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Heavy cycles in k-connected weighted graphs with large weighted degree sums

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Abstract

A weighted graph is one in which every edge e is assigned a nonnegative number $w(e)$, called the weight of e. The weight of a cycle is defined as the sum of the weights of its edges. The weighted degree of a vertex is the sum of the weights of the edges incident with it. In this paper, we prove that: Let G be a k-connected weighted graph with $k \geqslant 2$. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$, if G satisfies the following conditions: (1) The weighted degree sum of any $k+1$ pairwise nonadjacent vertices is at least m; (2) In each induced claw and each induced modified claw of G, all edges have the same weight. This generalizes an early result of Enomoto et al. on the existence of heavy cycles in k-connected weighted graphs.

MSC: 05C45; 05C38; 05C35

Keywords: Heavy cycle; Weighted degree (sum); Induced claw (modified claw)

1. Terminology and notation

We use Bondy and Murty [5] for terminology and notation not defined here and consider finite simple graphs only.

Let $G=(V,E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e. For a subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H)=\sum_{e\in E(H)} w(e).$$

For a vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v)=\sum_{h\in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively.

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An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a subgraph is simply the number of its edges.

An (x, y)-*path* is a path connecting two vertices x and y. Let H be a path or a cycle with a given orientation. By $\overleftarrow{H}$ we mean the same graph as H but with the reverse orientation. If v is a vertex of H, then v_H^{+1} and v_H^{-1} denote the immediate successor and immediate predecessor (if it exists) of v on H, respectively. In the following, we use v_H^+ for v_H^{+1} and v_H^- for v_H^{-1} for simplicity. For an integer $k \geqslant 2$, v_H^{+k} and v_H^{-k} are defined recursively by $v_H^{+k} = (v_H^{+(k-1)})^+$ and $v_H^{-k} = (v_H^{-(k-1)})^-$. If S is a set of vertices of H, then define $S_H^+ = \{s_H^+ | s \in S\}$. When no confusion occurs, we denote v_H^+, v_H^-, v_H^{+m}, v_H^{-m} and S_H^+ by v^+, v^-, v^{+m}, v^{-m} and S^+, respectively. For two vertices u and v of H, we use $H[u, v]$ to denote the segment of H from u to v. For a path $P[u, v]$, by $P(u, v)$, $P[u, v)$ and $P(u, v]$, we mean the path $P[u, v] - \{u, v\}$, $P[u, v] - \{v\}$ and $P[u, v] - \{u\}$, respectively.

The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. If G is noncomplete, then for a positive integer $k \leqslant \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k pairwise nonadjacent vertices. If G is complete, then both $\sigma_k(G)$ and $\sigma_k^w(G)$ are defined as ∞.

We call the graph $K_{1,3}$ a *claw*, and the graph obtained by joining a pendant edge to some vertex of a triangle a *modified claw*.

2. Results

There have been many results on the existence of long paths and cycles in unweighted graphs. In [3,4], Bondy and Fan generalized several classical theorems of Dirac and of Erdös and Gallai on paths and cycles to weighted graphs. A weighted generalization of Ore's theorem was obtained by Bondy et al. [2]. In [11], it was shown that if one wants to generalize Fan's theorem on the existence of long cycles to weighted graphs some extra conditions cannot be avoided. By adding two extra conditions, the authors gave a weighted generalization of Fan's theorem.

Among the many results on cycles in unweighted graphs, the following generalization of Ore's theorem is well-known.

Theorem A (*Fournier and Fraisse [8]*). *Let G be a k-connected graph where* $2 \leqslant k < \alpha(G)$, *such that* $\sigma_{k+1}(G) \geqslant m$. *Then G contains either a Hamilton cycle or a cycle of length at least* $2m/(k+1)$.

A natural question is whether Theorem A also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 1. *Let G be a k-connected weighted graph where* $2 \leqslant k < \alpha(G)$, *such that* $\sigma_{k+1}^w(G) \geqslant m$. *Is it true that G contains either a Hamilton cycle or a cycle of weight at least* $2m/(k+1)$.

It seems very difficult to settle this problem, even for the case $k = 2$. Motivated by the result in [11], Zhang et al. [10] proved that the answer to Problem 1 in the case $k = 2$ is positive with the two same extra conditions as in [11].

Theorem 1 (*Zhang et al. [10]*). *Let G be a* 2*-connected weighted graph which satisfies the following conditions*:

(1) $\sigma_3^w(G) \geqslant m$;
(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$;
(3) *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

In [7], after giving a characterization of the connected weighted graphs satisfying Conditions (2) and (3) of Theorem 1, Enomoto et al. proved that the answer to Problem 1 is positive for any $k \geqslant 2$ with these two extra conditions.

Theorem 2 (*Enomoto et al. [7]*). *Let G be a k-connected weighted graph where $k \geqslant 2$. Suppose that G satisfies the following conditions*:

(1) $\sigma^w_{k+1}(G) \geqslant m$;
(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$;
(3) *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$.

On the other hand, Fujisawa [9] gave so-called claw conditions for the existence of heavy cycles in weighted graphs, generalizing a result of Bedrossian et al. [1] on the existence of long cycles in unweighted graphs.

Theorem 3 (*Fujisawa [9]*). *Let G be a 2-connected weighted graph which satisfies the following conditions*:

(1) *For each induced claw and each induced modified claw of G, all its nonadjacent pair of vertices x and y satisfy* $\max\{d^w(x), d^w(y)\} \geqslant s/2$;
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least s.

A result similar to this theorem was obtained by Chen and Zhang [6]. It also generalizes Theorem 1.

Theorem 4 (*Chen and Zhang [6]*). *Let G be a 2-connected weighted graph which satisfies the following conditions*:

(1) $\sigma^w_3(G) \geqslant m$;
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

Clearly, Condition (2) of Theorem 4 is weaker than Conditions (2) and (3) of Theorem 2. Thus, we have the following problem: Can Conditions (2) and (3) in Theorem 2 be weakened by Condition (2) of Theorem 4? In this paper, we give a positive answer to this problem. Our result is a generalization of Theorem 2.

Theorem 5. *Let G be a k-connected weighted graph where $k \geqslant 2$. Suppose that G satisfies the following conditions*:

(1) $\sigma^w_{k+1}(G) \geqslant m$;
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$.

We postpone the proof of Theorem 5 to the next section.

3. Proof of Theorem 5

In this section we give a proof of Theorem 5. In order to make the proof easy to understand, we postpone the proof of two claims (Claims 4 and 5) to the next section.

We first present some lemmas. Lemma 1 can be proved by a minor modification of the proof of Lemma 5 in [4], while the proof of Lemma 2 is almost immediate.

Lemma 1. *Let G be a 2-connected weighted graph which is non-hamiltonian and P an (s, t)-path in G. Then there is a cycle $\tilde{C}$ in G with $w(\tilde{C}) \geqslant d^w(s) + d^w(t)$, if the following conditions are satisfied*:

(i) $N(s) \cup N(t) \subseteq V(P)$;
(ii) $N_P(s) \cap N_P(t)^+ = \emptyset$;
(iii) $w(x^-_P x) \geqslant w(sx)$ if $x \in N_P(s)$ and $w(x x^+_P) \geqslant w(xt)$ if $x \in N_P(t)$.

Lemma 2. *Let G be a k-connected weighted graph where $2 \leqslant k < \alpha(G)$, $\{u_1, u_2, \ldots, u_{k+1}\}$ an independent set of G. Then there exist u_i and u_j with $1 \leqslant i < j \leqslant k+1$ such that $d^w(u_i) + d^w(u_j) \geqslant 2/(k+1)\sigma^w_{k+1}(G)$.*

Lemma 3 (Fujisawa [9]). *Let G be a weighted graph satisfying Condition* (2) *of Theorem* 5. *If x_1yx_2 is an induced path with $w(x_1y) \neq w(x_2y)$ in G, then each vertex $x \in N(y)\backslash\{x_1, x_2\}$ is adjacent to both x_1 and x_2.*

Lemma 4 (Fujisawa [9]). *Let G be a weighted graph satisfying Condition* (2) *of Theorem* 5. *Suppose x_1yx_2 is an induced path such that $w_1 = w(x_1y)$ and $w_2 = w(x_2y)$ with $w_1 \neq w_2$, and yz_1z_2 is a path such that $\{z_1, z_2\} \cap \{x_1, x_2\} = \emptyset$* and $x_2z_2 \notin E(G)$. *Then*

(i) $\{z_1x_1, z_1x_2, z_2x_1\} \subseteq E(G)$,and $yz_2 \notin E(G)$. *Moreover, all edges in the subgraph induced by $\{x_1, y, x_2, z_1, z_2\}$, other than x_1y, have the same weight w_2.*
(ii) *Let Y be the component of $G - \{x_2, z_1, z_2\}$ with $y \in V(Y)$. For each vertex $v \in V(Y)\backslash\{x_1, y\}$, v is adjacent to all of x_1, x_2, y and z_2. Furthermore, $w(vx_1) = w(vx_2) = w(vy) = w(vz_2) = w_2$.*

Proof of Theorem 5. Let G be a k-connected weighted graph satisfying the conditions of Theorem 5. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least $2m/(k+1)$.

Choose a cycle C in G such that

(1) C is as long as possible;
(2) $w(C)$ is as large as possible, subject to (1).

Then from the assumption that G does not contain a Hamilton cycle, we can immediately see that $R = V(G)\backslash V(C) \neq \emptyset$. Choose $u_0 \in R$ such that $d^w(u_0) = \min\{d^w(u) | u \in R\}$ and denote by A_0 the component containing u_0 in $G - V(C)$. Since G is k-connected, there exist k paths $P_i = u_0 \cdots w_i v_i$ $(i = 1, 2, \ldots, k)$, such that $V(P_i) \cap V(C) = \{v_i\}$, and $V(P_i) \cap V(P_j) = \{u_0\}$ for $i \neq j$. Now, we assume that P_i has an orientation from u_0 to v_i.

Assign an orientation to C. Now let $u_i = v_i^+$. It is easy to see that $\{u_0, u_1, u_2, \ldots, u_k\}$ is an independent set of G by the choice of C.

If $N_R(u_i) \neq \emptyset$, choose a path $Q_i = u_i y_i \cdots z_i$ in $G[R \cup \{u_i\}]$ such that

(1) Q_i is as long as possible;
(2) $w(Q_i)$ is as large as possible, subject to (1).

Then from the choice of Q_i, we know that $N_R(z_i) \subseteq Q_i$. Let A_i be the component of $G - V(C)$ such that $y_i \in V(A_i)$. Without loss of generality, we can assume $N_R(u_i) = \emptyset$ for $i = 1, 2, \ldots, q$ and $N_R(u_i) \neq \emptyset$ for $i = q+1, q+2, \ldots, k$.

Claim 1. *Let P be an (s, t)-path with $|V(P)| > |V(C)|$. Then $N_P(s) \cap N_P(t)^+ = \emptyset$.*

Proof. Suppose $N_P(s) \cap N_P(t)^+ \neq \emptyset$. Let x be a vertex in $N_P(s) \cap N_P(t)^+$. Then we get a cycle $C' = sP[s, x^-]x^-t\overleftarrow{P}[t, x]xs$ which is longer than C, a contradiction. □

Claim 2. $A_0, A_{q+1}, \ldots, A_k$ are different components of $G - V(C)$.

Proof. If $A_0 = A_i$ for some $i \in \{q+1, q+2, \ldots, k\}$, then there exists a (w_i, y_i)-path P_i^* in this component. So we can get a cycle $C' = w_i P_i^* y_i u_i C[u_i, v_i] v_i w_i$ which is longer than C, a contradiction.

If $A_i = A_j$ for some i, j with $q+1 \leqslant i < j \leqslant k$, then there exists a (y_i, y_j)-path P_{ij}^* in this component. So we can get a cycle $C' = y_i P_{ij}^* y_j u_j C[u_j, v_i] v_i \overleftarrow{P_i} u_0 P_j v_j \overleftarrow{C}[v_j, u_i] u_i y_i$ which is longer than C, a contradiction. □

Claim 3. $\{u_0, u_1, u_2, \ldots, u_q, z_{q+1}, z_{q+2}, \ldots, z_k\}$ *is an independent set.*

Proof. As we noted before, $\{u_0, u_1, \ldots, u_k\}$ is an independent set. From Claim 2, we know that $\{u_0, z_{q+1}, z_{q+2}, \ldots, z_k\}$ is an independent set. The result follows from the assumption that $N_R(u_i) = \emptyset$, where $i = 1, 2, \ldots, q$, and $z_j \in R$, immediately. □

Claim 4. *Let i be an integer with $1 \leqslant i \leqslant k$ and let x be a vertex in $N_G(u_i) \cap V(C)$. Then $w(x_C^- x) \geqslant w(u_i x)$.*

Claim 5. *Let i and j be distinct integers with $1 \leqslant i, j \leqslant k$ and let x be a vertex in $N_G(u_i) \cap V(C)$.* If $x \in C(u_j, v_i)$, *then $w(x_C^+ x) \geqslant w(u_i x)$.*

The proofs of these two claims are somewhat complicated. So we leave them to the next section.

Claim 6. Let u be a vertex with $u \in V(G) \backslash V(C)$ and let x be a vertex in $N_G(u) \cap V(C)$. Then we have $w(xx_C^-) = w(xx_C^+) = w(ux)$.

Proof. By the choice of C, we have $ux_C^- \notin E(G)$ and $ux_C^+ \notin E(G)$. So $\{x, x_C^-, x_C^+, u\}$ induces a claw or a modified claw. Thus, $w(xx_C^-) = w(xx_C^+) = w(ux)$. $\square$

Applying Lemma 2 to the independent set $\{u_0, u_1, u_2, \ldots, u_q, z_{q+1}, z_{q+2}, \ldots, z_k\}$, there must be two vertices s and t in this set such that $d^w(s) + d^w(t) \geqslant 2m/(k+1)$.

We distinguish two cases:

Case 1. $u_0 \notin \{s, t\}$.

Case 1.1. $s = u_i$ and $t = u_j$ for some i and j with $1 \leqslant i < j \leqslant q$.

Consider the path $P = sC[s, v_j]v_j \overleftarrow{P_j} u_0 P_i v_i \overleftarrow{C}[v_i, t]t$. It is obvious that $V(C) \subset V(P)$. Then, from $N(s) \subset V(C)$ and $N(t) \subset V(C)$, we have $N(s) \cup N(t) \subset V(P)$; from $|V(P)| > |V(C)|$ and Claim 1, we have $N_P(s) \cap N_P(t)^+ = \emptyset$.

Let x be a vertex in $N_P(s)$. Since $N(s) \subset V(C)$, we have $x \in V(C)$. By Claim 3, it is clear that $x \neq u_j$. If $x = v_i$, then $x_P^- = w_i$. By Claim 6, we have $w(x_P^- x) = w(sx)$. If $x \neq v_i$, by Claims 4 and 5, we have $w(x_P^- x) \geqslant w(sx)$.

By the symmetry of s and t, we can prove that $w(xx_P^+) \geqslant w(xt)$ if $x \in N_P(t)$.

Now, by Lemma 1, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geqslant d^w(s) + d^w(t) \geqslant 2m/(k+1)$.

Case 1.2. $s = u_i$ for some i with $1 \leqslant i \leqslant q$ and $t = z_j$ for some j with $q+1 \leqslant j \leqslant k$.

Consider the path $P = sC[s, v_j]v_j \overleftarrow{P_j} u_0 P_i v_i \overleftarrow{C}[v_i, u_j]u_j Q_j t$. It is obvious that $V(C) \subset V(P)$. Then, from $N(s) \subset V(C)$ and $N(t) \subset (V(C) \cup V(Q_j))$, we have $N(s) \cup N(t) \subset V(P)$; from $|V(P)| > |V(C)|$ and Claim 1, we have $N_P(s) \cap N_P(t)^+ = \emptyset$.

Let x be a vertex in $N_P(s)$. Since $N(s) \subset V(C)$, we have $x \in V(C)$. By Claim 3, it is clear that $x \neq u_j$. If $x = v_i$, then $x_P^- = w_i$. By Claim 6, we have $w(x_P^- x) = w(sx)$. If $x \neq v_i$, by Claims 4 and 5, we have $w(x_P^- x) \geqslant w(sx)$.

Let x be a vertex in $N_P(t)$. If $x \in V(Q_j)$, then we have $w(xx_P^+) \geqslant w(xt)$ by the choice of Q_j. If $x \in (V(C) \backslash \{u_j\})$, by Claim 6, we have $w(xx_P^+) \geqslant w(xt)$.

Then, by Lemma 1, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geqslant d^w(s) + d^w(t) \geqslant 2m/(k+1)$.

Case 1.3. $s = z_i$ and $t = z_j$ for some i and j with $q+1 \leqslant i < j \leqslant k$.

Consider the path $P = s\overleftarrow{Q_i} u_i C[u_i, v_j]v_j \overleftarrow{P_j} u_0 P_i v_i \overleftarrow{C}[v_i, u_j]u_j Q_j t$. It is obvious that $V(C) \subset V(P)$. Then, from $N(s) \subset (V(C) \cup V(Q_i))$ and $N(t) \subset (V(C) \cup V(Q_j))$, we have $N(s) \cup N(t) \subset V(P)$; from $|V(P)| > |V(C)|$ and Claim 1, we have $N_P(s) \cap N_P(t)^+ = \emptyset$.

Let x be a vertex in $N_P(s)$. If $x \in V(Q_i)$, then we have $w(x_P^- x) \geqslant w(sx)$ by the choice of Q_i. If $x \in (V(C) \backslash \{u_i\})$, by Claim 6, we have $w(x_P^- x) \geqslant w(sx)$.

By the symmetry of s and t, we can prove that $w(xx_P^+) \geqslant w(xt)$ if $x \in N_P(t)$.

Then, by Lemma 1, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geqslant d^w(s) + d^w(t) \geqslant 2m/(k+1)$.

This completes the proof of Case 1.

Case 2. $u_0 \in \{s, t\}$.

Without loss of generality, we may assume that $t = u_0$.

Case 2.1. $s = u_i$ for some i with $1 \leqslant i \leqslant q$.

Choose a path $Q' = v_i y_0' \cdots z_0'$ in $G[V(A_0) \cup \{v_i\}]$ such that

(1) Q' is as long as possible;
(2) $w(Q')$ is as large as possible, subject to (1).

Then by the choice of Q' and u_0, we know that $N_R(z_0') \subseteq Q'$ and $d^w(z_0') \geqslant d^w(u_0)$.

Now consider the path $P = sC[s, v_i]v_i Q' z_0'$. It is obvious that $V(C) \subset V(P)$. Then, from $N(s) \subset V(C)$ and $N(z_0') \subset (V(C) \cup V(Q'))$, we have $N(s) \cup N(z_0') \subset V(P)$; from $|V(P)| > |V(C)|$ and Claim 1, we have $N_P(s) \cap N_P(z_0')^+ = \emptyset$.

Let x be a vertex in $N_P(s)$. Since $N(s) \subset V(C)$, we have $x \in V(C)$. By Claim 4, we have $w(x_P^- x) \geqslant w(sx)$.

Let x be a vertex in $N_P(z_0')$. If $x \in V(Q')$, then we have $w(xx_P^+) \geqslant w(xz_0')$ by the choice of Q'. If $x \in (V(C) \backslash \{v_i\})$, by Claim 6, we have $w(xx_P^+) \geqslant w(xz_0')$.

Now, we can see that the path $P = sC[s, v_i]v_i Q' z_0'$ satisfies the three conditions of Lemma 1. Therefore, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geqslant d^w(s) + d^w(z_0') \geqslant d^w(s) + d^w(t) \geqslant 2m/(k+1)$.

Case 2.2. $s = z_j$ for some j with $q + 1 \leqslant j \leqslant k$.

Choose a path $Q'' = v_j y_0'' \cdots z_0''$ in $G[V(A_0) \cup \{v_j\}]$ such that

(1) Q'' is as long as possible;
(2) $w(Q'')$ is as large as possible, subject to (1)

Then by the choice of Q'' and u_0, we know that $N_R(z_0'') \subseteq Q''$ and $d^w(z_0'') \geqslant d^w(u_0)$.

Consider the path $P = s\overleftarrow{Q_j} u_j C[u_j, v_j]\, v_j Q'' z_0''$. It is obvious that $V(C) \subset V(P)$. Then, from $N(s) \subset (V(C) \cup V(Q_j))$ and $N(z_0'') \subset (V(C) \cup V(Q''))$, we have $N(s) \cup N(z_0'') \subset V(P)$; from $|V(P)| > |V(C)|$ and Claim 1, we have $N_P(s) \cap N_P(z_0'')^+ = \emptyset$.

Let x be a vertex in $N_P(s)$. If $x \in V(Q_j)$, then we have $w(x_P^- x) \geqslant w(sx)$ by the choice of Q_j. If $x \in (V(C) \backslash \{u_j\})$, then by Claim 6, we have $w(x_P^- x) \geqslant w(sx)$. Let x be a vertex in $N_P(z_0'')$. If $x \in V(Q'')$, then we have $w(xx_P^+) \geqslant w(xz_0'')$ by the choice of Q''. If $x \in (V(C) \backslash \{v_j\})$, then by Claim 6, we have $w(xx_P^+) \geqslant w(xz_0'')$.

Now, we can see that the path $P = s\overleftarrow{Q_j} u_j C[u_j, v_j] v_j Q'' z_0''$ satisfies the three conditions of Lemma 1. Therefore, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geqslant d^w(s) + d^w(z_0'') \geqslant d^w(s) + d^w(t) \geqslant 2m/(k+1)$.

The proof of the theorem is complete. □

4. Proof of Claims 4 and 5

In the proof of Claims 4 and 5, we denote v_C^+, v_C^-, v_C^{+m} and v_C^{-m} by v^+, v^-, v^{+m} and v^{-m} for simplicity.

Proof of Claim 4. If $w(x^- x) = w(u_i x)$, then there is nothing to prove. So we make the following assumption.

Assumption 1. $w(x^- x) \neq w(u_i x)$.

We now prove that $w(x^- x) > w(u_i x)$.

Subclaim 1. *If $x^- v^- \in E(G)$ for some $v \in V(C(u_i, x))$, then $w_j v \notin E(G)$ for every j, $1 \leqslant j \leqslant k$; If $x^- v \in E(G)$ for some $v \in V(C(x, v_i))$, then $w_j v^- \notin E(G)$ for every j, $1 \leqslant j \leqslant k$.*

Proof. Suppose there exists a vertex $v \in V(C(u_i, x))$ such that $x^- v^- \in E(G)$ and $w_j v \in E(G)$ for some j with $1 \leqslant j \leqslant k$. Then we have another cycle $C' = v_i \overleftarrow{C}[v_i, x] x u_i C[u_i, v^-]\, v^- x^- \overleftarrow{C}[x^-, v] v w_j \overleftarrow{P_j} u_0 P_i w_i v_i$ which is longer than C, a contradiction.

Suppose there exists a vertex $v \in V(C(x, v_i))$ such that $x^- v \in E(G)$ and $w_j v^- \in E(G)$ for some j with $1 \leqslant j \leqslant k$. Then we have another cycle $C' = v_i \overleftarrow{C}[v_i, v]vx^- \overleftarrow{C}[x^-, u_i]u_i x\, C[x, v^-]v^- w_j \overleftarrow{P_j} u_0 P_i w_i v_i$ which is longer than C, a contradiction.

Subclaim 2. *$w_j x \notin E(G)$ and $w_j x^- \notin E(G)$ when $1 \leqslant j \leqslant k$.*

Proof. Suppose $w_j x \in E(G)$. By the choice of C, we have $w_j u_i \notin E(G)$ and $w_j x^- \notin E(G)$. Then $\{x, x^-, u_i, w_j\}$ induces a claw or a modified claw. So we have $w(x^- x) = w(u_i x)$, contradicting Assumption 1.

By Subclaim 1, it follows from $x^- x^{-2} \in E(G)$ that $w_j x^- \notin E(G)$ when $1 \leqslant j \leqslant k$.

Case 1. $u_i x^- \in E(G)$.

Subclaim 3. *Exactly one of $v_i x$ and $v_i x^-$ is an edge of G.*

Proof. If $v_i x \notin E(G)$ and $v_i x^- \notin E(G)$, then $\{u_i, x, x^-, v_i\}$ induces a modified claw. If $v_i x \in E(G)$ and $v_i x^- \in E(G)$, then by Subclaim 2 and the choice of C, both $\{v_i, x, x^-, w_i\}$ and $\{v_i, u_i, x, w_i\}$ induce modified claws. We can always get $w(x^- x) = w(u_i x)$, contradicting Assumption 1. $\square$

Case 1.1. $v_i x \notin E(G)$ and $v_i x^- \in E(G)$.

Subclaim 4. $w(u_i v_i) = w(v_i x^-)$.

Proof. By the choice of C and Subclaim 2, we can easily see that $\{v_i, u_i, x^-, w_i\}$ induces a modified claw. So $w(u_i v_i) = w(v_i x^-)$. $\square$

Consider the longest cycle $C' = u_i C[u_i, x^-]x^- v_i \overleftarrow{C}[v_i, x]x u_i$. By the choice of C, we have $w(u_i v_i) + w(x^- x) \geqslant w(u_i x) + w(v_i x^-)$. By Subclaim 4 and Assumption 1, we get $w(x^- x) > w(u_i x)$.

Case 1.2. $v_i x \in E(G)$ and $v_i x^- \notin E(G)$.

Subclaim 5. $w(x^- x) \neq w(v_i x)$.

Proof. Since $\{v_i, u_i, x, w_i\}$ induces a modified claw, $w(u_i x) = w(v_i x)$. By Assumption 1, we have $w(x^- x) \neq w(v_i x)$. $\square$

Let j be an integer with $1 \leqslant j \leqslant k$. By the choice of C, we have $w_j u_i \notin E(G)$ and $w_j v_i^- \notin E(G)$. By Subclaim 2, we have $w_j x^- \notin E(G)$ and $w_j x \notin E(G)$. So $v_j \notin \{u_i, v_i^-, x^-, x\}$.

Case 1.2.1. There exists $v_j \in V(C(u_i, x^-))$ for some $j \in \{1, \ldots, k\} \backslash \{i\}$.

Subclaim 6. $u_i^+ \in N(v_i) \cap N(x^-)$.

Proof. By Subclaim 1, it follows from $x^- u_i \in E(G)$ that $w_j u_i^+ \notin E(G)$. So we get $v_j \neq u_i^+$. Suppose $u_i^+ \notin N(v_i) \cap N(x^-)$. It is clear that w_i is a vertex of the component of $G - \{v_i, u_i, u_i^+\}$ containing x, and w_i is also a vertex of the component of $G - \{x^-, u_i, u_i^+\}$ containing x. If $v_i u_i^+ \notin E(G)$ or $x^- u_i^+ \notin E(G)$, by applying Lemma 4 (ii) to $\{x^-, x, v_i, u_i, u_i^+\}$, we can get that $w_i x \in E(G)$, contradicting Subclaim 2. $\square$

By Subclaim 1, it follows from $w_j v_j \in E(G)$ that $x^- v_j^- \notin E(G)$. This implies that there exists some vertex $u_i^{+p} \in V(C(u_i, v_j))$ such that $u_i^{+p} \notin N(v_i) \cap N(x^-)$. Choose the vertex u_i^{+p} such that p is as small as possible. By Subclaim 6, we have $p \geqslant 2$. Clearly, if $p = 2$, then $u_i^{+(p-2)} = u_i$.

Subclaim 7. *$w(u_i v_i) = w(v_i u_i^{+(p-1)})$ and $w(x^- u_i^{+(p-2)}) = w(u_i^{+(p-2)} u_i^{+(p-1)})$.*

Proof. By the choice of C and Subclaim 1, we have $w_iu_i \notin E(G)$ and $w_iu_i^{+(p-1)} \notin E(G)$. Then $\{v_i, u_i, u_i^{+(p-1)}, w_i\}$ induces a claw or a modified claw. So $w(u_iv_i) = w(v_iu_i^{+(p-1)})$.

If $p=2$, then by the choice of C and Subclaim 1, we have $w_iu_i^{+(p-2)} \notin E(G)$ and $w_iu_i^{+(p-1)} \notin E(G)$. If $p \geqslant 3$, then by the choice of u_i^{+p} and Subclaim 1, we have $w_iu_i^{+(p-2)} \notin E(G)$ and $w_iu_i^{+(p-1)} \notin E(G)$. Now $\{v_i, u_i^{+(p-1)}, u_i^{+(p-2)}, w_i\}$ induces a modified claw. So $w(v_iu_i^{+(p-2)}) = w(u_i^{+(p-2)}u_i^{+(p-1)})$.

Suppose $w(x^-u_i^{+(p-2)}) \neq w(u_i^{+(p-2)}u_i^{+(p-1)})$. Then $w(v_iu_i^{+(p-2)}) \neq w(x^-u_i^{+(p-2)})$. It is clear that w_i is a vertex of the component of $G - \{v_i, u_i^{+(p-1)}, u_i^{+p}\}$ containing $u_i^{+(p-2)}$, and w_i is also a vertex of the component of $G - \{x^-, u_i^{+(p-1)}, u_i^{+p}\}$ containing $u_i^{+(p-2)}$. By the choice of u_i^{+p}, we have $v_iu_i^{+p} \notin E(G)$ or $x^-u_i^{+p} \notin E(G)$. By applying Lemma 4(ii) to $\{x^-, u_i^{+(p-2)}, v_i, u_i^{+(p-1)}, u_i^{+p}\}$, we can get that $w_iu_i^{+(p-2)} \in E(G)$, which leads a contradiction. □

Let $C' = v_i \overleftarrow{C}[v_i, x]xu_iC[u_i, u_i^{+(p-2)}]u_i^{+(p-2)}x^- \overleftarrow{C}[x^-, u_i^{+(p-1)}]u_i^{+(p-1)}v_i$. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that

$$w(u_iv_i) + w(u_i^{+(p-2)}u_i^{+(p-1)}) + w(x^-x) \geqslant w(u_ix) + w(v_iu_i^{+(p-1)}) + w(x^-u_i^{+(p-2)}).$$

By Subclaim 7 and Assumption 1, we get $w(x^-x) > w(u_ix)$.

Case 1.2.2. Otherwise.

By Subclaim 1, it follows from $w_jv_j \in E(G)$ that $x^-u_j \notin E(G)$. This implies that there exists some vertex $x^{+p} \in V(C(x, u_j])$ such that $x^{+p} \notin N(v_i) \cap N(x^-)$. Choose the vertex x^{+p} such that p is as small as possible. By Subclaim 5 and Lemma 3, it follows from $x^+x \in E(G)$ that $x^+ \in N(v_i) \cap N(x^-)$. So we have $p \geqslant 2$. Clearly, if $p = 2$, then $x^{+(p-2)} = x$.

Subclaim 8. $w(u_iv_i) = w(v_ix^{+(p-2)})$ *and* $w(x^-x^{+(p-1)}) = w(x^{+(p-2)}x^{+(p-1)})$.

Proof. By Subclaim 1, it follows from $x^-x^{+(p-1)} \in E(G)$ that $w_ix^{+(p-2)} \notin E(G)$. By Subclaim 2, we have $w_ix^- \notin E(G)$. By the choice of x^{+p}, we have $\{x^{+(p-2)}, x^{+(p-1)}\} \subset \{N(v_i) \cap N(x^-)\}$.

Now $\{v_i, u_i, x^{+(p-2)}, w_i\}$ induces a claw or a modified claw, which implies that $w(u_iv_i) = w(v_ix^{+(p-2)})$.

Suppose $w_ix^{+(p-1)} \in E(G)$. Then $\{x^{+(p-1)}, x^{+(p-2)}, x^-, w_i\}$ induces a modified claw. So $w(x^-x^{+(p-1)}) = w(x^{+(p-2)}x^{+(p-1)})$. Suppose $w_ix^{+(p-1)} \notin E(G)$. Then $\{v_i, x^{+(p-2)}, x^{+(p-1)}, w_i\}$ induces a modified claw. On the other hand, $\{x^{+(p-1)}, x^{+p}, x^-, v_i\}$ induces a claw or a modified claw. So $w(x^{+(p-2)}x^{+(p-1)}) = w(v_ix^{+(p-1)})$ and $w(x^-x^{+(p-1)}) = w(v_ix^{+(p-1)})$. Thus we have $w(x^-x^{+(p-1)}) = w(x^{+(p-2)}x^{+(p-1)})$. □

Let $C' = v_i \overleftarrow{C}[v_i, x^{+(p-1)}]x^{+(p-1)}x^- \overleftarrow{C}[x^-, u_i]u_ixC[x, x^{+(p-2)}]x^{+(p-2)} v_i$. Then C' is a longest cycle different from C. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that $w(u_iv_i) + w(x^{+(p-2)}x^{+(p-1)}) + w(x^-x) \geqslant w(u_ix) + w(v_ix^{+(p-2)}) + w(x^-x^{+(p-1)})$. By Subclaim 8 and Assumption 1, we get $w(x^-x) > w(u_ix)$.

Case 2. $u_ix^- \notin E(G)$.

Suppose $v_ix^- \in E(G)$. By the choice of C and Subclaim 2, we have $w_iu_i \notin E(G)$ and $w_ix^- \notin E(G)$. Then $\{v_i, u_i, x^-, w_i\}$ induce a claw. So we have $w(v_ix^-) = w(u_iv_i)$. Let $C' = u_iC[u_i, x^-]x^-v_i \overleftarrow{C}[v_i, x]xu_i$. Then C' is a longest cycle different from C. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that $w(u_iv_i) + w(x^-x) \geqslant w(u_ix) + w(v_ix^-)$. Since $w(v_ix^-) = w(u_iv_i)$, by Assumption 1, we get $w(x^-x) > w(u_ix)$.

Now we make the following assumption.

Assumption 2. $v_ix^- \notin E(G)$.

Subclaim 9. $v_ix \notin E(G)$.

Proof. Suppose $v_i x \in E(G)$. Applying Lemma 3 to the induced path $u_i x x^-$ and v_i, we have $v_i \in N(u_i) \cap N(x^-)$, contradicting Assumption 2. $\square$

By Assumption 2, there exists a vertex $x^{+q} \in V(C(x, v_i])$ such that $x^{+q} \notin N(u_i) \cap N(x^-)$. Choose x^{+q} such that q is as small as possible. Applying Lemma 3 to the induced path $u_i x x^-$ and x^+, we know that $x^+ \in N(u_i) \cap N(x^-)$. So we have $q \geqslant 2$. Clearly, if $q = 2$, then $x^{+(q-2)} = x$.

Subclaim 10. $xx^{+q} \notin E(G)$.

Proof. If $xx^{+q} \in E(G)$, then from the choice of x^{+q}, $\{x, x^-, x^{+q}, u_i\}$ induces a claw or a modified claw. So $w(x^- x) = w(u_i x)$, contradicting Assumption 1. $\square$

Case 2.1. $xx^{+(q-1)} \in E(G)$.

Subclaim 11. $x^- x^{+q} \in E(G)$.

Proof. By Assumption 1, we have $w(u_i x) \neq w(x^- x)$. Suppose $x^- x^{+q} \notin E(G)$. It is clear that w_i is a vertex of the component of $G - \{x^-, x^{+(q-1)}, x^{+q}\}$ containing x. Applying Lemma 4 (ii) to $\{x^-, x, u_i, x^{+(q-1)}, x^{+q}\}$, we have $w_i x \in E(G)$, contradicting Subclaim 2, so $x^- x^{+q} \in E(G)$. $\square$

By Subclaim 11 and the choice of x^{+q}, we have $u_i x^{+q} \notin E(G)$. Since $u_i x^- \notin E(G)$, we have $x^{-2} \neq u_i$. Now x^{-2} is a vertex of the component of $G - \{u_i, x^{+(q-1)}, x^{+q}\}$ containing x, by applying Lemma 4 to $\{x^-, x, u_i, x^{+(q-1)}, x^{+q}\}$, we have $x^{-2} x \in E(G)$, and

$$w(x^{-2}x^-) = w(x^{-2}x) = w(x^- x^{+(q-1)}) = w(x^- x^{+q}) = w(x^{+(q-1)} x^{+q}) = w(u_i x). \tag{1}$$

Let $C' = x^{-2} \overleftarrow{C}[x^{-2}, x^{+q}] x^{+q} x^- x^{+(q-1)} \overleftarrow{C}[x^{+(q-1)}, x] x x^{-2}$. Then C' is another longest cycle. By the choice of C, we have

$$w(x^{-2}x^-) + w(x^- x) + w(x^{+(q-1)} x^{+q}) \geqslant w(x^{-2}x) + w(x^- x^{+q}) + w(x^- x^{+(q-1)}).$$

By (1), we get $w(x^- x) \geqslant w(x^- x^{+q})$. This implies that $w(x^- x) \geqslant w(u_i x)$. By Assumption 1, we have $w(x^- x) > w(u_i x)$.

Case 2.2. $xx^{+(q-1)} \notin E(G)$.

Now, it is clear that $q \geqslant 3$.

Subclaim 12. $w(u_i x) = w(x^- x^{+(q-1)})$.

Proof. By the choice of x^{+q}, $u_i x^{+(q-1)} \in E(G)$. By Subclaim 9, $\{u_i, v_i, x^{+(q-1)}, x\}$ induces a claw or a modified claw. So $w(u_i x) = w(u_i x^{+(q-1)})$. At the same time, by the choice of x^{+q}, $\{x^{+(q-1)}, x^{+q}, u_i, x^-\}$ induces a claw or a modified claw. So $w(u_i x^{+(q-1)}) = w(x^- x^{+(q-1)})$. This implies that $w(u_i x) = w(x^- x^{+(q-1)})$. $\square$

Subclaim 13. $x^{-2}x, x^{-2}x^{+(q-1)}, u_i x^{-2} \in E(G)$, $x^- x^{+q}, v_i x^{-2}, x^{-2} x^{+q}, x^{+(q-2)} x^{+q} \notin E(G)$.

Proof. By Subclaim 12 and Assumption 1, we get $w(x^- x) \neq w(x^- x^{+(q-1)})$. Applying Lemma 3 to the induced path $x x^- x^{+(q-1)}$ and the vertex x^{-2}, we get $x^{-2}x \in E(G)$ and $x^{-2} x^{+(q-1)} \in E(G)$. By applying Lemma 3 to the induced path $u_i x x^-$ and the vertex x^{-2}, we get $u_i x^{-2} \in E(G)$.

Suppose $x^- x^{+q} \in E(G)$. Applying Lemma 3 to the induced path $x x^- x^{+(q-1)}$ and the vertex x^{+q}, we get $x x^{+q} \in E(G)$, contradicting Subclaim 10.

Suppose $v_i x^{-2} \in E(G)$ ($x^{-2} x^{+q} \in E(G)$, or $x^{+(q-2)} x^{+q} \in E(G)$). Then since u_i is a vertex of the component of $G - \{x, x^{-2}, v_i\}$ ($G - \{x, x^{-2}, x^{+q}\}$, or $G - \{x, x^{+(q-2)}, x^{+q}\}$) containing x^-, by Subclaim 9 and Subclaim 10, applying Lemma 4 (ii) to $\{x^{+(q-1)}, x^-, x, x^{-2}, v_i\}$ ($\{x^{+(q-1)}, x^-, x, x^{-2}, x^{+q}\}$, or $\{x^{+(q-1)}, x^-, x, x^{+(q-2)}, x^{+q}\}$), we have $u_i x^- \in E(G)$, a contradiction. $\square$

Subclaim 14. $w(x^{-2}x^{-}) = w(x^{-2}x) = w(x^{-}x^{+(q-2)}) = w(x^{+(q-2)}x^{+(q-1)})$.

Proof. By the choice of x^{+q}, we have $x^{-}x^{+(q-1)} \in E(G)$ and $x^{-}x^{+(q-2)} \in E(G)$. By Subclaim 13, both $\{x^{+(q-1)}, x^{-2}, x^{-}, x^{+q}\}$ and $\{x^{+(q-1)}, x^{+(q-2)}, x^{-}, x^{+q}\}$ induce modified claws. So we have $w(x^{-}x^{+(q-1)}) = w(x^{-2}x^{-})$ and $w(x^{-}x^{+(q-1)}) = w(x^{-}x^{+(q-2)}) = w(x^{+(q-2)}x^{+(q-1)})$. By Subclaim 9 and Subclaim 13, $\{u_i, x^{-2}, x, v_i\}$ induces a modified claw. So $w(x^{-2}x) = w(u_i x)$. The result follows from Subclaim 12 immediately. $\square$

Let $C' = x^{-2}xC[x, x^{+(q-2)}]x^{+(q-2)}x^{-}x^{+(q-1)}C[x^{+(q-1)}, x^{-2}]x^{-2}$. Then C' is a longest cycle different from C. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that

$$w(x^{-2}x^{-}) + w(x^{+(q-2)}x^{+(q-1)}) + w(x^{-}x) \geqslant w(x^{-2}x) + w(x^{-}x^{+(q-2)}) + w(x^{-}x^{+(q-1)}).$$

It follows from Subclaim 14 that $w(x^{-}x) \geqslant w(x^{-}x^{+(q-1)})$. By Subclaim 12 and Assumption 1, we get $w(x^{-}x) > w(u_i x)$.

Proof of Claim 5. If $w(x^{+}x) = w(u_i x)$, then there is nothing to prove. So we make the following assumption.

Assumption 3. $w(x^{+}x) \neq w(u_i x)$.

We now prove that $w(x^{+}x) > w(u_i x)$.

Subclaim 15. *If $x^{+}v \in E(G)$ for some $v \in V(C(u_i, x))$, then $w_l v^{+} \notin E(G)$ and $w_l v^{-} \notin E(G)$ for every l, $1 \leqslant l \leqslant k$.*

Proof. Suppose there exists a vertex $v \in V(C(u_i, x))$ such that $x^{+}v \in E(G)$ and $w_l v^{+} \in E(G)$ for some l with $1 \leqslant l \leqslant k$. Then we have another cycle $C' = v_i \overleftarrow{C}[v_i, x^{+}]x^{+}v\overleftarrow{C}[v, u_i]u_i\, x\overleftarrow{C}[x, v^{+}]v^{+}w_l\overleftarrow{P_l}u_0 P_i w_i v_i$ which is longer than C, a contradiction.

Suppose there exists a vertex $v \in V(C(u_i, x))$ such that $x^{+}v \in E(G)$ and $w_l v^{-} \in E(G)$ for some l with $1 \leqslant l \leqslant k$. Then we have another cycle $C' = v_i \overleftarrow{C}[v_i, x^{+}]x^{+}vC[v, x]xu_iC[u_i, v^{-}]v^{-}\, w_l\overleftarrow{P_l}u_0 P_i w_i v_i$ which is longer than C, a contradiction. $\square$

Subclaim 16. *$w_l x \notin E(G)$ when $1 \leqslant l \leqslant k$.*

Proof. Suppose $w_l x \in E(G)$ for some l with $1 \leqslant l \leqslant k$. By the choice of C, we have $w_l u_i \notin E(G)$ and $w_l x^{+} \notin E(G)$. Then $\{x, x^{+}, u_i, w_l\}$ induces a claw or a modified claw. So $w(x^{+}x) = w(u_i x)$, contradicting Assumption 3. $\square$

Case 1. $u_i x^{+} \in E(G)$.

Subclaim 17. *$w_i x^{+} \notin E(G)$ and $w_i u_i^{+} \notin E(G)$.*

Proof. Suppose $w_i x^{+} \in E(G)$. By the choice of C and Subclaim 16, we have $w_i u_i \notin E(G)$ and $w_i x \notin E(G)$. Then $\{x^{+}, x, u_i, w_i\}$ induces a modified claw. So $w(x^{+}x) = w(u_i x)$, contradicting Assumption 3.

Suppose $w_i u_i^{+} \in E(G)$. Then we have another cycle $C' = u_i^{+}C[u_i^{+}, x]xu_ix^{+}C[x^{+}, v_i]v_i\, w_i u_i^{+}$ which is longer than C, a contradiction. $\square$

Subclaim 18. *Exactly one of $v_i x$ and $v_i x^{+}$ is an edge of G.*

Proof. If $v_i x \notin E(G)$ and $v_i x^{+} \notin E(G)$, then $\{u_i, x, x^{+}, v_i\}$ induces a modified claw. If $v_i x \in E(G)$ and $v_i x^{+} \in E(G)$, by Subclaim 16, Subclaim 17 and the choice of C, both $\{v_i, x, x^{+}, w_i\}$ and $\{v_i, u_i, x, w_i\}$ induce modified claws. We can always get $w(x^{+}x) = w(u_i x)$, contradicting Assumption 3. $\square$

Case 1.1. $v_i x \notin E(G)$ and $v_i x^{+} \in E(G)$.

Subclaim 19. $w(u_i x^{+}) = w(u_i v_i) = w(v_i x^{+})$.

Proof. By the choice of C and Subclaim 17, we can easily see that $\{v_i, u_i, x^+, w_i\}$ induces a modified claw. So $w(u_i x^+) = w(u_i v_i) = w(v_i x^+)$. $\square$

Subclaim 20. $v_i u_i^+ \in E(G)$.

Proof. Suppose $v_i u_i^+ \notin E(G)$. Then $\{u_i, x, v_i, u_i^+\}$ induced a claw or a modified claw. So $w(u_i v_i) = w(u_i x)$. By Subclaim 19 and Assumption 3, we have $w(v_i x^+) \neq w(x^+ x)$. It is clear that w_i is a vertex of the component of $G - \{v_i, u_i, u_i^+\}$ which contains x^+. Applying Lemma 4 (ii) to $\{x, x^+, v_i, u_i, u_i^+\}$, we get $w_i x^+ \in E(G)$, contradicting Subclaim 17. $\square$

Subclaim 21. $w(v_i u_i^+) = w(u_i u_i^+)$.

Proof. By Subclaim 17, Subclaim 20 and the choice of C, $\{v_i, u_i, u_i^+, w_i\}$ induces a modified claw. So we have $w(v_i u_i^+) = w(u_i u_i^+)$. $\square$

Let $C' = v_i \overleftarrow{C}[v_i, x^+]x^+ u_i x \overleftarrow{C}[x, u_i^+]u_i^+ v_i$. Then C' is a longest cycle different from C. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that $w(u_i v_i) + w(u_i u_i^+) + w(x^+ x) \geqslant w(u_i x^+) + w(v_i u_i^+) + w(u_i x)$. By Subclaim 19, Subclaim 21 and Assumption 3, we get $w(x^+ x) > w(u_i x)$.

Case 1.2. $v_i x \in E(G)$ and $v_i x^+ \notin E(G)$.

Subclaim 22. $w(x^+ x) \neq w(v_i x)$.

Proof. By the choice of C and Subclaim 16, $\{v_i, u_i, x, w_i\}$ induces a modified claw. So $w(u_i x) = w(v_i x)$. By Assumption 3, we have $w(x^+ x) \neq w(v_i x)$. $\square$

By Subclaim 15, it follows from $w_j v_j \in E(G)$ that $x^+ u_j \notin E(G)$. This implies that there exists some vertex $x^{-p} \in V(C[u_j, x))$ such that $x^{-p} \notin N(v_i) \cap N(x^+)$. Choose the vertex x^{-p} such that p is as small as possible. By Subclaim 22 and Lemma 3, it follows from $x^- x \in E(G)$ that $x^- \in N(v_i) \cap N(x^+)$. So we have $p \geqslant 2$. Clearly, if $p = 2$, then $x^{-(p-2)} = x$.

Subclaim 23. $w(u_i v_i) = w(v_i x^{-(p-2)})$ *and* $w(x^+ x^{-(p-1)}) = w(x^{-(p-2)} x^{-(p-1)})$.

Proof. By the choice of x^{-p}, we have $\{x^{-(p-2)}, x^{-(p-1)}\} \subset \{N(v_i) \cap N(x^+)\}$. By Subclaim 15, it follows from $x^+ x^{-(p-2)} \in E(G)$ and $x^+ x^{-(p-1)} \in E(G)$ that $w_i x^{-(p-1)} \notin E(G)$ and $w_i x^{-(p-2)} \notin E(G)$.

Now $\{v_i, u_i, x^{-(p-2)}, w_i\}$ induces a claw or a modified claw, which implies that $w(u_i v_i) = w(v_i x^{-(p-2)})$. At the same time, $\{v_i, x^{-(p-2)}, x^{-(p-1)}, w_i\}$ induces a modified claw and $\{x^{-(p-1)}, x^{-p}, x^+, v_i\}$ induces a claw or a modified claw. This implies that $w(x^{-(p-2)} x^{-(p-1)}) = w(v_i x^{-(p-1)})$ and $w(x^+ x^{-(p-1)}) = w(v_i x^{-(p-1)})$. Thus, $w(x^+ x^{-(p-1)}) = w(x^{-(p-2)} x^{-(p-1)})$. $\square$

Let $C' = v_i \overleftarrow{C}[v_i, x^+]x^+ x^{-(p-1)} \overleftarrow{C}[x^{-(p-1)}, u_i]u_i x \overleftarrow{C}[x, x^{-(p-2)}]x^{-(p-2)} v_i$. Then C' is a longest cycle different from C. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that $w(u_i v_i) + w(x^{-(p-2)} x^{-(p-1)}) + w(x^+ x) \geqslant w(u_i x) + w(v_i x^{-(p-2)}) + w(x^+ x^{-(p-1)})$. By Subclaim 23 and Assumption 3, we get $w(x^+ x) > w(u_i x)$.

Case 2. $u_i x^+ \notin E(G)$.

By Subclaim 15, it follows from $w_j v_j \in E(G)$ that $x^+ u_j \notin E(G)$. This implies that there exists some vertex $x^{-q} \in V(C[u_j, x))$ such that $x^{-q} \notin N(u_i) \cap N(x^+)$. Choose x^{-q} such that q is as small as possible. Applying Lemma 3 to the induced path $u_i x x^+$ and x^-, we know that $x^- \in N(u_i) \cap N(x^+)$, so $q \geqslant 2$. Clearly, if $q = 2$, then $x^{-(q-2)} = x$.

Subclaim 24. $x x^{-q} \notin E(G)$.

Proof. If $xx^{-q} \in E(G)$, then from the choice of x^{-q}, $\{x, x^+, x^{-q}, u_i\}$ induces a claw or a modified claw. So $w(x^+x) = w(u_i x)$, contradicting Assumption 3. □

Subclaim 25. $xx^{-(q-1)} \notin E(G)$.

Proof. Assume not. By the choice of x^{-q}, we know that $x^{-q} \notin N(u_i) \cap N(x^+)$.

Suppose $u_i x^{-q} \notin E(G)$ (or $x^+x^{-q} \notin E(G)$). It is clear that w_i is a vertex of the component of $G - \{u_i, x^{-(q-1)}, x^{-q}\}$ (or $G - \{x^+, x^{-(q-1)}, x^{-q}\}$) containing x. By applying Lemma 4(ii) to $\{x^+, x, u_i, x^{-(q-1)}, x^{-q}\}$, we can get that $w_i x \in E(G)$, contradicting Subclaim 16. □

By Subclaim 25, it is clear that $q \geqslant 3$.

Subclaim 26. $v_i x \notin E(G)$.

Proof. Suppose $v_i x \in E(G)$. By Assumption 3, we have $w(u_i x) \neq w(x^+x)$. By the choice of C, we have $w_i u_i \notin E(G)$. Since w_j is a vertex of the component of $G - \{u_i, v_i, w_i\}$ containing x, by applying Lemma 4(ii) to $\{x^+, x, u_i, v_i, w_i\}$, we have $w_j x \in E(G)$, contradicting Subclaim 16. □

Subclaim 27. $w(u_i x) = w(x^+x^{-(q-1)})$.

Proof. By the choice of x^{-q}, we have $u_i x^{-(q-1)} \in E(G)$. By Subclaims 25 and 26, $\{u_i, v_i, x^{-(q-1)}, x\}$ induces a claw or a modified claw. So $w(u_i x) = w(u_i x^{-(q-1)})$. At the same time, by the choice of x^{-q}, $\{x^{-(q-1)}, x^{-q}, u_i, x^+\}$ induces a claw or a modified claw. So $w(u_i x^{-(q-1)}) = w(x^+x^{-(q-1)})$. This implies that $w(u_i x) = w(x^+x^{-(q-1)})$. □

Subclaim 28. $x^{+2}x, x^{+2}x^{-(q-1)}, u_i x^{+2} \in E(G), x^+x^{-q}, v_i x^{+2}, x^{+2}x^{-q}, x^{-(q-2)}x^{-q} \notin E(G)$.

Proof. By Subclaim 27 and Assumption 3, we get $w(x^+x) \neq w(x^+x^{-(q-1)})$. By Subclaim 25, we have $xx^{-(q-1)} \notin E(G)$. Applying Lemma 3 to the induced path $xx^+x^{-(q-1)}$ and the vertex x^{+2}, we get $x^{+2}x \in E(G)$ and $x^{+2}x^{-(q-1)} \in E(G)$. By applying Lemma 3 to the induced path $u_i x x^+$ and the vertex x^{+2}, we get $u_i x^{+2} \in E(G)$.

Suppose $x^+x^{-q} \in E(G)$. By the choice of x^{-q}, we have $x^+x^{-(q-1)} \in E(G)$. Then, by Subclaims 24 and 25, $\{x^+, x^{-q}, x^{-(q-1)}, x\}$ induces a modified claw. So $w(x^+x) = w(x^+x^{-(q-1)})$. By Subclaim 27, we get $w(u_i x) = w(x^+x)$, contradicting Assumption 3.

Suppose $v_i x^{+2} \in E(G)$ ($x^{+2}x^{-q} \in E(G)$, or $x^{-(q-2)}x^{-q} \in E(G)$). It is clear that w_i is a vertex of the component of $G - \{x, x^{+2}, v_i\}$ ($G - \{x, x^{+2}, x^{-q}\}$, or $G - \{x, x^{-(q-2)}, x^{-q}\}$) containing x^+. By Subclaims 24 and 26, applying Lemma 4(ii) to $\{x^{-(q-1)}, x^+, x, x^{+2}, v_i\}$ ($\{x^{-(q-1)}, x^+, x, x^{+2}, x^{-q}\}$, or $\{x^{-(q-1)}, x^+, x, x^{-(q-2)}, x^{-q}\}$), we have $w_i x^+ \in E(G)$. Now there exists a cycle $C' = w_i v_i \overleftarrow{C}[v_i, x^{+2}]x^{+2}x u_i C[u_i, x^-]x^-x^+w_i$ which is longer than C, contradicting the choice of C. □

Subclaim 29. $w(x^{+2}x^+) = w(x^{+2}x) = w(x^+x^{-(q-2)}) = w(x^{-(q-2)}x^{-(q-1)})$.

Proof. By the choice of x^{-q}, we have $x^+x^{-(q-1)} \in E(G)$ and $x^+x^{-(q-2)} \in E(G)$. By Subclaim 28, both $\{x^{-(q-1)}, x^{+2}, x^+, x^{-q}\}$ and $\{x^{-(q-1)}, x^{-(q-2)}, x^+, x^{-q}\}$ induce modified claws. So we have $w(x^+x^{-(q-1)}) = w(x^{+2}x^+)$ and $w(x^+x^{-(q-1)}) = w(x^+x^{-(q-2)}) = w(x^{-(q-2)}x^{-(q-1)})$. By Subclaims 26 and 28, $\{u_i, x^{+2}, x, v_i\}$ induces a modified claw. So $w(x^{+2}x) = w(u_i x)$. The result follows from Subclaim 27 immediately. □

Let $C' = x^{+2}C[x^{+2}, x^{-(q-1)}]x^{-(q-1)}x^+x^{-(q-2)}C[x^{-(q-2)}, x]xx^{+2}$. Then C' is a longest cycle different from C. By the choice of C, we have $w(C') \geqslant w(C)$. This implies that

$$w(x^{+2}x^+) + w(x^{-(q-2)}x^{-(q-1)}) + w(x^+x) \geqslant w(x^{+2}x) + w(x^+x^{-(q-2)}) + w(x^+x^{-(q-1)}).$$

It follows from Subclaim 29 that $w(x^+x) \geqslant w(x^+x^{-(q-1)})$. By Subclaim 27 and Assumption 3, we get $w(x^+x) > w(u_i x)$.

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