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A new construction of the d-dimensional Buratti–Del Fra dual hyperoval

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ABSTRACT

The Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ is one of the four known infinite families of simply connected d-dimensional dual hyperovals over $\mathbf{F}_2$ with ambient space of vector dimension $(d+1)(d+2)/2$ (Buratti and Del Fra (2003) [1]). A criterion (Proposition 1) is given for a d-dimensional dual hyperoval over $\mathbf{F}_2$ to be covered by $\mathcal{D}_d(\mathbf{F}_2)$ in terms of the addition formula. Using it, we provide a simpler model of $\mathcal{D}_d(\mathbf{F}_2)$ (Proposition 3). We also give conditions (Lemma 4) for a collection $\mathcal{S}[B]$ of $(d+1)$-dimensional subspaces of $K \oplus K$ constructed from a symmetric bilinear form B on $K \cong \mathbf{F}_{2^{d+1}}$ to be a quotient of $\mathcal{D}_d(\mathbf{F}_2)$. For when d is even, an explicit form B satisfying these conditions is given. We also provide a proof for the fact that the affine expansion of $\mathcal{D}_d(\mathbf{F}_2)$ is covered by the halved hypercube (Proposition 10).

1. Introduction

For a natural number d, a *d-dimensional dual hyperoval* (*d-dual hyperoval*, for short) over a finite field $\mathbf{F}_q$ is a collection $\mathcal{S}$ of subspaces of a vector space U over $\mathbf{F}_q$ which satisfies the following conditions (h0)–(h3):

(h0) Each member X of $\mathcal{S}$ has vector dimension $n = d + 1$.
(h1) $\dim(X \cap Y) = 1$ for any two distinct members X and Y of $\mathcal{S}$.
(h2) $X \cap Y \cap Z = \{0\}$ for any three mutually distinct members X, Y, Z of $\mathcal{S}$.
(h3) $\mathcal{S}$ consists of $((q^n - 1)/(q - 1)) + 1$ members.

The subspace of U spanned by all members of $\mathcal{S}$ is called the *ambient space* of $\mathcal{S}$ and denoted as $\mathbf{A}(\mathcal{S})$. If $\mathbf{A}(\mathcal{S})$ has vector dimension $k+1$, we say that $\mathcal{S}$ is a *d-dual hyperoval in $PG(k, q)$*.

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For two d-dual hyperovals $\mathcal{S}_i$ over $\mathbf{F}_q$ ($i = 1, 2$), we say that $\mathcal{S}_1$ *covers* $\mathcal{S}_2$ (or $\mathcal{S}_2$ is a *quotient* of $\mathcal{S}_1$) if there is a surjective semilinear map ρ from $\mathbf{A}(\mathcal{S}_1)$ to $\mathbf{A}(\mathcal{S}_2)$ (thus if $q = 2$, ρ is just an $\mathbf{F}_2$-linear surjection) which sends each member of $\mathcal{S}_1$ bijectively to a member of $\mathcal{S}_2$. If $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$, then a surjective map ρ on $\mathbf{A}(\mathcal{S})$ with the above property is called an *automorphism* of $\mathcal{S}$. The set of all automorphisms of $\mathcal{S}$ forms a group with respect to the composition of maps, denoted as $\mathrm{Aut}(\mathcal{S})$.

A d-dual hyperoval $\mathcal{S}$ over $\mathbf{F}_q$ is said to be *simply connected* if any cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$ coincides with $\mathcal{S}$. For any d-dual hyperoval $\mathcal{S}$ over $\mathbf{F}_q$, it is known that $\dim \mathbf{A}(\mathcal{S}) \leq (d+1)(d+2)/2$ if $q > 2$. When $q = 2$, it is conjectured that the same result holds. (See [12, Subsection 2.4].) Currently, four infinite families of simply connected d-dual hyperovals in $PG(d(d+3)/2, q)$ are known: the Huybrechts dual hyperoval $\mathcal{H}_d(\mathbf{F}_2)$ over $\mathbf{F}_2$ [4], the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ over $\mathbf{F}_2$ (this is regarded as a certain deformation of $\mathcal{H}_d(\mathbf{F}_2)$); the Veronesean dual hyperoval $\mathcal{V}_d(\mathbf{F}_q)$ over $\mathbf{F}_q$ for any $q = 2^e$ [11], and its deformation $\mathcal{T}_d(\mathbf{F}_2)$ given by the first author [8] for when $q = 2$.

In a series of papers by the second author [14,16,15], it was shown that quadratic APN functions on $K \cong \mathbf{F}_{2^{d+1}}$, a remarkable class of functions with extremal nonlinearity, correspond (up to extended affine equivalence) to the dimensional dual hyperovals in $PG(2d+1, 2)$ which are obtained as some quotients of $\mathcal{H}_d(\mathbf{F}_2)$ (up to isomorphism of dual hyperovals). In particular, the classification of quadratic APN functions is reduced to the classification of some subspaces of $\mathbf{A}(\mathcal{H}_q(\mathbf{F}_2))$ satisfying strong restrictions.

This motivates the following project. Find explicit examples of quotients in $PG(2d+1, q)$ of known simply connected dual hyperovals in $PG(d(d+3)/2, q)$, and investigate relations between them and some classes of functions on a field $\mathbf{F}_{q^{d+1}}$ (or $\mathbf{F}_{q^{d+1}} \times \mathbf{F}_{q^{d+1}}$) which can be regarded as analogues of quadratic APN functions (or the associated alternating form). The authors made some contributions to this project for $\mathcal{V}_d(\mathbf{F}_q)$ [7,13,10]. However, the most interesting target that one should investigate next to $\mathcal{H}_d(\mathbf{F}_2)$ seems to be $\mathcal{D}_d(\mathbf{F}_2)$, because the latter is obtained from the former by a certain deformation.

This paper is the first attempt to find an explicit example of quotients in $PG(2d+1, 2)$ of $\mathcal{D}_d(\mathbf{F}_2)$. In Section 2, we give a general criterion for a d-dual hyperoval over $\mathbf{F}_2$ to be covered by $\mathcal{D}_d(\mathbf{F}_2)$ in terms of the addition formula (see Proposition 1). Using this criterion, we provide a simple and explicit model of $\mathcal{D}_d(\mathbf{F}_2)$ in which the members are described in a similar manner to those of $\mathcal{H}_d(\mathbf{F}_2)$ (see Proposition 3). In Section 3, we give conditions (see Lemma 4) for a certain collection $\mathcal{S}[B]$ of $(d+1)$-dimensional subspaces in $K \oplus K$ constructed from a symmetric bilinear map B on $K \cong \mathbf{F}_{2^{d+1}}$ (which corresponds to an alternating bilinear map in the case of a quadratic APN function) to be covered by $\mathcal{D}_d(\mathbf{F}_2)$. For d even, we found a single example of B satisfying these conditions: $B(x, y) = x^4y + xy^4 + xy + x^2y^2$. Notice that B is symmetric but not alternating. Hence $\mathcal{S}[B]$ for this map B is a quotient in $PG(2d+1, 2)$ of $\mathcal{D}_d(\mathbf{F}_2)$ (see Proposition 6). In Section 4, a proof is given for the fact that the affine expansion of the Buratti–Del Fra dual hyperoval is covered by the halved hypercube (see Proposition 10). We conclude the paper by proposing several questions in Section 5.

2. A simpler description of the Buratti–Del Fra dual hyperoval

In this section, we give a criterion for a d-dual hyperoval over $\mathbf{F}_2$ to be covered by the d-dimensional Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ in terms of the addition formula. Using it, we give a much simpler model of $\mathcal{D}_d(\mathbf{F}_2)$.

We first summarize the notation used in this section. We use letters n and d to denote natural numbers satisfying $n = d + 1 \geq 3$. The letter I is used to denote the set of integers i with $0 \leq i \leq d$, and we set $I_0 := I \setminus \{0\}$. The letter V denotes an n-dimensional vector space over $\mathbf{F}_2$ with a basis e_i ($i \in I$) containing a specified nonzero vector e_0. We shall use the symbol ξ to denote the characteristic function of $V' := V \setminus \{0, e_0\}$, namely ξ is a function from V to $\mathbf{F}_2$ defined by $\xi(x) = 1$ or 0 according as $x \in V'$ or not.

We give an elementary but important remark on the value of the characteristic function ξ, which will be frequently used below. Observe that the subset $V' = V \setminus \{0, e_0\}$ is invariant under the addition by e_0; namely, for $y \in V$, we have $y \in V'$ if and only if $y + e_0 \in V'$. This implies that

$$\xi(y) = \xi(y + \varepsilon e_0) \quad \text{for any } \varepsilon \in \mathbf{F}_2. \tag{1}$$

Before stating a characterization of d-dual hyperovals over $\mathbf{F}_2$ covered by the Buratti–Del Fra dual hyperoval, we give a brief overview of its construction. In [1], Buratti and Del Fra investigated an arbitrary d-dual hyperoval $S = \{X(t) \mid t \in K\}$ over $\mathbf{F}_2$ which satisfies the following equation:

$$a(s, t_1) + a(s, t_2) = a(s, s + t_1 + t_2 + \alpha(s, t_1, t_2)e_0), \tag{2}$$

where we denote by $a(s, t) = a(t, s)$ the unique nonzero element of $X(s) \cap X(t)$ for distinct $s, t \in V$ with the convention that $a(s, s) = 0$ for all $s \in V$ and $\alpha(x, y, z) := \xi(x + y) + \xi(y + z) + \xi(z + x)$ for the characteristic function ξ of $V' = V \setminus \{0, e_0\}$. We refer to Eq. (2) as the *addition formula* in S.

We set $\mathbf{B} := \{0, e_i \mid i \in I\}$. Buratti and Del Fra derived from the addition formula (2) an explicit expression for each $a(s, t)$ $(s, t \in K)$ as an $\mathbf{F}_2$-linear combination of $a(w, w')$, where (w, w') ranges over the pairs of distinct elements in $\mathbf{B}$ [1, Section 2, Formula (16)] (repeated as [3, Proposition 1]). Notice that this expression may not be uniquely determined in general, since the $a(w, w')$'s may be linearly dependent.

We embed V as a hyperplane of a vector space U of dimension $n + 1$ over $\mathbf{F}_2$, and pick a vector e_∞ of U outside V. In [3, Section 2], a d-dual hyperoval S satisfying the above addition formula (2) is constructed inside the exterior square $\wedge^2(U)$, in which the $a(w, w')$'s are linearly independent. (In fact, $a(w, w') = (e_\infty + w) \wedge (e_\infty + w')$.) This is the *Buratti–Del Fra dual hyperoval*.

To specify this specific d-dual hyperoval, we use the letter $\mathcal{D}_d(\mathbf{F}_2)$ in this paper. Its members are denoted by $\tilde{X}(t)$ $(t \in V)$ and the unique nonzero vector of $\tilde{X}(s) \cap \tilde{X}(t)$ for distinct s, t of V is denoted as $\tilde{b}(s, t)$. We set $\tilde{b}(s, s) = 0$ for $s \in V$. Then $\tilde{b}(s, t)$ $(s, t \in V)$ satisfy the addition formula (Eq. (2)) with $\tilde{b}(s, t)$ instead of $a(s, t)$. By construction, $n(n + 1)/2$ vectors $\tilde{b}(w, w')$ form a basis for the ambient space $\wedge^2(U)$ of $\mathcal{D}_d(\mathbf{F}_2)$. We have $\tilde{X}(t) = \{\tilde{b}(x, t) \mid x \in V\}$ for each $t \in V$.

Proposition 1. *Let $\mathcal{S} = \{X(s) \mid s \in V\}$ be a d-dual hyperoval over $\mathbf{F}_2$ consisting of subspaces $X(s)$ of the ambient space $\mathbf{A}(\mathcal{S})$ indexed by the elements of V. For $s, t \in V$, we denote by $b(s, t)$ the unique nonzero vector of $X(s) \cap X(t)$ or 0 according as $s \neq t$ or $s = t$.*

Then $\mathcal{S}$ is covered by the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ via an $\mathbf{F}_2$-linear surjection from $\wedge^2(U)$ onto $\mathbf{A}(\mathcal{S})$ sending a member $\tilde{X}(t)$ of $\mathcal{D}_d(\mathbf{F}_2)$ to a member $X(t)$ of $\mathcal{S}$ for each $t \in V$ if and only if the following formula is satisfied for any $s, t_1, t_2 \in V$:

$$b(s, t_1) + b(s, t_2) = b(s, s + t_1 + t_2 + \alpha(s, t_1, t_2)e_0),$$

where $\alpha(s, t_1, t_2) := \xi(s+t_1)+\xi(s+t_2)+\xi(t_1+t_2)$ with the characteristic function ξ of $V' = V\setminus\{0, e_0\}$.

Proof. This result is implicit in [1,3], but we give an expository account.

Consider any quotient $\mathcal{S}$ of $\mathcal{D}_d(\mathbf{F}_2)$, and let ρ be a linear surjection from $\wedge^2(U)$ to the ambient space $\mathbf{A}(\mathcal{S})$ of $\mathcal{S}$ which bijectively maps each member $\tilde{X}(t)$ of $\mathcal{D}_d(\mathbf{F}_2)$ to a member $X(t)$ of $\mathcal{S}$ for every $t \in V$. From the linearity of ρ, the vectors $b(s, t) := \rho(\tilde{b}(s, t))$ of $\mathbf{A}(\mathcal{S})$ for any $s, t \in V$ satisfy the same addition formula as (2) with $b(s, t)$ instead of $a(s, t)$.

Conversely, assume that $\mathcal{S} = \{X(s) \mid s \in V\}$ satisfies the conditions in the proposition: namely, denoting by $b(s, t)$ the unique nonzero vector $b(s, t)$ in $X(s) \cap X(t)$ for any distinct elements s and t of V with the convention $b(s, s) = 0$ for $s \in V$, the vectors $b(s, t)$ satisfy the addition formula (Eq. (2)) with $b(s, t)$ instead of $a(s, t)$. Then we define an $\mathbf{F}_2$-linear map ρ from $\wedge^2(U)$ to $\mathbf{A}(\mathcal{S})$ by first setting $\rho(\tilde{b}(w, w')) := b(w, w')$ $(w \neq w' \in \mathbf{B})$ on the basis of $\wedge^2(U)$, and then by extending it linearly to all vectors of $\wedge^2(U)$. As $\{\tilde{b}(s, t) \mid s, t \in V\}$ and $\{b(s, t) \mid s, t \in V\}$ satisfy the same addition formula (2), it follows from [1, Section 2, Formula (16)] that for each $s, t \in V$ both $b(s, t)$ and $\tilde{b}(s, t)$ can be expressed as linear combinations of $b(w, w')$'s and $\tilde{b}(w, w')$'s and that if $\tilde{b}(s, t) = \sum \alpha(w, w')\tilde{b}(w, w')$ then $b(s, t) = \sum \alpha(w, w')b(w, w')$. (There may be another expression for $b(s, t)$ as a linear combination of $b(w, w')$, but this is not a problem. The point is that we have *an* expression for $b(s, t)$ in which the coefficients are the same as those that appeared in the defining expression for $\tilde{b}(s, t)$.) Thus the linear map ρ sends $\tilde{b}(s, t)$ to $b(s, t)$. Hence ρ bijectively maps each member $\tilde{X}(t) = \{\tilde{b}(x, t) \mid x \in V\}$ of $\mathcal{D}_d(\mathbf{F}_2)$ to a member $X(t) = \{b(x, t) \mid x \in V\}$ of $\mathcal{S}$. This verifies that ρ gives a cover of $\mathcal{S}$ by $\mathcal{D}_d(\mathbf{F}_2)$ with the property stated in the proposition. $\square$

Applying the above criterion (Proposition 1), we now provide a new model of the Buratti–Del Fra dual hyperoval for which the ambient space has the form $V \oplus ((V \otimes V)/W)$ for some subspace W of the tensor product $V \otimes V$. Notice that the $e_i \otimes e_j$ $(i, j \in I)$ form a basis for $V \otimes V$.

Define W to be a subspace of $V \otimes V$ spanned by the following vectors:

$$(x \otimes y) + (y \otimes x) \text{ and } (x \otimes x) + \xi(x)(e_0 \otimes x) \text{ for all } x, y \in V.$$

Adopting the expressions $x = \sum_{i \in I} x_i e_i$ and $y = \sum_{j \in I} y_j e_j$ with $x_i, y_j \in \mathbf{F}_2$ $(i, j \in I)$, we have $(x \otimes y) + (y \otimes x) = \sum_{i,j \in I, i<j} x_i y_j (e_i \otimes e_j + e_j \otimes e_i)$. If $x \in \{0, e_0\}$, we have $(x \otimes x) + \xi(x)(e_0 \otimes x) = e_0 \otimes e_0$ or 0. For any $x \in V \setminus \{0, e_0\}$, we have $(x \otimes x) + \xi(x)(e_0 \otimes x) = (x \otimes x) + (e_0 \otimes x) = \sum_{i \in I_0} x_i (e_i \otimes e_i + e_0 \otimes e_i) + \sum_{i,j \in I, i<j} x_i x_j (e_i \otimes e_j + e_j \otimes e_i)$. This implies that W has the following vectors as a basis, and hence has dimension $\binom{d+1}{2} + (d+1) = (d+1)(d+2)/2$ over $\mathbf{F}_2$:

$$(e_i \otimes e_j) + (e_j \otimes e_i) \quad \text{for all } i, j \in I \text{ with } i < j,$$

$$e_0 \otimes e_0 \quad \text{and} \quad (e_i \otimes e_i) + (e_0 \otimes e_i) \quad \text{for all } i \in I_0. \tag{3}$$

We denote by $\overline{v}$ the image $v + W$ of a vector $v \in V \otimes V$ under the canonical projection of $V \otimes V$ onto $(V \otimes V)/W$. Then $\overline{x \otimes y} = \overline{y \otimes x}$ and $\overline{(x_1 + x_2) \otimes y} = \overline{x_1 \otimes y} + \overline{x_2 \otimes y}$ for any $x, x_1, x_2, y \in V$. Furthermore, the following vectors form a basis for $(V \otimes V)/W$ (of dimension $(d+1)^2 - (d+1)(d+2)/2 = d(d+1)/2$):

$$\overline{e_i \otimes e_j} \quad \text{for all } i, j \in I_0 \text{ with } i < j, \quad \text{and} \quad \overline{e_0 \otimes e_i} \quad \text{for all } i \in I_0. \tag{4}$$

Lemma 2. *For nonzero vectors u and x of V, we have $\overline{x \otimes u} = \overline{0}$ in $(V \otimes V)/W$ if and only if $x = u + \xi(u)e_0$.*

Proof. Expressing u and x as $u = \sum_{i \in I} u_i e_i$ and $x = \sum_{i \in I} x_i e_i$ with $u_i, x_i \in \mathbf{F}_2$, we obtain the expression for $\overline{x \otimes u}$ as an $\mathbf{F}_2$-linear combination of the basis for $(V \otimes V)/W$ in Eq. (4):

$$\begin{aligned}\overline{x \otimes u} &= \sum_{i,j \in I} x_i u_j \overline{e_i \otimes e_j} \\ &= \sum_{j \in I_0} (u_0 x_j + x_0 u_j + u_j x_j) \overline{e_0 \otimes e_j} + \sum_{i,j \in I_0, i<j} (x_i u_j + u_i x_j) \overline{e_i \otimes e_j}.\end{aligned}$$

(Observe that $\overline{e_i \otimes e_j} = \overline{e_j \otimes e_i}$ and $\overline{e_i \otimes e_i} = \overline{e_0 \otimes e_j}$ for any $i \in I$, from Eqs. (3).) Thus the condition $\overline{x \otimes u} = \overline{0}$ is equivalent to the following simultaneous equations in $\mathbf{F}_2$:

$$x_0 u_j = (u_0 + u_j) x_j \quad \text{and} \quad x_i u_j = u_i x_j \quad \text{for all } i, j \in I_0. \tag{5}$$

Assume that $u \neq e_0$. As $u \neq 0$, there exists some $j_0 \in I_0$ with $u_{j_0} = 1$. Then it follows from Eq. (5) that $x_0 = (u_0 + 1) x_{j_0}$ and $x_i = x_{j_0} u_i$ for all $i \in I_0$. Thus $x = \sum_{i \in I} x_i e_i = x_{j_0}(e_0 + \sum_{i \in I} u_i e_i) = x_{j_0}(e_0 + u)$. As $x \neq 0$, we have $x_{j_0} = 1$ and $x = e_0 + u$ in this case. Conversely, $x = u + e_0$ satisfies $x_0 = 1 + u_0$ and $x_j = u_j$ for all $j \in I_0$, and so Eq. (5) holds.

In the remaining case when $u = e_0$, we have $u_0 = 1$ but $u_j = 0$ for all $j \in I_0$. Thus it follows from Eq. (5) that $x_j = 0$ for all $j \in I_0$. Then $x = e_0$, as $x \neq 0$. Conversely, $x = e_0 = u$ satisfies $x_0 = u_0 = 1$ and $x_j = u_j = 0$ for all $j \in I_0$, and so Eq. (5) holds.

Using the characteristic function ξ, the resulting nonzero vector x in the above two cases can be uniformly written as $x = \xi(u)e_0 + u$. $\square$

We consider the direct sum $A := V \oplus ((V \otimes V)/W) := \{(x, \overline{v}) \mid x \in V, v \in V \otimes V\}$. The dimension of A is $(d+1) + (d(d+1)/2) = (d+1)(d+2)/2$. For each $t \in V$, we define a subset $X(t)$ of A by

$$X(t) := \{(x, \overline{x \otimes t}) \mid x \in V\}. \tag{6}$$

Then $X(t)$ is a subspace of A of dimension $d + 1$.

Proposition 3. *Under the above notation, $\mathcal{D} := \{X(t) \mid t \in V\}$ is a d-dual hyperoval over $\mathbf{F}_2$ with ambient space A, which is isomorphic to the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$.*

Proof. Take any two distinct vectors s and t of V. Then a nonzero vector $(x, \overline{y})$ lies in $X(s) \cap X(t)$ if and only if $\overline{y} = \overline{x \otimes s} = \overline{x \otimes t}$ with $x \neq 0$. Then x is a nonzero vector of V with $\overline{x \otimes (s+t)} = \overline{0}$. From Lemma 2, we have $x = s + t + \xi(s+t)e_0$. Thus the intersection $X(s) \cap X(t)$ contains a unique nonzero vector

$$b(s, t) := (s + t + \xi(s + t)e_0, \overline{(s + t + \xi(s + t)e_0) \otimes s}). \tag{7}$$

(In order to see that $b(s, t) \in X(t)$, notice that the second component of $b(s, t)$ can also be written as $\overline{(s + t + \xi(s + t)e_0) \otimes t}$, because $(s + t) \otimes (s + t) + \xi(s + t)((s + t) \otimes e_0)$ lies in W.) From Eq. (7), it is clear that $X(s) \cap X(t) \cap X(u) = \{(0, \overline{0})\}$ for any mutually distinct vectors s, t, u of V. Hence the collection $\mathcal{D}$ of 2^{d+1} subspaces $X(t)$ $(t \in V)$ is a d-dual hyperoval over $\mathbf{F}_2$. The ambient space of $\mathcal{D}$ coincides with $A = V \oplus ((V \oplus V)/W)$, because $X(0) = \{(x, \overline{0}) \mid x \in V\}$ and $(V \oplus V)/W$ is spanned by $\overline{x \otimes t}$ for all $x, t \in V$. Summarizing, we have verified that $\mathcal{D}$ is a d-dual hyperoval over $\mathbf{F}_2$ with ambient space A.

For distinct vectors s, t of V, the nonzero vector $b(s, t)$ is given by Eq. (7). We extend it by assuming that $b(s, s) = (0, \overline{0})$ for $s \in V$. We shall verify the addition formula; namely, for any $s, t_1, t_2 \in V$ we have

$$b(s, t_1) + b(s, t_2) = b(s, s + t_1 + t_2 + \alpha(s, t_1, t_2)e_0). \tag{8}$$

By Eq. (7), the first components of the left and the right hand sides of Eq. (8) are respectively calculated to be $t_1 + t_2 + (\xi(s+t_1) + \xi(s+t_2))e_0$ and $t_1 + t_2 + (\alpha(s, t_1, t_2) + \xi(t_1 + t_2 + \alpha(s, t_1, t_2)e_0))\, e_0$. Thus to verify Eq. (8) it suffices to show that $\alpha(s, t_1, t_2) = \xi(s+t_1) + \xi(s+t_2) + \xi(t_1 + t_2 + \alpha(s, t_1, t_2)e_0)$, or equivalently $\xi(t_1 + t_2) = \xi(t_1 + t_2 + \alpha(s, t_1, t_2)e_0)$. This follows from Eq. (1). Thus we have verified Eq. (8).

Then it follows from Proposition 1 that $\mathcal{D}$ is covered by the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$. Notice here that the ambient space A of $\mathcal{D}$ has the dimension $(d + 1)(d + 2)/2$, which is equal to the dimension of the ambient space of $\mathcal{D}_d(\mathbf{F}_2)$. Thus the covering map of $\mathcal{D}$ by $\mathcal{D}_d(\mathbf{F}_2)$ gives an isomorphism of $\mathcal{D}_d(\mathbf{F}_2)$ with $\mathcal{D}$. □

The original construction of the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ [3] is very complicated, because the members are described there as subspaces of the wedge product $\wedge^2(U)$ for a $(d + 2)$-dimensional space U containing V as a hyperplane. Notice that $\wedge^2(U)$ is isomorphic to $A = V \oplus ((V \otimes V)/W)$ as vector spaces, in view of their dimensions. The above description $\mathcal{D}$ of the Buratti–Del Fra hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ is much simpler—in it, the members are described as subspaces of A instead. This new description immediately shows that $\mathcal{D}_d(\mathbf{F}_2)$ *splits* in the sense of [15]; namely $Y := \{(0, \overline{v}) \mid v \in V \otimes V\}$ is a subspace of A of codimension $n = d + 1$ which intersects every member of $\mathcal{D}$ at the zero subspace.

The above construction also provides another account of why the Buratti–Del Fra dual hyperoval is regarded as a "deformation" of the Huybrechts dual hyperoval. Recall that the Huybrechts dual hyperoval $\mathcal{H}_d(\mathbf{F}_2)$ has the ambient space $V \oplus ((V \otimes V)/W')$ where W' is the subspace of $V \otimes V$ spanned by $(x \otimes y) + (y \otimes x)$ and $x \otimes x$ for all $x, y \in V$. In this case $(V \otimes V)/W'$ is regarded as the alternating square tensor product $\wedge^2(V)$ by identifying $(x \otimes t) + W'$ with $x \wedge t$ $(x, t \in V)$, and hence $V \oplus ((V \otimes V)/W')$ is isomorphic to $\wedge^2(U)$ via the map sending $(x, x \wedge t)$ to $(e_\infty + t) \wedge x$. In fact, each member $X(t)$ of $\mathcal{H}_d(\mathbf{F}_2)$ is defined as a subspace $X(t) := \{(x, x \wedge t) \mid x \in V\}$ of $V \oplus ((V \otimes V)/W')$.

We conclude this section by providing some automorphisms of the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$, in terms of the above model $\mathcal{D}$. For each $a \in V$, define a map $\tau(a)$ on $A = V \oplus ((V \otimes V)/W)$ by

$$(x, \overline{y})^{\tau(a)} := (x, \overline{y} + \overline{x \otimes a}), \quad x \in V,\ y \in V \otimes V. \tag{9}$$

It is easy to see that $\tau(a)$ is an $\mathbf{F}_2$-linear bijection on A. As $(x, \overline{x \otimes t})^{\tau(a)} = (x, \overline{x \otimes (t + a)})$ for each $t \in V$, $\tau(a)$ sends a member $X(t)$ of $\mathcal{D}$ to $X(t + a)$ for every $t \in V$. Thus $\tau(a)$ is an automorphism of $\mathcal{D}$. Furthermore, $\{\tau(a) \mid a \in V\}$ is a subgroup of $\mathrm{Aut}(\mathcal{D})$ which acts regularly on the members of $\mathcal{D}$.

For each $\mathbf{F}_2$-linear bijection α of V which fixes the specified element e_0 of V, define a map $\overline{\alpha}$ on $V \otimes V$ by $(e_i \otimes e_j)^{\overline{\alpha}} = e_i^{\alpha} \otimes e_j^{\alpha}$ on a basis $e_i \otimes e_j$ $(i, j \in I)$ for $V \otimes V$, and then extend it linearly. Then $\overline{\alpha}$ is an $\mathbf{F}_2$-linear bijection on $V \otimes V$ satisfying $(x \otimes y)^{\overline{\alpha}} = x^{\alpha} \otimes y^{\alpha}$ for $x, y \in V$. As 0 and e_0 are elements of V fixed by α, we have $\xi(x^{\alpha}) = \xi(x)$ and $(e_0 \otimes x)^{\overline{\alpha}} = e_0 \otimes x^{\alpha}$ for $x \in V$. In particular, $\overline{\alpha}$ maps generators $x \otimes y + y \otimes x$ and $x \otimes x + \xi(x)e_0 \otimes x$ $(x \in V)$ of the subspace W to $x^{\alpha} \otimes y^{\alpha} + y^{\alpha} \otimes x^{\alpha}$ and $x^{\alpha} \otimes x^{\alpha} + \xi(x^{\alpha})e_0 \otimes x^{\alpha}$ $(x \in V)$, which lie in W as well. Thus W is stabilized by $\overline{\alpha}$, and hence $\overline{\alpha}$ induces an $\mathbf{F}_2$-linear bijection on the quotient space $(V \otimes V)/W$, which we also denote by the same letter $\overline{\alpha}$. Then $\overline{x \otimes y}^{\overline{\alpha}} = \overline{x^{\alpha} \otimes y^{\alpha}}$ for all $x, y \in V$. Now define an $\mathbf{F}_2$-linear bijection $l(\alpha)$ on $A = V \oplus ((V \otimes V)/W)$ by

$$(x, \overline{y})^{l(\alpha)} := (x^{\alpha}, \overline{y}^{\overline{\alpha}}) \quad x \in V,\ y \in V \otimes V. \tag{10}$$

As $(x, \overline{x \otimes t})^{l(\alpha)} = (x^{\alpha}, \overline{x^{\alpha} \otimes t^{\alpha}})$ for $x, t \in V$, $l(\alpha)$ sends a member $X(t)$ of $\mathcal{D}$ to a member $X(t^{\alpha})$ of $\mathcal{D}$ $(t \in V)$. Hence $l(\alpha)$ is an automorphism of $\mathcal{D}$. Moreover, $\{l(\alpha) \mid \alpha \in GL(V), e_0^{\alpha} = e_0\}$ is a subgroup of $\mathrm{Aut}(\mathcal{D})$ which fixes $X(0)$ and $X(e_0)$ and acts transitively on the remaining members of $\mathcal{D}$, because the stabilizer of e_0 in $GL(V) \cong GL_{d+1}(2)$ acts transitively on $V \setminus \{0, e_0\}$.

From [3, Proposition 10], $\mathrm{Aut}(\mathcal{D})$ coincides with a semi-direct product of the normal subgroup $\{\tau(a) \mid a \in V\} \cong 2^{d+1}$ with a complement $\{l(\alpha) \mid \alpha \in GL(V), e_0^{\alpha} = e_0\} \cong 2^d : GL_d(2)$.

3. Quotients in $PG(2d+1, 2)$ of the Buratti–Del Fra d-dual hyperoval

In this section, we retain the notation of the previous section, but we take as V the vector space over $\mathbf{F}_2$ underlying the finite field $K \cong \mathbf{F}_{2^n}$, $n = d + 1 \geq 3$. We regard $K \oplus K$ as a vector space over $\mathbf{F}_2$ of dimension $2n$.

Lemma 4. *Let B be an $\mathbf{F}_2$-bilinear map from $K \times K$ to K. For each $t \in K$, define a subset $X(t)$ of $K \oplus K$ by*

$$X(t) := \{(x, B(x, t)) \mid x \in K\}.$$

Then a collection $\mathcal{S}[B]$ of $X(t)$ $(t \in K)$ is a d-dual hyperoval over $\mathbf{F}_2$ if and only if the following conditions are satisfied:

(1) *For each nonzero $t \in K$, there exists a unique nonzero element $\kappa(t)$ of K such that $B(\kappa(t), t) = 0$.*
(2) *The map κ sending t to $\kappa(t)$ is a bijection on $K^{\times}$.*

Furthermore, $S[B]$ is covered by the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$ via a linear surjection from $\mathbf{A}(\mathcal{D}_d(\mathbf{F}_2))$ to $\mathbf{A}(\mathcal{S}[B])$ which sends $\tilde{X}(t) \in \mathcal{D}_d(\mathbf{F}_2)$ to $X(t) \in \mathcal{S}[B]$ for each $t \in K$ if and only if the conditions (1), (2) above and the following condition (3) hold:

(3) *there exists a nonzero element e_0 of K together with an $\mathbf{F}_2$-linear map λ on K such that $\kappa(x) = \lambda(x + \xi(x)e_0)$ for every $x \in K$, where ξ denotes the characteristic function of $K \setminus \{0, e_0\}$.*

Proof. The bilinearity of B implies that each $X(t)$ is a subspace of $K \oplus K$ of dimension $d+1$. For distinct elements s and t of K, a nonzero vector (x, y) of $K \oplus K$ lies in both $X(s)$ and $X(t)$ if and only if x is a nonzero element with $y = B(x, s) = B(x, t)$. As the latter equation is equivalent to the condition $B(x, s + t) = 0$, there is a unique such $x \in K^{\times}$ if and only if the condition (1) in lemma holds. Under the condition (1), $X(s) \cap X(t)$ has a unique nonzero vector $b(s, t) := (\kappa(s + t), B(\kappa(s + t), s))$. Thus the condition (2) in the lemma is satisfied if and only if $X(s) \cap X(t) \cap X(u) = \{(0, 0)\}$ for any mutually distinct s, t, u of K. Thus we verified that $\mathcal{S}[B]$ is a d-dual hyperoval if and only if (1) and (2) hold.

We shall show the latter claim. As we saw above, $b(s, t) = (\kappa(s+t), B(\kappa(s+t), s))$ for any distinct $s, t \in K$. We extend the definition of $b(s, t)$ for all $s, t \in K$ by setting $b(s, s) = 0$. Accordingly, we define $\kappa(0) = 0$. It follows from Proposition 1 that the d-dual hyperoval $\mathcal{S}[B]$ is covered by $\mathcal{D}_d(\mathbf{F}_2)$ via a linear surjection from $\mathbf{A}(\mathcal{D}_d(\mathbf{F}_2))$ to $\mathbf{A}(\mathcal{S}[B])$ sending $\tilde{X}(t) \in \mathcal{D}_d(\mathbf{F}_2)$ to $X(t) \in \mathcal{S}[B]$ for every $t \in K$ if and only if the addition formula (Eq. (2)) holds for some $e_0 \in K^{\times}$. In view of the above shape of $b(s, t)$ in $\mathcal{S}[B]$, the addition formula in $\mathcal{S}[B]$ is equivalent to the following equation for any $s, t_1, t_2 \in K$: $\kappa(s + t_1) + \kappa(s + t_2) = \kappa(t_1 + t_2 + \alpha e_0)$, where $\alpha := \xi(s + t_1) + \xi(s + t_2) + \xi(t_1 + t_2)$. Setting $x = s + t_1, y = s + t_2$, this is equivalent to requiring that the following equation holds for every $x, y \in K$, where $\alpha(x, y) = \xi(x) + \xi(y) + \xi(x + y)$:

$$\kappa(x) + \kappa(y) = \kappa(x + y + \alpha(x, y)e_0). \tag{11}$$

Now let e_i ($i \in I$) be a basis for K including e_0, and denote by H the hyperplane of K spanned by e_i ($i \in I_0 = I \setminus \{0\}$). By the definition of ξ and its property (1), we have $\alpha(e_0, y) = \xi(e_0) + \xi(y) + \xi(y + e_0) = 0$. Then it follows from Eq. (11) with $x = e_0$ that

$$\kappa(e_0) + \kappa(y) = \kappa(e_0 + y) \tag{12}$$

for every $y \in H$. For an element v of H, we express v as $v = \sum_{i=1}^{d} a_i e_i$ with $a_i \in \mathbf{F}_2$ ($i \in I_0$). We set $\mathrm{Supp}(v) := \{i \in I_0 \mid a_i = 1\}$. Assume that $|\mathrm{Supp}(v)| \geq 2$. Pick any $j \in \mathrm{Supp}(v)$. Then none of $x = v, y = v + e_j$ and $x + y = e_j$ are contained in $\{0, e_0\}$, and hence $\alpha(x, y) = 1$. It follows from Eqs. (11) and (12) that $\kappa(v) = \kappa(v+e_j)+\kappa(e_j+e_0) = \kappa(v+e_j)+\kappa(e_j)+\kappa(e_0)$. Notice that $|\mathrm{Supp}(v+e_j)| = |\mathrm{Supp}(v)|-1$. Continuing in this way, we conclude that $\kappa(v) = \sum_{j\in \mathrm{Supp}(v)} \kappa(e_j)+(|\mathrm{Supp}(v)|-1)\kappa(e_0)$. This equation holds when $|\mathrm{Supp}(v)| = 1$ as well. This conclusion can also be stated as follows, because $|\mathrm{Supp}(v)| = \sum_{i=1}^{d} a_i$ modulo 2: if $\sum_{i=1}^{d} a_i e_i \neq 0$, then

$$\kappa\left(\sum_{i=1}^{d} a_i e_i\right) = \sum_{i=1}^{d} a_i \kappa(e_i) + \left(\sum_{i=1}^{d} a_i + 1\right)\kappa(e_0) = \sum_{i=1}^{d} a_i(\kappa(e_i) + \kappa(e_0)) + \kappa(e_0).$$

This equation does not hold for $\sum_{i=1}^{d} a_i e_i = 0$. However, since $\xi(v)$ for $v \in H$ is 0 or 1 according as $v = 0$ or $v \neq 0$, the following equation holds for any $v = \sum_{i=1}^{d} a_i e_i$ of V, including $v = 0$:

$$\kappa\left(\sum_{i=1}^{d} a_i e_i\right) = \sum_{i=1}^{d} a_i(\kappa(e_i) + \kappa(e_0)) + \xi\left(\sum_{i=1}^{d} a_i e_i\right)\kappa(e_0). \tag{13}$$

For $w = e_0 + v$ with $v \in H$, we then have $\kappa(e_0 + \sum_{i=1}^{d} a_i e_i) = \kappa(e_0) + \sum_{i=1}^{d} a_i(\kappa(e_i) + \kappa(e_0)) + \xi(\sum_{i=1}^{d} a_i e_i)\kappa(e_0) = \kappa(e_0) + \sum_{i=1}^{d} a_i(\kappa(e_i) + \kappa(e_0)) + \xi(e_0 + \sum_{i=1}^{d} a_i e_i)\kappa(e_0)$ from Eq. (12) and Property (1) of ξ. Hence, we conclude that for every element $\sum_{i=0}^{d} a_i e_i$ ($a_i \in \mathbf{F}_2, i = 0, \ldots, d$) of K the following equation holds:

$$\kappa\left(\sum_{i=0}^{d} a_i e_i\right) = a_0\kappa(e_0) + \sum_{i=1}^{d} a_i(\kappa(e_i) + \kappa(e_0)) + \xi\left(\sum_{i=0}^{d} a_i e_i\right)\kappa(e_0). \tag{14}$$

We now define an $\mathbf{F}_2$-linear map λ on K on the basis e_i ($i = 0, \ldots, d$) by $\lambda(e_0) := \kappa(e_0), \lambda(e_i) := \kappa(e_i) + \kappa(e_0)$ for $i = 1, \ldots, d$, and then extend it linearly on K. Thus for every element $x = \sum_{i=0}^{d} a_i e_i$ of K we have

$$\kappa\left(\sum_{i=0}^{d} a_i e_i\right) = \lambda\left(\sum_{i=0}^{d} a_i e_i + \xi\left(\sum_{i=0}^{d} a_i e_i\right)e_0\right). \tag{15}$$

Conversely, if κ is given as $\kappa(x) = \lambda(x + \xi(x)e_0)$ for some $\mathbf{F}_2$-linear map λ, then we can check Eq. (11) as follows: for every x and y of K we have $\kappa(x) + \kappa(y) = \lambda(x + y + (\xi(x) + \xi(y))e_0) = \lambda(x+y+(\alpha(x, y)+\xi(x+y))e_0) = \lambda(x+y+\alpha(x, y)e_0+\xi(x+y+\alpha(x, y)e_0)e_0) = \kappa(x+y+\alpha(x, y)e_0)$, using the linearity of λ, the definition of $\alpha(x, y)$ and Property (1) of ξ. $\square$

We now give an explicit example of a bilinear map B on $K \cong \mathbf{F}_{2^n}$ which satisfies the conditions (1)–(3) in Lemma 4 with $e_0 = 1$. Accordingly, ξ denotes the characteristic function of $K \setminus \{0, 1\}$. Consider the bilinear map B from $K \times K$ to K defined by

$$B(x, t) = x^4 t + x t^4 + (xt) + (xt)^2 \quad (x, t \in K). \tag{16}$$

Observe that B is symmetric (that is, $B(x, y) = B(y, x)$ for any $x, y \in K$) but not alternating (namely, $B(x, x) \neq 0$ for some $x \in K$). For each $t \in K$, define a subspace $X(t)$ of $K \oplus K$ by

$$X(t) := \{(x, B(x, t)) \mid x \in K\}.$$

It is easy to see that $X(t)$ is a subspace of $K \oplus K$ of dimension n for every $t \in K$. We shall investigate whether $\mathcal{S}[B] = \{X(t) \mid t \in K\}$ is a d-dimensional dual hyperoval ($d = n - 1$) over $\mathbf{F}_2$.

Lemma 5. (1) *Assume that n is odd. Then, for each nonzero element t of K, there is a unique nonzero $\kappa(t) \in K$ satisfying $B(\kappa(t), t) = 0$. Explicitly,*

$$\kappa(t) = t + \xi(t). \tag{17}$$

(2) *Assume that n is even. Then, for each nonzero element t of K not contained in the four-element subfield K_0 of K, there are exactly three nonzero elements x of K satisfying $B(x, t) = 0$.*

Proof. Fix a nonzero element t of K. Observe the following factorization of $B(x, t)$:

$$\begin{aligned} B(x, t) &= x^4 t + x t^4 + xt + x^2 t^2 = xt\{x^3 + tx + (t^3 + 1)\} \\ &= xt(x + t + 1)\{x^2 + (t + 1)x + (t^2 + t + 1)\}. \end{aligned}$$

Thus the set of solutions x in K for the equation $B(x, t) = 0$ consists of 0, $t + 1$ and the solutions in K for the quadratic equation $x^2 + (t + 1)x + (t^2 + t + 1) = 0$. In particular, if $t = 1$, we have $\{x \in K \mid B(x, t) = 0\} = \{0, 1\}$.

Notice that if $t + 1 \neq 0$, the quadratic equation $x^2 + (t + 1)x + (t^2 + t + 1) = 0$ has a root x in K if and only if $\mathrm{Tr}((t^2 + t + 1)/(t + 1)^2) = 0$, where Tr denotes the trace function for the extension $K/\mathbf{F}_2$. Since $(t^2 + t + 1)/(t + 1)^2 = 1 + (t/(t + 1)^2) = 1 + (1/(1 + t)) + (1/(t + 1))^2$, we have $\mathrm{Tr}(\frac{(t^2+t+1)}{(t+1)^2}) = \mathrm{Tr}(1) = n$.

Thus if n is odd, there is no $x \in K$ satisfying $x^2 + (t + 1)x + (t^2 + t + 1) = 0$ for every $t \in K \setminus \{0, 1\}$. Hence for every $t \in K \setminus \{0, 1\}$, the equation $B(x, t) = 0$ has exactly one nonzero solution $x = t + 1$. Using the characteristic function ξ for the subset $K' := K \setminus \{0, 1\}$, this can be written as $x = t + \xi(t)$. This formula holds if $t = 1$ as well, because $x = 1 = 1 + \xi(1)$ is the unique nonzero solution for $B(x, 1) = 0$. We have verified Claim (1).

On the other hand, if n is even, there are exactly two solutions x in K for the equation $x^2 + (t + 1)x + (t^2 + t + 1) = 0$ for every $t \in K'$. If t is not contained in K_0, we have $t^2 + t + 1 \neq 0$, whence none of these solutions is equal to $t + 1$. Thus if $t \notin K_0$, then there are exactly three solutions in K for the equation $B(x, t) = 0$. This verifies Claim (2). □

Proposition 6. *If $n = d + 1$ is odd, then $\mathcal{S}[B] = \{X(t) \mid t \in K\}$ is a d-dimensional dual hyperoval over $\mathbf{F}_2$ with ambient space $K \oplus K$, which is covered by the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$, while if n is even, $\mathcal{S}[B]$ is not a dimensional dual hyperoval.*

Proof. Choose two distinct elements s and t of K. A nonzero vector (x, y) of $K \oplus K$ lies in both $X(s)$ and $X(t)$ if and only if x is a nonzero element of K satisfying $B(x, t) = B(x, s)$. From the bilinearity of B, then x is a nonzero solution in K for the equation $B(x, s + t) = 0$.

Assume that n is even. As $n \geq 3$, there is an element u of K not contained in the four-element subfield K_0. It follows from Lemma 5 that there are three distinct elements x in K with $B(x, u) = 0$. Thus if n is even and $s + t \notin K_0$, $X(s) \cap X(t)$ is not a projective point. In particular, $\mathcal{S}[B]$ is not a dimensional dual hyperoval in this case.

Now assume that n is odd (so $d = n - 1$ is even). It follows from Lemma 5 that we have $x = (s+t)+\xi(s+t)$. With the notation of Lemma 4, this implies that $\kappa(a) = a+\xi(a)$ for $a \in K^\times$. We extend κ on K by setting $\kappa(0) = 0$. We shall verify that the function κ is bijective on K. Assume that $t_1 + \xi(t_1) = t_2 + \xi(t_2)$ for distinct $t_1, t_2 \in K$. Then we have $0 \neq t_1 + t_2 = \xi(t_1) + \xi(t_2) \in \{0, 1\}$. Thus $t_1 + t_2 = 1 = \xi(t_1) + \xi(t_2)$. Now it follows from Eq. (1) that we have $\xi(t_1) = \xi(t_2)$, as $t_1 + t_2 = 1$. However, this contradicts that $\xi(t_1) + \xi(t_2) = 1$. Hence the map κ is injective, and hence bijective. Now we have verified the conditions (1) and (2) in Lemma 4. Thus $\mathcal{S}[B]$ is a d-dual hyperoval if d is even. Moreover, as $\kappa(x) = x + \xi(x)$ $(x \in K)$, the condition (3) in Lemma 4 is satisfied for taking the identity map and the element 1 respectively as λ and e_0. Hence it follows from Lemma 4 that $\mathcal{S}[B]$ with d even is covered by the Buratti–Del Fra dual hyperoval $\mathcal{D}_d(\mathbf{F}_2)$.

It remains to verify that the ambient space of $\mathcal{S}[B]$ coincides with $K \oplus K$. As $X(0) = \{(x, 0) \mid x \in K\}$, the ambient space of $\mathcal{S}[B]$ has the shape $K \oplus H$, where H is a subspace of K spanned by $B(x, t)$ for all $x, t \in K$. If H is a proper subspace of K, it is contained in a hyperplane of K, so there is a nonzero element α of K such that $\mathrm{Tr}(\alpha B(x, t)) = 0$ for all $x, t \in K$. As

$$0 = \mathrm{Tr}(\alpha B(x, t)) = \mathrm{Tr}(\{\alpha t + (\alpha t^4)^4 + (\alpha t)^4 + (\alpha t^2)^2\} x^4)$$

for all $x \in K$, we have $0 = \alpha t + (\alpha t^4)^4 + (\alpha t)^4 + (\alpha t^2)^2 = \alpha^4 t^{16} + (\alpha^4 + \alpha^2)t^4 + \alpha t$ for all $t \in K$. Thus the polynomial $a(X) := \alpha^4 X^{16} + (\alpha^4 + \alpha^2)X^4 + \alpha X \in K[X]$ of degree at most 16 has at least $|K| = 2^n$ solutions. If $n \geq 5$, then $a(X)$ should be the zero polynomial, and hence $\alpha = 0$, which is a contradiction. If $n = 3$, we have $t^8 = t$, and then $\alpha^4 t^2 + (\alpha^4 + \alpha^2)t^4 + \alpha t = 0$ for all $t \in K$. Thus $b(X) := (\alpha^4 + \alpha^2)X^4 + \alpha^4 X^2 + \alpha X \in K[X]$ of degree at most 4 has at least $|K| = 8$ solutions, whence $b(X)$ is the zero polynomial. Then we have $\alpha = 0$, which is again a contradiction. Hence in any case we should have $H = K$ and the ambient space of $\mathcal{S}[B]$ is $K \oplus K$. $\square$

As we saw in the proof of Lemma 4, the unique nonzero vector $b(s, t)$ in $X(s) \cap X(t)$ is $b(s, t) = (s+t+\xi(s+t), B(s+t+\xi(s+t), s))$. With some calculations, we can verify that the second component of $b(s, t)$ is equal to $B(s, t) + (s^4 + s^2)(\xi(s + t) + 1)$.

4. The affine expansion of the Buratti–Del Fra dual hyperoval

In this section, we provide a proof for the fact that the affine expansion of the Buratti–Del Fra dual hyperoval is covered by the halved hypercube, one of the most important simply connected semibiplanes. This shows another similarity of $\mathcal{D}_d(\mathbf{F}_2)$ to the d-dimensional Huybrechts dual hyperoval $\mathcal{H}_d(\mathbf{F}_2)$, because the affine expansion of the latter is also covered by the halved hypercube (of index 2^{d+1}). Thus, while $\mathcal{H}_d(\mathbf{F}_2)$ and $\mathcal{D}_d(\mathbf{F}_2)$ are non-isomorphic simply connected d-dual hyperovals over $\mathbf{F}_2$, the incidence geometries constructed from them are controlled by the same simply connected incidence geometry. This fact was already known before: in fact, in 2006 when a survey article [12] by the second author was published, Pasini had already observed Lemma 8 and Del Fra and the second author noticed Lemma 9 (see a comment in the last paragraph of [12, Subsection 5.4]). However, no literature is available, as far as the authors know. Also our simpler model $\mathcal{D}$ for $\mathcal{D}_d(\mathbf{F}_2)$ in Section 2 provides an easy proof of Lemma 9. For these reasons, we shall give below expository proofs for these lemmas.

We first give a brief review of some basic facts about semibiplanes. For the details, see [6, Subsections 1.1]. A *semibiplane* is a connected finite incidence structure $\Pi = (\mathcal{P}, \mathcal{B}; *)$ consisting of a set $\mathcal{P}$ of *points* and a set $\mathcal{B}$ of *blocks*, in which every two distinct points (resp. blocks) are incident to exactly zero or two blocks (resp. points) in common. We say that Π has *index* q if every point (resp. block) is incident to exactly q blocks (resp. points). A *line* of Π is a quadruple $L = \{v_i, X_j \mid i, j \in \{0, 1\}\}$ of points v_i and blocks X_j such that $v_i * X_j$ for any $i, j \in \{0, 1\}$. We denote by $\mathcal{L}$ the set of lines of Π and extend the incidence $*$ between $\mathcal{L}$ and $\mathcal{P} \cup \mathcal{B}$ by inclusion. We identify Π with the resulting incidence geometry $(\mathcal{P}, \mathcal{L}, \mathcal{B}; *)$ of rank 3. For a semibiplane $\Pi' = (\mathcal{P}', \mathcal{B}'; *')$, we say that Π' *covers* Π (or Π is a *quotient* of Π') if the geometry $(\mathcal{P}', \mathcal{L}', \mathcal{B}'; *')$ is a (2-)cover of $(\mathcal{P}, \mathcal{L}, \mathcal{B}; *)$ in the sense of incidence geometry.

We shall give two examples of semibiplanes of index $q = 2^n$ for a fixed positive integer n. To define the first one, consider the row vector space $\mathbf{F}_2^q$. Define a point (resp. block) to be a vector $x = (x_i)_{i=1}^q$ of $\mathbf{F}_2^q$ with even (resp. odd) weight; namely $\sum_{i=1}^q x_i = 0$ (resp. 1) in $\mathbf{F}_2$. For vectors $x = (x_i)$ and $y = (y_i)$ of $\mathbf{F}_2^q$, we define $x * y$ whenever the Hamming distance between them is 1: ${}^\#\{i \in \{1, \ldots, q\} \mid x_i \neq y_i\} = 1$. Denote by $\mathcal{P}$ (resp. $\mathcal{B}$) the set of points (resp. blocks). The resulting incidence structure $\mathbf{H}(q) := (\mathcal{P}, \mathcal{B}; *)$ is a semibiplane of index q, called the *halved hypercube*. It is simply connected in the sense of incidence geometry. In $\mathbf{H}(q)$, a line L incident to a point $v_0 = 0$ is of the form $\{0, e_a + e_b; e_a, e_b\}$ for some $a \neq b$ in $\{1, \ldots, q\}$, where e_k denotes a vector of $\mathbf{F}_2^q$ with all entries 0 except 1 at the kth component.

The second example is constructed from an arbitrary d-dual hyperoval $\mathcal{S}$ over $\mathbf{F}_2$, where $d = n - 1$ [12, Subsection 2.7]. We embed the ambient space $A := \mathbf{A}(\mathcal{S})$ of $\mathcal{S}$ as a hyperplane of a vector space U over $\mathbf{F}_2$. In the sequel, we use the following convention in the vector space of U over $\mathbf{F}_2$. A projective point (a one-dimensional subspace of U) is identified with the unique nonzero vector contained in it. Accordingly, in a projective line (a two-dimensional subspace of U) $\langle p_0, p_1\rangle$ generated by two distinct projective points p_i $(i = 0, 1)$, the unique projective point on $\langle p_0, p_1\rangle$ distinct from p_0 and p_1 is written as $p_0 + p_1$. Now we call each projective point in $U \setminus A$ a point. A block is defined to be a $(d + 2)$-dimensional subspace X of U such that $X \cap A$ is a member of $\mathcal{S}$. The incidence structure $(\mathcal{P}, \mathcal{B}; *)$ consisting of a set $\mathcal{P}$ (resp. $\mathcal{B}$) of points (resp. blocks) together with the incidence $*$ given by inclusion is called the *affine expansion* of $\mathcal{S}$ and denoted $\mathrm{Af}(\mathcal{S})$. It is a semibiplane of index

$q = 2^n = 2^{d+1}$. In $\mathrm{Af}(\mathcal{S})$, each block X is generated by a member $X \cap A$ of $\mathcal{S}$ and a point $p \in \mathcal{P}$ contained in it. We refer to $B = X \cap A$ as the *base* of a block X. A line $L = \{v_i, X_j \mid i, j \in \{0, 1\}\}$ of $\mathrm{Af}(\mathcal{S})$ corresponds to the projective line $\langle v_0, v_1 \rangle = X_0 \cap X_1$. Thus L is determined by either the points v_i ($i = 0, 1$) or the blocks X_i ($j = 0, 1$). Notice that each member in L is uniquely determined by the other three members: $v_1 = v_0 + (B_0 \cap B_1)$ for bases B_j of X_j ($j = 0, 1$), and $X_1 = \langle v_0, B_1 \rangle = \langle v_1, B_1 \rangle$, where B_1 is the unique member of $\mathcal{S}$ distinct from $B_0 = X_0 \cap A$ which contains $\langle v_0, v_1 \rangle \cap A$. This remark about a line of $\mathrm{Af}(\mathcal{S})$ is repeatedly used in the proof of Lemma 8.

Now we recall a geometric condition concerning a d-dual hyperoval $\mathcal{S}$ in general [12, Subsection 2.6]. We say that $\mathcal{S}$ satisfies *Property* (T_1) if $X \cap \langle Y, Z \rangle$ has vector dimension 2 for all mutually distinct members X, Y, Z of $\mathcal{S}$, where $\langle Y, Z \rangle$ denotes the subspace of $\mathbf{A}(\mathcal{S})$ generated by Y and Z. Notice that this implies that $X \cap \langle Y, Z \rangle$ is the projective line spanned by two projective points $X \cap Y$ and $X \cap Z$.

For the convenience of the reader, we record the following observation with a proof, which is a special case of [12, Lemma 2.13].

Lemma 7. *Assume that $\mathcal{S}$ is a d-dual hyperoval over $\mathbf{F}_2$ satisfying Condition (T_1). Take any mutually distinct members B_i ($i = 0, 1, 2$) of $\mathcal{S}$. For any permutation (i, j, k) of $(0, 1, 2)$, let $c_i := (B_i \cap B_j) + (B_i \cap B_k)$. Then c_0, c_1, c_2 lie on a projective line contained in a member C of $\mathcal{S}$.*

Proof. Without loss of generality, we may assume that $(i, j, k) = (0, 1, 2)$. Let C be the unique member of $\mathcal{S} \setminus B_0$ containing c_0. Then C is distinct from B_i for any $i \in \{0, 1, 2\}$. By Property (T_1), we have $C \cap \langle B_0, B_1 \rangle = \langle c_0, C \cap B_1 \rangle$, which is a subspace of $\langle c_0, B_1 \rangle$. As $c_0 = C \cap B_0$ lies in the projective line $\langle B_0 \cap B_1, B_0 \cap B_2 \rangle \subseteq \langle B_1, B_2 \rangle$, the intersection $C \cap \langle B_0, B_1 \rangle$ lies in $\langle B_1, B_2 \rangle$. Thus $C \cap \langle B_0, B_1 \rangle \subseteq C \cap \langle B_1, B_2 \rangle$. As both subspaces of this inclusion relation have dimension 2 by Property (T_1), we conclude that $l := C \cap \langle B_0, B_1 \rangle = C \cap \langle B_1, B_2 \rangle$. Then $l \cap B_2 \subset \langle B_0, B_1 \rangle \cap B_2 = \langle B_0 \cap B_2, B_1 \cap B_2 \rangle$, which implies that $l \cap B_2 = c_2$. Then the above argument starting with c_2 shows that $l = C \cap \langle B_2, B_0 \rangle$ and $l \cap B_1 = c_1$. Hence the projective points $c_i = C \cap B_i$ ($i = 0, 1, 2$) lie on the line l contained in the member C of $\mathcal{S}$. □

Now we prove the following result, originally due to Pasini.

Lemma 8. *Assume that $\mathcal{S}$ is a d-dual hyperoval over $\mathbf{F}_2$ which satisfies Condition (T_1). Then $\mathrm{Af}(\mathcal{S})$ is covered by the halved hypercube.*

Proof. We invoke the theory of a wrapping number developed by Pasini and Pica [5]. By [5, Corollary 4.2], in order to establish the claim it suffices to show the *wrapping number* $w(\Pi)$ of the semibiplane $\Pi := \mathrm{Af}(\mathcal{S})$ is 1.

The calculation of $w(\Pi)$ is carried out as follows (see [6, Subsection 1.1]). In Π, take a point v_0 and a line $L = \{v_i, X_j \mid i, j \in \{0, 1\}\}$ incident to v_0. For each block X incident to v_0 but not to v_1, let (u_0, Y, u_1, Z) be a sequence of points and blocks obtained by the following procedure:

(1) Consider the line $\{v_0, u_0; X, X_0\}$ determined by X and X_0.
(2) Consider the line $\{u_0, v_1; X_0, Y\}$ determined by u_0 and v_1.
(3) Consider the line $\{v_1, u_1; Y, X_1\}$ determined by Y and X_1.
(4) Consider the line $\{u_1, v_0; X_1, Z\}$ determined by u_1 and v_0.

Then the map sending X to Z is a permutation $\gamma_{v_0,L}$ on the set $\mathcal{B}(v_0) \setminus \mathcal{B}(v_1)$ of blocks incident to v_0 but not to v_1. The wrapping number $w(\Pi)$ is given as the maximum number of the order of the permutation $\gamma_{v_0,L}$ when (v_0, L) ranges over all pairs of incident points and lines of Π.

We shall show that $Z = X$ by tracing the above procedure. Let B and B_i be the bases of blocks X and X_i respectively ($i = 0, 1$). The initial line L corresponds to the projective line $\langle v_0, v_1 \rangle = X_0 \cap X_1$ with three projective points v_0, v_1 and $B_0 \cap B_1$. In Step (1), it follows from the remarks given in the last part of the definition of $\mathrm{Af}(\mathcal{S})$ above that $u_0 = v_0 + (B_0 \cap B)$. In Step (2), similarly we have $Y = \langle u_0, C \rangle = \langle v_1, C \rangle$, where C is the unique member of $\mathcal{S} \setminus \{X_0\}$ containing $\langle u_0, v_1 \rangle \cap A$. This point is determined by observing the plane (the three-dimensional subspace over $\mathbf{F}_2$) $\langle v_0, v_1, u_0 \rangle$ generated by v_0, v_1 and u_0. As $\langle v_0, v_1 \rangle$ and $\langle v_0, u_0 \rangle$ intersect A at $B_0 \cap B_1$ and $B_0 \cap B$ respectively, the plane $\langle v_0, v_1, u_0 \rangle$ (contained in X_0) intersects A in the projective line $\langle B_0 \cap B_1, B_0 \cap B \rangle$. Thus we conclude that $\langle u_0, v_1 \rangle \cap A = (B_0 \cap B_1) + (B_0 \cap B)$.

Now we use the assumption that $\mathcal{S}$ satisfies Property (T_1). We set $c_0 := (B_0 \cap B_1) + (B_0 \cap B) = \langle u_0, v_1 \rangle \cap A$, $c_1 := (B_1 \cap B_0) + (B_1 \cap B)$ and $c := (B \cap B_0) + (B \cap B_1)$. It follows from Lemma 7 that the above C is a member of $\mathcal{S}$ containing c_0, c_1 and c.

In Step (3) in the procedure, we have $u_1 = v_1 + (B_1 \cap C)$. Notice that $B_1 \cap C = c_1$. In Step (4), $Z = \langle u_1, D \rangle = \langle v_0, D \rangle$, where D is the unique member of $\mathcal{S} \setminus \{B_1\}$ containing $\langle u_1, v_0 \rangle \cap A$. The last point is determined inside the plane $\langle v_0, v_1, u_1 \rangle$ as follows. As $\langle v_0, v_1 \rangle$ and $\langle v_1, u_1 \rangle$ intersect A at $B_0 \cap B_1$ and $B_1 \cap C = c_1$ respectively, the plane $\langle v_0, v_1, u_1 \rangle$ (contained in X_1) intersects A in the projective line $\langle B_0 \cap B_1, B_1 \cap C \rangle$. Then we conclude that $\langle u_1, v_0 \rangle \cap A = B_0 \cap B_1 + B_1 \cap C$. As $B_1 \cap C = c_1 = (B_1 \cap B_0) + (B_1 \cap B)$, we have $\langle u_1, v_0 \rangle \cap A = B_1 \cap B$. Thus the unique member D of $\mathcal{S} \setminus \{B_1\}$ containing this point should be B. Hence we have $Z = \langle v_0, B \rangle = X$, as desired.

As X is any block in $\mathcal{B}(v_0) \setminus \mathcal{B}(v_1)$, we have verified that the permutation $\gamma_{v_0,L}$ is the identity. This holds for any pair (v_0, L) of an incident point and a line of Π, whence the wrapping number of Π is 1. □

We now show the following result, using the model $\mathcal{D}$ for $\mathcal{D}_d(\mathbf{F}_2)$ and the automorphisms of $\mathcal{D}$ described in Section 2.

Lemma 9. *The Buratti–Del Fra dimensional dual hyperoval satisfies Condition* (T_1).

Proof. We will work in the model $\mathcal{D}$ of $\mathcal{D}_d(\mathbf{F}_2)$ in Section 2. Take any three mutually distinct members X, Y and Z of $\mathcal{D}_d(\mathbf{F}_2)$. Notice that $(X \cap \langle Y, Z \rangle)^g = X^g \cap \langle Y^g, Z^g \rangle$, whence $X \cap \langle Y, Z \rangle$ and $X^g \cap \langle Y^g, Z^g \rangle$ have the same dimension for any $g \in \mathrm{Aut}(\mathcal{D})$. Thus we may assume that $X = X(0) = \{(x, \bar{0}) \mid x \in V\}$ by applying the automorphism $\tau(a)$ defined in Eq. (9) for some $a \in V$. As the subgroup of $\mathrm{Aut}(\mathcal{D})$ consisting of $l(\alpha)$ (see Eq. (10)) for $\alpha \in GL(V)$ fixing e_0 has three orbits $\{X(0)\}$, $\{X(e_0)\}$ and $\{X(t) \mid t \in V \setminus \{0, e_0\}\}$ on $\mathcal{D}$, we may assume that $(Y, Z) = (X(a), X(b))$ for $(a, b) = (e_0, e_1)$, $(e_0, e_0 + e_1)$ or (e_1, e_2) with some fixed elements $e_1 \neq e_2 \in V \setminus \{0, e_0\}$ with $e_2 \neq e_0 + e_1$.

Observe that there is a basis e_i $(i = 0, \ldots, d)$ for V containing e_0, e_1, e_2. Fix such a basis. As $\langle X(a), X(b) \rangle = \{(x + y, \overline{x \otimes a} + \overline{y \otimes b}) \mid x, y \in V\}$ for $a \neq b \in V$, the intersection $X(0) \cap \langle X(a), X(b) \rangle$ consists of $(x + y, \bar{0})$ for $x, y \in V$ satisfying

$$\overline{x \otimes a} = \overline{y \otimes b}. \tag{18}$$

Assume first that $(X, Y, Z) = (X(0), X(a), X(b))$ with $a = e_1$ and $b = e_2$. Expressing $x = \sum_{i=0}^{d} x_i e_i$ and $y = \sum_{i=0}^{d} y_i e_i$ with $x_i, y_i \in \mathbf{F}_2$ $(i \in \{0, \ldots, d\})$, the condition (18) is written as follows, where we define $\overline{e_i \otimes e_j} = \bar{e}_{i,j}$, for short $(i, j \in I = \{0, \ldots, d\})$:

$$x_0\bar{e}_{0,1} + x_1\bar{e}_{1,1} + x_2\bar{e}_{1,2} + \sum_{i\geq 3} x_i\bar{e}_{1,i} = y_0\bar{e}_{0,2} + y_1\bar{e}_{1,2} + y_2\bar{e}_{2,2} + \sum_{i\geq 3} y_i\bar{e}_{2,i}.$$

Recall that $\bar{e}_{0,0} = \bar{0}$, $\bar{e}_{i,i} = \bar{e}_{0,i}$ for $i \in I$. Thus we have

$$(x_0 + x_1)\bar{e}_{0,1} + (y_0 + y_2)\bar{e}_{0,2} + (x_2 + y_1)\bar{e}_{1,2} + \sum_{i\geq 3} x_i\bar{e}_{1,i} + \sum_{i\geq 3} y_i\bar{e}_{2,i} = \bar{0}.$$

As $\bar{e}_{0,i}$ $(i \in I_0 = \{1, \ldots, d\})$, $\bar{e}_{i,j}$ $(i, j \in I_0, i < j)$ form a basis for $(V \otimes V)/W$, then we have $x_0 + x_1 = 0$, $y_0 + y_2 = 0$, $x_2 + y_1 = 0$ and $x_i = y_i = 0$ for all $i \in \{3, \ldots, d\}$. Thus $x = x_0(e_0 + e_1) + x_2 e_2$ and $y = y_0(e_0 + e_2) + x_2 e_1$, and then $x + y = (x_0 + x_2)(e_0 + e_1) + (x_2 + y_0)(e_0 + e_2)$. This implies that $(x + y, \bar{0})$ lies in the two-dimensional subspace spanned by $(e_1 + e_0, \bar{0})$ and $(e_2 + e_0, \bar{0})$ for any $x, y \in V$ satisfying Eq. (18). Hence $X(0) \cap \langle X(e_1), X(e_2) \rangle$ coincides with the two-dimensional subspace spanned by $(e_1 + e_0, \bar{0})$ and $(e_2 + e_0, \bar{0})$. (Notice that the last vectors are $b(0, e_1)$ and $b(0, e_2)$ in view of Eq. (7).)

Next consider the case where $(X, Y, Z) = (X(0), X(a), X(b))$ with $a = e_1$ and $b = e_0 + e_1$. In this case, the condition (18) for $x = \sum_{i=0}^{d} x_i e_i$ and $y = \sum_{i=0}^{d} y_i e_i$ $(x_i, y_i \in \mathbf{F}_2, i \in I)$ is equivalent to the following equation:

$$x_0\bar{e}_{0,1} + x_1\bar{e}_{1,1} + \sum_{i\geq 2} x_i\bar{e}_i = y_0(\bar{e}_{0,0} + \bar{e}_{0,1}) + y_1(\bar{e}_{0,1} + \bar{e}_{1,1}) + \sum_{i\geq 2} y_i\bar{e}_{0,i} + \sum_{i\geq 2} y_i\bar{e}_{1,i}.$$

As $\bar{e}_{0,0} = \bar{0}$ and $\bar{e}_{i,i} = \bar{e}_{0,i}$ for $i \in I$, we have

$$(x_0 + y_0 + x_1 + y_1)\bar{e}_{0,1} + \sum_{i\geq 2} y_i\bar{e}_{0,i} + \sum_{i\geq 2}(x_i + y_i)\bar{e}_{1,i} = \bar{0},$$

which implies $x_0+y_0 = x_1$ and $y_i = 0 = x_i$ for all $i \in \{2, \dots, d\}$. Thus $(x+y, \bar{0}) = (x_1(e_0+e_1)+y_1e_1, \bar{0})$ lies in the two-dimensional subspace spanned by $(e_0 + e_1, \bar{0})$ and $(e_1, \bar{0})$ for any $x, y \in V$. Hence $X(0)\cap\langle X(e_1), X(e_0 + e_1)\rangle$ is the two-dimensional subspace spanned by $(e_0+e_1, \bar{0})$ and $(e_1, \bar{0})$. (Notice that the last two vectors are $b(0, e_1)$ and $b(0, e_0 + e_1)$ in view of Eq. (7).)

In the remaining case where $(X, Y, Z) = (X(0), X(a), X(b))$ for $a = e_0$ and $b = e_1$, the condition (18) for $x = \sum_{i=0}^{d} x_ie_i$ and $y = \sum_{i=0}^{d} y_ie_i$ ($x_i, y_i \in \mathbf{F}_2, i \in I$) is

$$x_0\bar{e}_{0,0} + x_1\bar{e}_{0,1} + \sum_{i\geq 2} x_i\bar{e}_{0,i} = y_0\bar{e}_{0,1} + y_1\bar{e}_{1,1} + \sum_{i\geq 2} y_i\bar{e}_{1,i},$$

from which we have $x_1 + y_0 + y_1 = 0$ and $x_i = y_i = 0$ for all $i \in \{2, \dots, d\}$. Then $x + y = (x_0 + y_0)e_0 + y_0e_1$, and so $(x + y, \bar{0})$ lies in the two-dimensional subspace spanned by $(e_0, \bar{0})$ and $(e_0 + e_1, \bar{0})$ for any $x, y \in V$ satisfying Eq. (18). Hence $X(0) \cap \langle X(e_0), X(e_1)\rangle$ coincides with the two-dimensional subspace spanned by $(e_0, \bar{0})$ and $(e_1 + e_0, \bar{0})$. (Notice that the last vectors coincide with $b(0, e_0)$ and $b(0, e_1)$ in view of Eq. (7).)

Thus in either case, $X \cap \langle Y, Z\rangle$ is of dimension 2. This establishes that $\mathcal{D}$ satisfies Property (T_1). □

From Lemmas 8 and 9, we obtain the following result.

Proposition 10. *The affine expansion* $\mathrm{Af}(\mathcal{D}_d(\mathbf{F}_2))$ *of the Buratti–Del Fra dimensional dual hyperoval* $\mathcal{D}_d(\mathbf{F}_2)$ *is covered by the halved hypercube* $\mathbf{H}(2^{d+1})$.

5. Additional remarks and questions

We conclude the paper with some remarks and questions.
The first two questions are easy to state.

Question 1. Find a quotient in $PG(2d + 1, 2)$ of the Buratti–Del Fra d-dual hyperoval with d odd.

Question 2. Find relations between the universal cover of the affine expansion $\mathrm{Af}(\mathcal{V}_d(\mathbf{F}_2))$ of the Veronesean dual hyperoval $\mathcal{V}_d(\mathbf{F}_2)$ over $\mathbf{F}_2$ and the universal cover of $\mathrm{Af}(\mathcal{T}_d(\mathbf{F}_2))$ for $\mathcal{T}_d(\mathbf{F}_2)$, the deformation of $\mathcal{V}_d(\mathbf{F}_2)$ given by the first author [8].

Before stating the next question, we need to review some facts. Recall that for a quadratic APN function f on $K \cong \mathbf{F}_{2^n}, n = d + 1$, we construct a d-dual hyperoval $\mathcal{S}[f]$ with ambient space $K \oplus K$, which is a quotient of the Huybrechts dual hyperoval $\mathcal{H}_d(\mathbf{F}_2)$ [14]. There are two equivalence relations among (not necessarily quadratic) APN functions, called the *extended affine equivalence* and *CCZ equivalence*. Two APN functions are CCZ-equivalent if they are extended affine equivalent, but the converse is not true in general. The following are known [16, Propositions 2 and 6; Proposition 5]: two quadratic APN functions f and g on $K \cong \mathbf{F}_{2^n}$ are extended affine equivalent if and only if the associated d-dual hyperovals $\mathcal{S}[f]$ and $\mathcal{S}[g]$ are isomorphic, while they are CCZ-equivalent if and only if the associated semibiplanes $\mathrm{Af}(\mathcal{S}[f])$ and $\mathrm{Af}(\mathcal{S}[g])$ are isomorphic as incidence structures. In fact, the second author recently proved that for quadratic APN functions on K they are extended affine equivalent if and only if they are CCZ-equivalent [17]. In view of the above strong similarity of the Buratti–Del Fra dual hyperoval to the Huybrechts dual hyperoval, it is natural to ask whether a similar phenomenon holds for quotients in $K \oplus K$ of the Buratti–Del Fra dual hyperoval.

Question 3. Let B and B' be $\mathbf{F}_2$-bilinear maps on $K \cong \mathbf{F}_{2^{d+1}}$ which satisfy the conditions (1)–(3) in Lemma 4, so that $\mathcal{S}[B]$ and $\mathcal{S}[B']$ are both d-dual hyperovals with ambient spaces $K \oplus K$ covered by $\mathcal{D}_d(\mathbf{F}_2)$. Prove or give a counterexample for the following statement. The associated semibiplanes $\mathrm{Af}(\mathcal{S}[B])$ and $\mathrm{Af}(\mathcal{S}[B'])$ are isomorphic as incidence structures if and only if $\mathcal{S}[B]$ and $\mathcal{S}[B']$ are isomorphic as dimensional dual hyperovals.

We do not attempt to state the last two questions with rigorous mathematical formulations.

The shape of the bilinear map $B(x, y) = x^4y + xy^4 + xy + (xy)^2$ on $K \cong \mathbf{F}_{2^{d+1}}$, d even, in Section 3 looks quite similar to the product $x \cdot y = x^9y + xy^9 - xy + (xy)^3$ on $\mathbf{F}_{3^{2e+1}}$, $e > 1$, which gives the structure of a semifield on $\mathbf{F}_{3^{2e+1}}$, known as the Coulter–Matthews semifield [2].

Question 4. Are there any reasons for this similarity?

The recent construction by the first author [9] of some quotients of $\mathcal{D}_d(\mathbf{F}_2)$ in $PG(3d, 2)$ using quadratic APN functions on $\mathbf{F}_{2^d}$ may be regarded as another example of the similarity between $\mathcal{H}_d(\mathbf{F}_2)$ and $\mathcal{D}_d(\mathbf{F}_2)$. It is carried out by 'pasting' some quotients of $\mathcal{H}_{d-1}(\mathbf{F}_2)$ together. This suggests the following:

Question 5. Find general methods for constructing a d-dual hyperoval by pasting together e-dual hyperovals (which are quotients of the Huybrechts or the Veronesean dual hyperoval) with e smaller than d. Determine the universal cover of the resulting d-dual hyperoval.

References

[1] M. Buratti, A. Del Fra, Semi-Boolean Steiner quadruple systems and dimensional dual hyperovals, Advances in Geometry 3 (2003) S245–S253 (Special Volume).
[2] R.S. Coulter, R.W. Matthews, Planar functions and planes of Lenz–Barlotti class II, Designs, Codes and Cryptography 10 (1997) 167–184.
[3] A. Del Fra, S. Yoshiara, Dimensional dual hyperovals associated with Steiner systems, European Journal of Combinatorics 26 (2005) 173–194.
[4] C. Huybrechts, A. Pasini, Flag transitive extensions of dual affine spaces, Contributions to Algebra and Geometry 40 (1999) 503–532.
[5] A. Pasini, G. Pica, Wrapping polygons in polygons, Annals of Combinatorics 2 (1998) 325–349.
[6] A. Pasini, S. Yoshiara, On a new family of flag-transitive semibiplanes, European Journal of Combinatorics 22 (2001) 529–545.
[7] H. Taniguchi, On a family of dual hyperovals over $GF(q)$ with q even, European Journal of Combinatorics 26 (2005) 195–199.
[8] H. Taniguchi, A new family of dual hyperovals in $PG(d(d+3)/2, 2)$ with $d \geq 3$, Discrete Mathematics 309 (2009) 418–429.
[9] H. Taniguchi, On d-dimensional Buratti–Del Fra type dual hyperovals in $PG(3d, 2)$, Discrete Mathematics 310 (2010) 3633–3645.
[10] H. Taniguchi, S. Yoshiara, New quotients of d-dimensional Veronesean dual hyperoval in $PG(2d + 1, 2)$, Preprint (2010).
[11] J. Thas, H. van Maldeghem, Characterizations of the finite quadric Veroneseans $\mathcal{V}_n^{2^n}$, The Quarterly Journal of Mathematics 55 (2004) 99–113. Oxford.
[12] S. Yoshiara, Dimensional dual arcs—a survey, in: A. Hulpke, B. Liebler, T. Penttila, A. Seress (Eds.), Finite Geometries, Groups, and Computation, Walter de Gruyter, Berlin, New York, 2006, pp. 247–266.
[13] S. Yoshiara, Notes on Taniguchi's dimensional dual hyperovals, European Journal of Combinatorics 28 (2007) 674–684.
[14] S. Yoshiara, Dimensional dual hyperovals associated with quadratic APN functions, Innovations in Incidence Geometry 8 (2008) 147–169.
[15] S. Yoshiara, Notes on split dimensional dual hyperovals, Incomplete Manuscript, 2009.
[16] S. Yoshiara, Notes on APN functions, semibiplanes and dimensional dual hyperovals, Designs, Codes and Cryptography 56 (2010) 197–218.
[17] S. Yoshiara, Equivalences of quadratic APN functions, Journal of Algebraic Combinatorics (in press). http://dx.doi.org/10.1007/s10801-011-0309-1.