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Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Numerical solution of singularly perturbed convection–diffusion problem using parameter uniform B-spline collocation method

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ARTICLE INFO

Article history:

Received 15 April 2007

Available online 27 January 2009

Submitted by M.D. Gunzburger

Keywords:

Singular perturbation problem

Convection–diffusion problem

B-spline collocation

Shishkin mesh

Boundary layer

Uniform convergence

ABSTRACT

This paper is concerned with a numerical scheme to solve a singularly perturbed convection–diffusion problem. The solution of this problem exhibits the boundary layer on the right-hand side of the domain due to the presence of singular perturbation parameter ε . The scheme involves B-spline collocation method and appropriate piecewise-uniform Shishkin mesh. Bounds are established for the derivative of the analytical solution. Moreover, the present method is boundary layer resolving as well as second-order uniformly convergent in the maximum norm. A comprehensive analysis has been given to prove the uniform convergence with respect to singular perturbation parameter. Several numerical examples are also given to demonstrate the efficiency of B-spline collocation method and to validate the theoretical aspects.

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1. Introduction

Singularly perturbed convection–diffusion problems arise in various branches of science and engineering such as modelling of water quality problems in river networks [2], fluid flow at high Reynolds numbers [10], convective heat transport problem with large Péclet numbers [11], drift diffusion equation of semiconductor device modelling [21], electromagnetic field problem in moving media [8], financial modelling [3] and turbulence model [16]. Normally, boundary and interior layers are present in the solutions of such problems, when the singular perturbation parameter ε is small. These layers are the thin region of the independent variable, where the gradient of the dependent variable is steep or unacceptably large oscillations occur in the numerical solution as the singular perturbation parameter tends to zero. To resolve these layers, two approaches have generally been considered. The first of these involves deriving a method which reflects the nature of the solution in these layers. This method is referred to as fitted operator method. The second approach is to use layer adapted mesh. This approach falls under the class of fitted mesh methods. Layer adapted meshes were first introduced by Bakhvalov [1] in the context of reaction–diffusion problems. In the late 1970s and early 1980s, special meshes for convection–diffusion problem were investigated by Gartland [7], Liseikin [17], Vulcanović [31] and others in order to achieve uniform convergence. The discussion livened up by the introduction of special piecewise uniform meshes by Shishkin [24]. Because of their simple structure, they have attracted much attention and are now widely referred to as Shishkin mesh. Uniformly convergent numerical methods, independent of singular perturbation parameter, were developed over last 25 years (for more detailed discussion see [5,6,18,23] and references therein).

In this paper, we consider the following singularly perturbed convection–diffusion problem

$$Lu(x) \equiv -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0, 1), \quad (1.1a)$$

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with the boundary conditions

$$u(0) = A, \quad u(1) = B, \quad (1.1b)$$

where $0 < \varepsilon \leq 1$ is a small perturbation parameter and A and B are given constants. Further, it is assumed that the functions $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth with

$$a(x) \geq \alpha > 0, \quad x \in \bar{\Omega}, \quad (1.2)$$

$$b(x) \geq \beta > 0, \quad x \in \bar{\Omega}. \quad (1.3)$$

Under these conditions, singularly perturbed convection–diffusion problem (1.1) possesses a unique smooth solution with boundary layer on the right side of the domain $\bar{\Omega}$.

This type of problem has been extensively studied in the literature. O'Malley [19] examined the theoretical aspects, such as existence, uniqueness, and asymptotic behavior of the solution. Various numerical methods based on fitted mesh and the analysis of the uniform convergence with respect to ε have been considered in [5,6,18,23] (see also references given there). A brief survey of numerical techniques for solving singularly perturbed ordinary differential equations is given by Kadalbajoo and Patidar [13]. In this survey paper, they discussed some standard singular perturbation models and the numerical methods developed by numerous researchers during 1984–2000.

Spline approximation method for numerical solutions of singularly perturbed two-point boundary value problems have been studied by various researchers. Uzelac and Surla [29] constructed a uniformly accurate scheme using collocation with classical quadratic polynomial splines on Shishkin meshes. Stojanović [26] introduced the spline collocation method for singular perturbation problem using piecewise quadratic interpolating polynomials as an approximate. Sakai and Usmani [27] gave a concept of B-spline in terms of hyperbolic and trigonometric splines which are different from earlier ones. It is proved that the hyperbolic and trigonometric B-splines are characterized by a convolution of some special exponential functions and a characteristic function on the interval $[0, 1]$. Again Sakai and Usmani [28], considered an application of simple exponential splines to the numerical solution of singular perturbation problem. They found that computational effort involved in their collocation method was less than that required for other exponential type splines. Kadalbajoo and Patidar [12] derived uniformly convergent schemes of order two for these problems using splines in tension and splines in compression. Kadalbajoo and Aggarwal [14] gave the B-spline collocation method of order two with Shishkin mesh for self-adjoint singularly perturbed two-point boundary value problem.

In this paper we propose and analyse B-spline collocation method to solve problem (1.1) with piecewise uniform Shishkin mesh. The principal aim of this paper is to analyse the boundary layer behavior and to provide a layer-resolving parameter-uniform method with sufficient accuracy. Spline collocation methods are more economical and straightforward to use, since they require no numerical integrations as in finite element method or Galerkin approximate. Also, B-spline collocation method in solving differential equations leads to banded matrices with a small number of bands, as opposed to the full matrices one obtains using (say), polynomials, trigonometric functions, and other well-known nonpiecewise approximates. Moreover, the present method does not require any information about the asymptotic approximation of the solution and is easy to implement. In fact, we prove that the B-spline collocation method provides uniform convergence in ε .

The paper is structured as follows. Some analytical results for continuous problem, like comparison principle, stability estimates and a priori bounds for the derivative of the exact solution of the problem (1.1) are presented in Section 2. In Section 3, we introduce the nonuniform mesh of Shishkin type and use B-spline collocation method to solve singularly perturbed two-point boundary value problem. In Section 4, we prove the main theoretical result, namely ε -uniform convergence in the maximum norm. To demonstrate the applicability of the proposed method, some numerical examples have been solved and the results are presented by using piecewise uniform mesh in Section 5. Finally, discussion and conclusion is given at the end of the paper in Section 6. Throughout this paper, C denotes a generic positive constant that is independent of singular perturbation parameter ε and mesh parameter N .

2. Continuous problem

In this section classical bounds for the solution of problem (1.1) and its derivative are derived. We first use the comparison principle to show that the solution of (1.1) is bounded. For any given function $g(x) \in C^k(\bar{\Omega})$ (k a non-negative integer), $\|g\|$ is a global maximum norm over the domain $\bar{\Omega}$ given by

$$\|g\| = \max_{x \in \bar{\Omega}} |g(x)|.$$

The proof of the following comparison principle for the differential operator is standard.

Lemma 1 (Comparison Principle). *Let $y(x) \in C^2(\bar{\Omega})$ satisfying $y(0) \geq 0$, $y(1) \geq 0$, such that $Ly(x) \geq 0$, $\forall x \in \Omega$. Then $y(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Lemma 1 implies the uniqueness of the solution and since the concerned problem is linear, existence of the solution follows by its uniqueness. An immediate consequence of the comparison principle is the following stability estimate for the solution u .

Lemma 2. *If $u(x)$ is the solution of the problem (1.1), then $\forall \varepsilon > 0$ we have*

$$\|u(x)\| \leq \frac{\|f\|}{\alpha} + \max(|A|, |B|), \quad \forall x \in \bar{\Omega}.$$

Proof. Let us consider the two barrier functions $\Psi^\pm(x)$ defined by

$$\Psi^\pm(x) = x \frac{\|f\|}{\alpha} + \max(|A|, |B|) \pm u(x).$$

Then we have

$$\begin{aligned} \Psi^\pm(0) &= \max(|A|, |B|) \pm u(0) \\ &= \max(|A|, |B|) \pm A, \quad \text{since } u(0) = A \\ &\geq 0, \end{aligned}$$

$$\begin{aligned} \Psi^\pm(1) &= \frac{\|f\|}{\alpha} + \max(|A|, |B|) \pm u(1) \\ &= \frac{\|f\|}{\alpha} + \max(|A|, |B|) \pm B, \quad \text{since } u(1) = B \\ &\geq 0, \end{aligned}$$

and

$$L\Psi^\pm(x) = -\varepsilon(\Psi^\pm(x))'' + a(x)(\Psi^\pm(x))' + b(x)\Psi^\pm(x) = a(x)\frac{\|f\|}{\alpha} + b(x)\left[\max(|A|, |B|) + \frac{x\|f\|}{\alpha}\right] \pm f(x).$$

Since $a(x) \geq \alpha > 0$ and $\|f\| \geq f(x)$, we have $a(x)\alpha^{-1}\|f\| \pm f(x) \geq 0$. Using this inequality, we get

$$L\Psi^\pm(x) \geq 0, \quad \forall x \in \Omega.$$

Therefore, by the comparison principle (Lemma 1), we get $\Psi^\pm(x) \geq 0$, $\forall x \in \bar{\Omega}$, giving the desired estimate. \square

Further we derive the bounds for the solution u and its derivative by the following estimates.

Theorem 3. *If $u(x)$ is the solution of the boundary value problem (1.1) and a, b and $f \in C^2(\bar{\Omega})$, then we have*

$$|u^{(i)}(x)| \leq C[1 + \varepsilon^{-i} \exp(-\alpha(1-x)/\varepsilon)], \quad 0 \leq i \leq 3, \quad x \in \bar{\Omega}.$$

Proof. First we shall show that

$$|u^{(k)}(x)| \leq C\varepsilon^{-k} \quad \text{for } 1 \leq k \leq 3.$$

By the mean value theorem, there exists a point $z \in (1 - \varepsilon, 1)$, such that

$$u'(z) = \frac{u(1) - u(1 - \varepsilon)}{\varepsilon},$$

thus

$$|\varepsilon u'(z)| \leq 2\|u\|. \quad (2.1)$$

Integrating the differential equation (1.1a) from z to 1, we get

$$-\varepsilon u'(1) + \varepsilon u'(z) = \int_z^1 (f(t) - a(t)u'(t) - b(t)u(t)) dt, \quad (2.2)$$

where, right-hand side gives

$$\left| \int_z^1 (f(t) - a(t)u'(t) - b(t)u(t)) dt \right| \leq \|f\| |1 - z| + \int_z^1 |a(t)u'(t)| dt + \|b\| \|u\| |1 - z|. \quad (2.3)$$

Here

$$\int_z^1 a(t)u'(t) dt = a(t)u(t)|_z^1 - \int_z^1 a'(t)u(t) dt.$$

Taking modulus on both sides, we get

$$\int_z^1 |a(t)u'(t)| dt \leq (2\|a\| + \|a'\| |1 - z|) \|u\|. \quad (2.4)$$

Now, combining the inequality (2.4) with inequality (2.3), and using Lemma 2 for the bounds on u and $0 \leq |1 - z| \leq 1$, we have

$$\left| \int_z^1 (f(t) - a(t)u'(t) - b(t)u(t)) dt \right| \leq \|f\| + (2\|a\| + \|a'\| + \|b\|) \|u\|. \quad (2.5)$$

Using this inequality with inequalities (2.1) and (2.2), we get $|u'(1)| \leq C\varepsilon^{-1}$, where $C = \|f\| + (2 + 2\|a\| + \|a'\| + \|b\|) \|u\|$. Now using (2.2) with $z = x \in \bar{\Omega}$, we get

$$|u'(x)| \leq C\varepsilon^{-1}, \quad \forall x \in \bar{\Omega},$$

where $C = 2(\|f\| + (1 + 2\|a\| + \|a'\| + \|b\|) \|u\|)$ is independent of ε . Similarly, for $k = 2, 3$, one can easily obtain the bounds by repeatedly differentiating of (1.1a) and using the bounds on u and u' . Now we shall prove

$$|u^{(i)}(x)| \leq C[1 + \varepsilon^{-i} \exp(-\alpha(1 - x)/\varepsilon)], \quad 0 \leq i \leq 3, \quad x \in \bar{\Omega}.$$

We obtain these bounds by following the approach given in [15]. The proof follows by induction. From Lemmas 1 and 2 we have

$$|u(x)| \leq C, \quad \forall x \in \bar{\Omega}. \quad (2.6)$$

Differentiating i times both side of the original equation $Lu = f$ we have

$$Lu^{(i)} = f_i, \quad 1 \leq i \leq 3,$$

where $f_0 = f$ and f_i , $1 \leq i \leq 3$, depends on u, a, b, f and their derivatives of order up to and including i . Using induction hypotheses, the following estimates hold

$$|u^{(i)}(x)| \leq C[1 + \varepsilon^{-i} \exp(-\alpha(1 - x)/\varepsilon)], \quad x \in \bar{\Omega},$$

and

$$|f_i(x)| \leq C[1 + \varepsilon^{-i} \exp(-\alpha(1 - x)/\varepsilon)], \quad x \in \bar{\Omega}.$$

At the boundaries we have $|u^{(i)}(0)| \leq C(1 + \varepsilon^{-i} \exp(-\alpha/\varepsilon)) \leq C(1 + \varepsilon^{-(i-1)})$, because $e^{-\alpha/\varepsilon} \leq \varepsilon \leq 1$, and $|u^{(i)}(1)| \leq C(1 + \varepsilon^{-i})$. Therefore, we have

$$|u^{(i)}(0)| \leq C\varepsilon^{-(i-1)}, \quad |u^{(i)}(1)| \leq C\varepsilon^{-i}. \quad (2.7)$$

Let

$$\theta_i(x) = \frac{1}{\varepsilon} \int_x^1 f_i(t) e^{-(A(x) - A(t))/\varepsilon} dt,$$

where $A(x)$ is the indefinite integral of $a(x)$. The particular solution of the equation $Lu^{(i)} = f_i$ is given by

$$u_p^{(i)}(x) = - \int_x^1 \theta_i(t) dt.$$

Therefore, its general solution can be written as

$$u^{(i)} = u_p^{(i)} + u_h^{(i)},$$

where the homogeneous solution $u_h^{(i)}$ satisfies

$$Lu_h^{(i)} = 0, \quad u_h^{(i)}(0) = u^{(i)}(0) - u_p^{(i)}(0), \quad u_h^{(i)}(1) = u^{(i)}(1).$$

Now introducing the function

$$\phi(x) = \frac{\int_x^1 e^{-A(t)/\varepsilon} dt}{\int_0^1 e^{-A(t)/\varepsilon} dt}. \quad (2.8)$$

It is clear that $L\phi \geq 0$, $\phi(0) = 1$, $\phi(1) = 0$ and $0 \leq \phi(x) \leq 1$. Then $u_h^{(i)}$ is given by

$$u_h^{(i)}(x) = (u^{(i)}(0) - u_p^{(i)}(0))\phi(x) + u^{(i)}(1)(1 - \phi(x)).$$

Thus above leads to the expression for $u^{(i+1)}$ given by

$$u^{(i+1)}(x) = u_p^{(i+1)} + u_h^{(i+1)} = \theta_i(x) + (u^{(i)}(0) - u_p^{(i)}(0) - u^{(i)}(1))\phi'(x).$$

Bounds of $a(x)$ lead to the estimate

$$|\phi'(x)| \leq C\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon}. \quad (2.9)$$

Furthermore, using the estimate for f_i and bounds of $a(x)$ and then evaluating the integral, we have

$$|\theta_i(x)| \leq C(1 + \varepsilon^{-(i+1)}e^{-\alpha(1-x)/\varepsilon}). \quad (2.10)$$

Since $u_p^{(i)}(0) = -\int_0^1 \theta_i(t) dt$, it follows that $|u_p^{(i)}(0)| \leq C\varepsilon^{-i}$. But

$$|u^{(i+1)}(x)| \leq |\theta_i(x)| + (|u^{(i)}(0)| + |u_p^{(i)}(0)| + |u^{(i)}(1)|)|\phi'(x)|.$$

Therefore, from Eqs. (2.7), (2.9) (2.10) and above estimate for $u_p^{(i)}(0)$ we lead to

$$|u^{(i+1)}(x)| \leq C(1 + \varepsilon^{-(i+1)}e^{-\alpha(1-x)/\varepsilon}), \quad x \in \bar{\Omega},$$

which is the required estimate. \square

For numerical analysis below we shall need a decomposition of the solution into regular and singular components. Using the results in Lemma 1 and Lemma 2, we obtain the following estimates.

Theorem 4. If solution $u(x)$ of the problem (1.1) admits the decomposition $u(x) = v(x) + w(x)$, then for all i , $0 \leq i \leq 3$, the regular component $v(x)$ satisfies

$$|v^{(i)}(x)| \leq C[1 + \varepsilon^{-(i-2)} \exp(-\alpha(1-x)/\varepsilon)], \quad \forall x \in \bar{\Omega},$$

and the singular component $w(x)$ satisfies

$$|w^{(i)}(x)| \leq C\varepsilon^{-i} \exp(-\alpha(1-x)/\varepsilon), \quad \forall x \in \bar{\Omega}.$$

Proof. The regular component $v(x)$ can be written into three term asymptotic expansion as

$$v(x) = v_0 + \varepsilon v_1 + \varepsilon^2 v_2, \quad (2.11)$$

now, plugging the solution $u = v + w$ into Eq. (1.1), we obtain the following relation

$$\begin{aligned} a(x)v_0'(x) + b(x)v_0(x) &= f(x), & v_0(0) &= u(0), \\ a(x)v_1'(x) + b(x)v_1(x) &= v_0''(x), & v_1(0) &= 0, \\ -\varepsilon v_2'(x) + a(x)v_2'(x) + b(x)v_2(x) &= v_1''(x), & v_2(0) &= 0, & v_2(1) &= 0. \end{aligned} \quad (2.12)$$

Thus the smooth (regular) component $v(x)$ is the solution of

$$Lv(x) = f(x), \quad v(0) = u(0), \quad v(1) = v_0(1) + \varepsilon v_1(1),$$

and consequently, singular component $w(x)$ is the solution of the homogeneous problem

$$Lw(x) = 0, \quad w(0) = 0, \quad w(1) = u(1) - v(1). \quad (2.13)$$

Since v_0 is the solution of the reduced problem, which is the first order differential equation with bounded coefficient and independent of ε , we have

$$|v_0^{(i)}| \leq C, \quad 0 \leq i \leq 3. \quad (2.14)$$

Similarly, v_1 is independent of ε and we have

$$|v_1^{(i)}| \leq C, \quad 0 \leq i \leq 3. \quad (2.15)$$

Further, $v_2(x)$ is the solution of boundary value problem similar to the boundary value problem (1.1), therefore from Theorem 3, we have

$$|v_2^{(i)}(x)| \leq C(1 + \varepsilon^{-i} e^{-\alpha(1-x)/\varepsilon}), \quad 0 \leq i \leq 3. \quad (2.16)$$

Using Eqs. (2.14)–(2.16) with Eq. (2.11), we get desired bounds on regular component. To obtain the required bounds on the singular component $w(x)$, construct two barrier functions defined by

$$\psi^\pm(x) = |w(1)|e^{-\alpha(1-x)/\varepsilon} \pm w(x).$$

Clearly, $\psi^\pm(0) \geq 0$, $\psi^\pm(1) \geq 0$ and $L\psi^\pm(x) \geq 0$. Comparison principle (Lemma 1) gives $\psi^\pm(x) \geq 0$, $\forall x \in \bar{\Omega}$, which gives

$$|w(x)| \leq Ce^{-\alpha(1-x)/\varepsilon}, \quad \forall x \in \bar{\Omega}, \quad (2.17)$$

where $C = (|u(1)| + |v(1)|)$. Now $w(x)$ can be defined as

$$w(x) = C_1\phi(x) + C_2(1 - \phi(x)),$$

where function $\phi(x)$ is defined by Eq. (2.8) in Theorem 3. Using the values of $\phi(x)$ at $x = 0$ and $x = 1$, we get $C_1 = 0$, $C_2 = w(1) = u(1) - v(1)$. Thus

$$|w'(x)| \leq C|\phi'(x)| \leq C\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon}, \quad \forall x \in \bar{\Omega}. \quad (2.18)$$

Finally, the bounds for the second and third derivatives of $w(x)$ can be estimated immediately from the estimates of $w(x)$ and $w'(x)$. This completes the proof. \square

3. Discrete problem

In this section, we discretize boundary value problem (1.1) using B-spline collocation method on a piecewise uniform mesh of Shishkin type. Shishkin mesh is introduced as follows.

3.1. Shishkin mesh

Shishkin mesh is a piecewise uniform mesh, condensing in the boundary layer region at $x = 1$. The piecewise uniform mesh $\bar{\Omega}^N$ is designed by partitioning the interval $\bar{\Omega} = [0, 1]$ into two subintervals $\Omega_1 = [0, 1 - \tau]$ and $\Omega_2 = (1 - \tau, 1]$ such that $\bar{\Omega} = \Omega_1 \cup \Omega_2$, where the transition parameter τ is chosen such that

$$\tau \equiv \min \left\{ \frac{1}{2}, K\varepsilon \log N \right\}, \quad \text{with } K \geq \frac{1}{\alpha}.$$

It is assumed that $N = 2^r$, where $r \geq 2$ is an integer. This ensures that there is at least one point in the boundary layer region. Moreover, mesh spacing is defined by

$$\tilde{h} = \begin{cases} h_1 = h_i = 2(1 - \tau)/N, & \text{if } i = 1, 2, \dots, N/2, \\ h_2 = h_i = 2\tau/N, & \text{if } i = N/2 + 1, \dots, N, \end{cases} \quad (3.1)$$

where N is the number of discretization points and the set of mesh points $\bar{\Omega}_N = \{x_i\}_{i=0}^N$ with

$$x_i = \begin{cases} (2(1 - \tau)/N)i, & \text{if } i = 0, 1, 2, \dots, N/2, \\ (1 - \tau) + (2\tau/N)(i - N/2), & \text{if } i = N/2 + 1, \dots, N. \end{cases} \quad (3.2)$$

Thus, a uniform mesh is placed on each of these subintervals.

3.2. Methodology of B-spline collocation

We assume X is a linear subspace of $L_2(\bar{\Omega})$, the space of all square integrable functions defined on $\bar{\Omega}$. For $i = -1, 0, \dots, N+1$, the cubic B-splines are defined by [22]:

$$\phi_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x_{i-2} \leq x \leq x_{i-1}, \\ \tilde{h}^3 + 3\tilde{h}^2(x - x_{i-1}) + 3\tilde{h}(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \leq x \leq x_i, \\ \tilde{h}^3 + 3\tilde{h}^2(x_{i+1} - x) + 3\tilde{h}(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \leq x \leq x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \leq x \leq x_{i+2}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.3)$$

Each basis functions $\phi_i(x)$ is twice continuously differentiable, piecewise cubic on the partition $\bar{\Omega}_N : -1 = x_0 < x_1 < x_2 < \dots < x_N = 1$ and $\phi_i(x) \in X$. Let $\beta = \{\phi_{-1}, \phi_0, \dots, \phi_{N+1}\}$ and let $\Phi_3(\bar{\Omega}_N) = \text{span } \beta$. It has already been proven that all $\phi_i(x)$ are linearly independent, thus $\Phi_3(\bar{\Omega}_N)$ is $(N+3)$ -dimensional. Readers can find detailed description of B-spline functions in [20,22,25]. Let L be a linear operator whose domain is X and whose range is also in X . Let $\Phi_3(\bar{\Omega}_N)$ be an $(N+3)$ -dimensional subspace of X . Now suppose the approximate solution of Eq. (1.1) is given by

$$U(x) = \sum_{i=-1}^{N+1} c_i \phi_i(x), \quad (3.4)$$

where c_i are unknown real coefficients and $\phi_i(x)$ are cubic B-spline functions. Here we have introduced two extra cubic B-splines, ϕ_{-1} and ϕ_{N+1} to satisfy the boundary conditions. Furthermore, it is required that the approximate solution $U(x)$ satisfies the given problem (1.1) at mesh points $\bar{\Omega}_N$ as well as boundary conditions at $x = x_0$ and $x = x_N$. Therefore, we have

$$LU(x_i) = f(x_i), \quad 0 \leq i \leq N, \quad (3.5)$$

and

$$U(x_0) = A, \quad U(x_N) = B. \quad (3.6)$$

Solving the collocation equations (3.5), we obtain a system of $(N+1)$ linear equations in $(N+3)$ unknowns

$$\begin{aligned} c_{i-1}(-\varepsilon\phi''_{i-1}(x_i) + a_i\phi'_{i-1}(x_i) + b_i\phi_{i-1}(x_i)) + c_i(-\varepsilon\phi''_i(x_i) + a_i\phi'_i(x_i) + b_i\phi_i(x_i)) \\ + c_{i+1}(-\varepsilon\phi''_{i+1}(x_i) + a_i\phi'_{i+1}(x_i) + b_i\phi_{i+1}(x_i)) = f_i, \quad 0 \leq i \leq N. \end{aligned} \quad (3.7)$$

Furthermore, putting the values of B-spline functions ϕ_i and of derivatives at mesh points $\bar{\Omega}_N$, we get

$$(-6\varepsilon - 3a_i\tilde{h} + b_i\tilde{h}^2)c_{i-1} + (12\varepsilon + 4b_i\tilde{h}^2)c_i + (-6\varepsilon + 3a_i\tilde{h} + b_i\tilde{h}^2)c_{i+1} = f_i\tilde{h}^2, \quad 0 \leq i \leq N. \quad (3.8)$$

The given boundary conditions become

$$c_{-1} + 4c_0 + c_1 = A, \quad (3.9)$$

and

$$c_{N-1} + 4c_N + c_{N+1} = B. \quad (3.10)$$

Thus by Eqs. (3.8), (3.9) and (3.10) we obtain a $(N+3) \times (N+3)$ system with $(N+3)$ unknowns $\{c_{-1}, c_0, \dots, c_{N+1}\}$. Eliminating c_{-1} from first equation of (3.8) and from Eq. (3.9), we find

$$(36\varepsilon + 12a_0\tilde{h})c_0 + 6a_0\tilde{h}c_1 = f_0\tilde{h}^2 - A(-6\varepsilon - 3a_0\tilde{h} + b_0\tilde{h}^2). \quad (3.11)$$

Similarly, eliminating c_{N+1} from the last equation of (3.8) and from (3.10), we get

$$-6a_N\tilde{h}c_{N-1} + (36\varepsilon - 12a_N\tilde{h})c_N = f_N\tilde{h}^2 - B(-6\varepsilon + 3a_N\tilde{h} + b_N\tilde{h}^2). \quad (3.12)$$

Thus, we obtain a system of $(N+1)$ linear equations in $(N+1)$ unknowns

$$Tx^N = d^N, \quad (3.13)$$

where T is the matrix of the corresponding system and $x^N = (c_0, c_1, \dots, c_N)^T$ are the unknown real coefficients. The elements of the matrix T are given by

$$\begin{cases} t_{0,0} = 36\varepsilon + 12a_0\tilde{h}, \\ t_{0,1} = 6a_0\tilde{h}, \\ t_{i,i+1} = -6\varepsilon + 3a_i\tilde{h} + b_i\tilde{h}^2, & i = 1, 2, \dots, N-1, \\ t_{i,i} = 12\varepsilon + 4b_i\tilde{h}^2, & i = 1, 2, \dots, N-1, \\ t_{i,i-1} = -6\varepsilon - 3a_i\tilde{h} + b_i\tilde{h}^2, & i = 1, 2, \dots, N-1, \\ t_{N,N-1} = -6a_N\tilde{h}, \\ t_{N,N} = 36\varepsilon - 12a_N\tilde{h}, \\ t_{i,j} = 0, & \forall |i-j| > 1. \end{cases} \quad (3.14)$$

The entries of right-hand side column vector d^N are given by

$$\begin{cases} d_0^N = f_0\tilde{h}^2 - A(-6\varepsilon - 3a_0\tilde{h} + b_0\tilde{h}^2), \\ d_i^N = f_i\tilde{h}^2, & i = 1, 2, \dots, N-1, \\ d_N^N = f_N\tilde{h}^2 - B(-6\varepsilon + 3a_N\tilde{h} + b_N\tilde{h}^2). \end{cases} \quad (3.15)$$

It is easily seen that collocation matrix T is strictly diagonally dominant and hence nonsingular. Therefore, we can solve the linear system (3.13) uniquely for real unknowns c_0, c_1, \dots, c_N and then using the boundary conditions (3.9), (3.10) we obtain c_{-1} and c_{N+1} . Hence the method of collocation using a basis of cubic B-splines applied to problem (1.1) has a unique solution $U(x)$ given by (3.4).

4. Stability and convergence analysis

In this section we give the stability estimate for the approximate solution U and ε -uniform convergence estimate in the maximum norm and conclude that the present B-spline collocation method has a uniform convergence of order two with Shishkin mesh.

Lemma 5. *The third degree B-splines $\{\phi_{-1}, \phi_0, \dots, \phi_{N+1}\}$ satisfy the following inequality*

$$\sum_{i=-1}^{N+1} |\phi_i(x)| \leq 10, \quad 0 \leq x \leq 1.$$

Proof. The proof easily follows by the definition of third degree B-spline given by Eq. (3.3). \square

Now we shall prove the following stability estimate for the collocation approximate $U(x)$ to the solution $u(x)$ of the problem (1.1).

Theorem 6. *If $U(x)$ be the cubic B-spline collocation approximate from the space of cubic splines $\Phi_3(\bar{\Omega}_N)$ to the solution $u(x)$ of the problem (1.1), then for sufficiently small value of \tilde{h} and ε , we have*

$$|U(x)| \leq C, \quad x \in \bar{\Omega}.$$

Proof. In the previous section, we see that T is strictly diagonally dominant. Therefore, by a result in [30], for sufficiently small value of \tilde{h} and ε , we have

$$\|T^{-1}\| \leq \frac{1}{\min(2\beta\tilde{h}^2, 36\varepsilon + 6\alpha\tilde{h})} \leq \frac{C}{\tilde{h}^2},$$

where

$$a_i \geq \alpha > 0, \quad b_i \geq \beta > 0, \quad 0 \leq i \leq N.$$

Thus

$$\|x^N\| \leq \|T^{-1}\| \|d^N\| \leq C.$$

Due to boundary conditions, the coefficients c_{-1} and c_{N+1} are also bounded. Therefore,

$$\begin{aligned} |U(x)| &= \left| \sum_{i=-1}^{N+1} c_i \phi_i(x) \right| \leq \sum_{i=-1}^{N+1} |c_i| |\phi_i(x)|, \quad x \in \bar{\Omega}, \\ &\leq \max |c_i| \sum_{i=-1}^{N+1} |\phi_i(x)|, \quad x \in \bar{\Omega}. \end{aligned}$$

Since, by Lemma 5, we have

$$\sum_{i=-1}^{N+1} |\phi_i(x)| \leq 10, \quad x \in \bar{\Omega},$$

it follows that

$$|U(x)| \leq C, \quad x \in \bar{\Omega}. \quad \square$$

Now, ε -uniform convergence estimate is given in the following theorem.

Theorem 7. Let $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$ and $f(x)$ be sufficiently smooth function so that $u \in C^4[0, 1]$ be the solution of problem (1.1), and let U be the cubic B-spline collocation approximate on the piecewise uniform mesh. Then for sufficiently large value of N (independent of ε), error component satisfies the following error estimate

$$\sup_{0 < \varepsilon \leq 1} \|U - u\|_{\bar{\Omega}} \leq CN^{-2}(\log N)^2.$$

Proof. The solution U of the discrete problem is decomposed into the smooth component V and singular component W , as in the case of continuous problem. Thus

$$U = V + W,$$

where V is the solution of the inhomogeneous problem given by

$$LV = f, \quad V(0) = v(0), \quad V(1) = v(1),$$

and W is the solution of the homogeneous problem

$$LW = 0, \quad W(0) = w(0), \quad W(1) = w(1).$$

Thus the error can be written in the form

$$U - u = (V - v) + (W - w).$$

Here we use de Boor [4] and Hall [9] spline interpolation error estimates to derive ε -uniform error estimate. By using de Boor–Hall error estimates and matrix analysis and simplifying, we are lead to the following ε -uniform error estimate

$$\sup_{0 < \varepsilon \leq 1} \|U - u\|_{\bar{\Omega}} \leq Ch_c^2 \max_{\bar{\Omega}} |u''|, \quad (4.1)$$

where $h_c = \max\{h_1, h_2\}$. Noe the ε -uniform convergence estimate is obtained on each subinterval $\Omega_i = (x_{i-1}, x_i)$, $\forall i = 1, 2, \dots, N$, separately. Each finite subinterval Ω_i is covered by four cubic B-spline basis functions, therefore the B-spline collocation approximation U of u , on Ω_i , is given by

$$U = c_{i-2}\phi_{i-2} + c_{i-1}\phi_{i-1} + c_i\phi_i + c_{i+1}\phi_{i+1}.$$

It is obvious that on Ω_i

$$|U(x)| \leq \max_{\Omega_i} |u(x)|. \quad (4.2)$$

By ε -uniform error estimate (4.1), it is easy to see that

$$|U(x) - u(x)| \leq Ch_i^2 \max_{\Omega_i} |u''(x)|. \quad (4.3)$$

By Theorem 3, the following estimates holds for the solution u of (1.1) and its derivatives

$$|u^{(k)}(x)| \leq C\varepsilon^{-k}, \quad \text{for } 0 \leq k \leq 3.$$

Therefore, from Eq. (4.3), we have

$$|U(x) - u(x)| \leq C \frac{h_i^2}{\varepsilon^2}. \quad (4.4)$$

Furthermore, using Eqs. (4.2) and (4.3), on Ω_i , we have

$$\begin{aligned}
|U(x) - u(x)| &= |V(x) + W(x) - v(x) - w(x)| \\
&\leq |V(x) - v(x)| + |W(x)| + |w(x)| \\
&\leq Ch_i^2 \max_{\Omega_i} |v''(x)| + 2 \max_{\Omega_i} |w(x)| \\
&\leq C(h_i^2 + e^{-\alpha(1-x_i)/\varepsilon}),
\end{aligned} \tag{4.5}$$

since, by Theorem 4, we have $|v''(x)| \leq C[1 + \exp(-\alpha(1-x)/\varepsilon)] \leq C$ and $|w(x)| \leq C \exp(-\alpha(1-x)/\varepsilon)$, $\forall x \in \bar{\Omega}$. The required ε -uniform estimate depends on whether $K\varepsilon \log N \geq 1/2$ or $K\varepsilon \log N \leq 1/2$. In the first case mesh is uniform with mesh spacing $h_i = 1/N$ and $1/\varepsilon \leq C \log N$. Using this argument in Eq. (4.4), we easily obtain desired ε -uniform estimates.

In the second case $K\varepsilon \log N \leq 1/2$, so we have transition parameter $\tau = K\varepsilon \log N$. In this case mesh is piecewise uniform and $h_i = 2\tau/N$ for i satisfying $N/2 + 1 \leq i \leq N$ in the boundary layer region. Therefore

$$\frac{h_i}{\varepsilon} = \frac{2\tau}{N\varepsilon} = CN^{-1} \log N, \quad N/2 + 1 \leq i \leq N.$$

The result immediately follows combining this with Eq. (4.4). On the other hand, if i satisfies $1 < i \leq N/2$ in no boundary layer region, then $\tau \leq 1 - x_i$ and therefore

$$e^{-\alpha(1-x_i)/\varepsilon} \leq e^{-\alpha\tau/\varepsilon} = e^{-\alpha K \log N} = N^{-\alpha K} = N^{-2},$$

whenever $K = 2/\alpha$ in the definition of transition parameter τ . Using this in (4.5), we get the required estimates. Hence the method is uniformly convergent of order two in the discrete maximum norm. \square

5. Numerical examples and results

Two numerical examples are considered and solved to demonstrate the applicability of proposed method.

Example 1. This example corresponds to the following singularly-perturbed homogeneous boundary value problem:

$$-\varepsilon u''(x) + u'(x) + (1 + \varepsilon)u(x) = 0, \quad x \in (0, 1), \tag{5.1a}$$

$$u(0) = 1 + \exp[-(1 + \varepsilon)/\varepsilon], \quad u(1) = 1 + [1/\exp(1)]. \tag{5.1b}$$

Its exact solution is given by

$$u(x) = \exp[(1 + \varepsilon)(x - 1)/\varepsilon] + \exp(-x). \tag{5.2}$$

Since the problem has an analytical solution, therefore, for every ε the computed maximum pointwise errors are estimated by

$$E_\varepsilon^N = \max_{x_i \in \Omega_N} |u(x_i) - U^N(x_i)|, \tag{5.3}$$

where U^N denotes the numerical solution obtained by using N finite elements. The ε -uniform maximum pointwise error is estimated as

$$E^N = \max_\varepsilon E_\varepsilon^N. \tag{5.4}$$

Furthermore, the numerical order of convergence is obtained by

$$p_{\varepsilon, N} = \frac{\log(E_\varepsilon^N/E_\varepsilon^{2N})}{\log 2}. \tag{5.5}$$

The estimated maximum pointwise error and the numerical order of convergence are presented in Table 1 with piecewise uniform mesh.

Example 2. Now we consider the following nonhomogeneous singularly perturbed boundary value problem:

$$-\varepsilon u''(x) + u'(x) + u(x) = \cos \pi x, \quad x \in (0, 1), \tag{5.6a}$$

$$u(0) = 0, \quad u(1) = 0, \tag{5.6b}$$

which has the analytical solution given by

$$u(x) = a \cos \pi x + b \sin \pi x + Ae^{\lambda_0 x} + Be^{-\lambda_1(1-x)}, \tag{5.7}$$

Table 1

Maximum pointwise errors and order of convergence for Example 1.

$\varepsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^0	2.3803E-4	5.9435E-5	1.4854E-5	3.7133E-6	9.284E-7	2.321E-7	5.8022E-8
	2.0018	2.0004	2.0001	1.9999	2.0000	2.0001	
10^{-2}	2.9793E-2	1.1024E-2	3.1798E-3	1.2252E-3	4.1518E-4	1.3191E-4	4.4547E-5
	1.4343	1.7936	1.3759	1.5612	1.6542	1.5661	
10^{-4}	3.2912E-2	1.4195E-2	6.3383E-3	2.9970E-3	1.4339E-3	6.8495E-4	3.2200E-4
	1.2133	1.1632	1.0806	1.0636	1.0659	1.0889	
10^{-6}	3.2918E-2	1.4156E-2	6.2306E-3	2.8522E-3	1.3590E-3	6.7467E-4	3.4811E-4
	1.2174	1.1840	1.1273	1.0605	1.0103	0.9546	
10^{-8}	3.2918E-2	1.4156E-2	6.2292E-3	2.8489E-3	1.3518E-3	6.6094E-4	3.2841E-4
	1.2175	1.1843	1.1287	1.0755	1.0323	1.0090	
10^{-10}	3.2918E-2	1.4156E-2	6.2291E-3	2.8488E-3	1.3517E-3	6.6079E-4	3.2810E-4
	1.2175	1.1843	1.1287	1.0756	1.0326	1.0101	
10^{-12}	3.2925E-2	1.4175E-2	6.2236E-3	2.8524E-3	1.3544E-3	6.6086E-4	3.2835E-4
	1.2159	1.1875	1.1256	1.0746	1.0352	1.0091	

Table 2

Maximum pointwise errors and order of convergence for Example 2.

$\varepsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^0	1.2470E-4	3.1343E-5	7.8378E-6	1.9603E-6	4.901E-7	1.225E-7	3.06E-8
	1.9923	1.9996	1.9994	1.9999	2.0003	2.0012	
10^{-2}	3.7653E-2	1.6438E-2	6.6001E-3	2.4825E-3	6.4332E-4	1.6137E-4	4.0214E-5
	1.1957	1.3165	1.4107	1.9482	1.9952	2.0046	
10^{-4}	4.8420E-2	2.4642E-2	1.2649E-2	6.5826E-3	3.3112E-3	1.6136E-3	7.6340E-4
	0.9745	0.9621	0.9483	1.0810	0.9474	1.0798	
10^{-6}	4.8444E-2	2.4538E-2	1.2364E-2	6.2130E-3	3.1264E-3	1.5893E-3	8.2601E-4
	0.9812	0.9889	0.9927	0.9908	0.9761	0.9442	
10^{-8}	4.8444E-2	2.4537E-2	1.2360E-2	6.2044E-3	3.1086E-3	1.5561E-3	7.7904E-4
	0.9814	0.9893	0.9943	0.9970	0.9983	0.9982	
10^{-10}	4.8444E-2	2.4537E-2	1.2360E-2	6.2043E-3	3.1084E-3	1.5558E-3	7.7828E-4
	0.9814	0.9893	0.9943	0.9971	0.9985	0.9993	
10^{-11}	4.8444E-2	2.4537E-2	1.2360E-2	6.2043E-3	3.1083E-3	1.5557E-3	7.7827E-4
	0.9814	0.9893	0.9943	0.9971	0.9985	0.9993	
10^{-12}	4.8443E-2	2.4536E-2	1.2359E-2	6.2036E-3	3.1076E-3	1.5550E-3	7.7752E-4
	0.9814	0.9893	0.9944	0.9973	0.9989	1.0000	

where $\lambda_0(x) < 0$ and $\lambda_1(x) > 0$ are the real solutions of the characteristic equation

$$-\varepsilon\lambda^2(x) + \lambda(x) + 1 = 0,$$

and

$$a = \frac{\varepsilon\pi^2 + 1}{\pi^2 + (\varepsilon\pi^2 + 1)^2}, \quad b = \frac{\pi}{\pi^2 + (\varepsilon\pi^2 + 1)^2}, \quad A = -a \frac{1 + e^{-\lambda_1}}{1 - e^{\lambda_0 - \lambda_1}}, \quad B = a \frac{1 + e^{\lambda_0}}{1 - e^{\lambda_0 - \lambda_1}}.$$

The maximum pointwise and ε -uniform maximum pointwise errors and numerical order of convergence are calculated as in Example 1. The numerical results are displayed in Table 2 with piecewise uniform mesh.

6. Discussions and conclusions

A parameter uniform numerical method based on B-spline collocation with piecewise uniform mesh is presented to solve the boundary value problems for singularly perturbed differential equations of the convection–diffusion type. The solution of the problem exhibits the boundary layer on the right side of the domain. The width τ of boundary layer region plays an important role to solve singular perturbation problem, therefore transition parameter τ has to be defined with some care. The theoretical analysis is presented to show that the proposed method is parameter uniform of order two, i.e., the method converges independently of the singular perturbation parameter ε .

Numerical results presented in Tables 1 and 2 clearly indicate that the proposed method with Shishkin mesh is independent of mesh size h and singular perturbation parameter ε , and parameter uniform. To further corroborate the applicability of the proposed method, numerical solution profiles have been plotted in Figs. 1–4 for Examples 1 and 2 for the exact solution versus computed solution obtained at the value of $\varepsilon = 2^{-6}$, 2^{-8} and $N = 32$, 64 for uniform and piecewise uniform mesh respectively. It has been seen that exact and numerical solutions with uniform mesh are identical for most of the region of the domain except in the boundary layer regions. To control these deviations in the boundary layer region, we use Shishkin mesh to take more mesh points in the boundary layer region and the resulting behavior is seen in Figs. 1–4 with

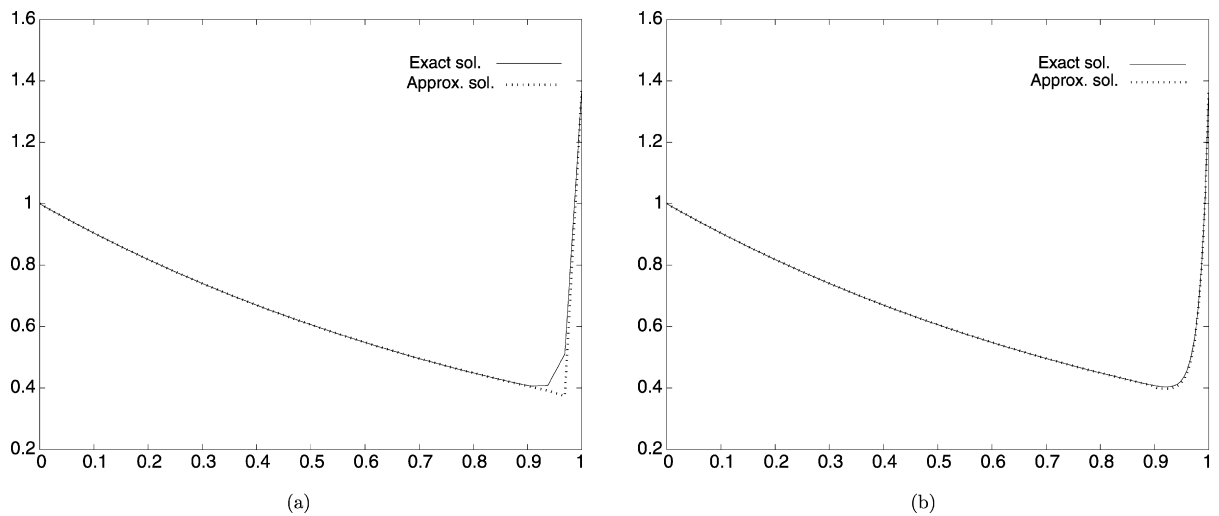


Fig. 1. Exact and approximate solutions of Example 1 for $\varepsilon = 2^{-6}$ and $N = 32$ with (a) uniform mesh and (b) nonuniform mesh.

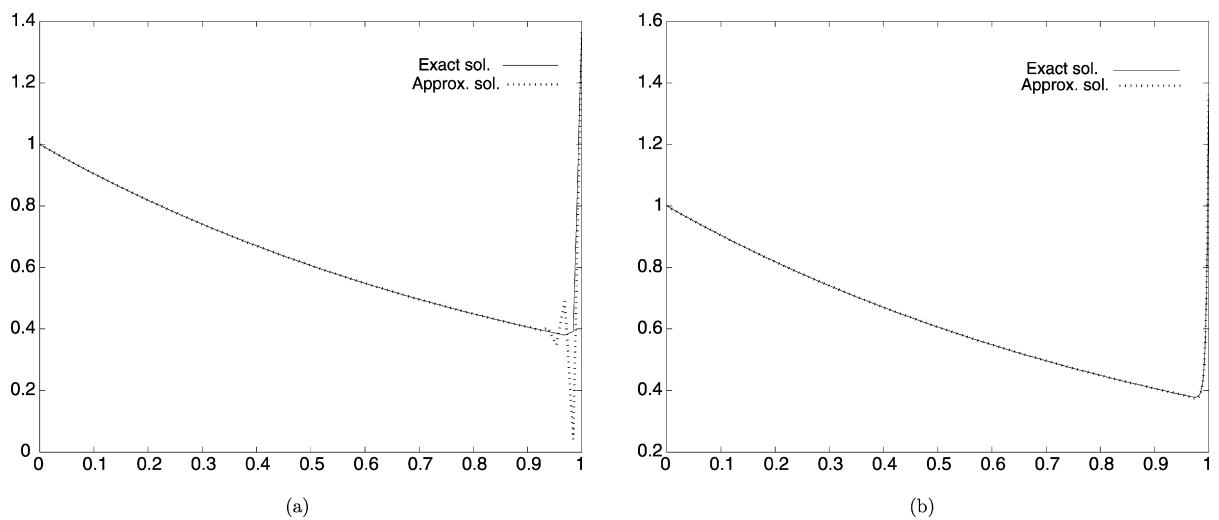


Fig. 2. Exact and approximate solutions of Example 1 for $\varepsilon = 2^{-8}$ and $N = 64$ with (a) uniform mesh and (b) nonuniform mesh.

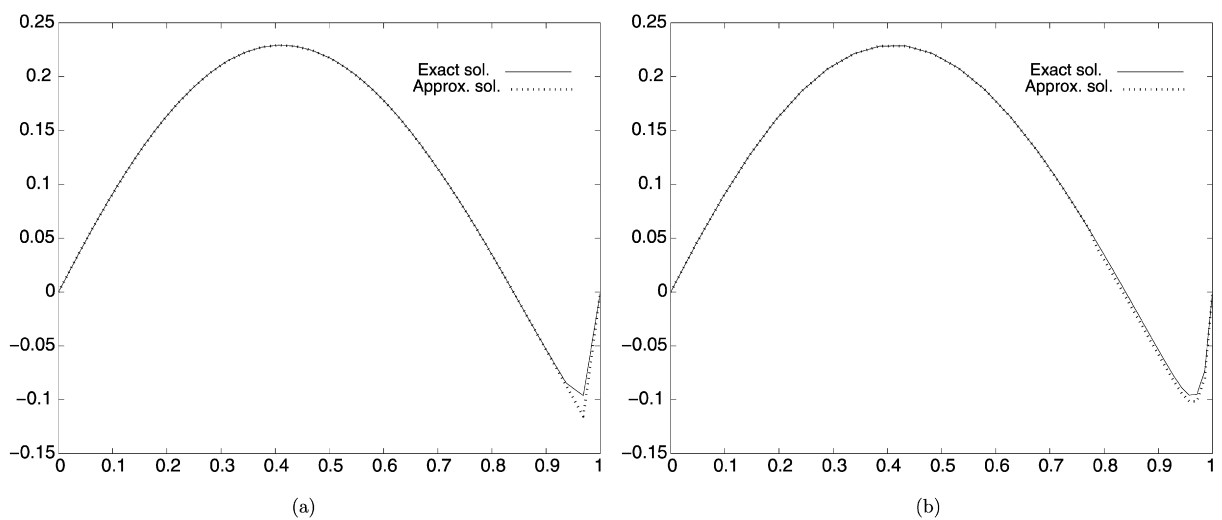


Fig. 3. Exact and approximate solutions of Example 2 for $\varepsilon = 2^{-6}$ and $N = 32$ with (a) uniform mesh and (b) nonuniform mesh.

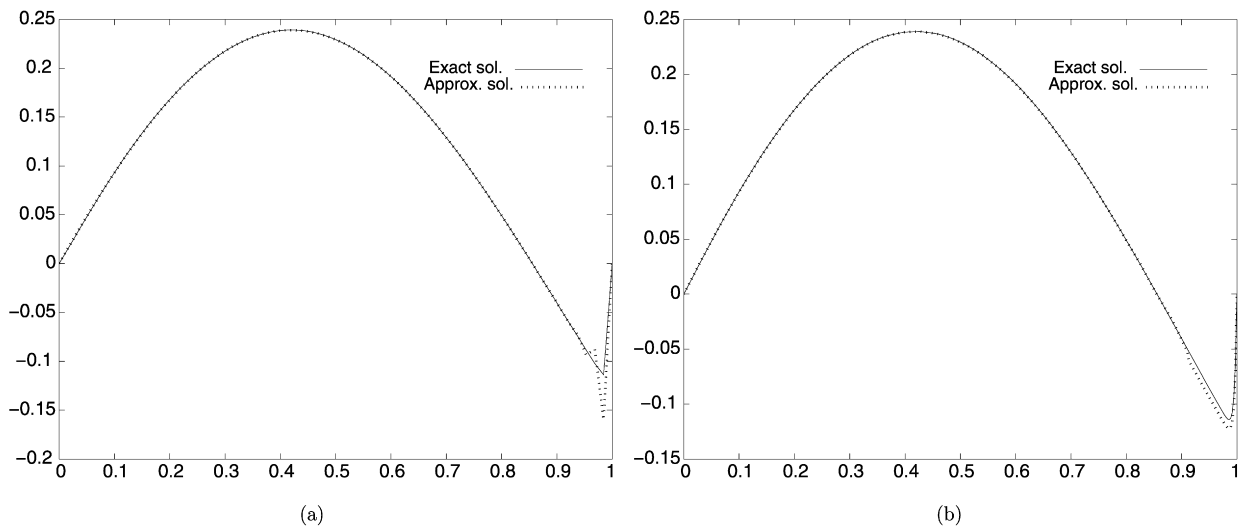


Fig. 4. Exact and approximate solutions of Example 2 for $\varepsilon = 2^{-8}$ and $N = 64$ with (a) uniform mesh and (b) nonuniform mesh.

nonuniform mesh. Also, it can be noticed that maximum pointwise errors arise near the transition point due to the abrupt changes in the mesh size.

Thus the present method is second order accurate and numerical results support the theoretical estimates. The proposed algorithms gives, in fact, more accurate results than many of other boundary layer resolving finite difference methods. Here we see that such collocation methods are closely related to Galerkin methods, hence to finite-element methods, but are much easier and more efficient in computation than Galerkin methods. The collocation matrix involves no numerical quadrature, which both increase the operation count and may result in some loss of accuracy to the matrix approximations. Therefore the collocation system is set up rather easily. Also this method ensure that the solution is, at least, continuous in the domain $\bar{\Omega}$, whereas the finite difference methods give the solution only at the chosen mesh points.

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