C-normality and solvability of groups

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Abstract

A subgroup $H$ is called $c$-normal in group $G$ if there exists a normal subgroup $N$ and $G$ such that $HN = G$ and $H \cap N \leq H_G$ where $H_G = \text{Core}(H) = \bigcap_{g \in G} H^g$ is the maximal normal subgroup of $G$ which is contained in $H$. We obtain some results about the $c$-normal subgroups and the solvability of groups.

1. Introduction

The relationship between the properties of maximal subgroups of a finite group $G$ and the structure of $G$ have been studied by many people. It is well known that a finite groups $G$ is nilpotent if and only if every maximal subgroup of $G$ is normal in $G$. As for the class of supersolvable groups, Huppert's well-known theorem shows that a finite group $G$ is supersolvable if and only if every maximal subgroup of $G$ has a prime index in $G$. In terms of normality, we have that $G$ is supersolvable if and only if every maximal subgroup of $G$ is weakly normal in $G$ [8], Theorem 1.8.7. Also, some people try to characterize group structure using as few maximal subgroups as possible [2–4, 7].

Definition 1.1. Let $G$ be a group. We call a subgroup $H$ $c$-normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $HN = G$ and $H \cap N \leq H_G$.

It is clear that a normal subgroup of $G$ is a $c$-normal subgroup of $G$ but the converse is not true. For example, $S_3 = C_3 \times C_2$, $C_2 \not\leq S_3$ but $C_2$ is $c$-normal in $S_3$.

Definition 1.2. We call a group $G$ $c$-simple if $G$ has no $c$-normal subgroup except the identity group $1$ and $G$. 
We can easily show that $G$ is $c$-simple if and only if $G$ is simple, which is useful in our discussion.

In this paper, we give some analogue properties of normal subgroups for $c$-normal subgroups. We prove that a finite group $G$ is solvable if and only if every maximal subgroup of $G$ is a $c$-normal in $G$. We also try to minimize the number of the maximal subgroups to characterize the structure of $G$.

Let $p$ be a prime and $p'$ the complementary set of primes. Let $G$ be a finite group. Then we denote $M < G$ to indicate that $M$ is a maximal subgroups of $G$. Also, $|G:M|_p$ denotes the $p$-part of $|G:M|$. Consider the following families of subgroups.

**Definition 1.3.** We define

- $\mathcal{F} = \{M: M < G\}$,
- $\mathcal{F} = \{M: M < G\}$ with $|G:M|$ is composite.
- $\mathcal{F}_p = \{M: M < G, |G:M|_p = 1\}$.
- $\mathcal{F}_{pc} = \mathcal{F}_p \cap \mathcal{F}_c$.
- $\mathcal{F}' = \{M: M < G, N_G(P) \leq M\}$ for a $P \in Syl_p(G)$.
- $\mathcal{F}^s = \bigcup_{P \in Syl_p(G)} \mathcal{F}_p$.
- $\mathcal{F}^cc = \mathcal{F}^s \cap \mathcal{F}_c$.

**Definition 1.4.**

- $\Phi_p(G) = \bigcap\{M: M \in \mathcal{F}_p\}$ if $\mathcal{F}_p$ is non-empty; otherwise $\Phi_p(G) = G$.
- $S_p(G) = \bigcap\{M: M \in \mathcal{F}_{pc}\}$ if $\mathcal{F}_{pc}$ is non-empty; otherwise $S_p(G) = G$.
- $\Phi^p(G) = \bigcap\{M: M \in \mathcal{F}_p\}$ if $\mathcal{F}_p$ is non-empty; otherwise $\Phi^p(G) = G$.
- $\Phi^s(G) = \bigcap\{M: M \in \mathcal{F}_s\}$ if $\mathcal{F}_s$ is non-empty; otherwise $\Phi^s(G) = G$.
- $S^p(G) = \bigcap\{M: M \in \mathcal{F}_{pc}\}$ if $\mathcal{F}_{pc}$ is non-empty; otherwise $S^p(G) = G$.
- $S^s(G) = \bigcap\{M: M \in \mathcal{F}^s\}$ if $\mathcal{F}^s$ is non-empty; otherwise $S^s(G) = G$.

It is clear that all the above subgroups are characteristic subgroups of $G$.

**2 Preliminaries**

**2.1 Basic properties**

**Lemma 2.1.** Let $G$ be a group. Then

1. If $H$ is normal in $G$, then $H$ is $c$-normal in $G$;
2. $G$ is $c$-simple if and only if $G$ is simple;
3. If $H$ is $c$-normal in $G$, $H \leq K \leq G$, then $H$ is $c$-normal in $K$;
4. Let $K \leq G$ and $K \leq H$. Then $H$ is $c$-normal in $G$ if and only if $H/K$ is $c$-normal in $G/K$. 
Proof. (1) \( HG = G \) and \( H \cap G = H \leq G \); hence \( H \) is c-normal in \( G \).

(2) By (1), we only need to prove the part "if". Assume that \( G \) is simplest but \( G \) is not c-simple. Then there exists a non-trivial subgroup \( H, 1 < H < G \) such that \( H \) is c-normal in \( G \). By definition, there exists \( N \leq G \) such that \( HN = G \), which yields that \( N \neq 1 \) and so \( N = G \). It follows that \( 1 \neq H = H \cap G \leq H \triangleleft G \), contrary to our assumption.

(3) \( HN = G, \ K = K \cap G = H(K \cap N) \). \( K \cap N \) is normal in \( K \) and \( H \cap N \cap K \leq H \triangleleft G \).

(4) Suppose that \( H/K \) is c-normal in \( G/K \). Then there exists \( N/K \leq G/K \) such that \( G/K = (H/K)(N/K) \) with \( (H/K) \cap (N/K) \leq (H/K)_{G/K} \). It is easy to see that \( G = H N \) and \( H \cap N \leq H G \). The converse is the same. 

Lemma 2.2. Let \( G \) be a finite group. Then

1. \( \Phi^p(G) \) is p-closed for every \( p \in \pi(G) \);
2. \( \Phi^s(G) \) is nilpotent;
3. \( S^p(G) \) is p-closed for the maximal prime divisor \( p \in \pi(S^p(G)) \);
4. \( S^s(G) \) has Sylow tower.

Proof. (1) Let \( P_1 \in Syl_p(\Phi^p(G)) \). By Sylow's theorem, there exists \( P \in Syl_p(G) \) such that \( P_1 = P \cap \Phi^p(G) \). If \( P_1 \not\leq G \), then there exists \( M < G \) such that \( N_G(P_1) \leq M \triangleleft G \) and so \( \Phi^p(G) \leq M \in \mathcal{F}^p \). By Frattini argument, \( G = \Phi^p(G)N_G(P_1) \leq M \triangleleft G \), a contradiction. Therefore \( P_1 \leq G \).

(2) It is clear that \( \Phi^s(G) = \bigcap_{p \in \pi(G)} \Phi^p(G) \). By (1), \( \Phi^s(G) \) is p-closed for every \( p \in \pi(G) \), this follows that \( \Phi^s(G) \) is nilpotent.

(3) Let \( P_1 \in Syl_p(S^p(G)) \). By Sylow's theorem, there exists \( P \in Syl_p(G) \) such that \( P_1 = P \cap S^p(G) \). If \( P_1 \not\leq G \), then there exists \( M < G \) such that \( N_G(P_1) \leq M \triangleleft G \). By the Frattini argument, \( G = S^p(G)N_G(P_1) \). If \( |G:M| = q \) is a prime, by Sylow's theorem, \( q = 1 + kp \). But \( q|S^p(G)| \) and hence \( q < p \), a contradiction. Hence \( |G:M| \) is composite and \( M \in \mathcal{F}^{ps} \). This yields that \( G = S^p(G)N_G(P_1) \leq M \triangleleft G \), a contradiction; therefore \( P_1 \leq G \).

(4) Let \( p \) be the maximal prime divisor or \( |S^s(G)| \). The same argument as (3) shows that \( S^s(G) \) is p-closed. Let \( P \in Syl_p(S^s(G)) \). Then \( P \leq char S^s(G) \triangleleft char G \) and it is easy to show that \( S^s(G/P) = S^s(G)/P \) by induction, \( S^s(G/P) \) has a Sylow tower and so does \( S^s(G) \).

Lemma 2.3. Let \( G \) be a finite group. Then

(a) \( G \) is nilpotent if and only if \( G = \Phi^s(G) \).
(b) \( G \) is nilpotent if and only if \( M \) is normal in \( G \) for every \( M \in \mathcal{F}^s \).
(c) \( G \) is nilpotent if and only if \( G/N \) is nilpotent for a normal subgroup \( N \) of \( G \) which is contained in \( \Phi^s(G) \).

Proof. (a) \( G = \Phi^s(G) \) if and only if \( \mathcal{F}^s = \emptyset \) if and only if \( N_G(P) = G \) for every Sylow \( p \)-subgroup of \( G \) and for every prime \( p \in \pi(G) \) if and only if \( G \) is nilpotent.
(b) By Frattini argument and (a).
(c) Suppose that \( G/N \) is nilpotent and \( M \) be a maximal subgroup of \( G \) and \( M \in \mathcal{F}^a \). Then \( M/N \leq G/N \) by (a) and hence \( M \leq G \). It follows from (b) that \( G \) is nilpotent.

Lemma 2.4. Let \( G \) be a finite group. Then

(a) \( G \) is supersolvable if and only if \( |G:M| \) is a prime for every \( M \in \mathcal{F}^a \).
(b) \( G \) is supersolvable if and only if \( S^a(G) \). \( G \) is nilpotent if and only if \( M \) is normal in \( G \) for every \( M \in \mathcal{F}^a \).
(c) \( G \) is supersolvable if and only if \( G/N \) is supersolvable for a normal subgroup \( N \) of \( G \) which is contained in \( S^a(G) \).

Proof. (a) By Huppert’s well-known theorem, we only need to prove the part “if”. Let \( p \) be the largest prime of \( \pi(G) \) and \( P \in \text{Syl}_p(G) \). Then \( P \leq G \). In fact, if it is false, then there exists a maximal subgroup \( M \) of \( G \) with \( N_G(P) \leq M \leq G \). By assumption, \( |G:M| = q \) for a prime \( q < p \), which yields that \( G/M \) is isomorphic to a subgroup of the symmetric group \( S_q \) and hence \( |G/M| \). In particular, \( P \leq M \). The Frattini argument yields that \( G = M_G N_G(P) \leq M_G \), a contradiction. It is easy to show that \( G/N \) satisfies the hypotheses of \( G \) for every minimal normal subgroup \( N \) and \( G \). Suppose that (a) is false and let \( G \) be a minimal counterexample. Then \( G \) has unique minimal normal subgroup \( N \). It is easy to prove that \( N = F(G) \) and \( G = N \times M \) with \( M \not< G \). Let \( q \) be the largest prime of \( \pi(M) \). Since \( M \) is supersolvable, we have that \( M \leq N_M(Q) \leq N_G(Q) \) for \( Q \in \text{Syl}_M(G) \cap \text{Syl}_G(G) \). Since \( M \) is a maximal subgroup of \( G \) and \( Q \not< G \), it follows that \( M = N_G(Q) \). By our assumption, \( |N| = |G:M| = p \) is a prime, which yields that \( G \) is supersolvable, contrary to our choice.
(b) By (a), \( G \) is supersolvable if and only if \( \mathcal{F}^{\infty} = \emptyset \), that is if and only if \( G = S^a(G) \).
(c) The same argument as Lemma 2.3(c). □

3. Theorems

Theorem 3.1. Let \( G \) be a finite group. Then \( G \) is solvable if and only if every maximal subgroup of \( G \) is \( c \)-normal in \( G \).

Proof. Suppose that every maximal subgroup \( M \) and \( G \) is \( c \)-normal in \( G \). We prove that \( G \) is solvable. Assume that it is false and let \( G \) be a minimal counterexample. If \( G \) is simple, the by Lemma 2.1(2), \( G \) is \( c \)-simple, it follows that \( M = 1 \) and \( G \) is a group of prime order, a contradiction. Hence, we assume that \( G \) is not simple. It is clear that the hypotheses of the theorem are satisfied by any quotient group \( G/K \) of \( G \). A trivial argument shows that \( G \) has unique minimal normal subgroup \( K \) with \( K \not< \Phi(G) \). Then there exists a maximal subgroup \( M \not< G \) such that \( K \not< M \), i.e. \( G = KM \). Since \( M \) is \( c \)-normal in \( G \), there exists \( N \not< G \) such that \( G = MN \) and \( N \not< M \). Since \( 1 \not= N \) it follows that \( K \not< N \) and so \( K \cap M = 1 \). Hence \( |N| = |G:M| = |K|, K = N \). For any maximal subgroup \( L \not< G \) with \( L_G = 1 \), we have \( KL = G \). Since \( L \) is \( c \)-normal.
in $G$, the same argument shows that $|G:L|=|K|$. By a result of Baer [1, Lemma 3], $K$ is solvable. It is clear that $G/K$ satisfies the hypotheses of $G$. The minimal choice of $G$ implies that $G/K$ is solvable. Now that both $K$ and $G/K$ are solvable follows that $G$ is solvable, a contradiction.

Conversely, suppose that $G$ is solvable and $M \triangleleft G$. If $M_G \neq 1$, consider $G/M_G$ and use induction on $|G|$, we get $M/M_G$ is c-normal in $G/M_G$. From Lemma 2.1 it follows that $M$ is c-normal in $G$. Assume $M_G = 1$. Let $N$ be a minimal normal subgroup of $G$ which is certainly abelian. Then $G = n M$ and $N \cap M \leq M_G = 1$. By definition, $M$ is c-normal in $G$. $\Box$

In the direction of limiting the number of maximal subgroups which we control, we prove the following result.

**Theorem 3.2.** Let $G$ be a finite group. Then $G$ is solvable if and only if there exists a solvable c-normal maximal subgroup $M$ of $G$.

**Proof.** Assume the theorem is false and let $G$ be a minimal counterexample. Let $M$ be a c-normal solvable maximal subgroup of $G$. Then $G$ must satisfy the following:

(a) $M$ is core-free. If $M_G \neq 1$, then $M/M_G$ is a solvable c-normal maximal subgroup of $G/M_G$, which yields that $G/M_G$ is solvable and hence $G$ is solvable, a contradiction.

(b) There exists a minimal normal subgroup $K$ of $G$ such that $G = K \rtimes M$. Since $M$ is c-normal in $G$, there exists a normal subgroup $N$ and $G$ such that $G = NM$ and $M \cap N \leq M_G = 1$. Let $L$ be a minimal normal subgroup of $M$, which is certainly abelian $p$-subgroup with $p \in \pi(M)$.

(c) $(p, |K|) = 1$ and $C_K(L) = 1$. In fact, $C_K(L)$ is normalized by both $M$ and $K$ and hence $C_K(L) \leq G$. If $C_K(L) = K$, then $1 \neq L \leq M_G$, contrary to (a). Therefore $C_K(L) = 1$. The orbit formula implies that $(p, |K|) = 1$.

(d) $K$ is a $q$-subgroup for a prime $q$.

By [5, Theorem 6.2.2] and (c), there exists an unique $L$-invariant Sylow $q$-subgroup $Q$ of $K$ for every prime $q \in \pi(K)$. For any element $m \in M$, $(Q^m)^L = (Q_L)^m = Q^m$, i.e. $Q^m$ is also a $L$-invariant $q$-Sylow subgroup of $K$. From the uniqueness it follows that $Q^m = Q$ and hence $Q$ is $M$-invariant. Since $M$ is a maximal subgroup of $G$, $Q \triangleright M = G = K \triangleright M$. This yields that $K = Q$ is a $q$-subgroup.

Now both $K$ and $G/K$ are solvable implies that $G$ is solvable, contrary to our choice. $\Box$

We can also discuss $p$-solvability in terms of c-normality.

**Theorem 3.3** Let $G$ be a finite group and $p$ be the maximal prime divisor of $|G|$. If $M$ is c-normal in $G$ for every non-nilpotent maximal subgroup $M \in \mathcal{P}^c$, then $G$ is $p$-solvable.
Proof. Assume that the theorem is false and \( G \) is a minimal counterexample. Then

(1) \( \mathcal{F}^{p^c} \neq \emptyset \). If \( \mathcal{F}^{p^c} = \emptyset \), then \( G = S^p(G) \) is \( p \)-closed by Lemma 2.2(3). Hence, \( P \leq G \) for Sylow \( p \)-subgroup \( P \) and \( G \) is \( p \)-solvable, a contradiction.

(2) \( M \) is \( c \)-normal in \( G \) for every \( M \in \mathcal{F}^{p^c} \). It is sufficient to prove that \( G \) has no nilpotent maximal subgroup \( M \) with \( M \in \mathcal{F}^{p^c} \). In fact, suppose that there exists \( M \in \mathcal{F}^{p^c} \) with \( M \) nilpotent. Since \( G \) is non-solvable, Thompson's theorem [5, 10.3.2] implies that \( M_2 \neq 1 \). If \( M \) is a 2-subgroup, then \( p = 2 \) and \( G \) is a 2-group, contrary to our choice. Hence, \( G \) is a non-solvable and \( M_2 \neq 1 \neq M_2 \). By [6, Theorem 1], \( M_2 \) is normal in \( G \). It is easy to show that \( G/M_2 \) satisfies the hypotheses of \( G \). The minimal choice of \( G \) yields that \( G/M_2 \) is \( p \)-solvable. Now \( M_2 \) is solvable implies that \( G \) is \( p \)-solvable, a contradiction.

(3) \( G \) has an unique minimal normal subgroup \( N \) and \( G/N \) is \( p \)-solvable. By (1) and Lemma 2.1(2), \( G \) is not simple. For every non-trivial normal subgroup \( N \) of \( G \), the minimal choice of \( G \) yields that \( G/N \) is \( p \)-solvable. Since \( p \)-solvable groups form a saturated formation, there exists an unique minimal normal subgroup \( N \) of \( G \).

If \( p \nmid |N| \) or \( |N| \) is a \( p \)-group, then \( N \) is \( p \)-solvable and then \( G \) is \( p \)-solvable, contrary to our choice. We assume that \( p \mid |N| \) and \( N = N_p \). The Frattini argument yields that \( G = N_G(N_p) \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \) such that \( N_p = P \cap N \). Since \( 1 \neq N_p \neq N, N_G(N_p) \neq G \). There exists \( M < G \) such that \( N_G(P) \leq N_G(N_p) \leq M \); hence of \( M \in \mathcal{F}^{p^c} \). From \( N \leq M \) it follows that \( M_G = 1 \). If \( |G:M| = q \) with \( q \) a prime, then \( q < p \) and \( |G|/q! \), a contradiction. Hence \( [G:M] \) is composite and \( M \in \mathcal{F}^{p^c} \). By (2), \( M \) is a \( c \)-normal in \( G \) and it follows that there exists a normal subgroup \( K \) such that \( N \cap M \leq K \cap M \leq M_G = 1 \). \( |G:M| \) yields that \( |N|_p = 1 \), a contradiction. There is no counterexample. □

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