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Vector-Valued Entire Functions Satisfying a Differential Equation

RANJAN ROY

Department of Mathematics, Beloit College, Wisconsin 53511

AND

S. M. SHAH

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

Submitted by R. P. Boas

We consider vector-valued functions, with components which are entire functions, for growth problems. All such functions, when they satisfy a class of differential equations, are of bounded index and exponential type, and their components are also of bounded index.

1. INTRODUCTION

Let $F: C^1 \to C^m$ be a vector-valued function whose components $f_k: C^1 \to C^1$ are all entire functions. We write

$$F(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix}$$

which for convenience in printing we shall write

$$F(z) = (f_1(z), \dots, f_m(z)).$$

We now define two norms for F: (i) the sup norm

$$\| F(z) \|_s = \max\{ | f_i(z) |;\ 1 \leqslant i \leqslant m \} \tag{1.1}$$

and (ii) the euclidean norm

$$\| F(z) \|_E = \left\{ \sum_{i=1}^{M} | f_i(z) |^2 \right\}^{1/2}. \tag{1.2}$$

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DEFINITION. A vector-valued entire function F is said to be of bounded index (BI) with sup norm if there exists an integer $N = N_s$ such that

$$\max_{0 \leqslant i \leqslant N} \frac{\| F^{(i)}(z) \|_s}{i!} \geqslant \frac{\| F^{(k)}(z) \|_s}{k!} \tag{1.3}$$

for all $z \in \mathbf{C}$ and $k = 0, 1, \ldots$. The least such integer N_s is called the index of F.

If we use in (1.3) the euclidean norm we will have a class of functions of BI with euclidean norm and index N_E. The following theorem shows that these two definitions of BI are equivalent.

THEOREM 1. *If F is of BI with supnorm then it is of BI with euclidean norm and vice versa. The two indices N_s and N_E may possibly be different.*

Note that this definition of BI for vector-valued entire functions is similar to that for scalar functions. See Lepson [7], and Shah [9]. For vector-valued functions of BI with sup norm see Heath [5]. In the following we will use the sup norm definition (1.3) of BI and write $N_s = N$.

We now show that even if F is of BI, the components f_k not be of BI.

EXAMPLE 1.1. Let $\{k_n\}_1^\infty$ be a strictly increasing sequence of positive integers and let $\{a_n\}_1^\infty$ be a strictly increasing sequence of positive numbers. Then

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right)^{k_n}$$

where $\sum k_n/a_n < \infty$ is an entire function of unbounded index for all such choices of $\{k_n\}$ and $\{a_n\}$. By suitably choosing such $\{k_n\}$ and $\{a_n\}$ we can show that (Shah [10]) $f(z) - c$, where $c \in \mathbf{C}$, $c \neq 0$, is of BI. Hence $F(z) = (f(z), f(z) - c)$ is of BI but one component f is not of BI. (See also Heath [5].)

If we assume that F satisfies a differential equation (DE) with coefficients which are matrices with entries in R (see Theorem 2 below) then we can show that each component is of BI.

Let R denote the class of all rational functions $r(z)$ bounded at infinity, and $Q_i(z)$ $(1 \leqslant i \leqslant n)$ denote an $m \times m$ matrix with entries in R. Write

$$Q_i(z) = (a_{pq,i}(z)), \qquad \lim_{z \to \infty} |a_{pq,i}(z)| = |A_{pq,i}| \tag{1.4}$$

and

$$\sup(|A_{pq,i}|, 1 \leqslant p, q \leqslant m) = |A_i|.$$

THEOREM 2. *Let $F\colon C^1 \to C^m$ be a vector-valued function whose components $f_1, f_2, ..., f_m$ are all entire functions. Suppose that F satisfies the DE*

$$L_n(W, z, Q) = W^{(n)}(z) + Q_1(z)\, W^{(n-1)}(z) + \cdots + Q_n(z)\, W(z) = 0. \quad (1.5)$$

Then each f_j satisfies a DE of the form (1.5) (*with poosibly different n and coefficients*) *and hence each f_j is of BI.*

2. GROWTH BOUNDS

Let

$$M(r, F) = \max_{|z| = r} \| F(z) \|.$$

THEOREM 3. *If $F(z)$is of BI with index N, then*

$$\| F(z) \| \leqslant A \exp((N+1)\, |z|), \quad (2.1)$$

where

$$A = \max_{0 \leqslant k \leqslant N} \| F^{(k)}(0) \| / (N+1)^k.$$

The result is sharp.

THEOREM 4. *If $X(z)$ is a vector-valued entire solution of the DE*

$$L_n(W, z, Q) = g(z) \quad (2.2)$$

where $L_n(W, z, Q)$ is as in (1.5) *and $g(z)$ is a vector-valued entire function of BI, then $X(z)$ is of BI.*

We give later an example to show that if the entries in Q_i are not in R, then F may not be of *BI*.

Note that if $g'(z)$ is of *BI*, then on differentiating (2.2) we see that $X'(z)$ is also of *BI*. We give an example to show that in general $F(z)$ may be of *BI* but $F'(z)$ may not be.

We next consider the DE (2.2) when $g(z) \equiv 0$ and obtain (i) bound on the index and (ii) bounds on $\| W \|$ and $M(r, W)$.

THEOREM 5. *Let $W(z) \neq 0$ be a vector-valued entire function satisfying the DE*

$$L_n(W, z, Q) = 0$$

where Q_i $(1 \leqslant i \leqslant n)$ are all matrices with constant elements, and $p \geqslant 0$ is any integer such that

$$m\left[\frac{|A_1|}{n+p}+\frac{|A_2|}{(n+p)(n+p-1)}+\cdots+\frac{|A_n|}{(n+p)\cdots(p+1)}\right]\leqslant 1 \quad (2.3)$$

then the index $N(W)$ of $W(z)$, is less than or equal to $n+p-1$. The bound on N is best possible.

THEOREM 6. *Let $W(z)$ be a vector-valued entire function satisfying the DE.*

$$L_n(W, z, Q)=0.$$

Then

$$\limsup_{r\to 0}\frac{\log M(r, W)}{r}\leqslant \max\left\{1, m\sum_1^n |A_i|\right\}. \quad (2.4)$$

THEOREM 7. *Let q be the least positive integer such that $m\{|A_1|\,(n+q-1)!+\cdots+|A_n|q!\}<(n+q)!$, where $|A_i|$ are as in Theorem 6. If $W(z)$ is a vector-valued entire function satisfying the DE.*

$$L_n(W, z, Q)=0,$$

then

$$\| W(z)\| \leqslant A\exp\{(n+q)\,|z|\} \quad (2.5)$$

where A is a constant.

3. PROOF OF THEOREM 1

(i) Suppose F is of BI with sup norm, BI(s), and index N_s. Write

$$\| F^{(k)}\|_s=\| F^{(k)}\| \qquad \text{and} \qquad N_s=N.$$

Then

$$\max_{0\leqslant k\leqslant N}\frac{\| F^{(k)}\|}{k!}\geqslant\frac{\| F^{(N+1)}\|}{(N+1)!}.$$

Now for all $k\geqslant 0$,

$$\| F^{(k)}\|\leqslant\| F^{(k)}\|_E\leqslant\sqrt{m}\,\| F^{(k)}\|.$$

Hence

$$\frac{\|F^{(N+1)}\|_E}{(N+1)!} \leqslant \sqrt{m}\frac{\|F^{(N+1)}\|}{(N+1)!} \leqslant \sqrt{m} \max_{0\leqslant k\leqslant N} \frac{\|F^{(k)}\|}{k!}$$

$$\leqslant \sqrt{m} \max_{0\leqslant k\leqslant N} \frac{\|F^{(k)}\|_E}{k!},$$

that is,

$$\|F^{(N+1)}\|_E \leqslant C_1 \max_{0\leqslant k\leqslant N} \frac{\|F^{(k)}\|_E}{k!} \leqslant C \max_{0\leqslant k\leqslant N} \|F^{(k)}\|_E. \tag{3.1}$$

Here C_1 and C are constants and we may suppose $C > 1$. Let $\alpha \in \mathbf{C}$, $|\alpha| = 1$. Fix α and consider for $x \geqslant 0$,

$$G(x) = \max_{0\leqslant j\leqslant N} \|F^{(k)}(z_0 + \alpha x)\|_E.$$

We use (3.1) and obtain (cf. Hayman [4, Theorem 2])

$$G(x) \leqslant G(0)\exp(Cx).$$

Writing $z = z_0 + \alpha x$, we have

$$\max_{0\leqslant j\leqslant N} \|F^{(k)}(z)\|_E \leqslant \exp(C|z - z_0|) \max_{0\leqslant j\leqslant N} \|F^{(k)}(z_0)\|_E. \tag{3.2}$$

Now for any component f_j and $R \geqslant 1$ we have

$$|f_j^{(n)}(z_0)| \leqslant \frac{1}{R^n} \max_{|z-z_0|=R} |f_j(z)|$$

$$\leqslant \frac{1}{R^n} \max_{|z-z_0|=R} \|F(z)\|_E,$$

and so by (3.2)

$$|f_j^{(n)}(z_0)| \leqslant \frac{1}{R^n}\exp(CR) \max_{0\leqslant j\leqslant N} \|F^{(k)}(z_0)\|_E.$$

This holds for all $n \geqslant 0$. Adding these inequalities we get

$$\|F^{(n)}(z_0)\|_E \leqslant \frac{\sqrt{m}}{R^n}\exp(CR) \max_{0\leqslant j\leqslant N} \|F^{(k)}(z_0)\|_E.$$

Choose $R = 2$ and n_0 such that

$$(\sqrt{m}\exp(2C))/2^{n_0} \leqslant n_0!/_{N!},$$

and we have

$$\frac{\|F^n(z_0)\|_E}{n!} \leqslant (2^{n_0-n})\frac{n_0!}{n!}\max_{0\leqslant k\leqslant N}\frac{\|F^{(k)}(z_0)\|_E}{k!}$$
$$\leqslant \max_{0\leqslant k\leqslant N}\frac{\|F^{(k)}(z_0)\|_E}{k!}$$

for all $n \geqslant n_0$. Hence F is of BI(E) with $N_E \leqslant n_0$.

(ii) Suppose now F is of BI(E). Then we have on writing $N = N_E$,

$$\|F^{(N+1)}\| \leqslant C \max_{0\leqslant k\leqslant N} \|F^{(k)}\|$$

for some $C > 1$; and the above argument shows that F is of BI(S).

We also get (cf. Hayman [4] for scalar functions).

COROLLARY 1.1. *$F(z)$ is of BI if and only if there exists a number C and an integer N such that*

$$\|F^{(N+1)}(z)\| \leqslant C \max_{0\leqslant i\leqslant N} \|F^{(i)}(z)\|$$

for all z.

4. Proof of Theorem 2

We consider first the case when $m = 2 = n$. Write

$$Q_1(z) = \begin{pmatrix} a_1(z), a_2(z) \\ a_3(z), a_4(z) \end{pmatrix}, \qquad Q_2(z) = \begin{pmatrix} b_1(z), b_2(z) \\ b_3(z), b_4(z) \end{pmatrix} \tag{4.1}$$

and

$$F(z) = (f(z), g(z)),$$

where $a_i, b_i \in R$. From (1.4) and (4.1) we have

$$f'' + a_1 f' + a_2 g' + b_1 f + b_2 g = 0, \tag{4.2}$$

$$g'' + a_3 f' + a_4 g' + b_3 f + b_4 g = 0. \tag{4.3}$$

Hence

$$f'' + a_1 f' + b_1 f = -a_2 g' - b_2 g. \tag{4.4}$$

Write the right side of (4.4) as $A_2 g' + B_2 g$ and the left side as $D_2(f)$. We differentiate (4.4) and use the equation for g'' and get

$$D_3(f) = (a_2 a_4 - a_2' - b_2) g' + (a_2 b_4 - b_2') g \equiv A_3 g' + B_3 g. \qquad (4.5)$$

Similarly we get

$$D_n(f) = A_n g' + B_n g \qquad (4.6)$$

where the following recurrence formulae connecting A_{n+1} and B_{n+1} hold.

$$A_{n+1} = -A_n a_4 + A_n' + B_n$$
$$B_{n+1} = B_n' - b_4 A_n.$$

We note that for each $n \geqslant 2$, $D_n(f)$ is of the form $f^{(n)} + R_{1n} f^{(n-1)} + \cdots + R_{nn} f$ where each R_{in} $(1 \leqslant i \leqslant n) \in R$; that is , $D_n(f)$ is a monic operator with coefficients in R.

Further for each $n \geqslant 2$, A_n and $B_n \in R$. Now we use the following lemma due to A. Sathaye.

LEMMA 2.1. *Let R denote the class of rational functions bounded at infinity. Let $\{F_n\}_{n \geqslant n_0}$ be a sequence of monic operators with coefficients in R such that the degrees of F_n form a strictly increasing sequence and*

$$F_n(f) = \sum_{i=1}^{r} \lambda_i^{(n)} v_i, \qquad n \geqslant n_0 \qquad (4.7)$$

where $\lambda_i^{(n)} \in R$ and $v_1, ..., v_r$ are all functions and $f \in C^\infty$. Then there exists a sequence of monic operators $\{G_n\}_{n \geqslant n_1}$ such that the degrees of G_n form a strictly increasing sequence and

$$G_n(f) = \sum_{i=1}^{r-1} \mu_i^{(n)} v_i, \qquad n \geqslant n_1, \qquad (4.8)$$

where $\mu_i^{(n)} \in R$. In particular if $r = 1$ then $G_n(f) = 0$ for $n \geqslant n_1$.

Proof. For $a \in R$, $\text{ord}_\infty a \equiv \text{ord}\, a$ denotes the order of a at infinity. Recall that $\text{ord}\, a = \infty \Leftrightarrow a = 0$.

Choose $m \geqslant n_0$ such that $\text{ord}\, \lambda_r^{(m)} = \min\{\text{ord}\, \lambda_r^{(n)} \mid n \geqslant m\}$.
If $\text{ord}\, \lambda_r^{(m)} = \infty$ then $\lambda_r^{(n)} = 0$ for all $n \geqslant m$, and if $m = n_1$, $G_n = F_n$ for $n \geqslant n_1$ and $\mu_i^{(n)} = \lambda_i^{(n)}$ for $i = 1, 2, ..., r-1$ then (4.8) is satisfied. Now assume $\lambda_r^{(m)} \neq 0$. Since

$$\text{ord}\, \lambda_r^{(n)} \geqslant \text{ord}\, \lambda_r^{(m)} \qquad (4.9)$$

we see that

$$\lambda_r^{(n)}/\lambda_r^{(m)} \in R.$$

For $n \geqslant m$ consider

$$F_n(f) - (\lambda_r^{(n)}/\lambda_r^{(m)})\, F_m(f)$$
$$= \sum_{i=1}^{r} \{\lambda_i^{(n)} - (\lambda_r^{(n)}/\lambda_r^{(m)})\,\lambda_i^{(m)}\}\, v_i.$$

Set $n_1 = m+1$,

$$G_n = F_n - (\lambda_r^{(n)}/\lambda_r^{(m)}) F_m \qquad \text{for} \quad n \geqslant n_1, \tag{4.10}$$

and

$$\mu_i^{(n)} = \lambda_i^{(n)} - (\lambda_r^{(n)}/\lambda_r^{(m)})\,\lambda_i^{(m)} \qquad \text{for} \quad i = 1, 2,..., r-1 \text{ and } n \geqslant n_1. \tag{4.11}$$

Then it is evident that

$$G_n(f) = \sum_{i=1}^{r-1} \mu_i^{(n)} v_i + \{\lambda_r^{(n)} - (\lambda_r^{(n)}/\lambda_r^{(m)})\,\lambda_r^{(m)}\}\, v_r$$
$$= \sum_{i=1}^{r-1} \mu_i^{(n)} v_i.$$

Thus (4.8) holds. Since $n \geqslant n_1 \geqslant m$, it is evident from (4.10) that G_n is monic and the degrees of G_n are strictly increasing for $n \geqslant n_1$. Also $\mu_i^{(n)} \in R$, because of (4.9) and (4.11). The remark about $r = 1$ is obvious; and the proof of the lemma is complete.

COROLLARY 2.2. *If F_n, f are as above then there exist monic operators H_n with coefficients in R such that*

$$H_n(f) = 0 \qquad \textit{for sufficiently large } n.$$

Proof. Use induction on r.

To complete the proof of the theorem, when $m = 2 = n$, observe that $r = 2$ and $v_1 = g$, $v_2 = g'$ and A_n, $B_n \in R$.

The proof for the general case when $m > 2$ and F satisfies the DE $L_p(W, z, Q) = 0$, $p \geqslant 1$, is similar. We follow the same process and obtain for any large n $(n > (m-1)\,p)$ a relation of the form (4.6), where f is replaced by one of the components, say f_1. The right side will consist of $(m-1)$ terms involving $f_j^{(k)}$ $(2 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant p-1)$ and the coefficients in R. Lemma 2.1 and Corollary 2.2 then give the required result.

5. Proof of Theorem 3

Since F is of BI N we have

$$\frac{\| F^{(N+1)}(z/N+1) \|}{(N+1)!} \leqslant \max_{0 \leqslant k \leqslant N} \frac{\| F^{(k)}(z/N+1) \|}{k!}.$$

Set $Y(z) = F(z/N+1)$. Then

$$\frac{\| Y^{(N+1)}(z) \| \, (N+1)^{(N+1)}}{(N+1)!} \leqslant \max_{0 \leqslant k \leqslant N} \frac{\| Y^{(k)}(z) \| \, (N+1)^k}{k!}$$

and hence

$$\| Y^{(N+1)}(z) \| \leqslant \max_{0 \leqslant k \leqslant N} \| Y^{(k)}(z) \|,$$

For a fixed real θ and $R > 0$, we set

$$g(t) = \max_{0 \leqslant k \leqslant n} \| Y^{(k)}(te^{i\theta}) \|, \qquad 0 \leqslant t < R$$

and obtain, as in the proof of Theorem 1,

$$\| Y(te^{i\theta}) \| \leqslant \max_{0 \leqslant k \leqslant N} \| Y^{(k)}(0) \| \, e^t.$$

This gives, on writing $z = te^{i\theta}$,

$$\| F(z/N+1) \| \leqslant \max_{0 \leqslant k \leqslant N} \frac{\| F^{(k)}(0) \|}{(N+1)^k} e^{|z|}. \tag{5.1}$$

We now replace $z/N+1$ by z. The proof is complete.

To show that the result is sharp we give:

Example 3.1. Let $F(z)$: $C^1 \to C^m$ have components which are all equal to $f(z) = \exp((N+1)z)$. For this function F, there is equality sign in (2.1).

Remark. For similar results for scalar functions see Hayman [4], Fricke and Shah [2].

6. Proof of Theorem 4

We require two lemmas:

Lemma 4.1. *If $X(z) \neq 0$ is an entire vector-valued function and T is a given positive number, then there exists an integer $k > 0$ such that for every z, $|z| \leqslant T$,*

$$\max \left\{ \| X(z) \|, \frac{\| X^{(1)}(z) \|}{1!}, \ldots, \frac{\| X^{(k)}(z) \|}{k!} \right\} \geqslant \frac{1}{2^k}.$$

We omit the proof of this lemma and the next one. See for similar lemmas for scalar functions, Shah [9].

LEMMA 4.2. *If $X(z)$ is an entire vector-valued function and T is a given positive number, then there is an integer L such that*

$$\max\left\{\|X(z)\|,..., \frac{\|X^{L}(z)\|}{L!}\right\} \geqslant \frac{\|X^{(j)}(z)\|}{j!}$$

for $|z| \leqslant T$ and $j = 1, 2,...$.

Proof of the Theorem. Since g is of BI, there exists by Corollary 1.1 an integer N and a number C such that

$$\|g^{(N+1)}(z)\| \leqslant C \max_{0 \leqslant j \leqslant N} \|g^{(j)}(z)\|$$

for all $z \in \mathbf{C}$. Thus from Eq. (2.2) we have (cf. Fricke and Shah [1])

$$\begin{aligned}
\|g^{(N+1)}(z)\| &= \left\| \sum_{j=0}^{n} \frac{d^{N+1}}{dz^{N+1}} Q_j(z)\, X^{(n-j)}(z) \right\| \\
&= \left\| \sum_{j=0}^{n} \sum_{k=0}^{N+1} \binom{N+1}{k} Q_j^{(k)}(z)\, X^{(n-j+N+1-k)}(z) \right\| \\
&\leqslant C \max_{0 \leqslant j \leqslant N} \|g^{(j)}(z)\| \\
&= C \max_{0 \leqslant j \leqslant N} \left\| \sum_{k=0}^{n} \sum_{t=0}^{j} \binom{j}{t} Q_k^{(t)}(z)\, X^{(n-k+j-t)}(z) \right\|.
\end{aligned}$$

Hence by a simple transposition

$$\begin{aligned}
\|X^{(n+N+1)}(z)\| \leqslant{}& \left\| \sum_{j=1}^{n} \sum_{k=0}^{N+1} \binom{N+1}{k} Q_j^{(k)}(z)\, X^{(n-j+N+1-k)}(z) \right\| \\
&+ C \max_{0 \leqslant j \leqslant N} \left\| \sum_{k=0}^{n} \sum_{t=0}^{j} \binom{j}{t} Q_k^{(t)}(z)\, X^{(n-k+j-1)}(z) \right\|.
\end{aligned}$$

Now, since the functions in R are all bounded at infinity, we may choose R_0 large enough so that (see (1.4)) for $|z| > R_0$ and $0 \leqslant k \leqslant N+n$,

$$|a_{pq,i}^{(k)}(z)| < M$$

for some constant M.

Thus, when $P = (C+1)\, m(n+1)(N+1)(N+2)$

$$\|X^{(n+N+1)}(z)\| \leqslant PM \max_{0 \leqslant j \leqslant N+n} \|X^{(j)}(z)\|,$$

for $|z| > R_0$. By Lemma 4.2, this inequality holds also for $|z| \leqslant R_0$, provided we replace PM by a suitable constant K. Hence by Corollary 1.1, $X(z)$ is of BI.

EXAMPLE 4.1. Consider the DE

$$W' - 2zIW = 0$$

where I is a unit matrix. This equation is satisfied by $W = (f_1, f_2, \ldots, f_m)$, where all components $f_j(z)$ are equal to $\exp(z^2)$. The coefficient is not in R and $W(z)$ is not of BI.

EXAMPLE 4.2. Let F be as defined in Example 1.1. Then F is of BI but F' is not.

7. PROOF OF THEOREM 5

We differentiate Eq. (1.5) p times and get

$$W^{(n+p)} = -\{Q_1 W^{(n+p-1)} + \cdots + Q_n W^{(p)}\}.$$

This implies

$$\frac{\| W^{(n+p)} \|}{(n+p)!} \leqslant \frac{m\,|A_1|}{n+p} \frac{\| W^{(n+p-1)} \|}{(n+p-1)!} + \cdots + \frac{m\,|A_n|}{(n+p)\cdots(p+1)} \frac{\| (W^{(p)}) \|}{p!}.$$

Choose p such that (2.3) holds. Then

$$\frac{\| W^{(n+p)}(z) \|}{(n+p)!} \leqslant \max_{0 \leqslant i \leqslant n+p-1} \frac{\| W^{(i)}(z) \|}{i!}$$

for all $z \in \mathbf{C}$. It is clear from the above argument that p can be replaced by $p+1$, $p+2, \ldots$ and hence

$$\max_{0 \leqslant i \leqslant n+p-1} \frac{\| W^{(i)}(z) \|}{i!} \geqslant \frac{\| W^{(j)}(z)}{j!}$$

for $j = 0, 1, 2, \ldots$ and all $z \in \mathbf{C}$. This gives

$$N(W) \leqslant n + p - 1.$$

To show that this bound is best possible, we consider:

EXAMPLE 5.1. Let

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} X.$$

This equation is satisfied by $W(z) = (f(z), f'(z))$, where

$$f(z) = e^{-z/2}\left(C_1 \operatorname{Cos} \frac{\sqrt{3}z}{2} + C_2 \operatorname{Sin} \frac{\sqrt{3}z}{2}\right)$$

and C_1, C_2 are any two constants. Choose $0 < C_1 < C_2\sqrt{3}$. Now

$$\| W(0) \| = \max \left\{ C_1, \frac{-C_1 + C_2\sqrt{3}}{2} \right\},$$

$$\| W'(0) \| = \frac{C_1 + C_2\sqrt{3}}{2} > \| W(0) \|.$$

Hence $N(W) \geqslant 1$. But since p is the smallest non-negative integer such that $2/(1+p) \leqslant 1$ we take $p = 1$, and so $N(W) \leqslant 1 + 1 - 1 = 1$. Hence $N(W) = 1$.

8. PROOF OF THEOREM 6

By Theorem 4, $W(z)$ is of BI. From the equation (1.5) we have

$$\| W^{(n)}(z) \| \leqslant \| Q_1(z)\, W^{(n-1)}(z) \| + \cdots + \| Q_n(z)\, W(z) \|.$$

Given $\varepsilon > 0$, we have, from (1.4), for $z \geqslant R_1 > R_0(\varepsilon)$,

$$\| W^{(n)}(z) \| \leqslant \left\{ m \sum_{i=1}^{n} |A_i| + \varepsilon \right\} \max_{0 \leqslant i \leqslant n-1} \| W^{(i)}(z) \|,$$

and consequently we have by the argument of Theorem 1,

$$\| W(z) \| \leqslant A \exp\left\{ \max\left(1, m \sum_{i=1}^{n} |A_i| + \varepsilon\right) |z| \right\}$$

for $|z| \geqslant R_1$. Since ε can be chosen arbitrarily small, this gives (2.4).

9. PROOF OF THEOREM 7

We have from Eq. (1.5)

$$\frac{\| W^{(n+q)} \|}{(n+q)!} \leqslant \frac{1}{(n+q)} \frac{\| Q_1 W^{(n+q-1)} \|}{(n+q-1)!} + \cdots + \frac{1}{(n+q) \cdots (q+1)} \frac{\| Q_n W^{(q)} \|}{q!}$$

$$+ \text{ similar terms involving derivatives of } Q_i \ (i = 1, 2, \ldots, n).$$

Hence for $|z| \geqslant R_2 > R_0(\varepsilon)$

$$\frac{\|W^{(n+q)}\|}{(n+q)!} \leqslant \frac{m(|A_1|+\varepsilon)}{(n+q)} \max_{0 \leqslant j \leqslant n+q-1} \frac{\|W^{(j)}\|}{j!} + \cdots + \frac{m(|A_n|+\varepsilon)}{(n+q)\cdots(q+1)} \max_{0 \leqslant j \leqslant n+q-1} \frac{\|W^{(j)}\|}{j!} + \varepsilon \max_{0 \leqslant j \leqslant n+q-1} \frac{\|W^{(j)}\|}{j!}$$

Since ε can be chosen arbitrarily small we have for $|z| \geqslant R_2 > R_0$

$$\frac{\|W^{(n+q)}\|}{(n+q)!} \leqslant \max_{0 \leqslant j \leqslant n+q-1} \frac{\|W^{(j)}\|}{j!}$$

And this gives $N(W) \leqslant n+q-1$. We now use Theorem 3 to get the result.

Remark. For similar and other results on scalar functions of BI, see Shah [11, 12] and Fricke, Roy and Shah [3]; and for functions of several complex variables see Krishna and Shah [6].

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