

# A Polynomial Counterexample to the Markus–Yamabe Conjecture

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We give a polynomial counterexample to both the Markus–Yamabe conjecture and the discrete Markus–Yamabe problem for all dimensions  $\geq 3$ . © 1997 Academic Press

## INTRODUCTION

The following conjecture was explicitly stated by Markus and Yamabe in [13]: let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $\mathcal{C}^1$ -class vector field with  $F(0) = 0$  and such that for all  $x \in \mathbb{R}^n$  all eigenvalues of  $JF(x)$  have negative real part, then 0 is a global attractor of the autonomous system  $\dot{x} = F(x)$ ; i.e., every solution tends to the origin as  $t$  tends to infinity.

A particular case of the above is the so-called Kalman conjecture; see [1]. The Markus–Yamabe conjecture has been studied by many authors

and several partial results are obtained (for a summary of results we refer to the papers [3, 8, 10, 14]).

The conjecture was solved affirmatively in the case  $n=2$  for polynomial vector fields by Meisters and Olech in 1988 [15]. In the same year Barabanov published a paper [1] containing ideas to construct a  $\mathcal{C}^1$ -counterexample to the Kalman conjecture for all  $n \geq 4$  and also to the Markus–Yamabe conjecture. In fact, in 1994 in [2], such a counterexample, even analytic, was constructed by Bernat and Llibre. Finally in 1993 the Markus–Yamabe conjecture was completely solved affirmatively for  $n=2$  independently by Fessler in [9] and by Gutierrez in [12]. In 1994 another proof of the same fact was given by Glutsuk in [11]. However, the Markus–Yamabe conjecture remained open for  $n=3$  and for polynomial vector fields for all  $n \geq 3$ .

We would also like to mention that it was pointed out in [5, 16] that an affirmative answer to the polynomial Markus–Yamabe conjecture would imply a positive answer to the famous Jacobian conjecture. Furthermore, recently a discrete version of the Markus–Yamabe conjecture was proposed by Cima, Gasull, and Mañosas in [4]; i.e., let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a polynomial map with  $F(0) = 0$  and such that for all  $x \in \mathbb{R}^n$  all eigenvalues of  $JF(x)$  have absolute value smaller than one, does it follow that for each  $x \in \mathbb{R}^n$   $F^k(x)$  tends to zero if  $k$  tends to infinity? It was shown in [4] that the answer is affirmative if  $n=2$ . However, in 1995 in [7], van den Essen and Hubbers gave a family of counterexamples to this question for all  $n \geq 4$ .

In this paper we show that for each  $n \geq 3$  there is a polynomial map, similar to the family in [7], which provides a counterexample to the polynomial Markus–Yamabe conjecture! In contrast with the  $\mathcal{C}^1$ -class counterexamples given in [1, 2] which have a periodic orbit, our example has orbits which escape to infinity if  $t$  tends to infinity. Finally we also give a counterexample to the discrete Markus–Yamabe problem for all  $n \geq 3$ .

## 1. THE COUNTEREXAMPLE

**THEOREM 1.1.** *Let  $n \geq 3$  and  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be given by*

$$F(x_1, \dots, x_n) = (-x_1 + x_3 d(x)^2, -x_2 - d(x)^2, -x_3, \dots, -x_n),$$

where  $d(x) = x_1 + x_3 x_2$ . Then  $F$  is a counterexample to the Markus–Yamabe conjecture. More precisely there exists a solution of  $\dot{x} = F(x)$  which tends to infinity if  $t$  tends to infinity.

*Proof.* One easily verifies that for all  $x \in \mathbb{R}^n$  all eigenvalues of  $JF(x)$  are equal to  $-1$ . Finally one checks that

$$\begin{aligned}x_1(t) &= 18e^t \\x_2(t) &= -12e^{2t} \\x_3(t) &= e^{-t} \\&\vdots \\x_n(t) &= e^{-t}\end{aligned}$$

is a solution of  $\dot{x} = F(x)$  which obviously tends to infinity as  $t$  tends to infinity. ■

To conclude this section we also give a new counterexample to the discrete Markus–Yamabe problem for all  $n \geq 3$ .

**THEOREM 1.2.** *Let  $n \geq 3$  and  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be given by*

$$F(x_1, \dots, x_n) = \left(\frac{1}{2}x_1 + x_3 d(x)^2, \frac{1}{2}x_2 - d(x)^2, \frac{1}{2}x_3, \dots, \frac{1}{2}x_n\right),$$

where  $d(x) = x_1 + x_3x_2$ . Then  $F$  is a counterexample to the discrete Markus–Yamabe question. More precisely, there exists an initial condition  $x^{(0)}$  such that the sequence  $x^{(n+1)} = F(x^{(n)})$  tends to infinity when  $n$  tends to infinity.

*Proof.* One easily verifies that for all  $x \in \mathbb{R}^n$  the eigenvalues of  $JF(x)$  are equal to  $\frac{1}{2}$ . Taking  $x^{(0)} = \left(\frac{147}{32}, \frac{-63}{32}, 1, 0, \dots, 0\right)$  it is easy to verify by induction that

$$x^{(n)} = \left(\frac{147}{32} \cdot 2^n, \frac{-63}{32} \cdot 2^{2n}, \left(\frac{1}{2}\right)^n, 0, \dots, 0\right)$$

which obviously tends to infinity as  $n$  tends to infinity. ■

## 2. CONCLUSION

Polynomial Markus–Yamabe conjectures (both continuous and discrete) have two main motivations: first their interest as knowledge of the global behaviour of a dynamical system and second for their relation to the Jacobian conjecture.

As follows from the survey made in the introduction and from the results of this paper only the second interest remains; in fact weaker versions of both polynomial Markus–Yamabe conjectures are still of importance for the study of the Jacobian conjecture. So to conclude this paper let us formulate these weaker versions:

*Conjecture 2.1* (see [5]). Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a polynomial map such that for all  $x \in \mathbb{R}^n$  all eigenvalues of  $JF(x)$  have negative real part, then  $F$  is injective.

*Conjecture 2.2* (see [4]). Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a polynomial map such that for all  $x \in \mathbb{R}^n$  all eigenvalues of  $JF(x)$  have absolute value less than one, then  $F$  has a unique fixed point.

It is shown in [5, 16] that the first one implies the Jacobian conjecture. The second one is equivalent to the Jacobian conjecture as is proved in [4].

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