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Smoothness and analyticity of perturbation expansions in qed

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Abstract

We consider the ground state of an atom in the framework of non-relativistic qed. We assume that the ultraviolet cutoff is of the order of the Rydberg energy and that the atomic Hamiltonian has a non-degenerate ground state. We show that the ground state energy and the ground state are k-times continuously differentiable functions of the fine structure constant and respectively the square root of the fine structure constant on some nonempty interval $[0, c_k)$.

Keywords: Quantum physics; Non-relativistic qed; Spectral analysis; Renormalization group

1. Introduction

Non-relativistic quantum electrodynamics (qed) is the theory describing the interactions between electrically charged non-relativistic quantum mechanical matter and the quantized electromagnetic field. Existence of ground states has been established under various physically reasonable assumptions [6,7,11]. In this paper we investigate expansions of the ground state

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and the ground state energy of an atom as functions of the fine structure constant α, as α tends to zero. In [3,4] it was proven that there exists an asymptotic expansion involving coefficients which depend on the coupling parameter α and have at most mild singularities. In [8,13,14] related expansions of the ground state energy were obtained and it was shown that logarithmic divergences can occur in non-relativistic qed. On the other hand it was shown that an atom in a dipole approximation of qed (which effectively leads to an infrared regularization) has a ground state and ground state energy which are analytic functions of the coupling constant [10].

This paper can be viewed as a continuation of [17], where it was shown that the ground state as well as the ground state energy of the atom are analytic functions of the coupling constant, g, which couples to the vector potential. Moreover in [17] it was shown that in an expansion in powers of g, the corresponding expansion coefficients are bounded as functions of a coupling constant, β, which originates from the coupling to the electrostatic potential. The main result of this paper states that these expansions coefficients are C^∞ functions of β, and we obtain satisfactory bounds on the first k derivatives with respect to β. We consider an atom which is coupled to the quantized radiation field in a scaling limit where the ultraviolet cutoff is measured in units of Rydberg. This scaling limit is a reasonable limit to study the properties of atoms. For example in this scaling limit estimates on the lifetimes of metastable states [20,7] were proven, which agree with experiment, see also [1]. Moreover, it was shown [12] that the ionization probability agrees with calculations done by physicists. As a corollary of the main result of this paper, we show that the ground state and the ground state energy have convergent power series expansions, with α dependent coefficients which are C^∞ functions of $\alpha \geqslant 0$. We show that the ground state energy as well as the ground state are k-times continuously differentiable functions of α respectively $\alpha^{1/2}$ on some nonempty interval $[0, c_k)$. Moreover, it follows that the ground state as well as the ground state energy are given as an asymptotic series in powers of $\alpha^{1/2}$ and α, respectively, with constant coefficients. These coefficients can be calculated by means of ordinary perturbation theory in a straightforward manner. As a consequence of our result it follows that in the scaling limit where the ultraviolet cutoff is of the order of the Rydberg energy no logarithmic terms occur. This clarifies an issue which was raised in [4], see the remark on page 1031 therein.

Let us now address the proof of the main result. It is well known that the ground state energy is embedded in the continuous spectrum. In such a situation regular perturbation theory is typically not applicable and other methods have to be employed. To prove the existence result as well as the analyticity result we use a variant of the operator theoretic renormalization analysis as introduced in [5]. An important ingredient of the proof is that by rotation invariance one can infer that in the renormalization analysis, terms which are linear in creation and annihilation operators do not occur. This is explained in [17] and [18]. In that case it follows that the renormalization transformation is a contraction even without infrared regularization. A similar idea was used in a paper to prove existence and analyticity of the ground state and ground state energy in the spin-boson model [16]. In the proof we will use results obtained in [16] and [17]. We note that similar ideas were used also in [10]. The main new ingredient in the proof is the control of derivatives with respect to the parameter β. The main estimates which control these derivatives are contained in Theorem 6.3 and Lemma 6.4 for the initial Feshbach transformation and in Lemma 7.10, Theorem 7.7, and Theorem 7.12(d) for the renormalization transformation. The most delicate estimates are used in the proof of Lemma 6.4 and Theorem 7.7, and can be considered as the key ingredients of the proof.

2. Model and statement of results

Let $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{\mathfrak{h}})$ be a Hilbert space. We introduce the direct sum of the n-fold tensor product of $\mathfrak{h}$ and set

$$\mathcal{F}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathfrak{h}), \quad \mathcal{F}^{(n)}(\mathfrak{h}) = \mathfrak{h}^{\otimes^n},$$

where we have set $\mathfrak{h}^{\otimes 0} := \mathbb{C}$. We introduce the vacuum vector $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}(\mathfrak{h})$. The space $\mathcal{F}(\mathfrak{h})$ is an inner product space where the inner product is induced from the inner product in $\mathfrak{h}$. That is, on vectors $\eta_1 \otimes \cdots \otimes \eta_n, \varphi_1 \otimes \cdots \otimes \varphi_n \in \mathcal{F}^{(n)}(\mathfrak{h})$ we have

$$\langle \eta_1 \otimes \cdots \otimes \eta_n, \varphi_1 \otimes \cdots \otimes \varphi_n \rangle := \prod_{i=1}^{n} \langle \eta_i, \varphi_i \rangle_{\mathfrak{h}}.$$

This definition extends to all of $\mathcal{F}(\mathfrak{h})$ by bilinearity and continuity. We introduce the bosonic Fock space

$$\mathcal{F}_s(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathcal{F}_s^{(n)}(\mathfrak{h}), \quad \mathcal{F}_s^{(n)}(\mathfrak{h}) := S_n \mathcal{F}^{(n)}(\mathfrak{h}),$$

where S_n denotes the orthogonal projection onto the subspace of totally symmetric tensors in $\mathcal{F}^{(n)}(\mathfrak{h})$. For $h \in \mathfrak{h}$ we introduce the so-called creation operator $a^*(h)$ in $\mathcal{F}_s(\mathfrak{h})$ which is defined on vectors $\eta \in \mathcal{F}_s^{(n)}(\mathfrak{h})$ by

$$a^*(h)\eta := \sqrt{n+1} S_{n+1}(h \otimes \eta). \tag{2.1}$$

The operator $a^*(h)$ extends by linearity to a densely defined linear operator on $\mathcal{F}(\mathfrak{h})$. One can show that $a^*(h)$ is closable, cf. [23], and we denote its closure by the same symbol. We introduce the annihilation operator by $a(h) := (a^*(h))^*$. For a closed operator $A \in \mathfrak{h}$ with domain $D(A)$ we introduce the operator $\Gamma(A)$ and $d\Gamma(A)$ in $\mathcal{F}(\mathfrak{h})$ defined on vectors $\eta = \eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{F}^{(n)}(\mathfrak{h})$, with $\eta_i \in D(A)$, by

$$\Gamma(A)\eta = A\eta_1 \otimes \cdots \otimes A\eta_n$$

and

$$d\Gamma(A)\eta = \sum_{i=1}^{n} \eta_1 \otimes \cdots \otimes \eta_{i-1} \otimes A\eta_i \otimes \eta_{i+1} \otimes \cdots \otimes \eta_n$$

and extended by linearity to a densely defined linear operator on $\mathcal{F}(\mathfrak{h})$. One can show that $d\Gamma(A)$ and $\Gamma(A)$ are closable, cf. [23], and we denote their closure by the same symbol. The operators $\Gamma(A)$ and $d\Gamma(A)$ leave the subspace $\mathcal{F}_s(\mathfrak{h})$ invariant, that is, their restriction to $\mathcal{F}_s(\mathfrak{h})$ is densely defined, closed, and has range contained in $\mathcal{F}_s(\mathfrak{h})$. To define qed, we fix

$$\mathfrak{h} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$$

and set $\mathcal{F} := \mathcal{F}_s(\mathfrak{h})$. We define the operator of the free field energy by

$$H_f := d\Gamma(M_\omega),$$

where $\omega(k,\lambda) := \omega(k) := |k|$ and M_φ denotes the operator of multiplication with the function φ. For $f \in \mathfrak{h}$ we write

$$a^*(f) = \sum_{\lambda=1,2} \int f(k,\lambda) a^*(k,\lambda), \qquad a(f) = \sum_{\lambda=1,2} \int \overline{f(k,\lambda)} a^*(k,\lambda),$$

where $a(k,\lambda)$ and $a^*(k,\lambda)$ are operator-valued distributions. They satisfy the following commutation relations, which are to be understood in the sense of distributions,

$$\left[a(k,\lambda), a^*(k',\lambda')\right] = \delta_{\lambda\lambda'} \delta(k-k'), \qquad \left[a^\#(k,\lambda), a^\#(k',\lambda')\right] = 0,$$

where $a^\#$ stands for a or a^*. For $\lambda = 1,2$ we introduce the so-called polarization vectors

$$\varepsilon(\cdot,\lambda) : S^2 := \left\{k \in \mathbb{R}^3 \mid |k| = 1\right\} \to \mathbb{R}^3$$

to be measurable maps such that for each $k \in S^2$ the vectors $\varepsilon(k,1), \varepsilon(k,2), k$ form an orthonormal basis of $\mathbb{R}^3$. We extend $\varepsilon(\cdot,\lambda)$ to $\mathbb{R}^3 \setminus \{0\}$ by setting $\varepsilon(k,\lambda) := \varepsilon(k/|k|,\lambda)$ for all nonzero k. For $x \in \mathbb{R}^3$ we define the field operator

$$A_\Lambda(x) = \sum_{\lambda=1,2} \int \frac{dk \kappa_\Lambda(k)}{\sqrt{2|k|}} \left[e^{-ik\cdot x} \varepsilon(k,\lambda) a^*(k,\lambda) + e^{ik\cdot x} \varepsilon(k,\lambda) a(k,\lambda)\right], \tag{2.2}$$

where the function κ_Λ serves as a cutoff, which satisfies $\kappa_\Lambda(k) = 1$ if $|k| \leqslant \Lambda$ and which is zero otherwise. $\Lambda > 0$ is an ultraviolet cutoff, which we assume to be finite. Next we introduce the atomic Hilbert space, which describes the configuration of N electrons, by

$$\mathcal{H}_{\rm at} := \left\{\psi \in L^2(\mathbb{R}^{3N}) \mid \psi(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = \operatorname{sgn}(\sigma) \psi(x_1, \ldots, x_N),\ \sigma \in \mathfrak{S}_N\right\},$$

where $\mathfrak{S}_N$ denotes the group of permutations of N elements, sgn denotes the signum of the permutation, and $x_j \in \mathbb{R}^3$ denotes the coordinate of the j-th electron. We will consider the following operator in $\mathcal{H} := \mathcal{H}_{\rm at} \otimes \mathcal{F}$,

$$H_{g,\beta} = :\sum_{j=1}^N \left(p_j + g A_\Lambda(\beta x_j)\right)^2: + V + H_f, \tag{2.3}$$

where $p_j = -i\partial_{x_j}$, $V = V(x_1, \ldots, x_N)$ denotes the potential, and $:(\cdot):$ stands for the Wick product. We will make the following assumptions on the potential V, which are related to the atomic Hamiltonian

$$H_{\rm at} := -\Delta + V,$$

which acts in $\mathcal{H}_{\rm at}$. We introduced the Laplacian $-\Delta := \sum_{j=1}^N p_j^2$.

Hypothesis (H). The potential V satisfies the following properties:

(i) V is symmetric under permutations and invariant under rotations.
(ii) V is infinitesimally operator bounded with respect to $-\Delta$.
(iii) $E_{\rm at} := \inf \sigma(H_{\rm at})$ is a non-degenerate isolated eigenvalue of $H_{\rm at}$.

Note that for the hydrogen, $N = 1$, the potential $V(x_1) = -|x_1|^{-1}$ satisfies Hypothesis (H). Moreover (ii) of Hypothesis (H) implies that $H_{g,\beta}$ is a self-adjoint operator with domain $D(-\Delta + H_f)$ and that $H_{g,\beta}$ is essentially self-adjoint on any operator core for $-\Delta + H_f$, see for example [21,15]. For a precise definition of the operator in (2.3), see Appendix A. We will use the notation $D_r(w) := \{z \in \mathbb{C} \mid |z - w| < r\}$ and $D_r := D_r(0)$. Let us now state the main result of the paper.

Theorem 2.1. *Assume Hypothesis* (H) *and let $k \in \mathbb{N}_0$. Then there exists a positive constant g_0 such that for all $g \in D_{g_0}$ and $\beta \in \mathbb{R}$ the operator $H_{g,\beta}$ has an eigenvalue $E_\beta(g)$ with eigenvector $\psi_\beta(g)$ and eigen-projection $P_\beta(g)$ satisfying the following properties.*

(i) *For $g \in \mathbb{R} \cap D_{g_0}$ we have $E_\beta(g) = \inf \sigma(H_{g,\beta})$, and for all $g \in D_{g_0}$ we have $P_\beta(g)^* = P_\beta(\overline{g})$.*
(ii) *$g \mapsto E_{(\cdot)}(g)$, $g \mapsto \psi_{(\cdot)}(g)$, and $g \mapsto P_{(\cdot)}(g)$ are analytic functions on D_{g_0} with values in $C_B^k(\mathbb{R})$, $C_B^k(\mathbb{R}; \mathcal{H})$, and $C_B^k(\mathbb{R}; \mathcal{B}(\mathcal{H}))$, respectively.*
(iii) *There exists a finite and positive C such that for all $g \in D_{g_0}$ we have*

$$\|E_{(\cdot)}(g)\|_{C^k(\mathbb{R})} \leqslant C, \qquad \|\psi_{(\cdot)}(g)\|_{C^k(\mathbb{R};\mathcal{H})} \leqslant C, \qquad \|P_{(\cdot)}(g)\|_{C^k(\mathbb{R};\mathcal{B}(\mathcal{H}))} \leqslant C.$$

Remark 2.2. Under the hypotheses of Theorem 2.1, non-degeneracy of $E_\beta(g)$ is known for $g \in \mathbb{R} \cap D_{g_0}$. This is shown in for example [6,24]; see also [19].

The next result states that the expansions coefficients of the eigenvalue, eigenfunction, and the corresponding eigen-projection are C^∞ as functions of β.

Corollary 2.3. *Assume Hypothesis* (H) *and let $k \in \mathbb{N}_0$. Then there exists a positive constant g_0 such that for all $g \in D_{g_0}$ and $\beta \in \mathbb{R}$ the operator $H_{g,\beta}$ has an eigenvalue $E_\beta(g)$ with eigenvector $\psi_\beta(g)$ and eigen-projection $P_\beta(g)$ satisfying the following properties. On D_{g_0} we have the convergent expansions*

$$E_\beta(g) = \sum_{n=0}^\infty E_\beta^{(2n)} g^{2n}, \qquad \psi_\beta(g) = \sum_{n=0}^\infty \psi_\beta^{(n)} g^n, \qquad P_\beta(g) = \sum_{n=0}^\infty P_\beta^{(n)} g^n. \tag{2.4}$$

There exist finite and positive constants C and r such that

$$\|E_{(\cdot)}^{(2n)}\|_{C^k(\mathbb{R})} \leqslant C r^{2n}, \qquad \|\psi_{(\cdot)}^{(n)}\|_{C^k(\mathbb{R};\mathcal{H})} \leqslant C r^n, \qquad \|P_{(\cdot)}^{(n)}\|_{C^k(\mathbb{R};\mathcal{B}(\mathcal{H}))} \leqslant C r^n.$$

The expansion coefficients are as functions of β in $C^\infty(\mathbb{R})$, $C^\infty(\mathbb{R}; \mathcal{H})$, and $C^\infty(\mathbb{R}; \mathcal{B}(\mathcal{H}))$, respectively.

Various conclusions can be drawn from Theorem 2.1. For instance, if we set $\beta = \alpha \geqslant 0$ and $g = \alpha^{3/2}$ then we obtain the following corollary. It states that the ground state and the ground state energy of an atom in qed, in a scaling limit where the ultraviolet cutoff is of the order of the Rydberg energy, can be differentiated arbitrarily many times as functions of α and $\alpha^{1/2}$, respectively, provided one chooses α sufficiently small (depending on the number of derivatives). As a conclusion it follows that no logarithmic terms appear in this scaling limit.

Corollary 2.4. *Assume Hypothesis* (H). *There exists a positive* α_0 *such that for* $0 \leqslant \alpha \leqslant \alpha_0$ *the operator* $H_{\alpha^{3/2},\alpha}$ *has a ground state* $\psi(\alpha^{1/2})$ *with ground state energy* $E(\alpha)$ *such that we have the convergent expansions on* $[0, \alpha_0)$

$$E(\alpha) = \sum_{n=0}^{\infty} E_\alpha^{(2n)} \alpha^{3n}, \qquad \psi\big(\alpha^{1/2}\big) = \sum_{n=0}^{\infty} \psi_\alpha^{(n)} \alpha^{3n/2}. \tag{2.5}$$

The coefficients $E_\alpha^{(n)}$ *and* $\psi_\alpha^{(n)}$ *are as functions of* α *in* $C^\infty([0,\infty))$ *and* $C^\infty([0,\infty);\mathcal{H})$, *respectively. For every* $k \in \mathbb{N}_0$ *there exists a positive* $\alpha_0^{(k)}$ *such that* $\psi(\cdot)$ *and* $E(\cdot)$ *are* k*-times continuously differentiable on* $[0, \alpha_0^{(k)})$.

In [3,4] it was shown that there exist coefficients of the type (2.5) which have slower growth than α^{-t} for any $t > 0$. Corollary 2.4 states that the coefficients $E_\alpha^{(n)}$ and $\psi_\alpha^{(n)}$ are in fact smooth. Let us note that Corollary 2.4 implies the following corollary which states that the ground state and the ground state energy can be written in terms of an asymptotic series with constant coefficients in the sense of [22].

Corollary 2.5. *Assume Hypothesis* (H). *There exist formal power series with constant coefficients* $\sum_{n=0}^{\infty} c^{(n)} \alpha^{n/2}$ *and* $\sum_{n=0}^{\infty} e^{(n)} \alpha^n$ *which are asymptotic to the ground state and the ground state energy of* $H_{\alpha^{3/2},\alpha}$ *as* $\alpha \downarrow 0$, *respectively.*

In view of Corollary 2.4 and the continuity in the infrared cutoff which has been established in [17] one can calculate $c^{(n)}$ and $e^{(n)}$ of Corollary 2.5 using for example ordinary Rayleigh Schrödinger perturbation theory to determine first $\psi_\alpha^{(n)}$ and $E_\alpha^{(2n)}$, in Eq. (2.5), and then using a Taylor expansion of these coefficients.

3. Outline of the proof

The main method used in the proof of Theorem 2.1 is operator theoretic renormalization [5,2] and the fact that renormalization preserves analyticity [10,16]. The renormalization procedure is an iterated application of the so-called smooth Feshbach map. The smooth Feshbach map is reviewed in Appendix B and necessary properties of it are summarized. In this paper we will use many results stated in the previous papers [16] and [17]. The generalization from the Fock space over $L^2(\mathbb{R}^3)$, as considered in [16], to a Fock space over $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is straightforward. To be able to show that the renormalization transformation is a suitable contraction we use a rotation invariance argument, as explained in [17]. This is used in Lemma 5.5. For a careful treatment of this issue, see [18]. The main new ingredient in this paper is to control derivatives with respect to β. The subtleties originate from the reparameterization of the spectral parameter. In Section 4 we define an $SO(3)$ action on the atomic Hilbert space and the Fock space, which leaves the

Hamiltonian invariant. In Section 5 we introduce spaces which are needed to define the renormalization transformation. In Section 6 we show that after an initial Feshbach transformation the Feshbach map is in a suitable Banach space. This allows us to perform a renormalization analysis, which is the content of Section 7. We use results from [16] and complement it with new estimates needed to control differentiation with respect to β. In Section 8 we prove the contraction property of the renormalization transformation. In Section 9 we put the pieces together and prove Theorem 2.1. The proof is based on Theorems 6.1 and 7.12.

We use the notation $\mathbb{R}_+ = [0, \infty)$. For a multi-index $\underline{m} \in \mathbb{N}_0^l$ we use the usual convention $|\underline{m}| = \sum_{i=1}^l m_i$ and $\underline{m}! = \prod_{i=1}^l (m_i!)$. We shall make repeated use of the so-called pull-through formula which is given in Lemma A.1, in Appendix A. We refer the reader to Appendix A for notation of function spaces and will use Lemma C.1. Finally, let us note that using an appropriate scaling we can assume without loss of generality that the distance between the lowest eigenvalue of H_{at} and the rest of the spectrum is one, i.e.,

$$E_{\mathrm{at},1} - E_{\mathrm{at}} = 1, \tag{3.1}$$

where $E_{\mathrm{at},1} := \inf\{\sigma(H_{\mathrm{at}}) \setminus \{E_{\mathrm{at}}\}\}$. Any Hamiltonian of the form (2.3) satisfying Hypothesis (H) is up to a positive multiple unitarily equivalent to an operator satisfying (3.1) and Hypothesis (H), but with a rescaled potential and with different values for Λ, β, and g, see [17].

4. Symmetries

Let us introduce a representation of $SO(3)$ on $\mathcal{H}_{\mathrm{at}}$ and $\mathfrak{h}$. For details see [17]. For $R \in SO(3)$ and $\psi \in \mathcal{H}_{\mathrm{at}}$ we define

$$\mathcal{U}_{\mathrm{at}}(R)\psi(x_1, \ldots, x_N) = \psi\big(R^{-1}x_1, \ldots, R^{-1}x_N\big).$$

To define an $SO(3)$ action on Fock space it is convenient to consider a different but equivalent representation of the Hilbert space $\mathfrak{h}$. We introduce the Hilbert space $\mathfrak{h}_0 := L^2(\mathbb{R}^3; \mathbb{C}^3)$. We consider the subspace of transversal vector fields

$$\mathfrak{h}_T := \big\{ f \in \mathfrak{h}_0 \,\big|\, k \cdot f(k) = 0 \big\}.$$

It is straightforward to verify that the map $\phi : \mathfrak{h} \to \mathfrak{h}_T$ defined by

$$(\phi f)(k) := \sum_{\lambda=1,2} f(k, \lambda)\varepsilon(k, \lambda)$$

establishes a unitary isomorphism with inverse

$$\big(\phi^{-1} f\big)(k, \lambda) = f(k) \cdot \varepsilon(k, \lambda).$$

We define the action of $SO(3)$ on $\mathfrak{h}_T$ by

$$\big(\mathcal{U}_T(R) f\big)(k) = R f\big(R^{-1}k\big).$$

The function $R \mapsto \phi^{-1}\mathcal{U}_T(R)\phi$ defines a representation of $SO(3)$ on $\mathfrak{h}$ which we denote by $\mathcal{U}_{\mathfrak{h}}$. This yields a representation on Fock space which we denote by $\mathcal{U}_{\mathcal{F}}$. It is characterized by

$$\mathcal{U}_{\mathcal{F}}(R)a^{\#}(f)\mathcal{U}_{\mathcal{F}}(R)^* = a^{\#}\big(\mathcal{U}_{\mathfrak{h}}(R)f\big), \qquad \mathcal{U}_{\mathcal{F}}(R)\Omega = \Omega.$$

It is straightforward to show that the Hamiltonian $H_{g,\beta}$ is $SO(3)$ invariant.

5. Banach spaces of Hamiltonians

In this section we introduce Banach spaces of integral kernels, which parameterize certain subspaces of the space of bounded operators on Fock space. These spaces are used to control the renormalization transformation. Then we introduce Banach spaces, which we call extended Banach spaces, which are used to control derivatives with respect to β.

The renormalization transformation will be defined on operators acting on the reduced Fock space $\mathcal{H}_{\mathrm{red}} := P_{\mathrm{red}}\mathcal{F}$, where we introduced the notation $P_{\mathrm{red}} := \chi_{[0,1]}(H_f)$. We will investigate bounded operators in $\mathcal{B}(\mathcal{H}_{\mathrm{red}})$ of the form

$$H(w) := \sum_{m+n \geqslant 0} H_{m,n}(w), \tag{5.1}$$

with

$$\begin{aligned} H_{m,n}(w) &:= H_{m,n}(w_{m,n}), \\ H_{m,n}(w_{m,n}) &:= P_{\mathrm{red}} \int_{\underline{B}_1^{m+n}} \frac{d\mu(K^{(m,n)})}{|K^{(m,n)}|^{1/2}} a^*\big(K^{(m)}\big) w_{m,n}\big(H_f, K^{(m,n)}\big) a\big(\widetilde{K}^{(n)}\big) P_{\mathrm{red}}, \quad m+n \geqslant 1, \\ H_{0,0}(w_{0,0}) &:= w_{0,0}(H_f), \end{aligned} \tag{5.2}$$

where $w_{m,n} \in L^\infty([0,1] \times \underline{B}_1^m \times \underline{B}_1^n)$ is an integral kernel for $m+n \geqslant 1$, $w_{0,0} \in L^\infty([0,1])$, and w denotes the sequence of integral kernels $(w_{m,n})_{m,n \in \mathbb{N}_0^2}$. We have used and will henceforth use the following notation. We set $K = (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, and write

$$\begin{gathered} \underline{X} := X \times \mathbb{Z}_2, \qquad B_1 := \big\{ x \in \mathbb{R}^3 \;\big|\; |x| < 1 \big\}, \\ K^{(m)} := (K_1, \ldots, K_m) \in \big(\mathbb{R}^3 \times \mathbb{Z}_2\big)^m, \qquad \widetilde{K}^{(n)} := (\widetilde{K}_1, \ldots, \widetilde{K}_n) \in \big(\mathbb{R}^3 \times \mathbb{Z}_2\big)^n, \\ K^{(m,n)} := \big(K^{(m)}, \widetilde{K}^{(n)}\big), \\ \int_{\underline{X}^{m+n}} dK^{(m,n)} := \int_{X^{m+n}} \sum_{(\lambda_1,\ldots,\lambda_m,\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_n) \in \mathbb{Z}_2^{m+n}} dk^{(m)}\, d\widetilde{k}^{(n)}, \\ dk^{(m)} := \prod_{i=1}^m d^3k_i, \qquad d\widetilde{k}^{(n)} := \prod_{j=1}^n d^3\widetilde{k}_j, \qquad dK^{(m)} := dK^{(m,0)}, \qquad d\widetilde{K}^{(n)} := dK^{(0,n)}, \\ d\mu\big(K^{(m,n)}\big) := (8\pi)^{-\frac{m+n}{2}}\, dK^{(m,n)}, \end{gathered}$$

$$a^*\big(K^{(m)}\big) := \prod_{i=1}^{m} a^*(K_i), \qquad a\big(\widetilde{K}^{(m)}\big) := \prod_{j=1}^{m} a(\widetilde{K}_j),$$

$$\big|K^{(m,n)}\big| := \big|K^{(m)}\big| \cdot \big|\widetilde{K}^{(n)}\big|, \qquad \big|K^{(m)}\big| := |k_1| \cdots |k_m|, \qquad \big|\widetilde{K}^{(m)}\big| := |\widetilde{k}_1| \cdots |\widetilde{k}_m|,$$

$$\Sigma\big[K^{(m)}\big] := \sum_{i=1}^{n} |k_m|.$$

Note that in view of the pull-through formula (5.2) is equal to

$$\int_{\underline{B}_1^{m+n}} \frac{d\mu(K^{(m,n)})}{|K^{(m,n)}|^{1/2}} a^*\big(K^{(m)}\big)\chi\big(H_f + \Sigma\big[K^{(m)}\big] \leqslant 1\big) \\ \times w_{m,n}\big(H_f; K^{(m,n)}\big)\chi\big(H_f + \Sigma\big[\tilde{K}^{(n)}\big] \leqslant 1\big)a\big(\tilde{K}^{(n)}\big). \tag{5.3}$$

Thus we can restrict attention to integral kernels $w_{m,n}$ which are essentially supported on the sets

$$\underline{Q}_{m,n} := \big\{\big(r, K^{(m,n)}\big) \in [0,1] \times \underline{B}_1^{m+n} \;\big|\; r \leqslant 1 - \max\big(\Sigma\big[K^{(m)}\big], \Sigma\big[\widetilde{K}^{(m)}\big]\big)\big\}, \quad m+n \geqslant 1.$$

Moreover, note that integral kernels can always be assumed to be symmetric. That is, they lie in the range of the symmetrization operator, which is defined as follows,

$$w_{M,N}^{(\mathrm{sym})}\big(r; K^{(M,N)}\big) := \frac{1}{N!M!} \sum_{\pi \in S_M} \sum_{\widetilde{\pi} \in S_N} w_{M,N}(r, K_{\pi(1)}, \ldots, K_{\pi(N)}, \widetilde{K}_{\widetilde{\pi}(1)}, \ldots, \widetilde{K}_{\widetilde{\pi}(M)}). \tag{5.4}$$

Note that (5.2) is understood in the sense of forms. It defines a densely defined form which can be seen to be bounded using Lemma A.2. Thus it uniquely determines a bounded operator which we denote by $H_{m,n}(w_{m,n})$. This is explained in more detail in Appendix A. We have the following lemma.

Lemma 5.1. *For $w_{m,n} \in L^\infty([0,1] \times \underline{B}_1^m \times \underline{B}_1^n)$ we have*

$$\big\|H_{m,n}(w_{m,n})\big\| \leqslant \|w_{m,n}\|_\infty (n!m!)^{-1/2}. \tag{5.5}$$

The proof follows using Lemma A.2 and the estimate

$$\int_{\underline{S}_{m,n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^2} \leqslant \frac{(8\pi)^{m+n}}{n!m!}, \tag{5.6}$$

where $\underline{S}_{m,n} := \{(K^{(m)}, \widetilde{K}^{(n)}) \in \underline{B}_1^{m+n} \mid \Sigma[K^{(m)}] \leqslant 1,\ \Sigma[\widetilde{K}^{(n)}] \leqslant 1\}$. The renormalization procedure will involve kernels which lie in the following Banach spaces. We denote the norm of the Banach space $L^\infty(\underline{B}_1^{m+n}; C[0,1])$ by $\|\cdot\|_{\underline{\infty}}$. We shall identify the space $L^\infty(\underline{B}_1^{m+n}; C[0,1])$ with a subspace of $L^\infty([0,1] \times \underline{B}_1^{m+n})$ by setting

$$w_{m,n}\big(r, K^{(m,n)}\big) := w_{m,n}\big(K^{(m,n)}\big)(r).$$

This identification is used for example in (i) and (ii) of Definition 5.2.

Definition 5.2. We define $\mathcal{W}^{\#}_{m,n}$ to be the Banach space consisting of functions $w_{m,n} \in L^\infty(\underline{B}_1^{m+n}; C^1[0,1])$ satisfying the following properties:

(i) $w_{m,n}(1 - \chi_{\underline{Q}_{m,n}}) = 0$, for $m + n \geqslant 1$,
(ii) $w_{m,n}(r, K^{(m)}, \widetilde{K}^{(n)})$ is totally symmetric in the variables $K^{(m)}$ and $\widetilde{K}^{(n)}$,
(iii) the following norm is finite

$$\|w_{m,n}\|^{\#} := \|w_{m,n}\|_{\underline{\infty}} + \|\partial_r w_{m,n}\|_{\underline{\infty}}.$$

For $0 < \xi < 1$, we define the Banach space

$$\mathcal{W}^{\#}_{\xi} := \bigoplus_{(m,n)\in\mathbb{N}_0^2} \mathcal{W}^{\#}_{m,n}$$

to consist of all sequences $w = (w_{m,n})_{m,n\in\mathbb{N}_0}$ satisfying

$$\|w\|^{\#}_{\xi} := \sum_{(m,n)\in\mathbb{N}_0^2} \xi^{-(m+n)} \|w_{m,n}\|^{\#} < \infty.$$

Given $w \in \mathcal{W}^{\#}_{\xi}$, we write $w_{\geqslant r}$ for the vector in $\mathcal{W}^{\#}_{\xi}$ given by

$$(w_{\geqslant r})_{m+n} = \begin{cases} w_{m,n}, & \text{if } m+n \geqslant r, \\ 0, & \text{otherwise.} \end{cases}$$

For $w \in \mathcal{W}^{\#}_{\xi}$, it is easy to see using (5.5) that $H(w) := \sum_{m,n} H_{m,n}(w)$ converges in operator norm with bounds

$$\left\|H(w_{\geqslant r})\right\| \leqslant \xi^r \|w_{\geqslant r}\|^{\#}_{\xi}. \tag{5.7}$$

We shall use the notation

$$W[w] := \sum_{m+n\geqslant 1} H_{m,n}(w).$$

We will use the following theorem, which is a straightforward generalization of a theorem proven in [2]. A proof can also be found in [16].

Theorem 5.3. *The map $H : \mathcal{W}^{\#}_{\xi} \to \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is injective and bounded. Moreover $\|H(w)\| \leqslant \|w\|^{\#}_{\xi}$.*

The integral kernels depend on the spectral parameter. To accommodate for this we introduce the Banach space $\mathcal{W}_{\xi} := C^{\omega}_B(D_{1/2}; \mathcal{W}^{\#}_{\xi})$ with norm

$$\|w\|_{\xi} := \sup_{z\in D_{1/2}} \left\|w(z)\right\|^{\#}_{\xi}.$$

Moreover, the integral kernels depend on the coupling constants. We introduce the following Banach space

$$\mathcal{W}_\xi^{(k)}(S) := C_B^{\omega,k}\big(S \times \mathbb{R}; \mathcal{W}_\xi^{\#}\big),$$

with the norm

$$\|w\|_{\xi,S}^{(k)} := \sup_{(s,\beta)\in S\times\mathbb{R}} \sum_{m,n} \xi^{-m-n} \max_{0\leqslant l\leqslant k} \big\|\partial_\beta^l w(\beta,s)_{m,n}\big\|^{\#}.$$

Observe that this norm is different but equivalent to the natural norm,

$$\max_{0\leqslant l\leqslant k} \sup_{(s,\beta)\in S\times\mathbb{R}} \sum_{m,n} \xi^{-m-n} \big\|\partial_\beta^l w(\beta,s)_{m,n}\big\|^{\#} \leqslant \|w\|_{\xi,S}^{(k)}$$
$$\leqslant k \max_{0\leqslant l\leqslant k} \sup_{(s,\beta)\in S\times\mathbb{R}} \sum_{m,n} \xi^{-m-n} \big\|\partial_\beta^l w(\beta,s)_{m,n}\big\|^{\#}.$$

For notational compactness we will use an abbreviation for the case $S = D_{1/2}$ and set $\mathcal{W}_\xi^{(k)} := \mathcal{W}_\xi^{(\omega,k)}(D_{1/2})$ and $\|\cdot\|_\xi^{(k)} := \|\cdot\|_{\xi,S}^{(k)}$. We introduce the Banach space

$$\mathcal{W}_\xi^{(\#,k)} := C_B^k\big(\mathbb{R}; \mathcal{W}_\xi^{\#}\big), \qquad \|\cdot\|_\xi^{(\#,k)},$$

with the norm

$$\|w\|_\xi^{(\#,k)} := \sup_{\beta\in\mathbb{R}} \sum_{m,n} \xi^{-m-n} \max_{0\leqslant l\leqslant k} \big\|\partial_\beta^l w(\beta)_{m,n}\big\|^{\#}.$$

For $w \in \mathcal{W}_\xi$ we will use the notation $w_{m,n}(z,\cdot) := (w_{m,n}(z))(\cdot)$. We extend the definition of $H(\cdot)$ to $\mathcal{W}_\xi$ in the natural way: for $w \in \mathcal{W}_\xi$, we set

$$\big(H(w)\big)(z) := H\big(w(z)\big)$$

and likewise for $H_{m,n}(\cdot)$ and $W[\cdot]$. We say that a kernel $w \in \mathcal{W}_\xi$ is symmetric if $w_{m,n}(\bar{z}) = \overline{w_{n,m}(z)}$ for all $z \in D_{1/2}$. Note that because of Theorem 5.3 we have the following lemma.

Lemma 5.4. *Let $w \in \mathcal{W}_\xi$. Then w is symmetric if and only if $H(w(\bar{z})) = H(w(z))^*$ for all $z \in D_{1/2}$.*

We define on the space of kernels $\mathcal{W}_{m,n}^{\#}$ a natural representation of $SO(3)$, $\mathcal{U}$, which is uniquely determined by

$$H\big(\mathcal{U}(R)w_{m,n}\big) = \mathcal{U}(R)H(w_{m,n})\mathcal{U}^*(R), \quad \forall R \in SO(3), \tag{5.8}$$

[17]. The representation on $\mathcal{W}_{m,n}^{\#}$ yields a natural representation on $\mathcal{W}_\xi^{\#}$, which is given by $(\mathcal{U}(R)w)_{m,n} = \mathcal{U}(R)w_{m,n}$ for all $R \in SO(3)$. It lifts to a representation on $\mathcal{W}_\xi$ by setting

$(\mathcal{U}(R)w)(z) = \mathcal{U}(R)w(z)$ for all $w \in \mathcal{W}_\xi$. We say that a kernel $w_{m,n} \in \mathcal{W}^\#_{m,n}$ is rotation invariant if $\mathcal{U}(R)w_{m,n} = w_{m,n}$ and we say a kernel $w \in \mathcal{W}^\#_\xi$ is rotation invariant if $\mathcal{U}(R)w = w$. We will use the following lemma which is proven in [17].

Lemma 5.5. (i) *Let $w_{m,n} \in \mathcal{W}^\#_{m,n}$. Then $H(w_{m,n})$ is rotation invariant if and only if $w_{m,n}$ is rotation invariant. Let $w \in \mathcal{W}^\#_\xi$. Then $H(w)$ is rotation invariant if and only if w is rotation invariant.* (ii) *If $w_{m,n} \in \mathcal{W}^\#_{m,n}$ with $m+n=1$ is rotation invariant, then $w_{m,n}=0$.*

We will use the following polydiscs to define the renormalization transformation.

$$\mathcal{B}^\#(\alpha,\beta,\gamma) := \bigl\{ w \in \mathcal{W}^\#_\xi \bigm| \|\partial_r w_{0,0} - 1\|_\infty \leqslant \alpha,\ \bigl|w_{0,0}(0)\bigr| \leqslant \beta,\ \|w_{\geqslant 1}\|^\#_\xi \leqslant \gamma \bigr\},$$

$$\mathcal{B}(\alpha,\beta,\gamma) := \Bigl\{ w \in \mathcal{W}_\xi \Bigm| \sup_{z \in D_{1/2}} \bigl\|\partial_r w_{0,0}(z) - 1\bigr\|_\infty \leqslant \alpha,$$

$$\sup_{z \in D_{1/2}} \bigl|w_{0,0}(z,0) + z\bigr| \leqslant \beta,\ \|w_{\geqslant 1}\|_\xi \leqslant \gamma \Bigr\},$$

$$\mathcal{B}_0(\alpha,\beta,\gamma) := \bigl\{ w \in \mathcal{B}(\alpha,\beta,\gamma) \bigm| w(z) \text{ is rotation invariant for all } z \in D_{1/2} \bigr\}.$$

To control the derivatives with respect to β, we introduce the following extended polydisc.

$$\mathcal{B}^{(\#,k)}(\alpha,\beta,\gamma) := \bigl\{ w \in \mathcal{W}^{(\#,k)}_\xi \bigm| \|\partial_r w_{0,0} - 1\|_{C^k(\mathbb{R};C_B[0,1])} \leqslant \alpha,$$

$$\bigl\|w_{0,0}(0)\bigr\|_{C^k(\mathbb{R})} \leqslant \beta,\ \|w_{\geqslant 1}\|^{(\#,k)}_\xi \leqslant \gamma \bigr\},$$

$$\mathcal{B}^{(k)}(\alpha,\beta,\gamma) := \Bigl\{ w \in \mathcal{W}^{(k)}_\xi \Bigm| \sup_{z \in D_{1/2}} \bigl\|\partial_r w_{0,0}(z) - 1\bigr\|_{C^k(\mathbb{R};C_B[0,1])} \leqslant \alpha,$$

$$\sup_{z \in D_{1/2}} \bigl\|w_{0,0}(z,0) + z\bigr\|_{C^k(\mathbb{R})} \leqslant \beta,\ \|w_{\geqslant 1}\|^{(k)}_\xi \leqslant \gamma \Bigr\},$$

$$\mathcal{B}^{(k)}_0(\alpha,\beta,\gamma) := \bigl\{ w \in \mathcal{B}^{(k)}(\alpha,\beta,\gamma) \bigm| w(z) \text{ is rotation invariant for all } z \in D_{1/2} \bigr\}.$$

6. Initial Feshbach transformation

In this section we shall assume that the assumptions of Hypothesis (H) hold. Without loss of generality, see Section 3, we assume that the distance between the lowest eigenvalue of H_{at} and the rest of the spectrum is one, that is

$$\inf\bigl(\sigma(H_{\mathrm{at}}) \setminus \{E_{\mathrm{at}}\}\bigr) - E_{\mathrm{at}} = 1. \tag{6.1}$$

Let χ_1 and $\overline{\chi}_1$ be two functions in $C^\infty(\mathbb{R}_+;[0,1])$ with $\chi_1^2 + \overline{\chi}_1^2 = 1$, $\chi_1 = 1$ on $[0,3/4)$, and $\operatorname{supp}\chi_1 \subset [0,1]$. We use the abbreviations $\chi_1 = \chi_1(H_f)$ and $\overline{\chi}_1 = \overline{\chi}_1(H_f)$. It should be clear from the context whether χ_1 or $\overline{\chi}_1$ denotes a function or an operator. By φ_{at} we denote a fix choice for a normalized eigenstate of H_{at} with eigenvalue E_{at} and by P_{at} we denote the eigenprojection of H_{at} corresponding to the eigenvalue E_{at}. By Hypothesis (H) the range of P_{at} is one dimensional. Thus to every $\psi \in \operatorname{Ran} P_{\mathrm{at}} \otimes P_{\mathrm{red}}$ there exists a unique $\iota(\psi) \in \mathcal{H}_{\mathrm{red}}$ such that $\psi = \varphi_{\mathrm{at}} \otimes \iota(\psi)$. It follows that $\iota : \operatorname{Ran} P_{\mathrm{at}} \otimes P_{\mathrm{red}} \to \mathcal{H}_{\mathrm{red}}$ is unitary and commutes with the $SO(3)$

action. We will use ι to identify the range of $P_{\mathrm{at}} \otimes P_{\mathrm{red}}$ with $\mathcal{H}_{\mathrm{red}}$. We define $\chi^{(I)}(r) := P_{\mathrm{at}} \otimes \chi_1(r)$ and $\overline{\chi}^{(I)}(r) = \bar{P}_{\mathrm{at}} \otimes 1 + P_{\mathrm{at}} \otimes \overline{\chi}_1(r)$, with $\bar{P}_{\mathrm{at}} = 1 - P_{\mathrm{at}}$. We set $\chi^{(I)} := \chi^{(I)}(H_f)$ and $\overline{\chi}^{(I)} := \overline{\chi}^{(I)}(H_f)$. It follows directly from the definition that ${\chi^{(I)}}^2 + {\overline{\chi}^{(I)}}^2 = 1$. We use an initial transformation based on the smooth Feshbach map and its associated auxiliary operator, see Appendix B.

Theorem 6.1. *Assume Hypothesis* (H). *Let $k \in \mathbb{N}$. For any $0 < \xi < 1$ and any positive numbers $\delta_1, \delta_2, \delta_3$ there exists a positive number g_0 such that the following is satisfied. For all $(g, \beta, z) \in D_{g_0} \times \mathbb{R} \times D_{1/2}$ the pair of operators $(H_{g,\beta} - z - E_{\mathrm{at}}, H_0 - z - E_{\mathrm{at}})$ is a Feshbach pair for $\chi^{(I)}$. The operator-valued map*

$$Q_{\chi^{(I)}}(g, \beta, z) := Q_{\chi^{(I)}}(H_{g,\beta} - z - E_{\mathrm{at}}, H_0 - z - E_{\mathrm{at}})$$

is uniformly bounded in (g, β, z) and the function $(g, z) \mapsto Q_{\chi^{(I)}}(g, \cdot, z)$ is in $C_B^{\omega}(D_{g_0} \times D_{1/2}; C_B^k(\mathbb{R}; \mathcal{B}(\mathcal{H}_{\mathrm{red}}, \mathcal{H})))$. There exists a unique kernel $w^{(0)}(g, \beta, z) \in \mathcal{W}_\xi^\#$ such that

$$H\big(w^{(0)}(g, \beta, z)\big) = \iota\big(F_{\chi^{(I)}}(H_{g,\beta} - z - E_{\mathrm{at}}, H_0 - z - E_{\mathrm{at}}) \restriction \operatorname{Ran} P_{\mathrm{at}} \otimes P_{\mathrm{red}}\big)\iota^{-1}. \tag{6.2}$$

Moreover, $w^{(0)}$ satisfies the following properties.

(a) *We have $w^{(0)}(g) := w^{(0)}(g, \cdot, \cdot) \in \mathcal{B}_0^{(k)}(\delta_1, \delta_2, \delta_3)$ for all $g \in D_{g_0}$.*
(b) *$w^{(0)}(g, \beta, \cdot)$ is a symmetric kernel for all $(g, \beta) \in (D_{g_0} \cap \mathbb{R}) \times \mathbb{R}$.*
(c) *The function $(g, z, \beta) \mapsto w^{(0)}(g, \beta, z)$ is in $C_B^{\omega,k}(D_{g_0} \times D_{1/2} \times \mathbb{R}; \mathcal{W}_\xi^\#)$.*

The remaining part of this section is devoted to the proof of Theorem 6.1. Throughout this section we assume that

$$z = \zeta - E_{\mathrm{at}} \in D_{1/2}. \tag{6.3}$$

To prove Theorem (6.1), we write the interaction part of the Hamiltonian in terms of integral kernels as follows,

$$\begin{aligned} H_{g,\beta} &= H_{\mathrm{at}} + H_f + {:}W_{g,\beta}{:}, \\ W_{g,\beta} &:= \sum_{m+n=1,2} W_{m,n}(g, \beta), \end{aligned} \tag{6.4}$$

where $W_{m,n}(g, \beta) := \underline{H}_{m,n}(w_{m,n}^{(I)}(g, \beta))$ with

$$\underline{H}_{m,n}(w_{m,n}) := \int_{(\underline{\mathbb{R}}^3)^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} a^*\big(K^{(m)}\big) w_{m,n}\big(K^{(m,n)}\big) a\big(\widetilde{K}^{(n)}\big), \tag{6.5}$$

and

$$w_{1,0}^{(I)}(g,\beta)(K) := 2g\sum_{j=1}^{N} p_j\cdot\varepsilon(k,\lambda)\frac{\kappa_\Lambda(k)e^{i\beta k\cdot x_j}}{\sqrt{2}},$$

$$w_{1,1}^{(I)}(g,\beta)(K,\widetilde{K}) := g^2\sum_{j=1}^{N}\varepsilon(k,\lambda)\cdot\varepsilon(\widetilde{k},\widetilde{\lambda})\frac{\kappa_\Lambda(k)e^{-i\beta k\cdot x_j}}{\sqrt{2}}\frac{\kappa_\Lambda(\widetilde{k})e^{i\beta\widetilde{k}\cdot x_j}}{\sqrt{2}},$$

$$w_{2,0}^{(I)}(g,\beta)(K_1,K_2) := g^2\sum_{j=1}^{N}\varepsilon(k_1,\lambda_1)\cdot\varepsilon(k_2,\lambda_2)\frac{\kappa_\Lambda(k_1)e^{-i\beta k_1\cdot x_j}}{\sqrt{2}}\frac{\kappa_\Lambda(k_2)e^{-i\beta k_2\cdot x_j}}{\sqrt{2}}, \quad (6.6)$$

$w_{0,1}^{(I)}(g,\beta)(\widetilde{K}) := w_{0,1}^{(I)}(\overline{g},\beta)(\widetilde{K})^*$, and $w_{0,2}^{(I)}(g,\beta)(\widetilde{K}_1,\widetilde{K}_2) := \overline{w_{2,0}^{(I)}(\overline{g},\beta)(\widetilde{K}_1,\widetilde{K}_2)}$. We note that (6.5) is understood in the sense of forms, cf. Appendix A. We set

$$w_{0,0}^{(I)}(z)(r) := H_{\mathrm{at}} - z + r.$$

By $w^{(I)}$ we denote the vector consisting of the components $w_{m,n}^{(I)}$ with $m+n=0,1,2$.

The next theorem establishes the Feshbach property. To state it, we denote by P_0 the orthogonal projection onto the closure of $\operatorname{Ran}\overline{\chi}^{(I)}$. We will use the convention that $(H_0-z)^{-1}\overline{\chi}^{(I)}$ stands for $((H_0-z)\restriction\operatorname{Ran}\overline{\chi}^{(I)})^{-1}\overline{\chi}^{(I)}$, and that $(H_0-z)^{-1}P_0$ stands for $((H_0-z)\restriction\operatorname{Ran}P_0)^{-1}P_0$. The proof of the Feshbach property is based on the fact that

$$\inf\sigma(H_0\restriction\operatorname{Ran}P_0)\geqslant E_{\mathrm{at}}+\frac{3}{4}, \quad (6.7)$$

which follows directly from the definition, and the fact that the interaction part of the Hamiltonian is bounded with respect to the free Hamiltonian. A proof can be found in [17].

Theorem 6.2. *Let $|E_{\mathrm{at}}-\zeta|<\frac{1}{2}$. Then*

$$\left\|\left((H_0-\zeta)\restriction\operatorname{Ran}P_0\right)^{-1}\right\|\leqslant 4. \quad (6.8)$$

There is a $C<\infty$ and $g_0>0$ such that for all β and $|g|<g_0$,

$$\left\|(H_0-\zeta)^{-1}\overline{\chi}^{(I)}W_{g,\beta}\right\|\leqslant C|g|, \qquad \left\|W_{g,\beta}(H_0-\zeta)^{-1}\overline{\chi}^{(I)}\right\|\leqslant C|g|, \quad (6.9)$$

and $(H_{g,\beta}-\zeta, H_0-\zeta)$ is a Feshbach pair for $\chi^{(I)}$.

Theorem 6.3. *For g_0 sufficiently small*

$$(g,z)\mapsto Q_{\chi^{(I)}}(g,\cdot,z) \quad (6.10)$$

is in $C_B^\omega(D_{g_0}\times D_{1/2}; C_B^k(\mathbb{R};\mathcal{B}(\mathcal{H}_{\mathrm{red}},\mathcal{H})))$.

We write

$$\langle x\rangle := \left(1+\sum_{j=1}^{N}|x_j|^2\right)^{1/2}.$$

We will use the Leibniz rule for higher derivatives

$$\partial_\beta^l(f_1\cdots f_L)=\sum_{\underline{n}\in\mathbb{N}_0^L:\ |\underline{n}|=l}\frac{l!}{\underline{n}!}f_1^{(n_1)}\cdots f_L^{(n_L)}. \tag{6.11}$$

Proof of Theorem 6.3. For notational simplicity we set $W=W_{g,\beta}$ and $Q_{\chi^{(I)}}=Q_{\chi^{(I)}}(g,\beta,z)$. We have

$$\begin{aligned} Q_{\chi^{(I)}} &= \chi^{(I)}-\sum_{n=0}^{\infty}(-1)^n\overline{\chi}^{(I)}\big((H_0-\zeta)^{-1}\overline{\chi}^{(I)}W\overline{\chi}^{(I)}\big)^n(H_0-\zeta)^{-1}\overline{\chi}^{(I)}W\chi^{(I)} \\ &= \chi^{(I)}-\sum_{n=0}^{\infty}(-1)^n\big(\overline{\chi}^{(I)}(H_0-\zeta)^{-1}\overline{\chi}^{(I)}W\big)^{n+1}\chi^{(I)}. \end{aligned} \tag{6.12}$$

Formally differentiating l times with respect to β, the n-th term under the summation sign generates $(n+1)^l$ terms, each of the form

$$\big(\overline{\chi}^{(I)}(H_0-\zeta)^{-1}\overline{\chi}^{(I)}\partial_\beta^{l_{n+1}}W\big)\cdots\big(\overline{\chi}^{(I)}(H_0-\zeta)^{-1}\overline{\chi}^{(I)}\partial_\beta^{l_1}W\big)\chi^{(I)}, \tag{6.13}$$

where $l_1+\cdots+l_{n+1}=l$. We write

$$\begin{aligned} \overline{\chi}^{(I)}(H_0-\zeta)^{-1}\overline{\chi}^{(I)} &= \big(\overline{\chi}^{(I)}\big)^2(H_0+2-\zeta)^{-1} \\ &\quad + 2(H_0-\zeta)^{-1}\big(\overline{\chi}^{(I)}\big)^2(H_0+2-\zeta)^{-1}. \end{aligned} \tag{6.14}$$

It is well known that $\|e^{\gamma_1\langle x\rangle}P_{\rm at}\|<\infty$ for some $\gamma_1>0$ [22]. Define

$$\gamma_{j+1}=\left(1-k^{-1}\sum_{t=1}^{j}(1-\delta_{l_t,0})\right)\gamma_1;\quad j=1,\dots,n.$$

Since $\sum_{j=1}^n(1-\delta_{l_j,0})\leqslant k$, $\gamma_{n+1}\geqslant 0$. With

$$G_j=\big(e^{\gamma_{j+1}\langle x\rangle}\overline{\chi}^{(I)}(H_0-\zeta)^{-1}\overline{\chi}^{(I)}e^{-\gamma_{j+1}\langle x\rangle}\big)\big(e^{\gamma_{j+1}\langle x\rangle}\partial_\beta^{l_j}We^{-\gamma_j\langle x\rangle}\big) \tag{6.15}$$

the expression in (6.13) can be written as

$$e^{-\gamma_{n+1}\langle x\rangle}(G_{n+1}\cdots G_1)e^{\gamma_1\langle x\rangle}\chi^{(I)}. \tag{6.16}$$

We claim that for small enough $\gamma_1>0$ (chosen independent of n), for $|g|\leqslant 1$, and for $\zeta\in D_{1/2}+E_{\rm at}$

$$\|G_j\| \leqslant C|g|, \tag{6.17}$$

where C is independent of j, ζ, β, and n. It is clear that

$$\left\|(H_0+i)^{-1}e^{\gamma_{j+1}\langle x\rangle}\partial_\beta^{l_j} W e^{-\gamma_j\langle x\rangle}\right\| \leqslant C_1|g|,$$

since if $l_j > 0$, $\gamma_j - \gamma_{j+1} = \gamma_1/k$. (There is a slight subtlety here with the term $W_{0,1}^{(I)}$ which contains $(e^{i\beta k\cdot x_j})(p_j \cdot \epsilon(k,\lambda))$. But note that the two terms in parentheses commute so that the bound is indeed independent of β.) It remains to show

$$e^{\gamma\langle x\rangle}\overline{\chi}^{(I)}(H_0-\zeta)^{-1}\overline{\chi}^{(I)}e^{-\gamma\langle x\rangle}(H_0+i)$$

is bounded with bound independent of γ for small γ and $\zeta \in D_{1/2} + E_{\mathrm{at}}$. We have

$$H_0(\gamma) := e^{\gamma\langle x\rangle}H_0 e^{-\gamma\langle x\rangle} = H_{\mathrm{at}}(\gamma) + H_f,$$
$$H_{\mathrm{at}}(\gamma) := H_{\mathrm{at}} + i\gamma\left(\frac{x}{\langle x\rangle}\cdot p + p\cdot\frac{x}{\langle x\rangle}\right) - \gamma^2\frac{|x|^2}{\langle x\rangle^2}$$

and thus for all small γ

$$\left\|\big(H_0(\gamma)+2-\zeta\big)^{-1}(H_0+i)\right\| \leqslant C_2.$$

For $\zeta \in D_{1/2} + E_{\mathrm{at}}$. Clearly $\|e^{\gamma\langle x\rangle}(\overline{\chi}^{(I)})^2 e^{-\gamma\langle x\rangle}\| \leqslant c_3$ for γ small so from (6.14) it remains to bound

$$e^{\gamma\langle x\rangle}(H_0-\zeta)^{-1}\big(\overline{\chi}^{(I)}\big)^2 e^{-\gamma\langle x\rangle}.$$

Since $(\overline{\chi}^{(I)})^2 = P_{\mathrm{at}} \otimes \overline{\chi}_1(H_f)^2 + \bar{P}_{\mathrm{at}} \otimes 1$ and

$$(H_0-\zeta)^{-1}P_{\mathrm{at}}\otimes 1 = (1\otimes H_f + E_{\mathrm{at}} - \zeta)^{-1}P_{\mathrm{at}}\otimes 1$$

we must only control

$$e^{\gamma\langle x\rangle}(H_0-\zeta)^{-1}(\bar{P}_{\mathrm{at}}\otimes 1)e^{-\gamma\langle x\rangle}.$$

We write

$$(H_{\mathrm{at}}+t-\zeta)^{-1}\bar{P}_{\mathrm{at}} = \frac{1}{2\pi i}\int_\Gamma (w+t-\zeta)^{-1}(w-H_{\mathrm{at}})^{-1}\,dw, \tag{6.18}$$

where Γ is the contour $\Gamma_- - \Gamma_+$ with

$$\Gamma_\pm(s) = E_{\mathrm{at}} + 3/4 + e^{\pm i\pi/4}s, \quad 0 \leqslant s < \infty.$$

Thus (using an analytic continuation argument)

$$e^{\gamma\langle x\rangle}(H_0-\zeta)^{-1}(\bar{P}_{\mathrm{at}}\otimes 1)e^{-\gamma\langle x\rangle} = \frac{1}{2\pi i}\int_\Gamma \big(w-H_{\mathrm{at}}(\gamma)\big)^{-1}\otimes(w+H_f-\zeta)^{-1}\,dw. \tag{6.19}$$

The expression (6.19) is bounded using a numerical range argument for large w and a perturbation argument for small w. These estimates require γ to be small. We have thus shown (6.17). Moreover, it follows from the estimates above and Taylor's theorem with remainder that the derivative with respect to β in (6.15) and thus (6.13) exists with respect to the operator norm topology. It follows that for $l \leqslant k$,

$$\big\|\partial_\beta^l\big(Q_{\chi^{(I)}}-\chi^{(I)}\big)\big\| \leqslant \sum_{n=0}^\infty (n+1)^l\big(C|g|\big)^{n+1}\big\|e^{\gamma_1\langle x\rangle}P_{\mathrm{at}}\big\| \tag{6.20}$$

for all $\beta\in\mathbb{R}$ and $\zeta\in D_{1/2}+E_{\mathrm{at}}$. If $g_0>0$ is sufficiently small, then (6.20) converges for $|g|<g_0$. The expression in (6.13) is complex differentiable in $\zeta\in D_{1/2}+E_{\mathrm{at}}$ and in g with respect to the operator norm topology. The bounds (6.17) and (6.20) imply uniform convergence and that $(g,z)\mapsto Q_{\chi^{(I)}}(g,\cdot,z)$ is in $C_B^\omega(D_{g_0}\times D_{1/2};C_B^k(\mathbb{R};\mathcal{B}(\mathcal{H}_{\mathrm{red}},\mathcal{H})))$. □

Next we want to show that there exists a $w^{(0)}(g,\beta,z)\in\mathcal{W}_\xi^\#$ such that (6.2) holds. Uniqueness will follow from Theorem 5.3. In view of Theorem 6.2 for $z=\zeta-E_{\mathrm{at}}\in D_{1/2}$ and g sufficiently small we can define the Feshbach map and express it in terms of a Neumann series.

$$\begin{aligned}
&F_{\chi^{(I)}}(H_{g,\beta}-\zeta,H_0-\zeta)\restriction X_{\mathrm{at}}\otimes\mathcal{H}_{\mathrm{red}}\\
&\quad=\big(T+\chi W\chi-\chi W\overline{\chi}\big(T+\overline{\chi}W\overline{\chi}\big)^{-1}\overline{\chi}W\chi\big)\restriction X_{\mathrm{at}}\otimes\mathcal{H}_{\mathrm{red}}\\
&\quad=\left(T+\chi W\chi-\chi W\overline{\chi}\sum_{n=0}^\infty\big(-T^{-1}\overline{\chi}W\overline{\chi}\big)^nT^{-1}\overline{\chi}W\chi\right)\restriction X_{\mathrm{at}}\otimes\mathcal{H}_{\mathrm{red}},
\end{aligned}$$

where here we used the abbreviations $T=H_0-\zeta$, $W=W_{g,\beta}$, $\chi=\chi^{(I)}$, $\overline{\chi}=\overline{\chi}^{(I)}$ and $X_{\mathrm{at}}=\operatorname{Ran}P_{\mathrm{at}}$. We normal order above expression, using the pull-through formula. To this end we use a generalized version of the Wick theorem, see [7], see also [17, Appendix B]. Moreover we will use the definition

$$\underline{W}_{p,q}^{m,n}[w]\big(K^{(m,n)}\big):=\int_{(\underline{\mathbb{R}}^3)^{p+q}}\frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}}a^*\big(X^{(p)}\big)w_{m+p,n+q}\big(K^{(m)},X^{(p)},\widetilde{K}^{(n)},\widetilde{X}^{(q)}\big)a\big(\widetilde{X}^{(q)}\big).$$

We obtain a sequence of integral kernels $\widetilde{w}^{(0)}$, which are given as follows. For $M+N\geqslant 1$,

$$\begin{aligned}
&\widetilde{w}_{M,N}^{(0)}(g,\beta,z)\big(r,K^{(M,N)}\big)\\
&\quad=(8\pi)^{\frac{M+N}{2}}\sum_{L=1}^\infty(-1)^{L+1}\sum_{\substack{(\underline{m},\underline{p},\underline{n},\underline{q})\in\mathbb{N}_0^{4L}:\\|\underline{m}|=M,|\underline{n}|=N,\\1\leqslant m_l+p_l+q_l+n_l\leqslant 2}}\prod_{l=1}^L\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\right\}
\end{aligned}$$

$$\times V_{(\underline{m},\underline{p},\underline{n},\underline{q})}\big[w^I(g,\beta,\zeta)\big]\big(r,K^{(M,N)}\big). \tag{6.21}$$

Furthermore,

$$\widetilde{w}_{0,0}^{(0)}(g,\beta,z)(r) = -z + r + \sum_{L=2}^{\infty}(-1)^{L+1} \sum_{(\underline{p},\underline{q})\in\mathbb{N}_0^{2L}:\ p_l+q_l=1,2} V_{(\underline{0},\underline{p},\underline{0},\underline{q})}\big[w^{(I)}(g,\beta,\zeta)\big](r).$$

Above we have used the definition

$$\begin{aligned} &V_{\underline{m},\underline{p},\underline{n},\underline{q}}[w]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big) \\ &\quad := \Bigg\langle \varphi_{\mathrm{at}}\otimes\Omega,\ F_0[w](H_f+r) \\ &\qquad \times \prod_{l=1}^{L}\big\{\underline{W}_{p_l,q_l}^{m_l,n_l}[w]\big(K^{(m_l,n_l)}\big)F_l[w](H_f+r+\widetilde{r}_l)\big\}\varphi_{\mathrm{at}}\otimes\Omega\Bigg\rangle, \end{aligned} \tag{6.22}$$

where for $l=0,L$ we set $F_l[w](r) := \chi_1(r)$, and for $l=1,\dots,L-1$ we set

$$F_l[w](r) := F[w](r) := \frac{\overline{\chi}^{(I)}(r)^2}{w_{0,0}(r)}.$$

Moreover, we used the notation

$$r_l := \Sigma\big[\widetilde{K}_1^{(n_1)}\big] + \cdots + \Sigma\big[\widetilde{K}_{l-1}^{(n_{l-1})}\big] + \Sigma\big[K_{l+1}^{(m_{l+1})}\big] + \cdots + \Sigma\big[K_L^{(m_L)}\big], \tag{6.23}$$

$$\widetilde{r}_l := \Sigma\big[\widetilde{K}_1^{(n_1)}\big] + \cdots + \Sigma\big[\widetilde{K}_l^{(n_l)}\big] + \Sigma\big[K_{l+1}^{(m_{l+1})}\big] + \cdots + \Sigma\big[K_L^{(m_L)}\big]. \tag{6.24}$$

We have $w^{(0)}(g,\beta,z) = (\widetilde{w}^{(0)})^{(\mathrm{sym})}(g,\beta,z)$. So far we have determined $w^{(0)}$ on a formal level only.

Lemma 6.4. *Let $k\in\mathbb{N}_0$. The function $(g,\zeta,\beta)\mapsto V_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(I)}(g,\beta,\zeta)]$ is in $C_B^{\omega,k}(\mathbb{C}\times D_{1/2}(E_{\mathrm{at}})\times\mathbb{R};\mathcal{W}^{\#}_{|\underline{m}|,|\underline{n}|})$. There exists a finite constant C such that for all $(g,\beta,\zeta)\in\mathbb{C}\times\mathbb{R}\times D_{1/2}(E_{\mathrm{at}})$ we have*

$$\max_{0\leqslant l\leqslant k}\big\|\partial_\beta^l V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}(g,\beta,\zeta)\big]\big\|^{\#} \leqslant L^{k+1}C^L|g|^{|\underline{m}|+|\underline{n}|+|\underline{p}|+|\underline{q}|}. \tag{6.25}$$

Proof. For compactness we shall drop the ζ and β dependence in the notation. We show

$$\big|\partial_\beta^s V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}(g)\big]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big)\big| \leqslant L^sC^L|g|^{|\underline{m}|+|\underline{n}|+|\underline{p}|+|\underline{q}|}, \tag{6.26}$$

$$\big|\partial_r\partial_\beta^s V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}(g)\big]\big(r,K^{|\underline{m}|,|\underline{n}|}\big)\big| \leqslant L^{s+1}C^L|g|^{|\underline{m}|+|\underline{n}|+|\underline{p}|+|\underline{q}|}. \tag{6.27}$$

Consider

$$\partial_\beta^s V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}\big]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big)$$
$$= \sum_{\substack{\underline{j}\in\mathbb{N}_0^L\\ |\underline{j}|=s}} \frac{s!}{\underline{j}!}\Big\langle \varphi_{\mathrm{at}}\otimes\Omega, F_0\big[w^{(I)}\big](H_f+r)$$
$$\times\prod_{l=1}^{L}\big\{\partial_\beta^{j_l}\underline{W}^{m_l,n_l}_{p_l,q_l}\big[w^{(I)}\big]\big(K^{(m_l,n_l)}\big)F_l\big[w^{(I)}\big](H_f+r+\widetilde{r}_l)\big\}\varphi_{\mathrm{at}}\otimes\Omega\Big\rangle. \tag{6.28}$$

To estimate (6.28) we will use the same technique as in Theorem 6.3. For $l=1,\ldots,L-1$ define

$$A_l^{j_l}=e^{\gamma_{l+1}\langle x\rangle}\partial_\beta^{j_l}\underline{W}^{m_l,n_l}_{p_l,q_l}\big[w^{(I)}\big]\big(K^{(m_l,n_l)}\big)F_l\big[w^{(I)}\big](H_f+r+\widetilde{r}_l)e^{-\gamma_l\langle x\rangle} \tag{6.29}$$

and similarly for $A_L^{j_L}$ except that we replace $F_L[w^{(I)}](H_f+r+\widetilde{r}_l)$ by $(H_0-E_{\mathrm{at}}+1)^{-1}$. Here

$$\gamma_{l+1}=\left(1-k^{-1}\sum_{t=1}^{l}(1-\delta_{j_t,0})\right)\gamma_1;\quad l=1,\ldots,L.$$

Note again that since $s\leqslant k$, $\gamma_{L+1}\geqslant 0$. It follows that

$$\big|\partial_\beta^s V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}\big]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big)\big|\leqslant\sum_{\substack{\underline{j}\in\mathbb{N}_0^L\\ |\underline{j}|=s}}\frac{s!}{\underline{j}!}\left(\prod_{l=1}^{L}\big\|A_l^{j_l}\big\|\right)\big\|e^{\gamma_1\langle x\rangle}\varphi_{\mathrm{at}}\big\|.$$

We will show the bound

$$\big\|A_l^{j_l}\big\|\leqslant C|g|^{m_l+p_l+n_l+q_l}, \tag{6.30}$$

which gives (6.26). We write for $l\leqslant L$,

$$A_l^{j_l}=e^{\gamma_{l+1}\langle x\rangle}\partial_\beta^{j_l}\underline{W}^{m_l,n_l}_{p_l,q_l}\big[w^{(I)}\big]\big(K^{(m_l,n_l)}\big)(H_0-E_{\mathrm{at}}+1)^{-1}e^{-\gamma_l\langle x\rangle}$$
$$\times e^{\gamma_l\langle x\rangle}(H_0-E_{\mathrm{at}}+1)F_l\big[w^{(I)}\big](H_f+r+\widetilde{r}_l)e^{-\gamma_l\langle x\rangle}. \tag{6.31}$$

First we estimate the second factor. To this end we write

$$F_l\big[w^{(I)}\big](H_f+r+\widetilde{r}_l)=(H_0-E_{\mathrm{at}}-z+r+\widetilde{r}_l)^{-1}\big(\overline{P}_{\mathrm{at}}\otimes 1+P_{\mathrm{at}}\otimes\chi_1^2(H_f+r+\widetilde{r}_l)\big).$$

Since $e^{\gamma_l\langle x\rangle}P_{\mathrm{at}}e^{-\gamma_l\langle x\rangle}$ is bounded for γ_1 small, it is clear that

$$\big\|e^{\gamma_l\langle x\rangle}(H_0-E_{\mathrm{at}}+1)F_l\big[w^{(I)}\big](H_f+r+\widetilde{r}_l)e^{-\gamma_l\langle x\rangle}\big\|$$
$$\leqslant C_1+(r+\widetilde{r}_l)\big\|e^{\gamma_l\langle x\rangle}(H_0-E_{\mathrm{at}}-z+r+\widetilde{r}_l)^{-1}\overline{P}_{\mathrm{at}}\otimes 1e^{-\gamma_l\langle x\rangle}\big\|.$$

For $u \geqslant 0$ we write

$$(H_{\mathrm{at}} - E_{\mathrm{at}} - z + u)^{-1} \overline{P}_{\mathrm{at}} = \frac{1}{2\pi i} \int_{\Gamma} (w - E_{\mathrm{at}} - z + u)^{-1} (w - H_{\mathrm{at}})^{-1} \, dw \tag{6.32}$$

where Γ is the contour $\Gamma = \Gamma_- - \Gamma_+$ with

$$\Gamma_{\pm}(t) = E_{\mathrm{at}} + 3/4 + e^{\pm i \frac{\pi}{4}} t, \quad 0 \leqslant t < \infty,$$

and obtain for λ small

$$\begin{aligned} & e^{\lambda \langle x \rangle} (H_{\mathrm{at}} - E_{\mathrm{at}} - z + u)^{-1} \overline{P}_{\mathrm{at}} e^{-\lambda \langle x \rangle} \\ & \quad = \frac{1}{2\pi i} \int_{\Gamma} (w - E_{\mathrm{at}} - z + u)^{-1} \big(w - H_{\mathrm{at}}(\lambda) \big)^{-1} \, dw, \end{aligned} \tag{6.33}$$

where $H_{\mathrm{at}}(\lambda)$ is given as in the proof of Theorem 6.3. As in that proof we estimate (6.33) for large $w \in \Gamma$ using a numerical range estimate to bound $\|(w - H_{\mathrm{at}}(\lambda))^{-1}\|$ while for small $w \in \Gamma$ the resolvent can be bounded using

$$\big\| \big(H_{\mathrm{at}}(\lambda) - H_{\mathrm{at}} \big) (-\Delta + 1)^{-1} \big\| = O\big(|\lambda| \big)$$

for small λ. Then using the spectral theorem, which allows us to substitute $u = H_f + r + \widetilde{r}_l$, we obtain for small γ_1

$$y \big\| e^{\gamma_l \langle x \rangle} (H_0 - E_{\mathrm{at}} - z + y)^{-1} \overline{P}_{\mathrm{at}} \otimes 1 e^{-\gamma_l \langle x \rangle} \big\| \leqslant C$$

independent of $y \geqslant 0$. In order to show (6.30) for $1 \leqslant l \leqslant L$ it remains to bound the first factor on the right-hand side of (6.31). Using $\|(\frac{x}{\langle x \rangle} \cdot p + p \cdot \frac{x}{\langle x \rangle})(-\Delta + 1)^{-1}\| < \infty$ and Hypothesis (H) we see that

$$\big\| (-\Delta \otimes 1 + 1 \otimes H_f + 1) e^{\gamma_l \langle x \rangle} (H_0 - E_{\mathrm{at}} + 1)^{-1} e^{-\gamma_l \langle x \rangle} \big\|$$

is bounded uniformly in L for small γ_1. Thus to prove (6.30) we need only bound

$$e^{\gamma_{l+1} \langle x \rangle} \partial_{\beta}^{j_l} \underline{W}_{p_l, p_l}^{m_l, n_l} \big[w^{(I)} \big] \big(K^{(m_l, n_l)} \big) e^{-\gamma_l \langle x \rangle} (-\Delta \otimes 1 + 1 \otimes H_f + 1)^{-1}$$

or carrying out the differentiations with respect to β (if any) we need to bound

$$\underline{W}_{p_l, q_l}^{m_l, p_l} \big(e^{\gamma_{l+1} \langle x \rangle} w^{(I, j_l)} e^{-\gamma_l \langle x \rangle} \big) \big(K^{(m_l, n_l)} \big) (-\Delta \otimes 1 + 1 \otimes H_f + 1)^{-1},$$

where $w^{(I, j_l)} := \partial_{\beta}^{j_l} w^{(I)}$. Referring to (6.6) we have

$$\big\| e^{\gamma_{l+1} \langle x \rangle} w_{1,0}^{(I, j_l)} (K) e^{-\gamma_l \langle x \rangle} (-\Delta + 1)^{-1/2} \big\|_{\mathcal{H}_{\mathrm{at}} \to \mathcal{H}_{\mathrm{at}}} \leqslant c_1 |g| \kappa_{\Lambda}(k), \tag{6.34}$$

and similarly for $w_{0,1}^{(I, j_l)}$, while for $m + n = 2$

$$\left\| e^{\gamma_{l+1}\langle x\rangle} w_{m,n}^{(I,j_l)}\big(K^{(m,n)}\big) e^{-\gamma_l\langle x\rangle} \right\|_{\mathcal{H}_{\mathrm{at}}\to\mathcal{H}_{\mathrm{at}}} \leqslant c_2|g|^2 \kappa_\Lambda\big(K^{(m)}\big)\kappa_\Lambda\big(\widetilde{K}^{(n)}\big), \tag{6.35}$$

where $\kappa_\Lambda(K^{(m)}) = \prod_{j=1}^m \kappa_\Lambda(k_j)$. Given (6.34) and (6.35) we need only consider $\underline{W}_{p_l,q_l}^{m_l,n_l}$ with $p_l + q_l \geqslant 1$. From Lemma A.2, if $m_l + n_l \leqslant 1$, $p_l = 1$, $q_l = 0$,

$$\begin{aligned} &\left\| \int \frac{dX}{|X|^{1/2}} a^*(X) e^{\gamma_{l+1}\langle x\rangle} w_{m_l+1,n_l}^{(I,j_l)}\big(K^{(m_l)}, X, \widetilde{K}^{(n_l)}\big) e^{-\gamma_l\langle x\rangle} (-\Delta+1)^{-1/2} \otimes (H_f+1)^{-1/2} \right\|^2 \\ &\quad \leqslant \int \frac{dX}{|X|^2} \sup_{r\geqslant 0} \left\| e^{\gamma_{l+1}\langle x\rangle} w_{m_l+1,n_l}^{(I,j_l)}\big(K^{(m_l)}, X, \widetilde{K}^{(n_l)}\big) e^{-\gamma_l\langle x\rangle} (-\Delta+1)^{-1/2} \right\|_{\mathcal{H}_{\mathrm{at}}\to\mathcal{H}_{\mathrm{at}}} \frac{r+|X|}{r+1} \\ &\quad \leqslant c|g|^{m_l+n_l+1}, \end{aligned}$$

and similarly if $p_l = 0$, $q_l = 1$. If $p_l = q_l = 1$

$$\begin{aligned} &\left\| \int \frac{dX^{(1,1)}}{|X^{(1,1)}|^{1/2}} a^*(X_1) e^{\gamma_{l+1}\langle x\rangle} w_{1,1}^{(I,j_l)}(X_1, \widetilde{X}_2) e^{-\gamma_l\langle x\rangle} a(X_2)(H_f+1)^{-1} \right\|^2 \\ &\quad \leqslant \int \frac{dX^{(1,1)}}{|X^{(1,1)}|^2} \sup_{r\geqslant 0} \left\| e^{\gamma_{l+1}\langle x\rangle} w_{1,1}^{(I,j_l)}(X_1, \widetilde{X}_2) e^{-\gamma_l\langle x\rangle} \right\|_{\mathcal{H}_{\mathrm{at}}\to\mathcal{H}_{\mathrm{at}}} \frac{(r+|X_1|)(r+|\widetilde{X}_2|)}{(r+|\widetilde{X}_2|)^2} \\ &\quad \leqslant c|g|^2, \end{aligned}$$

and similarly if $p_l = 2$, $q_l = 0$ or $p_l = 0$, $q_l = 2$. Since

$$\begin{aligned} \left\| (-\Delta+1)^{1/2} \otimes (H_f+1)^{1/2} (-\Delta \otimes 1 + 1 \otimes H_f + 1)^{-1} \right\| &= 1, \\ \left\| 1 \otimes (H_f+1)(-\Delta \otimes 1 + 1 \otimes H_f + 1)^{-1} \right\| &= 1 \end{aligned}$$

we have proven (6.26). A similar argument gives (6.27)

$$\left| \partial_r \partial_\beta^s V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}\big]\big(r, K^{|\underline{m}|,|\underline{n}|}\big) \right| \leqslant L^{k+1} C^L |g|^{|\underline{m}|+|\underline{n}|+|\underline{p}|+|\underline{q}|}. \tag{6.36}$$

One can use the same estimates as above to show that the β derivative in (6.29) exists in $L^\infty(\underline{B}_1^{(|\underline{m}|,|\underline{n}|)}; C^1([0,1]; \mathcal{B}(\mathcal{H})))$. To show this one replaces $w^{(I,j_l)}$ by its difference from the differential quotient, i.e., $(\Delta\beta)^{-1}(w^{(I,j_l-1)}(\beta+\Delta\beta) - w^{(I,j_l-1)}(\beta)) - w^{(I,j_l)}(\beta)$ and using the explicit expressions for $w^{(I)}$ it is straightforward to verify using Taylor's theorem with remainder that the right-hand side in the corresponding estimates converge to zero as $\Delta\beta$ tends to zero. Likewise one shows continuity in β. It now follows that the β derivative in (6.28) exists in $L^\infty(\underline{B}_1^{(|\underline{m}|,|\underline{n}|)}; C^1[0,1])$ The mapping $(g,z) \mapsto V_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(I)}(g, E_{\mathrm{at}}+z)]$ is in $C_B^\omega(\mathbb{C} \times D_{1/2}; C_B^k(\mathbb{R}; \mathcal{W}_{|\underline{m}|,|\underline{n}|}^\#))$. To this end, observe that for fixed z, $V_{\underline{m},\underline{p},\underline{n},\underline{q}}$ is a polynomial in g with coefficients in $C_B^{\omega,k}(\mathbb{R}; \mathcal{W}_{|\underline{m}|,|\underline{n}|}^\#)$. For fixed g it is straightforward to verify that $V_{\underline{m},\underline{p},\underline{n},\underline{q}}$ is differentiable with respect to z. To this end observe that only $w_{0,0}^{(I)}$ depends on z. □

Using Lemma 6.4 the proof of Theorem 6.1(a) is analogous to the proof of [17, Theorem 7.1(a)]. Below we summarize the main estimates of the proof. Let $S_{M,N}^L$ denote the set of tuples $(\underline{m}, \underline{p}, \underline{n}, \underline{q}) \in \mathbb{N}_0^{4L}$ with $|\underline{m}| = M$, $|\underline{n}| = N$, and $1 \leqslant m_l + p_l + q_l + n_l \leqslant 2$. We find, with $\widetilde{\xi} := (8\pi)^{-1/2}\xi$,

$$\begin{aligned}
&\big\| w^{(0)}_{\geqslant 1}(g,z)\big\|^{(\#,k)}_{\xi} \\
&\quad = \sup_{\beta\in\mathbb{R}} \sum_{M+N\geqslant 1} \xi^{-(M+N)} \max_{0\leqslant l\leqslant k} \big\| \partial^l_\beta \widetilde{w}_{M,N}(g,\beta,z)\big\|^{\#} \\
&\quad \leqslant \sum_{M+N\geqslant 1} \sum_{L=1}^{\infty} \sum_{(\underline{m},\underline{p},\underline{n},\underline{q})\in S^L_{M,N}} \widetilde{\xi}^{-(M+N)} 4^L \sup_{\beta\in\mathbb{R}} \max_{0\leqslant l\leqslant k} \big\| \partial^l_\beta V_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w^{(I)}(g,\beta,\zeta)\big]\big\|^{\#} \\
&\quad \leqslant \sum_{L=1}^{\infty} \sum_{M+N\geqslant 1} \sum_{(\underline{m},\underline{p},\underline{n},\underline{q})\in S^L_{M,N}} \widetilde{\xi}^{-|\underline{m}|-|\underline{n}|} L^{k+1} (4C)^L g^{|\underline{m}|+|\underline{n}|+|\underline{p}|+|\underline{q}|} \\
&\quad \leqslant \sum_{L=1}^{\infty} L^{k+1} 14^L \widetilde{\xi}^{-2L} \big(4C|g|\big)^L, \qquad (6.37)
\end{aligned}$$

for all $(g,z) \in D_1 \times D_{1/2}$. A similar but simpler estimate yields

$$\begin{aligned}
&\sup_{r\in[0,1]} \big\| \partial_r w^{(0)}_{0,0}(g,z)(r) - 1\big\|_{C^k(\mathbb{R})} \\
&\quad \leqslant \sum_{L=2}^{\infty} \sum_{(\underline{p},\underline{q})\in\mathbb{N}_0^{2L}:\ p_l+q_l=1,2} \sup_{\beta\in\mathbb{R}} \max_{0\leqslant l\leqslant k} \big\| \partial^l_\beta V_{\underline{0},\underline{p},\underline{0},\underline{q}}\big[w^{(I)}(g,\beta,\zeta)\big]\big\|^{\#} \\
&\quad \leqslant \sum_{L=2}^{\infty} 3^L L^{k+1} \big(C|g|\big)^L, \qquad (6.38)
\end{aligned}$$

for all $(g,z) \in D_1 \times D_{1/2}$. Analogously we have for all $(g,z) \in D_1 \times D_{1/2}$,

$$\begin{aligned}
\big\| w^{(0)}_{0,0}(g,z)(0) + z\big\|_{C^k(\mathbb{R})} &\leqslant \sum_{L=2}^{\infty} \sum_{(\underline{p},\underline{q})\in\mathbb{N}_0^{2L}:\ p_l+q_l=1,2} \sup_{\beta\in\mathbb{R}} \max_{0\leqslant l\leqslant k} \big\| \partial^l_\beta V_{\underline{0},\underline{p},\underline{0},\underline{q}}\big[w^{(I)}(g,\zeta)\big]\big\|^{\#} \\
&\leqslant \sum_{L=2}^{\infty} 3^L L^{k+1} \big(C|g|\big)^L. \qquad (6.39)
\end{aligned}$$

The right-hand sides in (6.37)–(6.39) can be made arbitrarily small for sufficiently small $|g|$. This implies that $w^{(0)}(g)$ is in $\mathcal{B}^{(k)}(\delta_1,\delta_2,\delta_3)$. Rotation invariance and the symmetry property have already been shown in [17, Theorem 7.1]. Theorem 6.1(c) follows from Lemma 6.4 and the convergence for small g established in (6.37)–(6.39).

7. Renormalization transformation

In this section we define the renormalization transformation as in [2]. It is a combination of the Feshbach transformation which cuts out higher photon energies, a rescaling of the resulting operator so that it acts on the fixed subspace $\mathcal{H}_{\text{red}}$ and a conformal transformation of the spectral parameter. Let $0 < \xi < 1$ and $0 < \rho < 1$. For $w \in \mathcal{W}_\xi$ we define the analytic function

$$E_\rho[w](z) := \rho^{-1} E[w](z) := -\rho^{-1} \langle \Omega, H\big(w(z)\big)\Omega \rangle$$

and the set

$$U[w] := \big\{ z \in D_{1/2} \mid \big|E[w](z)\big| < \rho/2 \big\}.$$

Lemma 7.1. *Let $0 < \rho \leqslant 1/2$. Then for all $w \in \mathcal{B}(\rho/8, \rho/8, \rho/8)$, the function $E_\rho[w] : U[w] \to D_{1/2}$ is an analytic bijection, $D_{3\rho/8} \subset U[w] \subset D_{5\rho/8}$, and for all $z \in D_{5\rho/8}$ we have*

$$\big|\partial_z E[w](z) - 1\big| \leqslant \frac{4\rho}{(4 - 5\rho)^2}. \tag{7.1}$$

If $w \in \mathcal{B}(\rho/32, \rho/32, \rho/32)$, then $D_{15\rho/32} \subset U[w] \subset D_{17\rho/32}$ and for all $z \in D_{17\rho/32}$ we have

$$\big|\partial_z E[w](z) - 1\big| \leqslant \frac{16\rho}{(16 - 17\rho)^2}. \tag{7.2}$$

For a proof of the lemma we apply following lemma with $r = \rho/2$ and $\epsilon = \rho/8$ respectively $\epsilon = \rho/32$. For a proof of Lemma 7.2 see [16, Lemma 6.2] or [2].

Lemma 7.2. *Let $0 < \epsilon < 1/2$, and let $E : D_{1/2} \to \mathbb{C}$ be an analytic function which satisfies*

$$\sup_{z \in D_{1/2}} \big|E(z) - z\big| \leqslant \epsilon.$$

Then for any $r > 0$ with $r + \epsilon < 1/2$ the following are true.

(a) *For $w \in D_r$ there exists a unique $z \in D_{1/2}$ such that $E(z) = w$.*
(b) *The map $E : U_r := \{z \in D_{1/2} \mid |E(z)| < r\} \to D_r$ is biholomorphic.*
(c) *We have $D_{r-\epsilon} \subset U_r \subset D_{r+\epsilon}$.*
(d) *If $z \in D_{r+\epsilon}$, then $|\partial_z E(z) - 1| \leqslant \frac{\epsilon}{2}(1/2 - (r + \epsilon))^{-2}$.*

If $0 < \rho \leqslant 1/4$, then for $w \in \mathcal{B}(\rho/32, \rho/32, \rho/32)$ we find using (7.2), that for all $z \in D_{17\rho/32}$

$$\big|\partial_z E_\rho[w]\big| \geqslant \frac{1}{\rho}\big(1 - |\partial_z E - 1|\big) \geqslant \frac{15}{16\rho}. \tag{7.3}$$

Let $I_\rho[w]$ denote the inverse of $E_\rho[w] : U[w] \to D_{1/2}$. It satisfies

$$E_\rho[w]\big(I_\rho[w](z)\big) = z, \tag{7.4}$$

for all $z \in D_{1/2}$. For notational compactness we shall occasionally drop the dependence on w and write E_ρ and I_ρ. In the previous section we introduced smooth functions χ_1 and $\overline{\chi}_1$. We set

$$\chi_\rho(\cdot) = \chi_1(\cdot/\rho), \qquad \overline{\chi}_\rho(\cdot) = \overline{\chi}_1(\cdot/\rho),$$

and use the abbreviation $\chi_\rho = \chi_\rho(H_f)$ and $\overline{\chi}_\rho = \overline{\chi}_\rho(H_f)$. It should be clear from the context whether χ_ρ or $\overline{\chi}_\rho$ denotes a function or an operator. The following theorem is proven in [2,16].

Lemma 7.3. *Let $0 < \rho \leqslant 1/2$. Then for all $w \in \mathcal{B}(\rho/8, \rho/8, \rho/8)$, and all $z \in D_{1/2}$ the pair of operators $(H(w(E_\rho^{-1}(z))), H_{0,0}(w(E_\rho^{-1}(z))))$ is a Feshbach pair for χ_ρ.*

The definition of the renormalization transformation involves a scaling transformation S_ρ which scales the energy value ρ to the value 1. For operators $A \in \mathcal{B}(\mathcal{F})$ we define

$$S_\rho(A) = \rho^{-1} \Gamma_\rho A \Gamma_\rho^*,$$

where Γ_ρ is the unitary dilation on $\mathcal{F}$ which is uniquely determined by

$$\Gamma_\rho a^\#(k) \Gamma_\rho^* = \rho^{-3/2} a^\#\big(\rho^{-1} k\big), \qquad \Gamma_\rho \Omega = \Omega.$$

It is easy to check that $\Gamma_\rho H_f \Gamma_\rho^* = \rho H_f$ and hence $\Gamma_\rho \chi_\rho \Gamma_\rho^* = \chi_1$. We are now ready to define the renormalization transformation, which in view of Lemmas 7.1 and 7.3 is well defined.

Definition 7.4. Let $0 < \rho \leqslant 1/2$. For $w \in \mathcal{B}(\rho/8, \rho/8, \rho/8)$ we define the renormalization transformation

$$\big(\mathcal{R}_\rho H(w)\big)(z) := S_\rho F_{\chi_\rho}\big(H\big(w\big(E_\rho^{-1}(z)\big)\big), H_{0,0}\big(w\big(E_\rho^{-1}(z)\big)\big)\big) \restriction \mathcal{H}_{\mathrm{red}},$$

where $z \in D_{1/2}$.

Theorem 7.5. *Let $0 < \rho \leqslant 1/2$ and $0 < \xi \leqslant 1/2$. For $w \in \mathcal{B}(\rho/8, \rho/8, \rho/8)$ there exists a unique integral kernel $\mathcal{R}_\rho(w) \in \mathcal{W}_\xi$*

$$\big(\mathcal{R}_\rho H(w)\big)(z) = H\big(\mathcal{R}_\rho(w)(z)\big).$$

If w is symmetric then also $\mathcal{R}_\rho(w)$ is symmetric. If $w(z)$ is invariant under rotations for all $z \in D_{1/2}$ then also $\mathcal{R}_\rho(w)(z)$ is invariant under rotations for all $z \in D_{1/2}$.

A proof of the existence of the integral kernel as stated in Theorem 7.5 can be found in [2] or [16, Theorem 8.1]. The uniqueness follows from Theorem 5.3. The statement about the symmetry and the rotation invariance follows from Lemmas 5.4 and 5.5 and the fact that the renormalization transformation preserves symmetry and rotation invariance, respectively. This is explained in detail in [17]. The renormalized kernels are given as follows. For $w \in \mathcal{W}^\#_{m+p,n+q}$ we define

$$\begin{aligned} &W^{m,n}_{p,q}[w]\big(r, K^{(m,n)}\big) \\ &:= P_{\mathrm{red}} \int_{B_1^{p+q}} \frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}} a^*\big(x^{(p)}\big) w_{p+m,q+n}\big(H_f + r, x^{(p)}, k^{(m)}, \widetilde{x}^{(q)}, \widetilde{k}^{(n)}\big) a\big(\widetilde{x}^{(q)}\big) P_{\mathrm{red}} \end{aligned}$$

which defines an operator for a.e. $K^{(m,n)} \in B_1^{m+n}$. In the case $m = n = 0$ we set $W_{m,n}^{0,0}[w](r) := W_{m,n}[w](r)$. For $w \in \mathcal{B}(\rho/8, \rho/8, \rho/8)$ we have

$$\mathcal{R}_\rho(w)(z) = \mathcal{R}_\rho^\#\big(w\big(I_\rho[w](z)\big)\big).$$

For $w \in \mathcal{W}_\xi^\#$ we define

$$\mathcal{R}_\rho^\#(w) := \widehat{w}^{(\mathrm{sym})},$$

where the kernels $\widehat{w}$ are given as follows. For $M + N \geqslant 1$,

$$\widehat{w}_{M,N}\big(r, K^{(M,N)}\big) := \sum_{L=1}^{\infty} (-1)^{L-1} \rho^{M+N-1} \sum_{\substack{(\underline{m},\underline{p},\underline{n},\underline{q}) \in \mathbb{N}_0^{4L}: \\ |\underline{m}|=M, |\underline{n}|=N, \\ m_l+p_l+n_l+q_l \geqslant 1}} \times \prod_{l=1}^{L} \left\{ \binom{m_l + p_l}{p_l} \binom{n_l + q_l}{q_l} \right\} v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w]\big(r, K^{(M,N)}\big), \tag{7.5}$$

and

$$\widehat{w}_{0,0}(r) := \rho^{-1} w_{0,0}(\rho r) + \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{(\underline{p},\underline{q}) \in \mathbb{N}_0^{2L}: \\ p_l+q_l \geqslant 1}} v_{\underline{0},\underline{p},\underline{0},\underline{q}}[w](r). \tag{7.6}$$

Moreover, we have introduced the expressions

$$\begin{aligned} & v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w]\big(r, K^{(|\underline{m}|,|\underline{n}|)}\big) \\ & \quad := \Big\langle \Omega, F_0[w]\big(H_f + \rho(r + \widetilde{r}_0)\big) \\ & \qquad \times \prod_{l=1}^{L} \big\{ W_{p_l,q_l}^{m_l,n_l}[w]\big(\rho(r + r_l), \rho K_l^{(m_l,n_l)}\big) F_l[w]\big(H_f + \rho(r + \widetilde{r}_l)\big) \big\} \Omega \Big\rangle, \end{aligned} \tag{7.7}$$

where $F_0[w](r) := \chi_\rho(r)$ and $F_L[w](r) := \chi_\rho(r)$, and for $l = 1, \ldots, L-1$

$$F_l[w](r) := F[w](r) := \frac{\overline{\chi}_\rho^2(r)}{w_{0,0}(r)}. \tag{7.8}$$

We used the notation introduced in (6.23) and (6.24). The next theorem states the contraction property.

Theorem 7.6. *For any positive numbers $\rho_0 \leqslant 1/4$ and $\xi_0 \leqslant 1/2$ there exist numbers ρ, ξ, ϵ_0 satisfying $\rho \in (0, \rho_0]$, $\xi \in (0, \xi_0]$, and $0 < \epsilon_0 \leqslant \rho/8$ such that the following property holds,*

$$\mathcal{R}_\rho : \mathcal{B}_0(\epsilon, \delta_1, \delta_2) \to \mathcal{B}_0(\epsilon + \delta_2/2, \delta_2/2, \delta_2/2), \quad \forall \epsilon, \delta_1, \delta_2 \in [0, \epsilon_0). \tag{7.9}$$

A proof of Theorem 7.6 can be found in [16, Theorem 9.1]. The proof given there relies on the fact that there are no terms which are linear in creation or annihilation operators. Since by rotation invariance and Lemma 5.5 there are no terms which are linear in creation and annihilation operators, Theorem 7.6 follows from the same proof. The contraction property allows us to iterate the renormalization transformation. To this end we introduce the following hypothesis.

(R) Let ρ, ξ, ϵ_0 be positive numbers such that the contraction property (7.9) holds and $\rho \leqslant 1/4$, $\xi \leqslant 1/4$ and $\epsilon_0 \leqslant \rho/8$.

Now we extend the renormalization transformation to $\mathcal{B}^{(0)}(\rho/8, \rho/8, \rho/8)$ by setting

$$\mathcal{R}_\rho(w)(\beta) = \mathcal{R}_\rho\big(w(\beta)\big)$$

for $w \in \mathcal{B}^{(0)}(\rho/8, \rho/8, \rho/8)$ and

$$\mathcal{R}_\rho^{\#}(w)(\beta) = \mathcal{R}_\rho^{\#}\big(w(\beta)\big)$$

for $w \in \mathcal{B}^{(\#,0)}(\rho/8, \rho/8, \rho/8)$. That is we have

$$\mathcal{R}_\rho(w)(\beta, z) = \mathcal{R}_\rho^{\#}\big(w\big(\beta, I_\rho(\beta, z)\big)\big).$$

The next theorem states that the extended renormalization transformation preserves the $\mathcal{B}_0^{(k)}$-balls and acts as a contraction on these balls in all but one dimension.

Theorem 7.7. *For $k \in \mathbb{N}_0$ and positive numbers $\rho_0 \leqslant 1/4$ and $\xi_0 \leqslant 1/4$ there exist numbers ρ, ξ, ϵ_0 satisfying $\rho \in (0, \rho_0]$, $\xi \in (0, \xi_0]$, and $0 < \epsilon_0 \leqslant \rho/32$ such that*

$$\mathcal{R}_\rho : \mathcal{B}_0^{(k)}(\epsilon, \delta_1, \delta_2) \to \mathcal{B}_0^{(k)}(\epsilon + \delta_2/4 + \delta_1/4, \delta_2/2, \delta_2/2), \quad \forall \epsilon, \delta_1, \delta_2 \in [0, \epsilon_0). \tag{7.10}$$

Theorem 7.7 will be shown below. The next theorem states that the extended renormalization transformation preserves analyticity.

Theorem 7.8. *Let $0 < \rho \leqslant 1/2$ and $0 < \xi \leqslant 1/2$. Let S be an open subset of $\mathbb{C}^\nu$ with $\nu \in \mathbb{N}$. Suppose the map $w(\cdot, \cdot) : S \times \mathbb{R} \to \mathcal{W}_\xi^{\#}$ is in $C^{\omega,k}(S \times \mathbb{R}; \mathcal{W}_\xi^{\#})$ and for all $s \in S$ we have $w(s, \cdot) \in \mathcal{B}^{(\#,k)}(\rho/32, 5\rho/8, \rho/32)$. Then*

$$(s, \beta) \mapsto \mathcal{R}_\rho^{\#}\big(w(s, \beta)\big)$$

is in $C_B^{\omega,k}(S \times \mathbb{R}; \mathcal{W}_\xi^{\#})$.

Theorem 7.9. *Let $0 < \rho \leqslant 1/2$ and $0 < \xi \leqslant 1/2$. Let S be an open subset of $\mathbb{C}$. Suppose*

$$w(\cdot,\cdot,\cdot): S \times D_{1/2} \times \mathbb{R} \to \mathcal{W}_\xi^\#$$
$$(s,z,\beta) \mapsto w(s,z,\beta)$$

is in $C^{\omega,k}(S \times D_{1/2} \times \mathbb{R}; \mathcal{W}_\xi^\#)$ and for all $s \in S$ we have $w(s,\cdot,\cdot) \in \mathcal{B}^{(k)}(\rho/32, \rho/32, \rho/32)$. Then

$$(s,z,\beta) \mapsto \big(\mathcal{R}_\rho\big(w(s,\cdot,\beta)\big)\big)(z)$$

is in $C_B^{\omega,k}(S \times D_{1/2} \times \mathbb{R}; \mathcal{W}_\xi^\#)$.

To show Theorems 7.7, 7.8, and 7.9 we will use the explicit expression for the renormalized integral kernels introduced above. For $w \in \mathcal{B}^{(0)}(\rho/8, \rho/8, \rho/8)$ we define

$$E_\rho(\beta,z) := E_\rho\big[w(\beta)\big](z), \qquad I_\rho(\beta,z) := I_\rho\big[w(\beta)\big](z).$$

The crucial point of that following estimate is that the constant C_L grows at most polynomially in L and that ρ^{-1} occurs to a power of at most $L-1$.

Lemma 7.10. *Let $0 \leqslant \rho \leqslant 1/4$ and let $w \in \mathcal{B}^{(\#,k)}(\rho/32, 5\rho/8, \cdot)$. Then for $(\underline{m}, \underline{p}, \underline{n}, \underline{q}) \in (\mathbb{N}_0^L)^4$ we have*

$$\max_{0\leqslant l\leqslant k} \big\| \partial_\beta^l v_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w(\beta)\big]\big\|^\# \leqslant C_L \left(\frac{1}{t}\right)^{L-1} \prod_{l=1}^{L} \frac{\max_{0\leqslant l\leqslant k} \|\partial_\beta^l w_{m_l+p_l,n_l+q_l}(\beta)\|^\#}{\sqrt{p_l!q_l!}}, \tag{7.11}$$

where $t := 3\rho/32$ and C_L is a constant which satisfies a bound

$$C_L \leqslant c\big(1 + \|\partial_r \chi_1\|_\infty\big)^k\big(1 + L^k\big),$$

where c is a finite numerical constant.

Proof. First we consider the case $k = 0$. Since in that case the β dependence is not relevant we drop the β dependence in the notation. Using

$$\big|\langle \Omega, A_1 A_2 \cdots A_n \Omega\rangle\big| \leqslant \|A_1\|_{\mathrm{op}} \|A_2\|_{\mathrm{op}} \cdots \|A_n\|_{\mathrm{op}}, \tag{7.12}$$

we find

$$\begin{aligned} &\operatorname*{ess\,sup}_{K^{(|\underline{m}|,|\underline{n}|)}} \sup_{r\in[0,1]} \big|v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w]\big(r, K^{(|\underline{m}|,|\underline{n}|)}\big)\big| \\ &\quad \leqslant \prod_{l=1}^{L} \operatorname*{ess\,sup}_{K^{(m_l,n_l)}} \sup_{r\in[0,1]} \big\| W_{p,q}^{m,n}[w]\big(r, K^{(m_l,n_l)}\big)\big\|_{\mathrm{op}}, \qquad \prod_{l=1}^{L-1} \big\|\overline{\chi}_\rho^2 / w_{0,0}\big\|_{C[0,1]}. \end{aligned}$$

To estimate the right-hand side we use

$$\operatorname*{ess\,sup}_{K^{(m,n)}} \sup_{r\in[0,1]} \left\| W_{p,q}^{m,n}[w]\big(r, K^{(m,n)}\big)\right\|_{\mathrm{op}} \leqslant \frac{\|w_{p+m,q+n},\|_{L^\infty(B_1^{m+n};C[0,1])}}{\sqrt{p!q!}}, \tag{7.13}$$

$$\left\|\overline{\chi}_\rho^2/w_{0,0}\right\|_{C[0,1]} \leqslant 1/t. \tag{7.14}$$

Inequality (7.13) can be shown using Lemma A.2 and (5.6). Inequality (7.14) can be shown as follows. For $r \geqslant \rho 3/4$ we have

$$\left|w_{0,0}(r)\right| \geqslant r - \left|r - \big(w_{0,0}(r) - w_{0,0}(0)\big)\right| - \left|w_{0,0}(0)\right| \geqslant r - r\frac{\rho}{32} - 5\rho/8 \geqslant \rho\frac{3}{32},$$

and thus

$$\Big[\inf_{r\in[\rho\frac{3}{4},1]} \left|w_{0,0}(r)\right|\Big]^{-1} \leqslant 1/t. \tag{7.15}$$

Next we calculate the derivative with respect to r. To this end first observe that using Lemma A.2 and dominated convergence one can show that for a.e. $K^{(m,n)}$ the partial derivative $\partial_r W_{p,q}^{m,n}[w](r, K^{(m,n)})$ exists with respect to the operator norm topology and equals $W_{p,q}^{m,n}[\partial_r w](r, K^{(m,n)})$. Thus

$$\operatorname*{ess\,sup}_{K^{(m,n)}} \sup_{r\in[0,1]} \left\| \partial_r W_{p,q}^{m,n}[w]\big(r, K^{(m,n)}\big)\right\|_{\mathrm{op}} \leqslant \frac{\|\partial_r w_{p+m,q+n},\|_{L^\infty(B_1^{m+n};C[0,1])}}{\sqrt{p!q!}}. \tag{7.16}$$

Furthermore,

$$D_r \frac{\overline{\chi}_\rho^2}{w_{0,0}} = -\frac{\overline{\chi}_\rho^2}{w_{0,0}^2}(\partial_r w_{0,0}) + \frac{2\overline{\chi}_\rho \partial_r \overline{\chi}_\rho}{w_{0,0}}$$

and thus for $s + \rho r \in [0, 1]$ we have

$$\left| D_r \frac{\overline{\chi}_\rho^2}{w_{0,0}}(s + \rho r)\right| \leqslant \frac{3}{2}\frac{\rho}{t^2} + \frac{2\|\chi_1'\|_\infty}{t}, \tag{7.17}$$

where we used $\|\partial_r w_{0,0}\|_{C[0,1]} \leqslant 3/2$. Calculating the derivative with respect to r using Leibniz and estimating the resulting expression with the help of (7.12), (7.13) (7.14), (7.16), and (7.17) the inequality (7.11) follows for $k = 0$.

Next we show (7.11) for $k \geqslant 1$. It follows from Lemma F.1(b) that $\beta \mapsto \frac{\overline{\chi}_\rho^2}{w_{0,0}(\beta)}$ is in $C^k(\mathbb{R}, \mathcal{W}_{0,0}^\#)$. We use (D.1) to calculate the derivative of $\overline{\chi}_\rho^2/w_{0,0}(\beta)$ with respect to β,

$$\partial_\beta^l \frac{\overline{\chi}_\rho^2}{w_{0,0}(\beta)} = \sum_{X\in P_l} |X|!(-1)^{|X|} \frac{\overline{\chi}_\rho^2}{(w_{0,0}(\beta))^{|X|+1}} \prod_{x\in X} \partial_\beta^{|x|} w_{0,0}(\beta). \tag{7.18}$$

The derivative in (7.18) is with respect to the $C[0, 1]$ norm. To estimate the right-hand side of (7.18) we use (7.15) that by assumption $\|\partial_\beta^j w_{0,0}(\beta)\|_{C[0,1]} \leqslant 5\rho/8$. It follows that there exists a finite constant, $C_{F,l}$, independent of ρ such that

$$\left\| \partial_\beta^l \frac{\overline{\chi}_\rho^2}{w_{0,0}(\beta)} \right\|_{C[0,1]} \leqslant \frac{C_{F,l}}{t}, \tag{7.19}$$

and $C_{F,0} = 1$. Using (D.1) we find

$$\begin{aligned} &D_r \partial_\beta^l \frac{\overline{\chi}_\rho^2}{w_{0,0}(\beta)} \\ &\quad = \sum_{X \in P_l} |X|!(-1)^{|X|} \frac{\overline{\chi}_\rho}{(w_{0,0}(\beta))^{|X|+1}} D\big(w_{0,0}(\beta), |X|, \overline{\chi}_1, \rho\big) \prod_{x \in X} \partial_\beta^{|x|} w_{0,0}(\beta) \\ &\qquad + \sum_{X \in P_l} |X|!(-1)^{|X|} \frac{\overline{\chi}_\rho}{(w_{0,0}(\beta))^{|X|+1}} \sum_{x \in X} \big(\partial_r \partial_\beta^{|x|} w_{0,0}(\beta)\big) \prod_{x' \in X, x' \neq x} \partial_\beta^{|x'|} w_{0,0}(\beta), \end{aligned} \tag{7.20}$$

where we wrote

$$D(w_{0,0}, m, \overline{\chi}_1, \rho) := \frac{2\partial_r \overline{\chi}_1(\cdot/\rho)}{\rho} - (m+1) \frac{\overline{\chi}_\rho}{w_{0,0}} \partial_r w_{0,0}.$$

We estimate

$$\big\| D(w_{0,0}, m, \overline{\chi}_1, \rho) \big\|_\infty \leqslant \frac{2}{\rho} \|\partial_r \overline{\chi}_1\|_\infty + (m+1) \frac{8}{\rho}, \tag{7.21}$$

where we used that by assumption it follows that $\|\partial_r w_{0,0}\|_\infty \leqslant 3/2$. The derivative in (7.20) is with respect to the $C[0,1]$ norm. Inserting (7.21) into (7.20) we find for $s + \rho r \in [0,1]$

$$\left| D_r \partial_\beta^l \left(\frac{\overline{\chi}_\rho^2}{w_{0,0}(\beta)} (s + \rho r) \right) \right| \leqslant t^{-1} C_{F,l} \big(2\|\partial_r \overline{\chi}_1\|_\infty + (l+1)8 \big) + t^{-1} l C_{F,l}. \tag{7.22}$$

Next observe that $v_{\underline{m},\underline{p},\underline{n},\underline{q}}[\cdot]$ is given as a multilinear expression of kernels $(w_{m,n})_{m+n \geqslant 1}$ and $\frac{\overline{\chi}_\rho}{w_{0,0}}$. It follows from Lemma F.1 that $\beta \mapsto v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w(\beta)]$ is in $C^k(\mathbb{R}; \mathcal{W}_{|\underline{m}|,|\underline{n}|})$ and that Leibniz rule for higher derivatives (6.11) is applicable to calculate derivatives $D_\beta^l v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w(\beta)]$. We thus apply (6.11) and estimate the resulting expression using (7.12). To this end we use

$$\operatorname*{ess\,sup}_{K^{(m,n)}} \sup_{r \in [0,1]} \sum_{s=0}^{1} \big\| \partial_r^s W_{p,q}^{m,n} \big[\partial_\beta^l w\big]\big(r, K^{(m,n)}\big) \big\|_{\mathrm{op}} \leqslant \frac{\|\partial_\beta^l w_{p+m,q+n}\|^\#}{\sqrt{p!q!}}, \tag{7.23}$$

which follows from (7.13) and (7.16). Using (7.19), (7.22), and (7.23) inequality (7.11) now follows from the following observation. The right-hand side of (6.11) contains L^k terms. Each term contains at most k factors involving a derivative. □

Proof of Theorem 7.8. First observe that by Lemma F.1(b)

$$\left[(s,\beta) \mapsto \frac{\overline{\chi}_\rho^2}{w_{0,0}(s,\beta)} \right] \in C^{\omega,k}\big(S \times \mathbb{R}, \mathcal{W}_{0,0}^\#\big). \tag{7.24}$$

It now follows from part (a) of the same lemma that the map $(s, \beta) \to v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w(s, \beta)]$ is in $C^{\omega,k}(S \times \mathbb{R}; \mathcal{W}^{\#}_{|\underline{m}|,|\underline{n}|})$. Using the estimate of Lemma 7.10 one can show the same way as in [16, Theorem 8.1] that $\mathcal{R}^{\#}_{\rho}(w(s, \beta))$ is given as a sum which is uniformly convergent on subsets which constitute an open covering of $\mathbb{R} \times S$ and that the sum is uniformly bounded. This is done in Appendix F. □

Lemma 7.11. *Let $0 < \rho \leqslant 1/4$ and assume $w \in \mathcal{B}^{(k)}(\cdot, \delta, \cdot)$, with $\delta \leqslant \rho/32$. Then $I_\rho \in C^{k,\omega}_B(\mathbb{R} \times D_{1/2})$ and*

$$\sup_{(\beta,z)\in\mathbb{R}\times D_{1/2}} \left|\partial_z I_\rho(\beta, z)\right| \leqslant \frac{16\rho}{15}. \tag{7.25}$$

Moreover, there exists a finite constant C_k depending only on k, such that

$$\max_{1\leqslant s\leqslant k} \sup_{(\beta,z)\in\mathbb{R}\times D_{1/2}} \left|\partial^s_\beta I_\rho(\beta, z)\right| \leqslant C_k \delta. \tag{7.26}$$

Proof. The assumption $w \in \mathcal{B}^{(k)}(\cdot, \delta, \cdot)$ implies that $E_\rho \in C^{k,\omega}(\mathbb{R} \times D_{1/2})$. By this and inequality (7.3) it follows from the inverse function theorem that I_ρ is in $C^{k,\omega}(\mathbb{R} \times D_{1/2})$. Let $(\beta, z) \in \mathbb{R} \times D_{1/2}$. From (7.4) we have

$$E_\rho\big(\beta, I_\rho(\beta, z)\big) = z. \tag{7.27}$$

Differentiating (7.27) with respect to z we find

$$\partial_z I_\rho(\beta, z) = -\frac{1}{\partial_2 E_\rho(\beta, I_\rho(\beta, z))},$$

where ∂_i denotes the derivative with respect to the i-th argument (note that ∂_1 is a real derivative and ∂_2 is a complex derivative). By this and (7.3) we obtain the bound (7.25). Now we show the remaining bounds. Differentiating (7.27) with respect to β, we find

$$\partial_\beta I_\rho(\beta, z) = -\frac{\partial_1 E(\beta, I_\rho(\beta, z))}{\partial_2 E(\beta, I_\rho(\beta, z))}, \tag{7.28}$$

with $E(\beta, z) = \rho E_\rho(\beta, z)$. This and (7.3) show (7.26) for $k = 1$. To show (7.26) for $k \geqslant 2$ we proceed by induction and use that the assumption $w \in \mathcal{B}^{(k)}(\cdot, \delta, \cdot)$ implies

$$\left|\partial^s_1 E(\beta, z)\right| \leqslant \delta \tag{7.29}$$

for all $1 \leqslant s \leqslant k$. Suppose (7.26) holds for $k = n$. We then show that it holds for $k = n + 1$. We differentiate (7.28) with respect to β. Using Leibniz we obtain

$$\partial^{n+1}_\beta I_\rho(\beta, z) = \sum_{p=0}^{n} \binom{n}{p} A_p B_{n-p},$$

where

$$A_p := D_\beta^p \partial_1 E\big(\beta, I_\rho(\beta, z)\big),$$
$$B_p := D_\beta^p i\big(\partial_2 E\big(\beta, I_\rho(\beta, z)\big)\big),$$

with $i(z) := -z^{-1}$. Now using (D.1), we find

$$A_p = \sum_{q=0}^{p} \binom{p}{q} \sum_{X \in P_q} \partial_1^{1+p-q} \partial_2^{|X|} E\big(\beta, I_\rho(\beta, z)\big) \prod_{x \in X} \partial_\beta^{|x|} I_\rho(\beta, z).$$

Using (7.29), analyticity of E_ρ in the second argument, and the induction hypothesis it follows that $|A_p| \leqslant C\delta$ for some finite constant, C, depending only on p. To this end we note that derivatives ∂_2 can be estimated using Cauchy's formula and $\operatorname{Ran} I_\rho \subset D_{17\rho/32}$, which follows from Lemma 7.1. Using (D.1) we find that

$$B_p = \sum_{X \in P_p} (-1)^{|X|+1} |X|! \big(\partial_2 E\big(\beta, I_\rho(\beta, z)\big)\big)^{-|X|-1} \prod_{x \in X} D_\beta^{|x|} \partial_2 E\big(\beta, I_\rho(\beta, z)\big).$$

By (7.3) and (7.29) we now see, similarly as for A_p, that $|B_p| \leqslant C$ for some finite constant C depending only on p. □

Proof of Theorem 7.9. By assumption it follows that $E_\rho \in C^{\omega,k}(S \times D_{1/2} \times \mathbb{R})$. By the inverse function theorem and (7.3) it follows that $I_\rho \in C^{\omega,k}(S \times D_{1/2} \times \mathbb{R})$. Moreover by Lemma 7.1

$$\operatorname{Ran} I_\rho \subset D_{17\rho/32}. \tag{7.30}$$

For $\zeta \in D_{17\rho/32}$ we have

$$\big\| w(s, \zeta, \beta) \big\|_{C^k(\mathbb{R})} \leqslant \big\| w(s, \zeta, \beta) + \zeta \big\|_{C^k(\mathbb{R})} + \|\zeta\|_{C^k(\mathbb{R})} \leqslant \frac{5\rho}{8}. \tag{7.31}$$

Thus we can apply Theorem 7.8 for $w|_{S \times D_{17\rho/32} \times \mathbb{R}}$ and conclude that

$$(s, \zeta, \beta) \mapsto \mathcal{R}_\rho^\#\big(w(s, \zeta, \beta)\big)$$

is in $C^{\omega,k}(S \times D_{17\rho/32} \times \mathbb{R}; \mathcal{W}_\xi^\#)$. By (7.30) it follows from the chain rule that

$$(s, z, \beta) \mapsto \mathcal{R}_\rho\big(w(s, \beta)\big)(z) = \mathcal{R}_\rho^\#\big(w(s, \zeta, \beta)\big)\big|_{\zeta = I_\rho(s, z, \beta)}$$

is in $C^{\omega,k}(S \times D_{1/2} \times \mathbb{R}; \mathcal{W}_\xi^\#)$. □

Theorem 7.7, which is proven in Section 8, allows us to iterate the extended renormalization transformation on the extended balls. Let us introduce the following hypothesis.

($R^{(k)}$) Let ρ, ξ, ϵ_0 be positive numbers such that the contraction property (7.10) holds and $\rho \leqslant 1/4$, $\xi \leqslant 1/4$ and $\epsilon_0 \leqslant \rho/32$.

Recall that by Theorem 7.7 and Theorem 7.6 there exists a nonempty set of parameters for which Hypotheses (R) and ($R^{(k)}$) are satisfied.

Theorem 7.12. *Let $k \in \mathbb{N}_0$. Assume Hypotheses (R) and ($R^{(k)}$). Then for $\epsilon_0 > 0$ and $\rho > 0$ sufficiently small there exist functions*

$$e_{(0)}[\cdot] : \mathcal{B}_0(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2) \to D_{1/2},$$
$$\psi_{(0)}[\cdot] : \mathcal{B}_0(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2) \to \mathcal{F}$$

such that the following hold.

(a) *For all $w \in \mathcal{B}_0(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2)$,*

$$\dim\ker\bigl\{H\bigl(w\bigl(e_{(0)}[w]\bigr)\bigr)\bigr\} \geqslant 1,$$

and $\psi_{(\infty)}[w]$ is a nonzero element in the kernel of $H(w(e_{(0,\infty)}[w]))$.

(b) *If w is symmetric and $-1/2 < z < e_{(0)}[w]$, then $H(w(z))$ is bounded invertible.*

(c) *The function $\psi_{(0)}[\cdot]$ is uniformly bounded with bound*

$$\sup_{w \in \mathcal{B}_0(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2)} \bigl\|\psi_{(0)}[w]\bigr\| \leqslant 4e^4.$$

If $H(w(z)) = H_f - z$, then $\psi_{(0)}[w] = \Omega$.

(d) *Suppose $w \in \mathcal{B}_0^{(k)}(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2)$. Then $\beta \to e_{(0)}[w(\beta)]$ and $\beta \to \psi_{(0)}[w(\beta)]$ are in $C_B^k(\mathbb{R})$ and $C_B^k(\mathbb{R}; \mathcal{F})$, respectively.*

(e) *Let S be an open subset of $\mathbb{C}$. Suppose we are given a mapping $(s, z, \beta) \mapsto w(s, z, \beta)$ in $C_B^{\omega,k}(S \times D_{1/2} \times \mathbb{R}; \mathcal{W}_\xi^\#)$ such that for all $s \in S$ we have $w(s, \cdot, \cdot) \in \mathcal{B}_0^{(k)}(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2)$. Then $s \mapsto (\beta \mapsto e_{(0)}[w(s)(\beta)])$ and $s \mapsto (\beta \mapsto \psi_{(0)}[w(s)(\beta)])$ are in $C_B^\omega(S; C_B^k(\mathbb{R}))$ and $C_B^\omega(S; C_B^k(\mathbb{R}; \mathcal{F}))$, respectively.*

Assumption (R) allows us to iterate the renormalization transformation as follows,

$$\mathcal{B}_0\left(\frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0\right) \xrightarrow{\mathcal{R}_\rho} \mathcal{B}_0\left(\left[\frac{1}{2} + \frac{1}{4}\right]\epsilon_0, \frac{1}{4}\epsilon_0, \frac{1}{4}\epsilon_0\right) \xrightarrow{\mathcal{R}_\rho} \cdots$$
$$\xrightarrow{\mathcal{R}_\rho} \mathcal{B}_0\left(\sum_{l=1}^{n} \frac{1}{2^l}\epsilon_0, \frac{1}{2^n}\epsilon_0, \frac{1}{2^n}\epsilon_0\right) \xrightarrow{\mathcal{R}_\rho} \cdots.$$

For $w \in \mathcal{B}_0(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2)$ and $n \in \mathbb{N}_0$, we define

$$w^{(n)} := \mathcal{R}_\rho^n(w) \in \mathcal{B}_0\left(\epsilon_0, \frac{1}{2^n}\epsilon_0, \frac{1}{2^n}\epsilon_0\right).$$

We introduce the definitions

$$E_{n,\rho}[w] := E_\rho\big[w^{(n)}\big] = \rho^{-1} E[w],$$
$$U_n[w] := U\big[w^{(n)}\big] := \big\{z \in D_{1/2} \,\big|\, \big|E\big[w^{(n)}\big](z)\big| < \rho/2\big\}.$$

By Lemma 7.1 the map

$$J_n[w] := E_{n,\rho}[w] : U_n[w] \to D_{1/2}, \quad z \mapsto E_{n,\rho}[w](z)$$

is an analytic bijection and $J_n[w]^{-1} : D_{1/2} \to U_n[w] \subset D_{1/2}$. For $0 \leqslant n \leqslant m$ we define

$$e_{(n,m)}[w] := J_n[w]^{-1} \circ \cdots \circ J_m[w]^{-1}(0).$$

It has been shown in [2], see also [16], that the following limit exists

$$e_{(n,\infty)}[w] := \lim_{m\to\infty} e_{(n,m)}[w]. \tag{7.32}$$

We define the vectors in $\mathcal{F}$, of

$$\psi_{(n,m)}[w] = Q_n[w]\varGamma_\rho^* Q_{n+1}[w]\varGamma_\rho^* \cdots Q_{m-1}\Omega,$$

with

$$Q_n[w] = \chi_\rho - \overline{\chi}_\rho\big(H_n[w]\big)^{-1}_{\overline{\chi}_\rho}\overline{\chi}_\rho W_n[w]\chi_\rho,$$

where

$$H_n[w] := H\big(w^{(n)}\big(e_{(n,\infty)}[w]\big)\big),$$
$$T_n[w] := w^{(n)}_{0,0}\big(e_{(n,\infty)}[w]\big)(H_f),$$
$$W_n[w] := H_n[w] - T_n[w].$$

It has been shown in [2], see also [16], that the following limit exists

$$\psi_{(n,\infty)}[w] := \lim_{m\to\infty} \psi_{(n,m)}[w] \tag{7.33}$$

and that $H_n[w]\psi_{(n,\infty)}[w] = 0$. This implies part (a) of Theorem 7.12, with $e_{(0)}[w] = e_{(0,\infty)}[w]$ and $\psi_{(0)}[w] = \psi_{(0,\infty)}[w]$. Part (b) has been shown in [16]. Moreover, in [16], the bound $\sup_{w\in\mathcal{B}_0(\epsilon_0/2,\epsilon_0/2,\epsilon_0/2)} \|\psi_{(0)}[w]\| \leqslant 4e^4$ was shown. The second part of (c) is a direct consequence of the definition of $\psi_{(0)}$. Now let us show (d). Assumption ($R^{(k)}$) allows us to iterate the renormalization transformation as follows,

$$\mathcal{B}_0^{(k)}\left(\frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0\right) \xrightarrow{\mathcal{R}_\rho} \mathcal{B}_0^{(k)}\left(\left[\frac{1}{2}+\frac{1}{4}\right]\epsilon_0, \frac{1}{4}\epsilon_0, \frac{1}{4}\epsilon_0\right) \xrightarrow{\mathcal{R}_\rho} \cdots$$
$$\xrightarrow{\mathcal{R}_\rho} \mathcal{B}_0^{(k)}\left(\sum_{l=1}^{n}\frac{1}{2^l}\epsilon_0, \frac{1}{2^n}\epsilon_0, \frac{1}{2^n}\epsilon_0\right) \xrightarrow{\mathcal{R}_\rho} \cdots.$$

We view $w \in \mathcal{B}_0^{(k)}(\frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0)$ as a function of β. Now $e_{(n,m)}[w(\beta)]$ and $\psi_{(n,m)}[w(\beta)]$ are functions of β as well as their limits as m tends to infinity. First we show that $e_{(n,m)}[w(\beta)] \to e_{(n,\infty)}[w(\beta)]$ converges uniformly in $C^k(\mathbb{R})$ for any n. This will then imply that $e_{(n,\infty)}$ is in C^k. We introduce for $\gamma, \delta > 0$ the balls

$$\mathcal{E}(\gamma,\delta) := \left\{ f \in C^k(\mathbb{R};\mathbb{C}) \;\middle|\; \|f\|_\infty < \gamma,\ \max_{1\leqslant l\leqslant k}\left\|\partial_1^l f\right\|_\infty < \delta \right\}.$$

Let $w \in \mathcal{B}^{(k)}(\cdot,\epsilon,\cdot)$ with $\epsilon \leqslant \rho/32$. We define a mapping $K[w]$ on $\mathcal{E}(1/2,\delta)$ by

$$\big(K[w](f)\big)(\beta) := I_\rho\big(\beta, f(\beta)\big).$$

From Lemma 7.1 it follows that $K[w](\mathcal{E}(1/2,\delta)) \subset \mathcal{E}(3/8,\infty)$. Using Faa di Bruno's formula we find

$$D_\beta^s I_\rho\big(\beta, f(\beta)\big) = \sum_{p=0}^{s}\binom{s}{p}\sum_{X\in P_p}\partial_1^{s-p}\partial_2^{|X|} I_\rho\big(\beta, f(\beta)\big)\prod_{x\in X}\partial_\beta^{|x|} f(\beta).$$

We use this to estimate the following difference

$$\begin{aligned} &D_\beta^s I_\rho\big(\beta, f(\beta)\big) - D_\beta^s I_\rho\big(\beta, g(\beta)\big) \\ &\quad= \sum{}' \partial_1^{s-p}\partial_2^{|X|}\big[I_\rho\big(\beta, f(\beta)\big) - I_\rho\big(\beta, g(\beta)\big)\big]\prod_{x\in X}\partial_\beta^{|x|} f(\beta) \\ &\qquad+ \sum{}' \partial_1^{s-p}\partial_2^{|X|} I_\rho\big(\beta, g(\beta)\big)\left[\prod_{x\in X}\partial_\beta^{|x|} f(\beta) - \prod_{x\in X}\partial_\beta^{|x|} g(\beta)\right], \end{aligned} \tag{7.34}$$

where we used the abbreviation $\sum{}' = \sum_{p=0}^{s}\binom{s}{p}\sum_{X\in P_p}$. To estimate (7.34) we use that

$$\big|I_\rho\big(\beta, f(\beta)\big) - I_\rho\big(\beta, g(\beta)\big)\big| \leqslant \sup_{z\in D_{1/2}}\big|\partial_2 I_\rho(\beta,z)\big|\big|f(\beta) - g(\beta)\big| \tag{7.35}$$

and that for $f, g \in \mathcal{E}(1/2, 1)$ we have

$$\left|\prod_{x\in X}\partial_\beta^{|x|} g(\beta) - \prod_{x\in X}\partial_\beta^{|x|} f(\beta)\right| \leqslant C_{|X|}\|f-g\|_{C^k(\mathbb{R})}, \tag{7.36}$$

for some constant depending only on the number of elements of the partition X. On the other hand by Lemma 7.11 there exists a constant C such that for all $(\beta, z) \in \mathbb{R}\times D_{3/8}$, we have

$$\max_{1\leqslant l'\leqslant k+1}\left|\partial_z^{l'} I_\rho(\beta,z)\right| \leqslant C\rho, \qquad \max_{1\leqslant l\leqslant k}\max_{0\leqslant l'\leqslant k+1}\left|\partial_\beta^{l}\partial_z^{l'} I_\rho(\beta,z)\right| \leqslant C\epsilon, \tag{7.37}$$

where we used the analyticity of I_ρ in its second argument. Using (7.37)–(7.35) to estimate (7.34) it follows that for ϵ and ρ sufficiently small we have

$$K[w]\big(\mathcal{E}(3/8,1)\big) \subset \mathcal{E}(3/8,1), \qquad \big\|K[w]f - K[w]g\big\|_{C^k(\mathbb{R})} \leqslant \frac{1}{2}\|f-g\|_{C^k(\mathbb{R})} \tag{7.38}$$

for all $f,g \in \mathcal{E}(3/8,1)$. For the sequence of kernels $w^{(l)} \in \mathcal{B}_0^{(k)}(\cdot, 2^{-l}\epsilon_0, \cdot)$ define $K_l := K[w^{(l)}]$. By definition we have

$$e_{(n,m)} = K_n \circ K_{n+1} \circ \cdots \circ K_m(0),$$

where 0 denotes the zero function. Thus if we choose ρ and ϵ_0 sufficiently small, then it follows from (7.38) that

$$\|e_{(n,m)} - e_{(n,m+l)}\|_{C^k(\mathbb{R})} \leqslant 2^{-(m-n)-1},$$

and thus $e_{(n,m)} \to e_{(n,\infty)}$ uniformly in $C^k(\mathbb{R})$ as $m \to \infty$ for any n. Since $e_{(n,n)} = 0$ it follows that

$$\|e_{(n,\infty)}\|_{C^k(\mathbb{R})} \leqslant 2. \tag{7.39}$$

Thus $e_{(n,m)}[w(\beta)] \to e_{(n,\infty)}[w(\beta)]$ converges uniformly in $C^k(\mathbb{R})$ for any n. Next we show that the ground state eigenvector $\psi_{(0,\infty)}[w(\beta)]$ is C^k in β. For notational compactness we write $\psi_{(n,m)}(\beta)$ for $\psi_{(n,m)}[w(\beta)]$ and similarly $e_{(n,m)}(\beta)$ for $e_{(n,m)}[w(\beta)]$. We set $\tilde{W}_n(\beta,z) := W[w^{(n)}(\beta,z)]$ with $w^{(n)}(\beta,z) = w^{(n)}(\beta)(z)$. Observe that with this notation $W_n(\beta) := W_n[w(\beta)] = \tilde{W}_n(\beta, e_{(0,\infty)}(\beta))$. We use analogous definitions for T_n, W_n, and Q_n. Let $(\beta,z) \in \mathbb{R}\times D_{1/2}$. We estimate the derivatives with respect to β of

$$\psi_{(n,m+1)} - \psi_{(n,m)} = Q_n \Gamma_\rho^* Q_{n+1}\cdots Q_{m-1}\Gamma_\rho^*(Q_m - \chi_\rho)\Omega.$$

Let

$$A_n := (T_n + \overline{\chi}_\rho W_n \overline{\chi}_\rho)|_{\operatorname{Ran}\overline{\chi}_\rho}.$$

Observe that

$$\big\|A_n^{-1}\big\| \leqslant 16/\rho. \tag{7.40}$$

This can be seen using $\|W_n\| \leqslant 2^{-n-1}\epsilon_0 \leqslant \rho/16$, see [2,16] for details. We have already proven estimates of the form

$$\big|\partial_\beta^l e_{(n,\infty)}\big| \leqslant c_l$$

for $n \in \mathbb{N}_0$ which we will use without comment. We also have estimates of the form

$$\left\| w_{\geqslant 1}^{(n)} \right\|_{\xi}^{(k)} \leqslant \frac{\epsilon_0}{2^n}, \tag{7.41}$$

$$\left\| w_{0,0}^{(n)} \right\|^{(k)} \leqslant \frac{\epsilon_0}{2^n} + \frac{1}{2} + \epsilon_0 + 1 \leqslant 2\epsilon_0 + \frac{3}{2}. \tag{7.42}$$

By the inequality given in Theorem 5.3 and the differentiability of the integral kernels it follows that T_n and W_n are differentiable functions of β (w.r.t. the operator norm topology) with uniformly bounded derivatives, and hence also Q_n and $\psi_{(n,m)}$. We have

$$D_\beta^l (Q_n - \chi_\rho) = - \sum_{l_1 + l_2 = l} \frac{l!}{l_1! l_2!} \overline{\chi}_\rho \left[D_\beta^{l_1} A_n^{-1} \right] \overline{\chi}_\rho D_\beta^{l_2} W_n \chi_\rho .$$

It is straightforward to verify that for all $l \leqslant k$,

$$\left\| D_\beta^l A_n^{-1} \right\| \leqslant C.$$

To see this we note that taking inverses is a differentiable mapping with respect to the operator norm topology, the first k derivatives of T_n and W_n with respect to β are uniformly bounded, and (7.40). Since

$$D_\beta W_n|_\beta = \left(\frac{\partial \tilde{W}_n}{\partial \beta} + \partial_z \tilde{W}_n \partial_\beta e_{(n,\infty)} \right) \bigg|_{(\beta, e_{(n,\infty)}(\beta))}$$

it is clear that if we can show that for $l, l' \leqslant k$

$$\left\| \partial_\beta^l \partial_z^{l'} \tilde{W}_n \big(\beta, e_{(n,\infty)}(\beta) \big) \right\| \leqslant \frac{c_l}{2^n}, \tag{7.43}$$

it will follow that for $l \leqslant k$,

$$\left\| D_\beta^l (Q_n - \chi_\rho) \right\| \leqslant \frac{c_l}{2^n}. \tag{7.44}$$

The Cauchy integral formula gives

$$\partial_\beta^l \partial_z^{l'} \tilde{W}_n(\beta, z) = \frac{l'!}{2\pi i} \int_{|\zeta| = 1/2 - \epsilon} \frac{\partial_\beta^l \tilde{W}_n(\beta, \zeta)}{(\zeta - z)^{l'+1}} \, d\zeta .$$

If $|z| < 1/2 - \epsilon$. Since $e_{(n,\infty)} \in D_{5\rho/8}$ we obtain from (7.41)

$$\left\| \big(\partial_\beta^l \partial_z^{l'} \tilde{W}_n \big) \big(\beta, e_{(n,\infty)}(\beta) \big) \right\| \leqslant \frac{(l')! \epsilon_0}{2^{n+1} (1/2 - 5\rho/8)^{l'+1}} \leqslant \frac{c}{2^n}.$$

Thus we have shown (7.44). Using this inequality we find for $l \leqslant k$, with $p = m - n + 1$,

$$
\begin{aligned}
&\left\| D_\beta^l (\psi_{(n,m+1)} - \psi_{(n,m)}) \right\| \\
&\quad = \left\| D_\beta^s Q_n \Gamma_\rho^* Q_{n+1} \cdots Q_{m-1} \Gamma_\rho^* (Q_m - \chi_\rho) \Omega \right\| \\
&\quad = \sum_{\underline{l} \in \mathbb{N}_0^p:\ |\underline{l}| = l} \frac{l!}{\underline{l}!} \big(D_\beta^{l_1} Q_n\big) \Gamma_\rho^* \cdots \big(D_\beta^{l_{p-1}} Q_{m-1}\big) \Gamma_\rho^* \big(D_\beta^{l_p} (Q_m - \chi_\rho)\big) \Omega \\
&\quad \leqslant (m-n+1)^l \prod_{j=n}^{m-1} \left(1 + \frac{C}{2^j}\right) \frac{C}{2^m} \leqslant (m+1)^l C 2^{-m} \exp\left(C \sum_{j=1}^\infty 2^{-j} \right).
\end{aligned} \tag{7.45}
$$

This implies that $\psi_{(n,m)}[w(\beta)] \to \psi_{(n,\infty)}[w(\beta)]$ converges uniformly in $C^k(\mathbb{R})$ for any n. Since $\psi_{(n,n)} = \Omega$, it follows that

$$
\|\psi_{(n,\infty)}\|_{C^k(\mathbb{R})} \leqslant 1 + C e^{2C} \sum_{m=0}^\infty (m+1)^k 2^{-m}. \tag{7.46}
$$

Now (d) follows.

To show (e) first observe by Theorem 7.9 $(s, z, \beta) \mapsto w^{(n)}(s, z, \beta) = \mathcal{R}_\rho^n(w(s, \beta))(z)$ is in $C_B^{\omega,k}(S \times D_{1/2} \times \mathbb{R}; \mathcal{W}_\xi^\#)$. It follows by (7.3) that $J_n^{-1} \in C_B^{\omega,k}(S \times D_{1/2} \times \mathbb{R})$. Thus $e_{(n,m)} \in C_B^{\omega,k}(S \times \mathbb{R}) \cong C_B^\omega(S; C_B^k(\mathbb{R}))$. It follows from the uniform convergence established in (d) that $e_{(n,\infty)} \in C_B^\omega(S; C_B^k(\mathbb{R}))$. It now follows from the bound in Theorem 5.3 and the chain rule that $H_n[w], W_n[w]$ are in $C_B^{\omega,k}(S \times \mathbb{R}; \mathcal{B}(\mathcal{H}_{\mathrm{red}}))$. Since $H_n[w]$ is bounded invertible on the range of $\overline{\chi}_\rho$ it follows from the bound (7.40) that $Q_n[w] \in C_B^{\omega,k}(S \times \mathbb{R}; \mathcal{B}(\mathcal{H}_{\mathrm{red}}))$. Thus $\psi_{(n,m)} \in C_B^{\omega,k}(S \times \mathbb{R}; \mathcal{H}_{\mathrm{red}}) \cong C_B^\omega(S; C^k(\mathbb{R}; \mathcal{H}_{\mathrm{red}}))$. By the uniform convergence established in (7.45) it follows that $\psi_{(n,\infty)} \in C_B^\omega(S; C_B^k(\mathbb{R}; \mathcal{H}_{\mathrm{red}}))$.

8. Contraction estimate

In this section we prove Theorem 7.7. By Lemma 7.10 we know that there exists a constant C_θ which is greater than 1 such that for $w \in \mathcal{B}^{(\#,k)}(\rho/32, 5\rho/8, \rho/32)$. We have

$$
\max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^l v_{\underline{m},\underline{p},\underline{n},\underline{q}}[w(\beta)] \right\|^\# \leqslant C_\theta \left(\frac{16}{\rho}\right)^{L-1} \prod_{l=1}^L \frac{\max_{0 \leqslant l' \leqslant k} \|\partial_\beta^{l'} w_{m_l+p_l, n_l+q_l}(\beta)\|^\#}{\sqrt{p_l! q_l!}}. \tag{8.1}
$$

The crucial point of Eq. (8.1) is that ρ^{-1} occurs to a power of at most $L-1$. This allows us to prove Theorem 7.7 using similar estimates as in the proof of [16, Theorem 9.1] or [2, Theorem 3.8]. There is an additional complication due to the β dependence of the reparameterization of the spectral parameter. We introduce the constant $D_k = \sum_{l=0}^k \binom{k}{l} \sum_{X \in P_l} 1$.

Let $0 < \rho \leqslant (k! 16 C_\theta D_k C_k^k)^{-1}$, $0 < \xi \leqslant \min(1/2, (\frac{C_\theta}{64} \tau C_k^k D_k)^{-1/4})$, and $0 < \epsilon_0 \leqslant \min(\frac{\rho}{32}, \frac{1}{D_k 8^{k+1} k! C_k^k})$.

We assume that $w \in \mathcal{B}^{(k)}(\epsilon, \delta_1, \delta_2)$ with $\epsilon, \delta_1, \delta_2 \in [0, \epsilon_0)$. Then the following estimates hold.

Step 1. We have

$$\left\|\mathcal{R}_\rho(w)_{\geqslant 2}\right\|_\xi^{(k)} \leqslant \frac{1}{2}\|w_{\geqslant 2}\|_\xi^{(k)}.$$

By definition $(\mathcal{R}_\rho w)(\beta, z) = \mathcal{R}_\rho^{\#}(w(\beta, I_\rho(\beta, z)))$. Taking the derivative with respect to β we obtain

$$D_\beta^l(\mathcal{R}_\rho w)(\beta, z) = \partial_\beta^l \mathcal{R}_\rho^{\#}\big(w(\beta,\zeta)\big)\big|_{\zeta = I_\rho(\beta,z)} + \sum_{p=1}^{l}\binom{l}{p}\sum_{X\in P_p}\partial_\beta^{l-p}\partial_\zeta^{|X|}\mathcal{R}_\rho^{\#}\big(w(\beta,\zeta)\big)\prod_{x\in X}\partial_\beta^{|x|}I_\rho(\beta,z)|_{\zeta=I_\rho(\beta,z)}. \tag{8.2}$$

Let us first estimate the first term on the right-hand side. To this end let $u \in D_{19\rho/32}$. Then $w(\beta,u) \in \mathcal{B}^{(\#,k)}(\rho/32, 5\rho/8, \rho/32)$ as the following estimate shows

$$\left\|w_{0,0}(\cdot,u)\right\|_{C^k(\mathbb{R})} \leqslant \left\|w_{0,0}(\cdot,u)+u\right\|_{C^k(\mathbb{R})} + |u| \leqslant 5\rho/8.$$

By (7.5) we find for $M + N \geqslant 2$,

$$\begin{aligned}
&\left\|\partial_\beta^l \mathcal{R}_\rho^{\#}\big(w(\beta,u)\big)_{M,N}\right\|^{\#} \\
&\quad\leqslant \sum_{L=1}^{\infty}\sum_{\substack{(\underline{m},\underline{p},\underline{n},\underline{q})\in\mathbb{N}_0^{4L}:\\ |\underline{m}|=M,|\underline{n}|=N,m_l+p_l+n_l+q_l\geqslant 1}}\rho^{|\underline{m}|+|\underline{n}|-1} \\
&\qquad\times\prod_{l=1}^{L}\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\left\|\partial_\beta^l v_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[w(\beta,u)\big]\right\|^{\#}.
\end{aligned}$$

Inserting this below and using (8.1), we find with $\tau := 16/\rho$,

$$\begin{aligned}
&\left\|\partial_\beta^l \mathcal{R}_\rho^{\#}\big(w(\beta,u)\big)_{\geqslant 2}\right\|_\xi^{\#} \\
&\quad= \sum_{M+N\geqslant 2}\xi^{-(M+N)}\max_{0\leqslant l\leqslant k}\left\|\partial_\beta^l \mathcal{R}_\rho^{\#}\big(w(\beta,u)\big)_{M,N}\right\|^{\#} \\
&\quad\leqslant \sum_{L=1}^{\infty}\sum_{\substack{(\underline{m},\underline{p},\underline{n},\underline{q})\in\mathbb{N}_0^{4L}:\\ |\underline{m}|+|\underline{n}|\geqslant 2,m_l+p_l+n_l+q_l\geqslant 1}}\rho^{-1}(2\rho)^{|\underline{m}|+|\underline{n}|}(2\xi)^{-(|\underline{m}|+|\underline{n}|)}C_\theta\tau^{L-1} \\
&\qquad\times\prod_{l=1}^{L}\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\frac{\max_{0\leqslant l'\leqslant k}\|\partial_\beta^{l'}w_{m_l+p_l,n_l+q_l}(\beta,u)\|^{\#}}{\sqrt{p_l!q_l!}}\right\} \\
&\quad\leqslant \frac{C_\theta}{16}[2\rho]^2\sum_{L=1}^{\infty}\tau^L\sum_{\substack{(\underline{m},\underline{p},\underline{n},\underline{q})\in\mathbb{N}_0^{4L}:\\ m_l+p_l+n_l+q_l\geqslant 1}}
\end{aligned}$$

$$\times \prod_{l=1}^{L} \left\{ \binom{m_l + p_l}{p_l} \binom{n_l + q_l}{q_l} \xi^{p_l+q_l} 2^{-(m_l+n_l)} \xi^{-(m_l+p_l+n_l+q_l)} \right.$$

$$\left. \times \max_{0 \leqslant l' \leqslant k} \left\| \partial_\beta^{l'} w_{m_l+p_l,n_l+q_l}(\beta, u) \right\|^{\#} \right\}$$

$$\leqslant \frac{C_\theta}{4} \rho^2 \sum_{L=1}^{\infty} \tau^L \left[\sum_{m+p+n+q \geqslant 1} \binom{m+p}{p} \binom{n+q}{q} \xi^{p+q} 2^{-(m+n)} \xi^{-(m+p+n+q)} \right.$$

$$\left. \times \max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^{l} w_{m+p,n+q}(\beta, u) \right\|^{\#} \right]^L$$

$$\leqslant \frac{C_\theta}{4} \rho^2 \sum_{L=1}^{\infty} \tau^L \left[\sum_{l+k \geqslant 1} \xi^{-(l+k)} \max_{0 \leqslant l' \leqslant k} \left\| \partial_\beta^{l'} w_{l,k}(\beta, u) \right\|^{\#} \right]^L$$

$$\leqslant \frac{C_\theta}{4} \rho^2 \sum_{L=1}^{\infty} \tau^L \left(\|w_{\geqslant 2}\|_\xi^{(k)} \right)^L$$

$$\leqslant 8 C_\theta \rho \|w_{\geqslant 2}\|_\xi^{(k)}, \tag{8.3}$$

where in the third last inequality we used the binomial formula and $0 < \xi \leqslant 1/2$ and we used $\tau \|w_{\geqslant 2}\|_\xi^{(k)} \leqslant 1/2$ in the last inequality. Now we estimate the terms involving derivatives with respect to ζ. By Cauchy we have for $\zeta \in U[w] \subset D_{17\rho/32}$

$$\partial_\beta^l \partial_\zeta^s \mathcal{R}_\rho^{\#}\big(w(\beta, \zeta)\big) = \frac{s!}{2\pi i} \int_{|\mu| = 18\rho/32} \frac{\partial_\beta^l \mathcal{R}_\rho^{\#}(w(\beta, \mu))}{(\mu - \zeta)^{s+1}} \, d\mu.$$

Using this and (8.3), we obtain the bound

$$\left\| \partial_\zeta^s \big(\mathcal{R}_\rho^{\#} w \big)_{\geqslant 2} \right\|_\xi^{(k)} \leqslant \left(\frac{32}{\rho} \right)^s s! 8 C_\theta \rho \|w_{\geqslant 2}\|_\xi^{(k)}.$$

Now by Lemma 7.11 we know that for $1 \leqslant l \leqslant k$ there exists a finite constant C_k such that

$$\sup_{(\beta,z) \in \mathbb{R} \times D_{1/2}} \left| \partial_\beta^l I_\rho(\beta, z) \right| \leqslant C_k \frac{\rho}{32}.$$

This and (8.2) imply that the ρ's cancel out. Collecting the above estimates we arrive at the bound

$$\left\| (\mathcal{R}_\rho w)_{\geqslant 2} \right\|_\xi^{(k)} \leqslant k! 8 C_\theta \rho D_k C_k^k \|w_{\geqslant 2}\|_\xi^{(k)}.$$

Step 2.

$$\sup_{z\in D_{1/2}} \big\| \partial_r (\mathcal{R}_\rho w)_{0,0}(z) - 1 \big\|_{C^k(\mathbb{R};C_B[0,1])}$$
$$\leqslant \sup_{z\in D_{1/2}} \big\| \partial_r w_{0,0}(z) - 1 \big\|_{C^k(\mathbb{R};C_B[0,1])} + \frac{1}{4} \sup_{z\in D_{1/2}} \big\| w_{0,0}(0,z) + z \big\|_{C^k(\mathbb{R})} + \frac{1}{4} \| w_{\geqslant 1} \|_\xi^{(k)}.$$

By (7.6) we have

$$\partial_r \big(\mathcal{R}_\rho w(\beta)\big)_{0,0}(z,r) - 1 = (\partial_r w_{0,0})\big(\beta, I_\rho(\beta,z), \rho r\big) - 1 + \partial_r T\big[w\big(\beta, I_\rho(\beta,z)\big)\big](r), \tag{8.4}$$

where we defined

$$T[w] := \rho^{-1}(-1)^{L-1} \sum_{L=2}^{\infty} \sum_{\substack{(\underline{p},\underline{q})\in\mathbb{N}_0^{2L}:\\ p_l+q_l\geqslant 1}} v_{\underline{0},\underline{p},\underline{0},\underline{q}}[w].$$

We need to estimate the derivative with respect to β. For the first term in (8.4) we find for $1 \leqslant l \leqslant k$, using (D.1)

$$\begin{aligned} &D_\beta^l (\partial_r w_{0,0})\big(\beta, I_\rho(\beta,z), \rho r\big) \\ &\quad = \partial_\beta^l (\partial_r w_{0,0})\big(\beta, I_\rho(\beta,z), \rho r\big) \\ &\qquad + \sum_{p=1}^{l} \binom{l}{p} \sum_{X\in P_p} \partial_\beta^{l-p} \partial_\zeta^{|X|} (\partial_r w_{0,0})(\beta,\zeta,\rho r)|_{\zeta=I_\rho(\beta,z)} \prod_{x\in X} \partial_\beta^{|x|} I_\rho(\beta,z). \end{aligned} \tag{8.5}$$

We use analyticity, Cauchy, and that $\zeta = I_\rho(\beta,z) \in D_{3/8}$ to estimate the derivatives with respect to the spectral parameter. We have

$$\partial_\zeta^s \big((\partial_r w_{0,0})(\beta,\zeta,\rho r) - 1\big) = \lim_{\eta\downarrow 0} \frac{s!}{2\pi i} \int_{|\mu|=1/2-\eta} \frac{(\partial_r w_{0,0})(\beta,\mu,\rho r) - 1}{(\mu-\zeta)^{s+1}}\, d\mu.$$

This yields for $1 \leqslant l \leqslant k$ or $0 \leqslant l \leqslant k$ and $1 \leqslant s$,

$$\big|\partial_\beta^l \partial_\zeta^s (\partial_r w_{0,0})(\beta,\zeta,\rho r)\big| \leqslant 8^s s! a, \quad \forall \zeta \in D_{3/8}, \tag{8.6}$$

where $a := \|\partial_r w_{0,0} - 1\|_{C^k(\mathbb{R};C_B[0,1])}$. Using estimate (8.6) and the estimate of Lemma 7.11 to bound the last line of (8.5) we find

$$\begin{aligned} &\sup_{0\leqslant l\leqslant k} \big| D_\beta^l \big((\partial_r w_{0,0})\big(\beta, I_\rho(\beta,z), \rho r\big) - 1\big)\big| \\ &\quad \leqslant \|\partial_r w_{0,0} - 1\|_{C^k(\mathbb{R};C_B[0,1])} + D_k 8^k k! C_k^k a \big\| w_{0,0}(0,z) + z \big\|_{C^k(\mathbb{R})}. \end{aligned} \tag{8.7}$$

The second term in (8.4) is estimated as follows. For $u \in D_{19\rho/32}$ and $0 \leqslant l \leqslant k$ we estimate

$$\begin{aligned}\left|\partial_\beta^l \partial_r T\left[w(\beta,u)\right](r)\right| &\leqslant \rho^{-1}\sum_{L=2}^{\infty} C_\theta \tau^{L-1} \sum_{\substack{(\underline{p},\underline{q})\in\mathbb{N}_0^{2L}:\\ p_l+q_l\geqslant 2}} \prod_{l=1}^{L} \frac{\max_{0\leqslant l'\leqslant k}\|\partial_\beta^{l'} w_{p_l,q_l}(\beta,u)\|^{\#}}{\sqrt{p_l!q_l!}} \\ &\leqslant \frac{C_\theta}{16}\sum_{L=2}^{\infty}\left[\tau\xi^2\right]^L\left[\sum_{p+q\geqslant 2}\xi^{-(p+q)}\max_{0\leqslant l'\leqslant l}\left\|\partial_\beta^{l'} w_{p,q}(\beta,u)\right\|^{\#}\right]^L \\ &\leqslant \frac{C_\theta}{16}\xi^4\sum_{L=2}^{\infty}\left[\tau\|w_{\geqslant 2}\|_\xi^{(k)}\right]^L \\ &\leqslant \frac{C_\theta}{16}\xi^4\tau\|w_{\geqslant 2}\|_\xi^{(k)} \end{aligned} \tag{8.8}$$

where in the last estimate we used $\tau\|w_{\geqslant 1}\|_\xi^{(k)} \leqslant 1/2$. Now using a contour estimate as in Step 1 one can show that

$$\left\|D_\beta^l \partial_r T\left[w\left(\beta, I_\rho(\beta,z)\right)\right]\right\|_{C[0,1]} \leqslant k!\frac{C_\theta}{16}\xi^4\tau C_k^k D_k\|w_{\geqslant 2}\|_\xi^{(k)}. \tag{8.9}$$

Now estimates (8.7) and (8.9) yield Step 2.

Step 3.

$$\sup_{z\in D_{1/2}}\left\|(\mathcal{R}_\rho w)_{0,0}(z,0)+z\right\|_{C^k(\mathbb{R})} \leqslant \frac{1}{4}\|w_{\geqslant 1}\|_\xi^{(k)}.$$

By (7.6) we have

$$\left(\mathcal{R}_\rho w(\beta)\right)_{0,0}(z,0)+z = T\left[w\left(\beta, I_\rho(\beta,z)\right)\right](0).$$

We estimate for $u\in D_{19\rho/32}$ and $0\leqslant l\leqslant k$ the same way as (8.8)

$$\left|\partial_\beta^l T\left[w(\beta,u)\right](0)\right| \leqslant \frac{C_\theta}{16}\xi^4\tau\|w_{\geqslant 1}\|_\xi^{(k)}.$$

As above one calculates the derivative with respect to β and estimates the derivatives with respect to the spectral parameter using a contour integral as in Step 1. As a result

$$\sup_{z\in D_{1/2}}\left\|(\mathcal{R}_\rho w)_{0,0}(z,0)+z\right\|_{C^k(\mathbb{R})} \leqslant k!\frac{C_\theta}{16}\xi^4\tau C_k^k D_k\|w_{\geqslant 2}\|_\xi^{(k)}.$$

Step 3 now follows.

9. Main theorem

In this section, we prove Theorem 2.1, the main result of this paper. Its proof is based on Theorems 6.1 and 7.12.

Proof of Theorem 2.1. Choose ρ, ξ, ϵ_0 such that the assertions of Theorem 7.12 hold. Choose g_0 such that the conclusions of Theorem 6.1 hold for $\delta_1 = \delta_2 = \delta_3 = \epsilon_0/2$. Let $g \in D_{g_0}$. It follows from Theorem 7.12(a) that $\psi_{(0)}[w^{(0)}(g,\beta)]$ is a nonzero element in the kernel of $H^{(0)}_{g,\beta}(e_{(0)}[w^{(0)}(g,\beta)])$. From the Feshbach property, Theorem B.2, it follows that

$$\psi_\beta(g) := Q_{\chi^{(I)}}\big(g, \beta, e_{(0)}\big[w^{(0)}(g,\beta)\big]\big)\psi_{(0)}\big[w^{(0)}(g,\beta)\big] \tag{9.1}$$

is nonzero and an eigenvector of $H_{g,\beta}$ with eigenvalue $E_\beta(g) := E_{\mathrm{at}} + e_{(0)}[w^{(0)}(g,\beta)]$.

By Theorem 6.1, we know that $g \mapsto w^{(0)}(g,\cdot,\cdot)$ is an analytic $\mathcal{W}^{(k)}_\xi$-valued function, with values in the ball $\mathcal{B}^{(k)}(\epsilon_0/2, \epsilon_0/2, \epsilon_0/2)$. By Theorem 7.12(d) it follows that the functions $g \mapsto \psi_{(0)}[w^{(0)}(g,\cdot)]$ and $g \mapsto E_{(\cdot)}(g)$ are in $C^\omega_B(D_{g_0}; C^k_B(\mathbb{R}; \mathcal{F}))$ and $C^\omega_B(D_{g_0}; C^k_B(\mathbb{R}))$, respectively. From Theorem 6.1 we know that the function $(g,z) \mapsto Q_{\chi^{(I)}}(g,\cdot,z)$ is in $C^\omega_B(D_{g_0} \times D_{1/2}; C^k_B(\mathbb{R}; \mathcal{B}(\mathcal{H}_{\mathrm{red}}; \mathcal{H})))$. It now follows from (9.1) that $g \mapsto \psi_{(\cdot)}(g)$ is in $C^\omega_B(D_{g_0}; C^k_B(\mathbb{R}; \mathcal{H}))$. By possibly restricting to a smaller ball than D_{g_0} we can ensure that the projection operator

$$P_\beta(g) := \frac{|\psi_\beta(g)\rangle\langle\psi_\beta(\overline{g})|}{\langle\psi_\beta(\overline{g}), \psi_\beta(g)\rangle} \tag{9.2}$$

is well defined for all $(g,\beta) \in D_{g_0} \times \mathbb{R}$, which is shown as follows. First observe that the denominator of (9.2) is for each β an analytic complex-valued function of g. By Theorem 7.12(c) we have $\langle\psi_\beta(0), \psi_\beta(0)\rangle = 1$. If we estimate the remainder of the Taylor expansion of the denominator of (9.2) using analyticity and the uniform bound on $\psi_{(\cdot)}$, it follows, by possibly choosing g_0 smaller but still positive, that there exists a positive constant c_0 such that $|\langle\psi_\beta(\overline{g}), \psi_\beta(g)\rangle| \geqslant c_0$ for all $|g| \leqslant g_0$. Using already established properties of $\psi_\beta(g)$, it follows from (9.2) that $g \mapsto P_{(\cdot)}(g)$ is in $C^\omega_B(D_{g_0}; C^k_B(\mathbb{R}; \mathcal{B}(\mathcal{H})))$. If $g \in D_{g_0} \cap \mathbb{R}$, then by definition (9.2) we see that $P_\beta(g)^* = P_\beta(\overline{g})$. The kernel $w^{(0)}(g,\beta)$ is symmetric for $g \in D_{g_0} \cap \mathbb{R}$, see Theorem 6.1. Exactly the same way as in the proof of [17, Theorem 2.1] one can show that $E_\beta(g) = \inf \sigma(H_{g,\beta})$ for real $g \in D_{g_0} \cap \mathbb{R}$. □

Proof of Corollary 2.4. We use Cauchy's formula. For any positive r which is less than g_0, we have

$$E^{(n)}_\beta = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{E_\beta(z)}{z^{n+1}}\,dz, \qquad \psi^{(n)}_\beta = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{\psi_\beta(z)}{z^{n+1}}\,dz,$$

$$P^{(n)}_\beta = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{P_\beta(z)}{z^{n+1}}\,dz. \tag{9.3}$$

The first equation of (9.3) implies that $\beta \mapsto E_\beta^{(n)}$ is in $C_B^k(\mathbb{R})$ and that $\|E_{(\cdot)}^{(n)}\|_{C^k(\mathbb{R})} \leqslant r^{-n}\|E_{(\cdot)}\|_{C_B^\omega(D_{g_0};C_B^k(\mathbb{R}))}$. Similarly we conclude by (9.3) that $\psi_\beta^{(n)}$ and $P_\beta^{(n)}$ are as functions of β in $C_B^k(\mathbb{R};\mathcal{H})$ and $C_B^k(\mathbb{R};\mathcal{B}(\mathcal{H}))$, respectively, and that there exists a finite constant C such that $\|\psi_{(\cdot)}^{(n)}\|_{C^k(\mathbb{R};\mathcal{H})} \leqslant Cr^{-n}$ and $\|P_{(\cdot)}^{(n)}\|_{C^k(\mathbb{R};\mathcal{B}(\mathcal{H}))} \leqslant Cr^{-n}$. Finally observe that $(-1)^N H_{g,\beta}(-1)^N = H_{-g,\beta}$ where N is the linear operator on $\mathcal{F}$ with $N \restriction \mathcal{F}^{(n)}(\mathfrak{h}) = n$. This implies that the ground state energy $E_\beta(g)$ cannot depend on odd powers of g. $\square$

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Appendix A. Elementary estimates and the pull-through formula

To give a precise meaning to expressions which occur in (5.2) and (6.5), we introduce the following. For $\psi \in \mathcal{F}$ having finitely many particles we have

$$\big[a(K_1)\cdots a(K_m)\psi\big]_n(K_{m+1},\ldots,K_{m+n}) = \sqrt{\frac{(m+n)!}{n!}}\,\psi_{m+n}(K_1,\ldots,K_{m+n}), \qquad \text{(A.1)}$$

for all $K_1,\ldots,K_{m+n} \in \underline{\mathbb{R}}^3 := \mathbb{R}^3 \times \mathbb{Z}_2$, and using Fubini's theorem it is elementary to see that the vector-valued map $(K_1,\ldots,K_m) \mapsto a(K_1)\cdots a(K_m)\psi$ is an element of $L^2((\underline{\mathbb{R}}^3)^m;\mathcal{F})$. The following lemma states the well-known pull-through formula. For a proof see for example [5,16].

Lemma A.1. *Let $f:\mathbb{R}_+ \to \mathbb{C}$ be a bounded measurable function. Then for all $K \in \mathbb{R}^3 \times \mathbb{Z}_2$*

$$f(H_f)a^*(K) = a^*(K)f\big(H_f + \omega(K)\big), \qquad a(K)f(H_f) = f\big(H_f + \omega(K)\big)a(K).$$

Let $w_{m,n}$ be function on $\mathbb{R}_+ \times (\underline{\mathbb{R}}^3)^{n+m}$ with values in the linear operators of $\mathcal{H}_{\mathrm{at}}$ or the complex numbers. To such a function we associate the quadratic form

$$q_{w_{m,n}}(\varphi,\psi) := \int_{(\underline{\mathbb{R}}^3)^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \big\langle a\big(K^{(m)}\big)\varphi, w_{m,n}\big(H_f, K^{(m,n)}\big)a\big(\widetilde{K}^{(n)}\big)\psi\big\rangle,$$

defined for all φ and ψ in $\mathcal{H}$ respectively $\mathcal{F}$, for which the right-hand side is defined as a complex number. To associate an operator to the quadratic form we will use the following lemma.

Lemma A.2. *Let $\underline{X} = \mathbb{R}^3 \times \mathbb{Z}_2$. Then*

$$\big|q_{w_{m,n}}(\varphi,\psi)\big| \leqslant \|w_{m,n}\|_\sharp \|\varphi\|\|\psi\|, \qquad \text{(A.2)}$$

where

$$\|w_{m,n}\|_\sharp^2 := \int_{\underline{X}^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^2} \sup_{r\geqslant 0}\Bigg[\big\|w_{m,n}\big(r, K^{(m,n)}\big)\big\|^2 \prod_{l=1}^{m}\big\{r+\Sigma\big[K^{(l)}\big]\big\}\prod_{\widetilde{l}=1}^{n}\big\{r+\Sigma\big[\widetilde{K}^{(\widetilde{l})}\big]\big\}\Bigg].$$

Proof. We set $P[K^{(n)}] := \prod_{l=1}^{n}(H_f+\Sigma[K^l])^{1/2}$ and insert 1's to obtain the trivial identity

$$\big|q_{w_{m,n}}(\varphi,\psi)\big| = \Bigg|\int_{\underline{X}^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|}\big\langle P\big[K^{(m)}\big]P\big[K^{(m)}\big]^{-1}\big|K^{(m)}\big|^{1/2}a\big(K^{(m)}\big)\varphi, w_{m,n}\big(H_f, K^{(m,n)}\big)$$

$$\times P\big[\widetilde{K}^{(n)}\big]P\big[\widetilde{K}^{(n)}\big]^{-1}\big|\widetilde{K}^{(n)}\big|^{1/2}a\big(\widetilde{K}^{(n)}\big)\psi\big\rangle\Bigg|.$$

The lemma now follows using the Cauchy–Schwarz inequality and the following well-known identity for $n \geqslant 1$ and $\phi \in \mathcal{F}$,

$$\int_{\underline{X}^n} dK^{(n)}\big|K^{(n)}\big|\Bigg\|\prod_{l=1}^{n}\big[H_f+\Sigma\big[K^{(l)}\big]\big]^{-1/2}a\big(K^{(n)}\big)\phi\Bigg\|^2 = \big\|P_\Omega^\perp\phi\big\|^2, \tag{A.3}$$

where $P_\Omega^\perp := |\Omega\rangle\langle\Omega|$. A proof of (A.3) can for example be found in [16, Appendix A]. □

Provided the form $q_{w_{m,n}}$ is densely defined and $\|w_{m,n}\|_\sharp$ is a finite real number, then the form $q_{w_{m,n}}$ determines uniquely a bounded linear operator $\underline{H}_{m,n}(w_{m,n})$ such that

$$q_{w_{m,n}}(\varphi,\psi) = \big\langle\varphi, \underline{H}_{m,n}(w_{m,n})\psi\big\rangle,$$

for all φ, ψ in the form domain of $q_{w_{m,n}}$. Moreover, $\|\underline{H}_{m,n}(w_{m,n})\| \leqslant \|w_{m,n}\|_\sharp$. Using the pull-through formula and Lemma A.2 it is easy to see that for $w^{(I)}$, defined in (6.6), with $m+n = 1, 2$, the form

$$q_{m,n}^{(I)}(\varphi,\psi) := q_{w_{m,n}^{(I)}}\big(\varphi, (H_f+1)^{-\frac{1}{2}(m+n)}(-\Delta+1)^{-\frac{1}{2}\delta_{1,m+n}}\psi\big)$$

is densely defined and bounded. Thus we can associate a bounded linear operator $L_{m,n}^{(I)}$ such that $q_{m,n}^{(I)}(\varphi,\psi) = \langle\varphi, L_{m,n}^{(I)}\psi\rangle$. This allows us to define

$$\underline{H}_{m,n}\big(w_{m,n}^{(I)}\big) := L_{m,n}^{(I)}(H_f+1)^{\frac{1}{2}(m+n)}(-\Delta+1)^{\frac{1}{2}\delta_{1,m+n}}$$

as an operator in $\mathcal{H}$.

Appendix B. Smooth Feshbach property

In this appendix we follow [2,9]. We introduce the Feshbach map and its auxiliary operator and state basic isospectrality properties. Let χ and $\overline{\chi}$ be commuting, nonzero bounded operators, acting on a separable Hilbert space $\mathcal{H}$ and satisfying $\chi^2+\overline{\chi}^2 = 1$. A *Feshbach pair* (H, T) for χ

is a pair of closed operators with the same domain,

$$H, T : D(H) = D(T) \subset \mathcal{H} \to \mathcal{H}$$

such that H, T, $W := H - T$, and the operators

$$W_\chi := \chi W \chi, \qquad W_{\overline{\chi}} := \overline{\chi} W \overline{\chi},$$
$$H_\chi := T + W_\chi, \qquad H_{\overline{\chi}} := T + W_{\overline{\chi}},$$

defined on $D(T)$ satisfy the following assumptions:

(a) $\chi T \subset T \chi$ and $\overline{\chi} T \subset T \overline{\chi}$,
(b) $T, H_{\overline{\chi}} : D(T) \cap \operatorname{Ran} \overline{\chi} \to \operatorname{Ran} \overline{\chi}$ are bijections with bounded inverse,
(c) $\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a bounded operator.

Remark B.1. By abuse of notation we write $H_{\overline{\chi}}^{-1} \overline{\chi}$ for $(H_{\overline{\chi}} \upharpoonright \operatorname{Ran} \overline{\chi})^{-1} \overline{\chi}$ and likewise $T^{-1} \overline{\chi}$ for $(T \upharpoonright \operatorname{Ran} \overline{\chi})^{-1} \overline{\chi}$.

We call an operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ *bounded invertible* in a subspace $V \subset \mathcal{H}$ (V not necessarily closed), if $A : D(A) \cap V \to V$ is a bijection with bounded inverse. Given a Feshbach pair (H, T) for χ, the operator

$$F_\chi(H, T) := H_\chi - \chi W \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi \tag{B.1}$$

on $D(T)$ is called the *Feshbach map of H*. The auxiliary operator

$$Q_\chi := Q_\chi(H, T) := \chi - \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi \tag{B.2}$$

is by conditions (a), (c) bounded, and Q_χ leaves $D(T)$ invariant. The Feshbach map is isospectral in the sense of the following theorem.

Theorem B.2. *Let (H, T) be a Feshbach pair for χ on a Hilbert space $\mathcal{H}$. Then the following holds. $\chi \ker H \subset \ker F_\chi(H, T)$ and $Q_\chi \ker F_\chi(H, T) \subset \ker H$. The mappings*

$$\chi : \ker H \to \ker F_\chi(H, T), \qquad Q_\chi : \ker F_\chi(H, T) \to \ker H$$

are linear isomorphisms and inverse to each other.

The proof of Theorem B.2 can be found in [2,9]. The next lemma gives sufficient conditions for two operators to be a Feshbach pair. It follows from a Neumann expansion [9].

Lemma B.3. *Conditions* (a)*,* (b)*, and* (c) *on Feshbach pairs are satisfied if:*

(a′) $\chi T \subset T \chi$ *and* $\overline{\chi} T \subset T \overline{\chi}$*,*
(b′) T *is bounded invertible in* $\operatorname{Ran} \overline{\chi}$*,*
(c′) $\|T^{-1} \overline{\chi} W \overline{\chi}\| < 1$*,* $\|\overline{\chi} W T^{-1} \overline{\chi}\| < 1$*, and* $T^{-1} \overline{\chi} W \chi$ *is a bounded operator.*

Appendix C. Function spaces

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. By $\mathcal{B}(X, Y)$ we denote the Banach space of bounded linear operators from X to Y. We set $\mathcal{B}(X) := \mathcal{B}(X, X)$. Let (M, μ) be a measure space. We say that a function $f : M \to X$ is measurable if there exists a sequence $(f_j)_{j\in\mathbb{N}_0}$ of simple functions from M to X, such that $\|f_j(m) - f(m)\|_X \to 0$ as $j \to \infty$, for a.e. $m \in M$. We define $L^\infty(M; X)$ to be the Banach space of measurable functions from M to X with norm

$$\|f\|_{L^\infty(M;X)} := \operatorname*{ess\,sup}_{m\in M} \big\| f(m) \big\|_X .$$

Let $[a, b]$ be a closed interval of $\mathbb{R}$. For $p \in \mathbb{N}_0$ we define the space $C^p[a, b]$ to be the space of functions $f : (a, b) \to \mathbb{C}$ such that for all $q = 0, \dots, p$ the partial derivatives $\partial_1^q f$ exist and are uniformly continuous on bounded subsets of (a, b). We define the norm

$$\|f\|_{C^p[a,b]} := \max_{0\leqslant q\leqslant p} \sup_{r\in(a,b)} \big|\partial_r^q f(r)\big|. \tag{C.1}$$

By $C_B^p[a, b]$ we denote the Banach space with norm $\|\cdot\|_{C^p[a,b]}$ which consists of elements in $C^p[a, b]$ for which the norm $\|\cdot\|_{C^p[a,b]}$ is finite. We denote by $C^k(\mathbb{R}; X)$ the space of strongly (w.r.t. the norm in X) k-times continuously differentiable functions. The norm is given by

$$\|f\|_{C^k(\mathbb{R};X)} := \max_{0\leqslant s\leqslant k} \sup_{x\in\mathbb{R}} \big\|\partial_x^s f(x)\big\|_X .$$

Let $C_B^k(\mathbb{R}; X)$ denote the set of functions f in $C^k(\mathbb{R}; X)$ for which the norm $\|f\|_{C^k(\mathbb{R};X)}$ is finite. Let $U \subset \mathbb{C}^n$ be a domain. We define the space $C^\omega(U; X)$ to consist of all strongly analytic functions $f : U \to X$. We define the norm

$$\|f\|_{C^\omega(U;X)} := \sup_{z\in U} \big\|f(z)\big\|_X .$$

By $C_B^\omega(U; X)$ we denote the Banach space with norm $\|\cdot\|_{C^\omega(U;X)}$ which consists of elements in $C^\omega(U; X)$ for which the norm $\|\cdot\|_{C^\omega(U;X)}$ is finite. We define the space $C^{\omega,k}(U \times \mathbb{R}; X)$ to consist of all functions $f : U \times \mathbb{R} \to X$ such that all partial derivatives $\partial_x^l \partial_{z_i}^t f$, with $l \in \mathbb{N}_0$, $l \leqslant k$, $i = 1, \dots, n$, and $t = 0, 1$, exist and are continuous. We define the norm

$$\|f\|_{C^{\omega,k}(U\times\mathbb{R};X)} := \sup_{z\in U} \max_{0\leqslant l\leqslant k} \sup_{x\in\mathbb{R}} \big\|\partial_x^l f(z, x)\big\|_X .$$

By $C_B^{\omega,k}(U \times \mathbb{R}; X)$ we denote the Banach space with norm $\|\cdot\|_{C^\omega(U\times\mathbb{R};X)}$ which consists of elements in $C^\omega(U \times \mathbb{R}; X)$ for which the norm $\|\cdot\|_{C^\omega(U;X)}$ is finite. In the case where $X = \mathbb{C}$ we will drop the X dependence in the notation. We introduce the polydiscs $D_{\underline{r}} = \prod_{i=1}^n D_{r_i}$ with $\underline{r} \in (0, \infty)^n$.

Lemma C.1. *We have the canonical isomorphism of Banach spaces*

$$C_B^{\omega,k}(D_{\underline{r}} \times \mathbb{R}; X) \cong C_B^\omega\big(D_{\underline{r}}; C_B^k(\mathbb{R}; X)\big). \tag{C.2}$$

Proof. Let $f \in C_B^{\omega,k}(D_{\underline{r}} \times \mathbb{R}; X)$. Then for every $x \in \mathbb{R}$ the function $z \mapsto f(z,x)$ is analytic on $D_{\underline{r}}$ and bounded. Thus for $\epsilon > 0$ sufficiently small

$$f(z,x) = \sum_{\underline{n}} c_{\underline{n}}(x) z^{\underline{n}}$$

with

$$c_{\underline{n}}(x) = \frac{1}{(2\pi i)^n} \int_{D_{\underline{r}-\underline{\epsilon}}} \frac{f(\zeta,x)}{\zeta^{\underline{n}+\underline{1}}} \prod_{j=1}^{n} d\zeta_j,$$

where the integral is a strong Riemann integral in X and we used the notation $\underline{1} = (1,\dots,1)$ and $\underline{\epsilon} = \epsilon \underline{1}$. It follows that $\|c_{\underline{n}}\|_{C^k(\mathbb{R};X)} \leqslant \prod_{j=1}^{n} r_j^{-n_j} \|f\|_{C^{\omega,k}(D_{\underline{r}} \times \mathbb{R};X)}$. This implies that the function $\widehat{f} : z \mapsto f(z,\cdot)$ is in $C_B^{\omega}(D_{\underline{r}}; C_B^k(\mathbb{R};X))$. Moreover,

$$\sup_{z \in D_{\underline{r}}} \|\widehat{f}(z)\|_{C_B^k(\mathbb{R};X)} = \sup_{z \in D_{\underline{r}}} \max_{0 \leqslant l \leqslant k} \sup_{x \in \mathbb{R}} \|\partial_x^l f(z,x)\|_X = \|f\|_{C^{\omega,k}(D_{\underline{r}} \times \mathbb{R};X)}.$$

Now suppose $g \in C_B^{\omega}(D_{\underline{r}}; C_B^k(\mathbb{R};X))$. Then

$$g(z) = \sum_{\underline{n}} a_{\underline{n}} z^{\underline{n}}$$

with

$$a_{\underline{n}} = \frac{1}{(2\pi i)^n} \int_{D_{\underline{r}-\underline{\epsilon}}} \frac{g(\zeta)}{\zeta^{\underline{n}+\underline{1}}} \prod_{j=1}^{n} d\zeta_j,$$

where the integral is a strong Riemann integral in $C_B^k(\mathbb{R};X)$. It follows that

$$\|a_{\underline{n}}\|_{C^k(\mathbb{R};X)} \leqslant \prod_{j=1}^{n} r_j^{-n_j} \|g\|_{C^{\omega}(D_{\underline{r}}; C_B^k(\mathbb{R};X))}. \tag{C.3}$$

We define

$$\tilde{g}(x,z) := \sum_{\underline{n}} a_{\underline{n}}(x) z^{\underline{n}}.$$

It follows from (C.3) that $\tilde{g} \in C_B^{\omega,k}(D_{\underline{r}} \times \mathbb{R}; X)$. Moreover,

$$\|\widetilde{g}\|_{C^{\omega,k}(D_{\underline{r}} \times \mathbb{R};X)} = \sup_{z \in D_{\underline{r}}} \max_{0 \leqslant l \leqslant k} \sup_{x \in \mathbb{R}} \|\partial_x^l \widetilde{g}(z,x)\|_X = \sup_{z \in D_{\underline{r}}} \|g(z)\|_{C^k(\mathbb{R};X)}. \qquad \square$$

Appendix D. Faa di Bruno's formula

Let P_n denote the set of all partitions of $\{1, \ldots, n\}$. Then

$$(f \circ g)^{(n)} = \sum_{X \in P_n} f^{(|X|)} \circ g \cdot \prod_{x \in X} g^{(|x|)}, \tag{D.1}$$

where $|X|$ and $|x|$ stand for the cardinality of the sets X and x, respectively.

Appendix E. Uniform convergence

Let $(s_0, \beta_0) \in S \times \mathbb{R}$. Then for every $\epsilon > 0$ there is an open set $U \subset S \times \mathbb{R}$ containing (s_0, β_0) such that

$$\sup_{(\beta,s) \in U} \max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^l w(\beta, s) - \partial_\beta^l w(s_0, \beta_0) \right\|_\xi^\# < \epsilon.$$

This implies

$$\sup_{(\beta,s) \in U} \max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^l w(\beta, s)_{m,n} \right\|^\# \leqslant \max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^l w(s_0, \beta_0)_{m,n} \right\|^\# + \xi^{m+n} \epsilon =: E_{m,n}.$$

By Lemma 7.10,

$$\sup_{(\beta,s) \in U} \max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^l v_{\underline{m},\underline{p},\underline{n},\underline{q}} \big[w(\beta, s) \big] \right\|^\# \leqslant C_L t^{-L+1} \prod_{l=1}^{L} \frac{E_{m_l+p_l, n_l q_l}}{\sqrt{p_l! q_l!}}, \tag{E.1}$$

where we used the notation introduced in that lemma. We estimate

$$\begin{aligned} &\sum_{M+N \geqslant 0} \sum_{L=1}^{\infty} \sum_{\substack{(\underline{m},\underline{p},\underline{n},\underline{q}) \in \mathbb{N}_0^{4L} \\ |\underline{m}|=M, |\underline{n}|=N \\ m_l+p_l+n_l+q_l \geqslant 1}} \xi^{-|\underline{m}|-|\underline{n}|} \rho^{|\underline{m}|+|\underline{n}|} \\ &\quad \times \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \right\} \sup_{(\beta,s) \in U} \max_{0 \leqslant l \leqslant k} \left\| \partial_\beta^l v_{\underline{m},\underline{p},\underline{n},\underline{q}} \big[w(\beta, s) \big] \right\|^\# \\ &\leqslant \sum_{L=1}^{\infty} C_L t^{1-L} G^L, \end{aligned} \tag{E.2}$$

where we used Eq. (E.1) and the definition

$$G := \sum_{m+p+n+q \geqslant 1} \binom{m+p}{p} \binom{n+q}{q} \xi^{p+q} (1/2)^{m+n} \xi^{-m-p-n-q} \frac{E_{m+p,n+q}}{\sqrt{p!q!}}.$$

Below we will show that

$$G \leqslant \left\| w(s_0, \cdot)_{\geqslant 1} \right\|_\xi^{(k,\#)} + \epsilon 16 e^4. \tag{E.3}$$

Since $t^{-1}G < 1$ for ϵ sufficiently small inequality (E.3) implies the convergence of (E.2), for small ϵ. To show (E.3), we will use the following estimate

$$\sum_{m+p\geqslant 0} \binom{m+p}{p} \xi^p (1/2)^m \frac{1}{\sqrt{p!}} \leqslant \sum_{m+p\geqslant 0} \binom{m+p}{p} (1/4)^p (1/2)^m e^{8\xi^2} = 4e^{8\xi^2} \leqslant 4e^2, \tag{E.4}$$

where in the first inequality we used the trivial estimate $(16\xi^2)^p/p! \leqslant e^{16\xi^2}$. Now (E.3) is seen by inserting the definition of $E_{m,n}$ into the definition of G. This yields two terms, which one has to estimate. The second term, involving ϵ, is estimated using (E.4), and the first term, involving $w_{m,n}(s_0, \beta_0)$, is estimated using the binomial formula, i.e.,

$$\begin{aligned}
&\sum_{m+p+n+q\geqslant 1} \binom{m+p}{p}\binom{n+q}{q} \xi^{p+q} (1/2)^{m+n} \xi^{-m-p-n-q} \max_{0\leqslant l\leqslant k} \left\| \partial_\beta^l w(\beta_0, s_0)_{m+p,n+q} \right\| \\
&\quad = \sum_{i+j\geqslant 1} (\xi + 1/2)^i (\xi + 1/2)^j \xi^{-i-j} \max_{0\leqslant l\leqslant k} \left\| \partial_\beta^l w(\beta_0, s_0)_{i,j} \right\|.
\end{aligned}$$

Appendix F. Differentiability

Lemma F.1.

(a) *The mapping*

$$\begin{aligned}
\widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}[\cdot] : \left(\mathcal{W}_\xi^\#\right)^L \times \left(\mathcal{W}_{0,0}^\#\right)^{L+1} &\to \mathcal{W}_{|\underline{m}|,|\underline{n}|}^\# \\
(w_1, \ldots, w_L, G_0, \ldots, G_L) &\mapsto \widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w_1, \ldots, w_L, G_0, \ldots, G_L]
\end{aligned}$$

defined by

$$\begin{aligned}
&\widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w_1, \ldots, w_L, G_0, \ldots, G_L]\big(r, K^{(|\underline{m}|,|\underline{n}|)}\big) \\
&\quad := \Big\langle \Omega, G_0\big(H_f + \rho(r + \widetilde{r}_0)\big) \\
&\qquad \times \prod_{l=1}^L \big\{ W_{p_l,q_l}^{m_l,n_l}[w_l]\big(\rho(r + r_l), \rho K_l^{(m_l,n_l)}\big) G_l\big(H_f + \rho(r + \widetilde{r}_l)\big)\big\} \Omega \Big\rangle
\end{aligned}$$

is continuous and multilinear.

(b) *The following mapping is in C^∞.*

$$\begin{aligned}
\Big\{ t \in \mathcal{W}_{0,0}^\# \,\Big|\, \inf_{r\in[\rho\frac{3}{4},1]} \big|t(r)\big| > \epsilon \Big\} &\to \mathcal{W}_{0,0}^\# \\
t &\mapsto \frac{\overline{\chi}_\rho^2}{t}.
\end{aligned}$$

Proof. (a) Using (7.12) we find

$$\begin{aligned}&\operatorname*{ess\,sup}_{K^{(|\underline{m}|,|\underline{n}|)}} \sup_{r\in[0,1]} \left|\widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w_1,\dots,w_L,G_0,\dots,G_L]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big)\right| \\ &\quad\leqslant \prod_{l=1}^{L} \operatorname*{ess\,sup}_{K^{(m_l,n_l)}} \sup_{r\in[0,1]} \left\| W^{m,n}_{p,q}[w]\big(r,K^{(m_l,n_l)}\big)\right\|_{\mathrm{op}}, \qquad \prod_{l=0}^{L}\|G_l\|_{C[0,1]}.\end{aligned}$$

To estimate the right-hand side we use

$$\operatorname*{ess\,sup}_{K^{(m,n)}} \sup_{r\in[0,1]} \left\| W^{m,n}_{p,q}[w]\big(r,K^{(m,n)}\big)\right\|_{\mathrm{op}} \leqslant \frac{\|w_{p+m,q+n},\|_{L^\infty(B_1^{m+n};C[0,1])}}{\sqrt{p!q!}}. \tag{F.1}$$

Inequality (F.1) can be shown using Lemma A.2 and (5.6). Next we calculate the derivative with respect to r. To this end first observe that using Lemma A.2 and dominated convergence one can show that for a.e. $K^{(m,n)}$ the partial derivative $\partial_r W^{m,n}_{p,q}[w](r,K^{(m,n)})$ exists and equals $W^{m,n}_{p,q}[\partial_r w](r,K^{(m,n)})$. Using Leibniz we obtain

$$\begin{aligned}&\partial_r \widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w_1,\dots,w_L,G_0,\dots,G_L]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big) \\ &\quad= \rho\sum_{j=1}^{2L+1} \widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}\big[\partial_r^{\delta_{1,j}}w_1,\dots,\partial_r^{\delta_{L,j}}w_L,\partial_r^{\delta_{L+1,j}}G_0,\dots,\partial_r^{\delta_{2L+1,j}}G_L\big]\big(r,K^{(|\underline{m}|,|\underline{n}|)}\big).\end{aligned}$$

Using again (7.12) and (F.1) to estimate this we find

$$\left\|\widetilde{v}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w_1,\dots,w_L,G_0,\dots,G_L]\right\|^{\#} \leqslant \prod_{l=1}^{L}\|w_l\|_\xi^{\#}\prod_{l=0}^{L}\|G_l\|^{\#}.$$

This yields (a).

(b) It is straightforward to verify that the mapping $t\mapsto \overline{\chi}_\rho^2/t$ is differentiable with derivative $-\overline{\chi}_\rho^2/t^2$, see [16, Lemma 8.6(b)]. Using the product rule one can now show iteratively that the function is in C^∞. $\square$

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