5	−10	10	6.0415	5.9541	1.47
100	200	300	400	−50	50	244.9653	239.6455	2.22

Table 3
Comparison of approximate and "Exact" frequencies for case 3.

Constant Parameters						Approximate solutions	Exact solution	$\lvert\omega-\omega_{ex}\rvert/\omega_{ex}$ (%)
m_1	m_2	k_1	k_2	X_0	Y_0	ω_{MMA}	ω_{ex}	
1	2	5	1	−4	1	5.9687	5.8892	1.35
3	5	2	5	5	−5	14.1798	13.8752	2.20
1	5	5	1	5	−5	9.7980	9.6119	1.94
10	5	10	10	20	30	15.0997	14.7806	2.16
5	10	50	−0.01	−20	40	2.6268	2.5468	3.14
100	1	10	5	20	25	10.2366	10.0564	1.79
50	100	50	100	100	25	112.5067	110.0633	2.22
1000	100	200	300	400	200	314.6461	307.8115	2.17

Moreover, by substituting $\alpha = (k_1 + 2\,k_2)/m$ and $\beta = 2\,k_3/m$ into Eqs. (24) and (25), we can obtain the approximate solution of the second case in Eqs. (26) and (27)

$$\omega_1 = \frac{\sqrt{k_1 + 2k_2 + 1.5k_3A^2}}{m}, \tag{26}$$

$$T_1 = \frac{\pi\sqrt{8m}}{\sqrt{2\,k_1 + 4\,k_2 + 3A^2\,k_3}}. \tag{27}$$

Similarly, by choosing the $\alpha = k_1(m_1 + m_2)/m_1m_2$ and $\beta = k_2(m_1 + m_2)/m_1m_2$, the following frequency and period values are obtained for case 3:

$$\omega_1 = \sqrt{\frac{(m_1 + m_2)}{m_1m_2}\left(k_1 + \frac{3}{4}A^2k_2\right)}, \tag{28}$$

$$T_1 = \frac{2\pi\sqrt{m_1m_2}}{\sqrt{(m_1 + m_2)\left(k_1 + \frac{3}{4}A^2k_2\right)}}. \tag{29}$$

To illustrate and verify accuracy of MMA, comparisons with the exact solutions are given in Tables 1–3. The exact frequency of nonlinear differential equation in the cubic form is [36]:

$$\omega_{ex}(A) = \frac{\pi\sqrt{\alpha + \beta A^2}}{2}\left(\int_0^{\pi/2}\frac{\mathrm{d}t}{1 - \delta\sin^2 t}\right)^{-1}, \quad \delta = \frac{\beta\,A^2}{2(\alpha + \beta\,A^2)}. \tag{30}$$

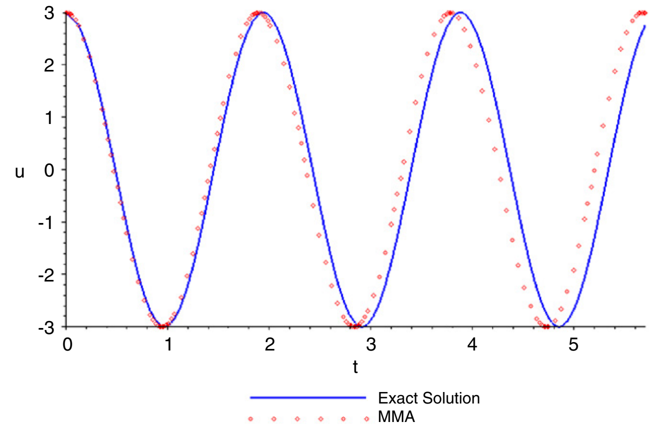

Fig. 4. Comparison of approximate periodic solutions of bucking of a column (case 1) with the exact one for $m = 1.0$, $l = 1.5$, $P = 5.0$, $k_1 = 5.0$ and $k_3 = 6.0$ with $u(0) = 3.0$.

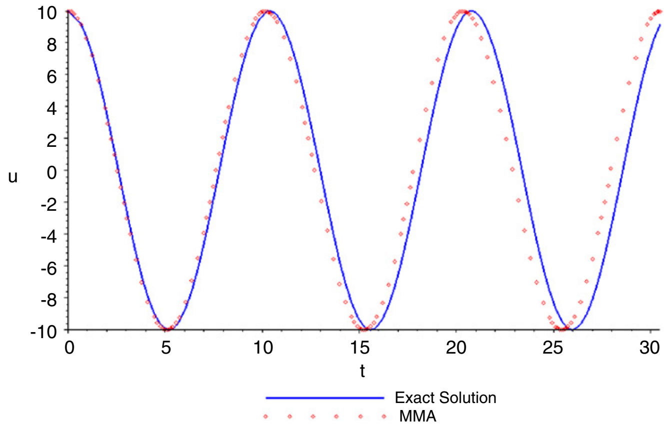

Fig. 5. Comparison of approximate periodic solutions of bucking of a column (case 1) with the exact one for $m = l = P = k_1 = 10.0$, and $k_3 = 50.0$ with $u(0) = 10.0$.

Substituting presented α and β values into Eq. (30) give the exact frequencies for case 1, 2, and 3 in the form of Eqs. (31)–(33) respectively:

$$\omega_{ex}(A) = \frac{\pi}{2}\sqrt{\frac{k_1 l^3 - 2Pl^2 + A^2 k_3 l^3 - 2A^2 P}{ml^3}}\left(\int_0^{\pi/2} \frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \quad \delta = \frac{(l^3 k_3 - 2P)A^2}{2(k_1 l^3 - 2Pl^2 + A^2 k_3 l^3 - 2A^2 P)} \tag{31}$$

$$\omega_{ex}(A) = \frac{\pi}{2}\sqrt{\frac{(k_1+2k_2)+2A^2 k_3}{m}}\left(\int_0^{\pi/2} \frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \quad \delta = \frac{2k_3 A^2}{2(k_1+2k_2)+2k_3 A^2}. \tag{32}$$

$$\omega_{ex}(A) = \frac{\pi}{2}\sqrt{\frac{(m_1+m_2)}{m_1 m_2}(k_1+k_2 A^2)}\left(\int_0^{\pi/2} \frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \quad \delta = \frac{k_2(m_1+m_2)A^2}{2(k_1(m_1+m_2)+k_2(m_1+m_2)A^2)}. \tag{33}$$

To illustrate and verify accuracy of this analytical approach, comparisons of analytical and exact results for the practical cases are presented in Tables 1–3 and Figs. 4–6. The following parameters and initial values have been used for assessing the accuracy of the presented method: The required parameters for three cases are listed as m, P, l, k_1, k_3, A for case 1, $m, k_1, k_2, k_3, X_0, Y_0$ for case 2 and $m_1, m_2, k_1, k_2, X_0, Y_0$ for case 3, respectively.

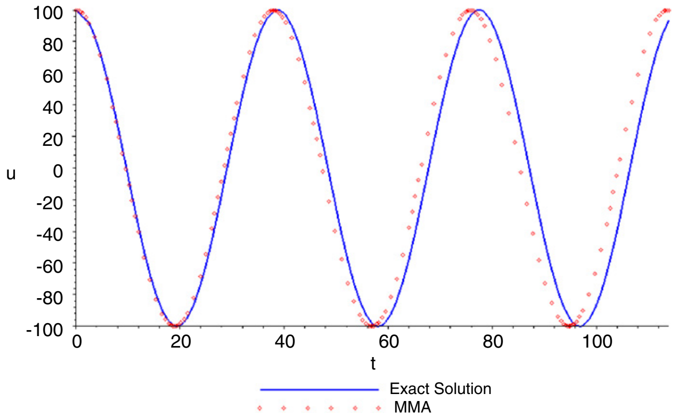

Fig. 6. Comparison of approximate periodic solutions of bucking of a column (case 1) with the exact one for $m = 100.0$, $l = 50.0$, $P = 150.0$, $k_1 = 70.0$ and $k_3 = 20.0$ with $u(0) = 100.0$.

6. Conclusions

The Max–Min Approach (MMA) has been utilized to obtain the first and second-order approximate frequencies and periods for single and two degrees of freedom (SDOF and TDOF) systems. The Max–Min method arose from an ancient Chinese inequality, called He Chengtian's Inequality and proposed by Ji-Huan He. Three nonlinear oscillation systems have been investigated representing a column to study the static stability by determining the nature of the singular point at $u = 0$ of the dynamic equations. Results were given and discussion was done; it was demonstrated that excellent agreement of the approximate solutions with the exact solution are achieved by (MMA).

References

[1] S. Natsiavas, On the dynamics of oscillators with bi-linear damping and stiffness, International Journal of Non-Linear Mechanics 25 (5) (1990) 535.
[2] L. Zhang, X.H. Ding, Solution of the prey–predator problem by He's parameterized perturbation methods, International Journal of Nonlinear Sciences and Numerical Simulation 10 (9) (2009) 1223–1226.
[3] X.H. Ding, L. Zhang, Applying He's parameterized perturbation method for solving differential–difference equation, International Journal of Nonlinear Sciences and Numerical Simulation 10 (9) (2009) 1249–1252.
[4] N.H. Sweilam, M.M. Khader, Application of He's parameter-expansion method for the nonlinear differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 10 (2) (2009) 265–272.
[5] A.O. Gilchrist, The free oscillations of conservative quasilinear systems with two-degrees-of-freedom, International Journal of Mechanical Science 3 (1961) 286.
[6] G.J. Efstathiades, Combination tones in single mode motion of a class of non-linear systems with two-degrees-of-freedom, Journal of Sound and Vibration 34 (1974) 379.
[7] G. Chen, Applications of a generalized Galerkin's method to non-linear oscillations of two-degree-of-freedom systems, Journal of Sound and Vibration 119 (1987) 225.
[8] L.N. Zhang, L. Xu, Determination of the limit cycle by He's parameter-expansion for oscillators in a $u(3)/(1 + u(2))$ potential, Zeitschrift für Naturforschung A 62 (7–8) (2007) 396–398.
[9] L. Cveticanin, Vibrations of a coupled two-degree-of-freedom system, Journal of Sound and Vibration 247 (2001) 279.
[10] K. Shin, M.J. Brennan, J.E. Oh, C.J. Harris, Analysis of disk brake noise using of a two-degree-of-freedom model, Journal of Sound and Vibration 245 (5) (2002) 837.
[11] L. Cveticanin, The motion of a two-mass system with non-linear connection, Journal of Sound and Vibration 252 (2002) 361.
[12] J.H. He, Homotopy perturbation method for bifurcation on nonlinear problems, International Journal of Non-Linear Sciences and Numerical Simulation 6 (2005) 207.
[13] S.S. Ganji, D.D. Ganji, S. Karimpour, H. Babazadeh, Applications of He's homotopy perturbation method to obtain second-order approximations of the coupled two-degree-of-freedom systems, International Journal of Non-Linear Sciences and Numerical Simulation 10 (3) (2009) 303.
[14] A. Beléndez, C. Pascual, S. Gallego, M. Ortuño, C. Neipp, Application of a modified He's homotopy perturbation method to obtain higher-order approximations of an $x^{1/3}$ force nonlinear oscillator, Physics Letters A 371 (2007) 421.
[15] S.S. Ganji, S. Karimpour, D.D. Ganji, Z.Z. Ganji, Periodic solution for strongly nonlinear vibration systems by energy balance method, Acta Applicandae Mathematicae. doi:10.1007/s10440-008-9283-6.
[16] S.S. Ganji, D.D. Ganji, S. Karimpour, He's energy balance and He's variational methods for nonlinear oscillations in engineering, International Journal of Modern Physics B 23 (3) (2009) 461.
[17] A. Barari, M. Omidvar, A.R. Ghotbi, D.D. Ganji, Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations, Acta Applicandae Mathematicae 104 (2) (2008) 161–171.
[18] M.G. Sfahani, A. Barari, M. Omidvar, S.S. Ganji, G. Domairry, Dynamic response of inextensible beams by improved energy balance method, Proceedings of the Institution of Mechanical Engineers, Part K Journal of multi-body Dynamics (2010) doi:10.1177/2041306810392113.
[19] M. Momeni, N. Jamshidi, A. Barari, D.D. Ganji, Application of He's energy balance method to duffing harmonic oscillators, International Journal of Computer Mathematics 88 (1) (2011) 135–144.
[20] S.S. Ganji, D.D. Ganji, M.G. Sfahani, S. Karimpour, Application of AFF and HPM to the systems of strongly nonlinear oscillation, Current Applied Physics 10 (5) (2010) 1317–1325.
[21] S.S. Ganji, A. Barari, L.B. Ibsen, G. Domairry, Differential transform method for mathematical modeling of jamming transition problem in traffic congestion flow, Central European Journal of Operations Research (2010) doi:10.1007/s10100-010-0154-7.
[22] L.B. Ibsen, A. Barari, A. Kimiaeifar, Analysis of highly nonlinear oscillation systems using He's max–min method and comparison with homotopy analysis and energy balance methods, Sadhana 35 (2010) 1–16.

[23] S.S. Ganji, D.D. Ganji, A.G. Davodi, S. Karimpour, Analytical solution to nonlinear oscillation system of the motion of a rigid rod rocking back using max–min approach, Applied Mathematical Modelling 34 (9) (2010) 2676–2684.
[24] M.G. Sfahani, S.S. Ganji, A. Barari, H. Mirgolbabaei, G. Domairry, Analytical solutions to nonlinear conservative oscillator with fifth-order non-linearity, Earthquake Engineering and Engineering Vibration 9 (3) (2010) 367–374. doi:10.1007/s11803-010-0021-5.
[25] M. Bayat, M. Shahidi, A. Barari, G. Domairry, Analytical evaluation of the nonlinear vibration of coupled oscillator systems, Zeitschrift für Naturforschung A 66a (1) (2011) 1–16.
[26] W.H. Jefferys, Some dynamical systems of two degrees of freedom in Clestial mechanics, The Astronomical Journal 71 (5) (1966).
[27] S.M. Megahed, A.K. Abd El-Razik, Vibration control of two degrees of freedom system using variable inertia vibration absorbers: modeling and simulation, Journal of Sound and Vibration 329 (23) (2010) 4841–4865.
[28] T.C. Harris, Periodic motions of arbitrarily long periods in non-linear spring-mass systems, International Journal of Non-Linear Mechanics 5 (1970) 491–500.
[29] G.W. Luo, J.N. Yu, J.G. Zhang, Periodic-impact motions and bifurcations of a dual component system, Nonlinear Analysis: Real World Applications 7 (2006) 813–828.
[30] G.W. Luo, X.H. Lv, L. Ma, Periodic-impact motions and bifurcations in dynamics of a plastic impact oscillator with a frictional slider, European Journal of Mechanics: A/Solids 27 (2008) 1088–1107.
[31] SHA. Hashemi Kachapi, R.V. Dukkipati, S.G. Hashemi Kachapi, S.Mey. Hashemi Kachapi, S.Meh. Hashemi Kachapi, S.K. Hashemi Kachapi, Analysis of the nonlinear vibration of a two-mass-spring system with linear and nonlinear stiffness, Nonlinear Analysis: Real World Applications 11 (2010) 1431–1441.
[32] E.V. Abrarova, Orbital steady motions of a system of two bodies with an elastic connection, Aerospace Science and Technology 8 (1998) 525–535.
[33] F.L. Chernousko, The optimum rectilinear motion of a two-mass system, Journal of Applied Mathematics and Mechanics 66 (2002) 1–7.
[34] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, Wiley-Interscience, New York, NY, USA, 1979.
[35] J.H. He, Application of He Chengtian's interpolation to Bethe equation, Computers & Mathematics with Applications 58 (2009) 2427–2430.
[36] S.S. Ganji, M. Ghalami Sfahani, S.M. Modares Tonekaboni, A.K. Moosavi, D.D. Ganji, Higher-order solutions of coupled systems using the parameter expansion method, Mathematical Problems in Engineering, 2009. 20 pages, doi:10.1155/2009/327462, Article ID 327462.