Hilbert functions of graded algebras over Artinian rings

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Abstract

In this paper we give an effective characterization of Hilbert functions and polynomials of standard algebras over an Artinian equicharacteristic local ring; the cohomological properties of such algebras are also studied. We describe algorithms to check the admissibility of a given function or polynomial as a Hilbert function or polynomial, and to produce a standard algebra with a given Hilbert function. © 1998 Elsevier Science B.V.

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0. Introduction

Let \( (R_0, m, k) \) be an Artinian local ring, \( R = R_0[X_1, \ldots, X_b] \) with \( \deg(X_i) = 1 \) for all \( 1 \leq i \leq b \) and \( I \subseteq R_+ = \bigoplus_{n \geq 1} R_n \), a homogeneous ideal. A standard \( R_0 \)-algebra is a graded algebra of the form \( S = R/I \); we will denote by \( H_S(n) = \lambda_{R_0}(S_n) \) the Hilbert function of \( S \). The study of Hilbert functions goes back a century in time; its origin is the celebrated result due to Hilbert:

[Hilbert 1890] If \( R_0 \) is a field, then \( H_S \) is asymptotically polynomial.

Later on Macaulay characterized Hilbert functions in the case \( R_0 \) is a field:

[Macaulay 1927] Let \( H : \mathbb{N} \to \mathbb{N} \) be a numerical function and \( k \) a field; then \( H \) is the Hilbert function of a standard \( k \)-algebra if and only if \( H(0) = 1 \) and \( H(n + 1) \leq (H(n))_+ \) for all \( n \geq 1 \), see [1] for a proof. Afterwards, Samuel and Serre extended Hilbert’s result to the Artinian case. In view of this situation, we found it natural to consider the following problems:

(P1) Extension of Macaulay’s characterization to the Artinian case,
(P2) Characterization of the Hilbert polynomials of standard \( R_0 \)-algebras.

Another interesting result in this line is Gotzmann's regularity theorem, see [4, 5]. This theorem, according to Green's presentation, provides us with an alternative expression of Hilbert polynomials that is more combinatorial than the usual one. In fact, it is deeply related to the study of the behaviour of Hilbert functions under hyperplane section.

(Gotzmann 1978, Green 1989). If \( R_0 \) is a field, there exist uniquely determined integers \( c_1 \geq c_2 \geq \cdots \geq c_s \geq 0 \) such that the Hilbert polynomial of \( S = R_0[X_1, \ldots, X_s]/I \) can be written as

\[
h_S(X) = \binom{X + c_1}{c_1} + \binom{X + c_2 - 1}{c_2} + \cdots + \binom{X + c_s - (s - 1)}{c_s}.
\]

Furthermore, the ideal sheaf \( \mathcal{I} \) associated to \( I \) is \( s \)-regular.

Hence an additional problem we have considered is:

(P3) Extension of Gotzmann's regularity theorem to the Artinian case and its relation with Castelnuovo–Mumford's regularity of the local cohomology \( H^i_{S_0}(S) \).

The aim of this work is to study problems (P1)-(P3) in the case where \( R_0 \) is an Artinian \( k \)-algebra. Besides of being an interesting object of study in its own right, the theory of Hilbert functions of standard algebras over Artinian rings is the natural framework to study Hilbert functions of \( m \)-primary ideals in local rings, the Hilbert scheme and infinitesimal deformations.

In order to study the combinatorics of \( R \) we introduce an ordered set of submodules of \( R \) which considers both the combinatorics of the monomials and the structure of the base ring \( R_0 \). This set plays the role of the usual reverse lexicographic ordering in \( k[X_1, \ldots, X_s] \). This will be especially neat when \( R_0 \) is a ring of deformations, i.e. \( R_0 = k[s] \).

The characterization theorem obtained in problem (P1) involves the embedding dimension \( b \) of the standard algebra \( S \) such that \( H = H_S \). This result is deeper than the mere generalization of Macaulay's theorem, since the only information this straightforward generalization provides about the least possible value of \( b \) is \( b_{\min} \leq H(1) \).

Since in the general case we know only that \( b_{\min} \geq H(1)/\mu_{R_0}(R_0) \), the conditions on \( H \) need to be refined to determine \( b_{\min} \).

The extension of Gotzmann's result will provide us with bounds, computed in terms of the Hilbert–Samuel coefficients, for the annihilation of the local cohomology \( H^i_{S_0}(S) \). For instance, we recover Hoa's result \( a(S) \leq e(S) - \dim(S) - 1 \), see Remark 3.10, and we show that the value \( s \) appearing in Gotzmann's result is a polynomial function of the Hilbert–Samuel coefficients.

We have been strongly concerned about the effectiveness of the results obtained. For instance: Macaulay's characterization as it is formulated is not an effective result, since there is no way to check the condition \( H(n + 1) \leq (H(n))_n \) for all \( n \in \mathbb{N} \). We will describe an algorithm to check these conditions in a finite number of steps, for any asymptotically polynomial function \( H \). Specifically, we give algorithms to determine:

(i) whether a polynomial \( P \in \mathbb{Q}[X] \) is a Hilbert polynomial,

(ii) whether an asymptotically polynomial function is a Hilbert function,
(iii) the minimal embedding dimension for a realizing algebra of a Hilbert function, and we also compute a generating system, that will be minimal in the case $R_0$ is a field, for the ideal $I$ such that a realizing algebra is $S = R/I$. In the general case, the generating system will depend on a composition series on $R_0$. Nevertheless, we can always obtain a composition series via Gröbner basis, see [2, Proposition 1(ii) in Ch. 5, Section 3].

1. Notations

Let $R = \bigoplus_{n \geq 0} R_n$ be a $d$-dimensional graded ring such that $R_0$ is an Artinian local ring and $R$ is an $R_0$-algebra finitely generated by $R_1$; we will call such a ring a standard $R_0$-algebra. We will denote by $R_+ = \bigoplus_{n \geq 1} R_n$. Let $H_R(n) := \lambda_{R_0}(R_n)$ be the Hilbert function of $R$. For $n \gg 0$, $H_R$ coincides with a polynomial $h_R$ of degree $d - 1$ which is called the Hilbert polynomial of $R$. This and the fact that $H_R(n) \in \mathbb{N}$ for all $n$ suggest the definitions

$$Q[X; Z] = \{P \in Q[X] \mid P(n) \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}\},$$

$$Q[X; N] = \{P \in Q[X; Z] \mid P(n) \geq 0 \text{ for all } n \gg 0\}.$$ 

Clearly, $Q[X; Z]$ is the set of polynomials in $Q[X; Z]$ which have a positive leading coefficient. Since the set $\{\binom{X+i}{i} \mid i \geq 0\}$, where $\binom{X+i}{i} = (X+i)(X+i-1)\cdots(X+1)/i!$ and $\binom{X}{0} = 1$, is a $Q$-basis of $Q[X]$, any $P \in Q[X]$ can be written uniquely as

$$P(X) = e_0\binom{X+c}{c} - e_1\binom{X+c-1}{c-1} + \cdots + (-1)^{c-1}e_{c-1}\binom{X+1}{1} + (-1)^ce_c.$$

The coefficients $e_i - e_i(P) \in Q$ are called the normalized Hilbert–Samuel coefficients of $P$. It is known that $P \in Q[X; Z]$ if and only if $e_i(P) \in \mathbb{Z}$ for all $i$. In the case $P = h_R$ we denote $e_i(h_R) = e_i(R)$ and call them the normalized Hilbert–Samuel coefficients of $R$.

If $H : \mathbb{N} \to \mathbb{Z}$ is an asymptotically polynomial function, the regularity index of $H$ is $i(H) = \min\{k \in \mathbb{N} \mid H(n) = P(n) \forall n \geq k\}$, where $P \subseteq Q[X; Z]$ is the polynomial eventually equal to $H$.

2. Characterization of Hilbert functions

The main goal of this section is to obtain a characterization theorem for Hilbert functions of graded algebras over Artinian equicharacteristic rings, see Theorem 2.9. We remark that our result is stronger than the straightforward generalization of the classical Macaulay’s result. This straightforward generalization is obtained as Corollary 2.11 and does not determine the embedding dimension of the realizing algebra when the base ring is not a field. This is why the conditions which characterize Hilbert functions of algebras with a given embedding dimension need to be refined.
We need some combinatorics to proceed: Given integers \( n, d \geq 1 \), it is known that there exist uniquely determined integers \( k_d > k_{d-1} > \cdots > k_\delta \geq \delta \geq 1 \) such that

\[
n = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}.
\]

This is called the \( d \)-binomial expansion of \( n \). We define then

\[
(n_d)^+ = \binom{k_d+1}{d} + \binom{k_{d-1}+1}{d-1} + \cdots + \binom{k_\delta+1}{\delta},
\]

\[
(n_d)^- = \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_\delta-1}{\delta},
\]

\[
(n_d)^+ = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d} + \cdots + \binom{k_\delta+1}{\delta+1},
\]

\[
(n_d)^- = \binom{k_d-1}{d-1} + \binom{k_{d-1}-1}{d-2} + \cdots + \binom{k_\delta-1}{\delta-1},
\]

with the convention that \( \binom{i}{j} = 0 \) if \( i < j \) and \( \binom{i}{0} = 1 \) for all \( i \geq 0 \). We also define \((0_d)^+ = (0_d)^- = (0_d)^0 = (0_d)^\pm = 0 \) for all \( d \geq 1 \). Notice that we immediately obtain the \( d \) and \( (d+1) \)-binomial expansions of \((n_d)^+\) and \((n_d)^0\), respectively. We will use as simplified notation \(((n_d)^-)^+\) instead of \(((n_d)^-)^0\). We refer the reader to \([11]\) for some properties of these functions that will be used in the sequel. However, for the reader’s convenience we list here the most used ones.

**Lemma 2.1.** Let \( n = \binom{k_n}{d} + \binom{k_{n-1}}{d} + \cdots + \binom{k_1}{d} \) and \( m = \binom{l_m}{d} + \binom{l_{m-1}}{d} + \cdots + \binom{l_1}{d} \) be the \( d \)-binomial expansions of \( n, m \geq 1 \). Then we have:

1. Define \( k_{d-1} = \cdots = k_1 = 0, l_{d-1} = \cdots = l_1 = 0 \). Then \( n \leq m \) if and only if \((k_d, \ldots, k_1) \leq (l_d, \ldots, l_1)\) in the lexicographic order.
2. If \( k_\delta > \delta \), then it holds that \((n-1)_d^- < (n_d)^-\).
3. If \( n < m \) then \((n_d)^+ < (m_d)^+\), \((n_d)^- \leq (m_d)^-\), \((n_d)^+ < (m_d)^+\) and \((n_d)^- \leq (m_d)^-\).

(i) and (iii) can be found in \([11, Section 4]\), while (ii) is Lemma 4.2.11(b) in \([1]\).

The following result will assure us of the existence of “good” linear forms that will allow us to perform the inductive step in Theorems 2.3 and 2.9. The elements of the set \( U_R(d, V) \) described below are the best possible: if \( V = I_d, I \) a homogeneous ideal, they are the elements in \( R_1 \cap (k')^b \) which are closest to being non-zero divisors in \( R/I \). This will be made clear in Lemma 3.2.

Let \((R_0, m)\) be an Artinian local equicharacteristic ring, \( R = R_0[X_1, \ldots, X_\delta] \). From \([10, Theorem 28.3]\), since \( R_0 \) is a complete local equicharacteristic ring, we get that it contains a coefficient field \( k' \): \( k' \subseteq R_0 \) is a subfield that maps isomorphically into \( R_0/m \) via the canonical projection \( R_0 \rightarrow R_0/m \). Hence, if \( R_0 \) has infinite residue field, \( k' \) is
infinite. Denote by $R_1 \cap (k')^b$ the set of linear forms in $R$ having all their coefficients in $k'$. It can be naturally identified with $(k')^b$, and we will consider it as a topological space endowed with the Zariski topology. For all $d \geq 1$ and any $R_0$-submodule $V \subseteq R_d$ let us consider the set

$$U_R(d, V) = \{h \in R_1 \cap (k')^b \mid \lambda_{R_0}(V + hR_{d-1}) \text{ is maximal}\}.$$ 

Then we have

**Proposition 2.2.** $U_R(d, V)$ is a Zariski open set in $R_1 \cap (k')^b$.

**Proof.** For any finitely generated $R_0$-module $M$ the inclusion $k' \subseteq R_0$ induces a $k'$-vector space structure in $M$ and $\dim_{k'}(M) = \lambda_{R_0}(M)$. If $m$ is the maximal value of $\dim_{k'}((V + hR_{d-1})/V)$ when $h \in R_1 \cap (k')^b$, then $U_R(d, V)$ is the set of linear forms $h \in R_1 \cap (k')^b$ such that $\dim_{k'}((V + hR_{d-1})/V) = m$. Now consider for $h \in R_1 \cap (k')^b$ the $k'$-linear map given by multiplication $\cdot h : R_{d-1} \to R_d/V$. Let $h = a_1X_1 + \cdots + a_bX_b$; given $k'$-bases of $R_{d-1}$ and $R_d/V$, we can describe this map by a matrix $M$ whose entries are polynomial functions on $a_1, \ldots, a_b$. Since, the image of $\cdot h$ is $(V + hR_{d-1})/V$, the complement of $U_R(d, V)$ in $(k')^b$ is the variety given by the ideal of $m \times m$ minors $Z_{m}(M)$. \qed 

**Theorem 2.3.** Let $(R_0, m)$ be an Artinian local equicharacteristic ring with infinite residue field and $R = R_0[X_1, \ldots, X_b]$. Let $\lambda_{R_0}(R_d/V) = (d+b-1)q + r$ be the Euclidean division of $\lambda_{R_0}(R_d/V)$ by $(d+b-1)$. For $h \in U_R(d, V)$, we put $R = R/(h)$, $\bar{V} = (V + hR_{d-1})/hR_{d-1}$. Then we have

$$\lambda_{R_0}(\bar{R}_d/\bar{V}) \leq \left(\frac{d + b - 2}{b - 2}\right) q + (r_d)^-.$$

**Proof.** We will proceed by induction on $(b, d)$ in the lexicographic order. Let $s = \lambda_{R_0}(R_0)$. If $b = 1$ the result is obvious; if $d = 1$, then $V \subseteq R_1 = R_0(X_1, \ldots, X_b)$ and $\lambda_{R_0}(R_1/V) = bq + r$ with $0 \leq r < b$; notice that $q \leq s$. Since $h \notin m[X_1, \ldots, X_b]$, the multiplication by $h$ induces an isomorphism $R_0 \cong R_0 h$; from this and the isomorphism $(V + R_0 h)/V \cong (R_0 h)/(V \cap R_0 h)$ we deduce that $U_R(1, V) = \{h \in R_1 \cap (k')^b \mid \lambda_{R_0}(V \cap R_0 h) \text{ is minimal}\}$. Let us distinguish two cases:

1. There exists $h_0 \in R_1 \cap (k')^b$ such that $\lambda_{R_0}(V \cap R_0 h_0) < s - q$. Then for all $h \in U_R(1, V)$ we must have $\lambda_{R_0}(V \cap R_0 h) < s - q$, and therefore $\lambda_{R_0}(\bar{V}) \geq (b - 1)s - (b - 1)q + (r_1)^-$. For all $h \in R_1 \cap (k')^b$ we have $\lambda_{R_0}(V \cap R_0 h) \geq s - q$ for all $1 \leq i \leq b$, hence $bq + r \leq \sum_{i=1}^{b} \lambda_{R_0}(R_0 X_i/(V \cap R_0 X_i)) \leq bq$. So, $r = 0$ and in particular $\lambda_{R_0}(V \cap R_0 X_i) = s - q$ for all $1 \leq i \leq b$. Since this is by hypothesis the least possible value, $U_R(1, V) = \{h \in R_1 \cap (k')^b \mid \lambda_{R_0}(V \cap R_0 h) = s - q\}$. Then if $h \in U_R(1, V)$, it holds $\lambda_{R_0}(\bar{V}) = (b - 1)q = (b - 1)q + (0_1)^-$. Finally, assume $b, d \geq 2$. Let $h \in U_R(d, V)$; we will denote by an overline the equivalence modulo $h$ and by $\pi : R \to \bar{R}$ the projection. Notice that, after a change of
variables, we can consider $\mathcal{R}$ as a polynomial ring in $b - 1$ variables. Then we have
\[ \pi(\mathcal{R}_1 \cap (k')^b) = \mathcal{R}_1 \cap (k')^{b-1}. \]

Let us define $(V : h) = \{ f \in \mathcal{R}_{d-1} \mid hf \in V \}$ and consider the Zariski open subset of
\[ B = U_R(d - 1, (V : h)) \cap \pi^{-1}(U_R(d, \overline{V})). \]

Since these are Zariski open sets and $k'$ is infinite, $B \neq \emptyset$. Pick $l \in B$, and denote by a hat accent the classes modulo $l$. Define $((V : h) : l) = \{ f \in \mathcal{R}_{d-2} \mid lf \in (V : h) \}$ and consider the exact sequences
\[
\begin{align*}
0 &\rightarrow \mathcal{R}_{d-1}/(V : h) \rightarrow \mathcal{R}_d/V \rightarrow \widehat{\mathcal{R}}_d/\overline{V} \rightarrow 0, \quad (1) \\
0 &\rightarrow \mathcal{R}_{d-1}/(V : l) \rightarrow \mathcal{R}_d/V \rightarrow \widehat{\mathcal{R}}_d/\overline{V} \rightarrow 0, \quad (2) \\
0 &\rightarrow \mathcal{R}_{d-2}/((V : l) : h) \rightarrow \mathcal{R}_{d-1}/(V : l) \rightarrow \widehat{\mathcal{R}}_{d-1}/(\overline{V} : l) \rightarrow 0, \quad (3) \\
0 &\rightarrow \mathcal{R}_{d-2}/((V : h) : l) \rightarrow \mathcal{R}_{d-1}/(V : h) \rightarrow \widehat{\mathcal{R}}_{d-1}/(\overline{V} : h) \rightarrow 0, \quad (4) \\
\overline{\mathcal{R}}_{d-1}/(\overline{V} : l) &\rightarrow \widehat{\mathcal{R}}_d/\overline{V} \rightarrow \widehat{\mathcal{R}}_d/\overline{V} \rightarrow 0, \quad (5)
\end{align*}
\]

(5) being obtained from (2) modulo $h$. Let us consider the following Euclidean divisions:
\[
\begin{align*}
\hat{\lambda}_{R_0}(\mathcal{R}_d/V) &= \left( \begin{array}{c} d + b - 1 \\ b - 1 \end{array} \right) q + r \quad \text{with } q \geq 0 \text{ and } 0 \leq r < \left( \begin{array}{c} d + b - 1 \\ b - 1 \end{array} \right), \\
\hat{\lambda}_{R_0}(\overline{\mathcal{R}}_d/\overline{V}) &= \left( \begin{array}{c} d + b - 2 \\ b - 2 \end{array} \right) \overline{q} + \overline{r} \quad \text{with } \overline{q} \geq 0 \text{ and } 0 \leq \overline{r} < \left( \begin{array}{c} d + b - 2 \\ b - 2 \end{array} \right), \\
\hat{\lambda}_{R_0}(\mathcal{R}_{d-1}/(V : h)) &= \left( \begin{array}{c} d - 1 + b - 1 \\ b - 1 \end{array} \right) \hat{q} + \hat{r} \\
\text{with } \hat{q} &\geq 0 \text{ and } 0 \leq \hat{r} < \left( \begin{array}{c} d - 1 + b - 1 \\ b - 1 \end{array} \right).
\end{align*}
\]

Since $h \in U_R(d, V)$ and $l \in \mathcal{R}_1 \cap (k')^b$, we have $\lambda_{R_0}(\mathcal{R}_d + l\mathcal{R}_{d-1}) \leq \lambda_{R_0}(\mathcal{R}_d + h\mathcal{R}_{d-1})$, and then from (1) and (2) we obtain $\lambda_{R_0}(\mathcal{R}_{d-1}/(V : l)) \leq \lambda_{R_0}(\mathcal{R}_{d-1}/(V : h))$. Applying this and the fact that $((V : l) : h) = ((V : h) : l)$ to (3) and (4) we get $\lambda_{R_0}(\overline{\mathcal{R}}_{d-1}/(\overline{V} : l)) \leq \lambda_{R_0}(\overline{\mathcal{R}}_{d-1}/(\overline{V} : h))$. Then by (5) we have
\[
\hat{\lambda}_{R_0}(\overline{\mathcal{R}}_d/\overline{V}) \leq \hat{\lambda}_{R_0}(\widehat{\mathcal{R}}_{d-1}/(\overline{V} : l)) + \hat{\lambda}_{R_0}(\widehat{\mathcal{R}}_d/\overline{V}) \leq \lambda_{R_0}(\overline{\mathcal{R}}_{d-1}/(\overline{V} : h)) + \lambda_{R_0}(\widehat{\mathcal{R}}_d/\overline{V}).
\]

Since $l \in U_R(d - 1, (V : h))$ we can apply the induction hypothesis (on $d$) to the first term. Since $\overline{l} \in U_R(d, \overline{V})$ we can apply the induction hypothesis (on $b$) to the second one. Therefore,
\[
\left( \begin{array}{c} d + b - 2 \\ b - 2 \end{array} \right) \overline{q} + \overline{r} \leq \left( \begin{array}{c} d - 1 + b - 2 \\ b - 2 \end{array} \right) \hat{q} + (\hat{r}_{d-1})^- + \left( \begin{array}{c} d + b - 3 \\ b - 3 \end{array} \right) \hat{q} + (\hat{r}_d)^-.
\]
From [11, Corollary 4.6(a)], we have \( r - (\overline{d}_d)^- = (\overline{d}_d)^- \), so this is equivalent to
\[
\left( \frac{d + b - 3}{b - 2} \right) \bar{q} + (\overline{d}_d)^- \leq \left( \frac{d - 1 + b - 2}{b - 2} \right) \bar{q} + (\overline{d}_d)^-.
\]

Finally, from the exact sequence (1) it holds
\[
(d + b - 1)q + r = \left( \frac{d + b - 2}{b - 2} \right) \bar{q} + \bar{r} + (d + b - 1)\frac{\bar{r}}{b - 1}.
\]

Then the claim follows from the following lemma, whose proof is a tedious computation with binomial coefficients and Euclidean divisions.

**Lemma 2.4.** Let \( b, d \geq 2, q, \bar{q}, \bar{r} \geq 0 \) and \( 0 \leq r < \binom{d + b - 1}{b - 1}, 0 \leq \bar{r} < \binom{d + b - 2}{b - 2} \), be integers such that
\[
(d + b - 1)q + r = \left( \frac{d + b - 2}{b - 2} \right) \bar{q} + \bar{r} + \left( \frac{d + b - 1}{b - 1} \right) \bar{q} + \bar{r}.
\]

Then \( \left( \frac{d + b - 2}{b - 2} \right) \bar{q} + \bar{r} \leq \left( \frac{d + b - 2}{b - 2} \right) q + (rd)^- \).

As a corollary we obtain a result which was proved by Green in the case \( R_0 \) is a field.

**Corollary 2.5.** Let \( (R_0, m) \) be an Artinian local equicharacteristic ring with infinite residue field and \( R = R_0[x_1, \ldots, x_k] \). Let \( d \geq 1 \) be an integer and \( V \subseteq R_d \) an \( R_0 \)-submodule; for all \( h \in U_R(d, V) \) let \( \overline{R} = R/H \) and \( \overline{V} = (V + hR_{d-1})/hR_{d-1} \). Then \( \lambda_{R_0}(\overline{R}_d/\overline{V}) = (\lambda_{R_0}(R_d/V))^- \).

**Proof.** Let \( c \geq 1 \) be an integer such that \( \lambda_{R_0}(R_d/V) < \binom{d + c - 1}{c - 1} \). If \( c < b \) then \( \lambda_{R_0}(R_d/V) < \binom{d + b - 1}{b - 1} \), so in the Euclidean division of \( \lambda_{R_0}(R_d/V) \) by \( \binom{d + b - 1}{b - 1} \) we get \( q = 0 \) and \( r = \lambda_{R_0}(R_d/V) \). Then by Theorem 2.3 we obtain the result.

If \( b < c \), consider \( R = R_0[x_1, \ldots, x_b] \subseteq R' = R_0[x_1, \ldots, x_c] \) and define \( K = (x_{b+1}, \ldots, x_c)_d \). Since \( V \cap K = K \), we have an isomorphism of \( R_0 \)-modules \( R_d/V' \cong R_d/V \).

Let for \( l \in R_1 \cap (k')^b \), \( l = h + f \) with \( h \in R_1 \cap (k')^b \) and \( f \in (x_{b+1}, \ldots, x_c) \). Then \( V' + lR_{d-1} = V + hR_{d-1} + K \) and therefore \( l \in U_{R'}(d, V') \) if and only if \( h \in U_R(d, V) \). In particular, if \( h \in U_R(d, V) \) then \( h \in U_{R'}(d, V') \). Moreover, \( \overline{R}_d/V' \cong \overline{R}_d/\overline{V} \), hence \( \lambda_{R_0}(\overline{R}_d/\overline{V}) = (\lambda_{R_0}(\overline{R}_d/\overline{V})d)^- = (\lambda_{R_0}(\overline{R}_d/\overline{V}))^- \) by the former case \( c = b \).

Let us define the functions we seek to characterize.
Definition 2.6. Given an Artinian equicharacteristic local ring \((R_0, m)\) and \(H : \mathbb{N} \to \mathbb{N}\), we will say that \(H\) is \textit{admissible} if there exists a standard \(R_0\)-algebra \(S\) such that \(H = H_S\). For \(b \geq 1\) we will say that \(H\) is \(b\)-admissible if there exists a homogeneous ideal \(I \subseteq R_+\), where \(R = R_0[X_1, \ldots, X_b]\), such that \(H = H_{R/I}\).

The following theorem is the main result of this section and characterizes when a function is \(b\)-admissible. For the constructive part of the proof we will need to consider an order in \(R = R_0[X_1, \ldots, X_b]\) which, in addition to the combinatorics of the monomials, also takes into account the structure of \(R_0\). Let us begin by fixing a suitable order in the set of monomials of \(R\). Given a multiindex \(\nu = (\nu_1, \ldots, \nu_b)\), let \(X^\nu = X_1^{\nu_1} \cdots X_b^{\nu_b}\) and \(|\nu| = \nu_1 + \cdots + \nu_b\). We have chosen the \textit{degree reverse lexicographic} order.

Definition 2.7. For \(X^\nu, X^{\mu}\) monomials in \(R\), we will say that \(X^\nu > X^{\mu}\) if \(|\nu| > |\mu|\) or \(|\nu| = |\mu|\) and the last nonzero entry of \((\nu_1 - \mu_1, \ldots, \nu_b - \mu_b)\) is negative.

Let \(\mathcal{J} = \{0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_s = R_0\}\), where \(s = \lambda_{R_0}(R_0)\), be a composition series in \(R_0\) and consider for all \(n \geq 1\) the set of the following \(s(n_b+1)\) \(R_0\)-submodules of \(R_n\):

\[
\mathcal{M}_n(\mathcal{J}) = \{J_i X^\nu \mid 1 \leq i \leq s \text{ and } |\nu| = n\}.
\]

Definition 2.8. We define a total ordering, \(\mathcal{J}\)-

\textit{reverse lexicographic order}, in \(\mathcal{M}_n(\mathcal{J})\) by \(J_i X^\nu < J_j X^{\nu'}\) if and only if \(i < j\) or \(i = j\) and \(X^\nu < X^{\nu'}\), where the order in the set of monomials in \(X_1, \ldots, X_b\) is the degree reverse lexicographic order.

Theorem 2.9 (Characterization of Hilbert functions). Let \((R_0, m)\) be an Artinian local equicharacteristic ring, \(b \geq 1\), \(R = R_0[X_1, \ldots, X_b]\) and \(H : \mathbb{N} \to \mathbb{N}\). For all \(n > 0\) consider the Euclidean division \(H(n) = (q(n), r(n)) = (q(n), r(n))_+ + (r(n), r(n))_+\) for all \(n \geq 0\).

Proof. Assume (ii), i.e. \((0, 0) \leq (q(n + 1), r(n + 1)) \leq (q(n), (r(n), r(n))_+) \leq (s, 0)\) in the lexicographic order, for all \(n \geq 0\). Let \(\mathcal{J} = \{0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_s = R_0\}\) be a composition series in \(R_0\). For \(n \geq 1\), if \(N = (n_b+1)\) and \(X^\nu > X^{\nu_1} > \cdots > X^{\nu_b}\) are the ordered monomials of degree \(n\) in \(X_1, \ldots, X_b\), define the following \(R_0\)-submodule of \(R_n\):

\[
I_n = J_{s-q(n)-1}X^{\nu_1} + \cdots + J_{s-q(n)-1}X^{\nu_{r(n)-1}} + J_{s-q(n)}X^{\nu_{r(n)}} + J_{s-q(n)}X^{\nu_{r(n)+1}} + \cdots + J_{s-q(n)}X^{\nu_b}.
\]

We have \(R_n/I_n \cong (R_0/J_{s-q(n)-1})^{(r(n))} \oplus (R_0/J_{s-q(n)})^{(N-r(n))}\). From \(\lambda_{R_0}(R_0/J_i) = s - i\) we get \(\lambda_{R_0}(R_n/I_n) = H(n)\). Hence, it suffices to show that \(I = \bigoplus_{n \geq 1} I_n\) is an ideal of \(R\).

Consider for all \(n \geq 1\) the ordered set \(\mathcal{M}_n(\mathcal{J})\) of \(R_0\)-submodules of \(R_n\) and let us write the elements of \(\mathcal{M}_n(\mathcal{J})\) ordered from greatest to least as the entries of the matrix

\[
(J_{s-1}X^{\nu})_{0 \leq i \leq s-1, 1 \leq j \leq N}.
\]
read from left to right and from top to bottom. Then \( I_n \) can be seen graphically by deleting the first \( q(n) \) rows in \( M_n(\mathcal{J}) \) and the first \( r(n) \) elements in the \((q(n)+1)\)th row, and keeping the remaining elements as generators of \( I_n \). What we have to prove is \( R_1 I_n \subseteq I_{n+1} \). Since \( R_1 \cdot (\text{ith row of } M_n(\mathcal{J})) \subseteq (\text{ith row of } M_{n+1}(\mathcal{J})) \) and \( q(n+1) < q(n) \), in the case \( q(n+1) < q(n) \) we have that the generators of \( R_1 I_n \) are contained in the generators of \( I_{n+1} \). On the other hand, if \( q(n) = q(n+1) \) we are deleting the same rows in \( M_n(\mathcal{J}) \) and in \( M_{n+1}(\mathcal{J}) \), and we must prove \( R_1 \cdot \frac{J_s \cdot X^{\text{remaining}}}{J_s \cdot X^{\text{remaining}}} \subseteq \frac{J_s \cdot X^{\text{remaining}}}{J_s \cdot X^{\text{remaining}}} \), where \( M = \binom{n+1+b-1}{b-1} \). We can ignore \( J_s \cdot X^{\text{remaining}} \), so it is enough to show that \( \lambda \cdot X^1 \cdot X^{r(n)+1} \cdot \ldots \cdot X^b \leq \lambda \cdot X^1 \cdot X^{r(n)+1} \cdot \ldots \cdot X^{r(n)+1} \).

Reciprocally, assume (i) and let \( s = \lambda R_0(R_0) \). Without loss of generality, we may assume that \( R_0 \) has infinite residue field. For \( n = 0 \) we have \( q(0) = s \) and \( r(0) = 0 \), and clearly \( H(1) \leq bs \). For \( n \geq 1 \) we have \( R_1 I_n \subseteq I_{n+1} \), therefore \( H(n+1) - \lambda R_0(R_{n+1}/I_{n+1}) \leq \lambda R_0(R_{n}/I_n) \). So, it will be enough to show that for an \( R_0 \)-submodule \( V \subseteq R_n \), if \( \lambda R_0(R_n/V) = \binom{n+b-1}{b-2} q + r \) is the Euclidean division, then \( \lambda R_0(R_{n+1}/R_1 V) \leq \binom{n+1+b-1}{b-2} q + (r_n)^+ \). We will proceed by induction on \( b \). In the case \( b = 1 \) we have \( \lambda R_0(R_n/V) = q \) and \( r = 0 \). Then \( V = a X^1 \subseteq R_0(X^1) \), with \( a \subseteq R_0 \) an ideal. So \( R_1 V = a X^1 \cdot X^{r(n)+1} \), and then \( \lambda R_0(R_{n+1}/R_1 V) = \lambda R_0(R_0/a) = q \cdot 0(n)_+ \).

In the case \( b \geq 2 \), pick \( h \in U(r,n) \) and consider the exact sequence

\[
R_n/V \xrightarrow{h} R_{n+1}/R_1 V \rightarrow R_{n+1}/(R_1 V + hR_n) \rightarrow 0.
\]

Let \( \overline{V} = (V + hR_n)/hR_n \) and consider the Euclidean division \( \lambda R_0(\overline{R}_n/\overline{V}) = \binom{n+b-2}{b-2} q + \overline{r} \). By Theorem 2.3 we have \( (n+b-2) q + \overline{r} \leq (n+b-2) q + (r_n)^- \), and both sides are Euclidean divisions. So we have \( (\overline{q},(r_n)^-) \leq (q, (r_n)^-) \) in the lexicographic order. Applying the induction hypothesis to \( \overline{R}_n/\overline{V} \) we obtain \( \lambda R_0(\overline{R}_n/\overline{R}_1 V) - \lambda R_0(\overline{R}_n/\overline{V}) \leq (n+1+b-2) q + (\overline{r}_n)^+ \).

Since \( 0 \leq \overline{r}, (r_n)^- < \binom{n+b-2}{b-2}, \) we obtain \( 0 \leq (\overline{r}_n)^+ < (n+1+b-2) \). On the other hand the inequality \( (\overline{q},\overline{r}) \leq (q, (r_n)^-) \) implies \( (\overline{q},(r_n)^-) \leq (q, (r_n)^-) \). Hence we have an inequality of Euclidean divisions \( (n+1+b-2) q + (\overline{r}_n)^+ \leq (n+b-2) q + (r_n)^+ \).

By the exact sequence we have \( \lambda R_0(R_{n+1}/(R_1 V)) \leq \lambda R_0(R_n/V) + \lambda R_0(\overline{R}_n/\overline{R}_1 V) \leq \binom{n+b-1}{b-1} q + r + (n+1+b-2) q + (r_n)^+ = (n+1+b-1) q + r + (r_n)^+ \). From [11, Proposition 4.8], we have \( (r_n)^+ = (r_n)^- + r \), and so we get the result. \( \square \)

Let us give a name to the ideals constructed in the proof of Theorem 2.9.

**Definition 2.10.** A homogeneous ideal \( I \subseteq R_+ = R_0[X_1, \ldots, X_b]_+ \) will be called a \( \mathcal{J} \)-segment ideal if for all \( n \geq 1 \) \( I_n \) is generated as an \( R_0 \)-module by the \( s(n+b-1)_{b-1} \) smallest elements in \( M_n(\mathcal{J}) \).

For each \( b \)-admissible function \( H \), there exists a unique \( \mathcal{J} \)-segment ideal \( I \subseteq R_0[X_1, \ldots, X_b] \), such that \( H = H_{R_0/I} \); it will be denoted by \( I_{H,\mathcal{J}} \). When \( R_0 \) is a field
the only composition series is the trivial one, hence the $\mathcal{J}$-reverse lexicographic order and the $\mathcal{J}$-segment ideals coincide with the usual degree reverse lexicographic order and segment ideals. Then we will write $I_{H,J} = I_H$.

The results in Section 3 will allow us to effectively apply the characterization theorem; see Section 5 for the algorithms.

The following corollary is the direct generalization of the classical version of Macaulay's theorem:

**Corollary 2.11.** Let $(R_0, \mathfrak{m})$ be an Artinian local equicharacteristic ring and $H : \mathbb{N} \rightarrow \mathbb{N}$ a function. Then the following conditions are equivalent:

- (i) There exists a standard $R_0$-algebra $S$ such that $H = H_S$.
- (ii) $H(0) = \lambda_{R_0}(R_0)$ and $H(n + 1) \leq (H(n)_n^+)$ for all $n \geq 1$.

**Proof.** Let $S = R/I$, where $R = R_0[X_1, \ldots, X_b]$ and $I \subseteq R_+$ is a homogeneous ideal with $H_0 = H$. The first part of (ii) is immediate. For the second part, if $H(1) < b$ and $H(n) = (\binom{n+b-1}{b-1})q(n) + r(n)$ is the Euclidean division, we must have $q(1) = 0$. Then by Theorem 2.9, $q(n) = 0$ and $r(n) = H(n)$ for all $n \geq 1$, and again by Theorem 2.9 we have (ii). If $H(1) \geq b$, by the same argument as in Corollary 2.5, we can reduce to the case $H(1) < b$.

Reciprocally, assume (ii) and let $b = H(1)$. From the condition $H(n+1) \leq (H(n)_n^+)$, for all $n \geq 1$ we get $H(n) \leq (\binom{n+b-1}{b-1})$. Let $R = R_0[X_1, \ldots, X_b]$ and consider the ordered monomials of degree $n \geq 1$ in $X_1, \ldots, X_b : X_{1^1} \succ X_{1^2} \succ \cdots \succ X_{1^n}$, where $N = (\binom{n+b-1}{b-1})$. Let $I_n \subseteq R_n$ be the $R_0$-submodule $I_n = \mathfrak{m}X_{1^1} + \cdots + \mathfrak{m}X_{1^n} + R_0X_{1^{n+1}} + \cdots + R_0X_{1^n}$. We have $\lambda_{R_0}(R_n/I_n) = H(n)$, and $I = \bigoplus_{n \geq 1} I_n$ is an ideal as in the proof of Theorem 2.9. \qed

**Corollary 2.12.** Let $H : \mathbb{N} \rightarrow \mathbb{N}$ be an admissible function. Then

- (i) If $H$ is $b$-admissible then it is $b'$-admissible for all $b' \geq b$.
- (ii) $H$ is $b$-admissible for all $b \geq H(1)$.
- (iii) $H$ is not $b$-admissible for all $b < H(1)/\lambda_{R_0}(R_0)$.

### 3. Gotzmann developments of Hilbert polynomials

The main result in this section, Theorem 3.3, is an improved version of Gotzmann's regularity theorem in the Artinian equicharacteristic case, see [5]. It gives an alternative expression of Hilbert polynomials, better suited than the usual one to deal with the combinatorial properties of Hilbert functions. For example, this will allow us to characterize Hilbert polynomials and to encode an entire Hilbert function in a finite amount of data, see Sections 4 and 5. Furthermore, it also provides us with information about the local cohomology of the ring. Let us begin by recalling some facts about local cohomology; see [6, Section 35], as a reference.
Let \((R_0, m)\) be an Artinian local ring, \(R\) a standard \(R_0\)-algebra, \(M = \bigoplus_{n \geq 0} M_n\) a finitely generated graded \(R\)-module. We will denote by \(H^q_{R_+}(M) = \bigoplus_{n \in \mathbb{Z}} H^q_{R_+}(M)_n\) the \(q\)th local cohomology module of \(M\) with respect to \(R_+\). Since \(\text{rad}(R_+) = \mathfrak{M} = m \oplus R_+\), we have \(H^q_{R_+}(M) = H^q_{\mathfrak{M}}(M)\) for all \(q\). Since these modules are Artinian and \(H^q_{R_+}(M)_n\) is a finitely generated \(R_0\)-module for all \(q, n, \) we can define

\[
a_q(M) = \min\{n \in \mathbb{N} \mid H^q_{R_+}(M)_n \neq 0\} < +\infty.
\]

It holds that \(H^q_{R_+}(M) = 0\) for \(q < \text{depth}_{R_+}(M)\) and \(q > \dim(M)\). We will adopt the convention that \(a_q(M) = -\infty\) for \(q < \text{depth}_{R_+}(M)\). The relationship between local cohomology and Hilbert functions is given by the following result, see [9, Lemma 1.3] for an algebraic proof.

**Proposition 3.1** (Grothendieck’s formula). Let \((R_0, m)\) be an Artinian local ring, \(R\) a standard \(R_0\)-algebra, \(M\) a finitely generated graded \(R\)-module; then for all \(n \in \mathbb{Z}\)

\[
H_M(n) - h_M(n) = \sum_{i \geq 0} (-1)^i \lambda_{R_0}(H^i_{R_+}(M)_n).
\]

We will begin by stating a preliminary lemma.

**Lemma 3.2.** Let \((R_0, m)\) be an Artinian local equicharacteristic ring with infinite residue field, \(k' \subseteq R_0\) a coefficient field for \(R_0\), \(R = R_0[Y_1, \ldots, Y_s]\) and \(I \subseteq R_+\) a homogeneous ideal such that \(\dim(R/I) \geq 1\). Then

(i) \(H^0_{R_+}(R/I) = 0\) if and only if there exists \(h \in R_1 \cap (k')^b\) such that \(\bar{h} \notin z(R/I)\).

(ii) If \(\text{depth}(R/I) \geq 1\), then \(\cap_{n \geq 1} U_{R}(n, I_n)\) is the set of all non-zero divisors in \(R_1 \cap (k')^b\); in particular it is nonempty.

**Proof.** (i) The condition is sufficient: then \(\text{depth}(R/I) \geq 1\) and hence \(H^0_{R_+}(R/I) = 0\). Reciprocally, assume that \(H^0_{R_+}(R/I) = 0\) and let \(\mathfrak{P}_1, \ldots, \mathfrak{P}_s\) be the associated primes of \(R/I\), so \(z(R/I) = \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_s\). For all \(i\) we have \(\mathfrak{P}_i = \text{Ann}(\bar{f}_i)\) with \(0 \neq \bar{f}_i \in R/I\) homogeneous. Since \(H^0_{R_+}(R/I) = 0\), in particular \(R_+ \bar{f}_i \neq 0\), that is, \(R_+ \not\subseteq \mathfrak{P}_i\). Therefore, \(X_1, \ldots, X_b\) can not simultaneously belong to \(\mathfrak{P}_i\), and so \(R_1 \cap (k')^b \neq (\mathfrak{P}_i)_1 \cap (k')^b\) for all \(1 \leq i \leq s\). In other words, \((\mathfrak{P}_i)_1 \cap (k')^b\) are proper vector subspaces of \(R_1 \cap (k')^b\). Since \(k'\) is infinite, we deduce that \((\mathfrak{P}_1)_1 \cap (k')^b \cup \cdots \cup (\mathfrak{P}_s)_1 \cap (k')^b \neq R_1 \cap (k')^b\), hence there exists \(h \in R_1 \cap (k')^b\) such that \(h \notin \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_s = z(R/I)\).

(ii) If \(\text{depth}(R/I) \geq 1\), pick \(h \in R_1 \cap (k')^b\) such that \(\bar{h}\) is a non-zero divisor in \(R/I\). Then \((I_n : h) = I_{n-1}\) for all \(n \geq 1\). If \(l\) is any element of \(R_1 \cap (k')^b\) we have an exact sequence

\[
0 \to (I_n : l) \to R_{n-1} \xrightarrow{l} (I_n + lR_{n-1})/I_n \to 0.
\]

So \(\lambda_{R_0}((I_n + lR_{n-1})/I_n)\) is maximal if and only if \(\lambda_{R_0}(I_n : l)\) is minimal. Since \(I_{n-1} \subseteq (I_n : l)\) and we have equality for all \(n \geq 1\) when \(l = h\), we get that \(l \in \bigcap_{n \geq 1} U_R(n, I_n)\) if and only if \((I_n : l) = I_{n-1}\) for all \(n \geq 1\), that is, \(l\) is not a zero divisor in \(R/I\). Hence,
\[ \bigcap_{n \geq 1} U_R(n, I_n) \] is precisely the set of all non-zero divisors in \( R_1 \cap (k')^b \), in particular it contains \( h \). \[ \square \]

We are ready now to prove the main theorem in this section:

**Theorem 3.3.** Let \((R_0, \mathfrak{m})\) be an Artinian local equicharacteristic ring, \( R = R_0[X_1, \ldots, X_b] \) and \( I \subseteq R_0 \) a homogeneous ideal such that \( \dim(R/I) \geq 1 \). Then:

(i) There exist integers \( b - 1 > c'_1 \geq \cdots \geq c'_p \geq 0 \), \( p \geq 0 \), and \( 0 \leq q \leq \lambda_{R_0}(R_0) \) such that

\[
h_{R/I}(X) = q \left( \frac{X + b - 1}{b - 1} \right) + \left( \frac{X + c'_1}{c'_1} \right) + \left( \frac{X + c'_2 - 1}{c'_2} \right) + \cdots + \left( \frac{X + c'_p - (p - 1)}{c'_p} \right);
\]

this equality gives a Euclidean division if \( X = n \) is an integer \( n \geq \max\{p - 1, 0\} \).

(ii) Let \( p_i = \# \{ j \mid c'_j \geq i - 1 \} \); then for \( i \geq 1 \)

\[
H^i_{R/I}(R/I)_n = 0 \quad \text{for all } n \geq p_i - i + 1 \quad \text{if } q > 0,
\]

\[
\left\{ \begin{array}{ll}
& \text{for all } n \geq p_i - i \quad \text{if } q = 0.
\end{array} \right.
\]

(iii) The regularity index of \( H_{R/I} \) verifies

\[
i(H_{R/I}) \leq \begin{cases} 
\max\{a_0(R/I) + 1, p\} & \text{if } q > 0, \\
\max\{a_0(R/I) + 1, p - 1\} & \text{if } q = 0.
\end{cases}
\]

**Proof.** We may assume that \( R_0 \) has infinite residue field. Let \( k' \subseteq R_0 \) be a coefficient field and \( H^0_{R_k}(R/I) = J/I \). By replacing \( R_0 \) by \( R_0/I_0 \) and \( R \) by \( (R_0/I_0)[X_1, \ldots, X_b] \) and \( I \) by \( J/I_0R \) we may assume that \( H^0_{R_k}(R/I) = 0 \). Then in (iii) we have to prove

\[
i(H_{R/I}) \leq \begin{cases} 
p & \text{if } q > 0, \\
p - 1 & \text{if } q = 0.
\end{cases}
\]

Let \( r = \lambda_{R_0}(R_0) \). We will proceed by induction on \( b \). In the case \( b = 1 \) the condition \( H^0_{R_k}(R/I) = 0 \) implies that \( I = 0 \). Then the theorem is easily deduced from Grothendieck's formula (Proposition 3.1).

In the case \( b \geq 2 \), by Lemma 3.2 we can choose \( h \in R_1 \cap (k')^b \) such that \( \tilde{h} \notin z(R/I) \). Let \( S = R/(h) \) and \( J = (I + (h))/(h) \); we have an exact sequence of graded \( R \)-modules

\[
(*) \quad 0 \rightarrow R/I(-1) \xrightarrow{h} R/I \rightarrow S/J \rightarrow 0.
\]

If \( \dim(S/J) = 0 \) then \( \dim(R/I) = 1 \) and \( h_{R/I} \) has degree 0, say \( h_{R/I} = t \). Since \( b - 1 > 1 \), (i) holds taking \( q = 0 \), \( p = t \) and \( c'_1 = \cdots = c'_p = 0 \). We need to show \( H^i_{R_k}(R/I)_n = 0 \) for all \( n \geq p - 1 \) and \( i(R/I) \leq p - 1 \). The last one is a well-known fact since \( R/I \) is Cohen–Macaulay, and then the first claim follows again from Grothendieck’s formula.
Assume now \( \dim(S/J) \geq 1 \). Consider \( S' \) and \( J' \) obtained from \( S \) and \( J \) as in the reduction to the case \( H^1_{R^+}(R/I) = 0 \); we have \( \text{rank}_{R_0}(S') = \text{rank}_{R_0}(S) = b - 1 \) and \( \lambda_{S_0'}(S'_0) \leq r \). Since \( H^0_{S_0'}(S'/J') = 0 \), the induction hypothesis applies to \( S'/J' \), therefore,

\[
\begin{align*}
  h_{S/J}(n) &= h_{S'/J'}(n) = q \left( \frac{n + b - 2}{b - 2} \right) + \left( \frac{n + b_1'}{b_1'} \right) \\
  &\quad + \left( \frac{n + b_2'-1}{b_2'} \right) + \cdots + \left( \frac{n + b_v' - (v - 1)}{b_v'} \right) \\
  \end{align*}
\]

for all \( n \gg 0 \), with \( 0 \leq q \leq \lambda_{S_0'}(S'_0) \leq \lambda_{R_0}(R_0) \) and \( b - 2 > b_1' \geq \cdots \geq b_v' \geq 0 \). Moreover, \( H^i_{S_0'}(S/J)_n \cong H^i_{S_0'}(S'/J')_n = 0 \) for all \( i \geq 1 \), \( n \geq v_i - i \), where \( v_i = \# \{ j \mid b_j' \geq i - 1 \} \).

To prove (i), fix \( n_0 \geq i(H_{R/I}, i(H_{S'/J'}), a_0(S/J)) + 1, v \). Then, since for all \( n \geq n_0 \) we have \( H_{R/I}(n) - H_{R/I}(n-1) = H_{S/J}(n) = H_{S/J'}(n) \), we obtain for all \( n \geq n_0 \)

\[
\begin{align*}
  h_{R/I}(n) &= \sum_{j=n_0}^n h_{S/J}(j) + H_{R/I}(n_0 - 1) = q \left( \frac{n + b - 1}{b - 1} \right) + \left( \frac{n + c_1'}{c_1'} \right) + \left( \frac{n + c_2' - 1}{c_2'} \right) \\
  &\quad + \cdots + \left( \frac{n + c_v' - (v - 1)}{c_v'} \right) \end{align*}
\]

where \( c_i' = b_i' + 1 \) and \( \rho = H_{R/I}(n_0 - 1) - \sum_{i=1}^v (n_0 + b_i' - (i-1)) - q \left( \frac{n_0 + b - 2}{b - 1} \right) \) is an integer independent of \( n \). If \( \rho \geq 0 \), taking \( p = v + \rho \) and \( c_{v+1}' = \cdots = c_\rho' = 0 \) we will have (i). Assume then \( \rho < 0 \); then for \( n \gg 0 \) we would have

\[
H_{R/I}(n) < q \left( \frac{n + b - 1}{b - 1} \right) + \left( \frac{n + c_1'}{c_1'} \right) + \left( \frac{n + c_2' - 1}{n - 1} \right) + \cdots + \left( \frac{n + c_v' - (v - 1)}{n - (v - 1)} \right).
\]

Notice that for \( n \geq v \) this is a Euclidean division. By Lemma 3.2 (ii), since \( h \notin z(R/I) \), we can apply Theorem 2.3 to get for all \( n \gg 0 \)

\[
\begin{align*}
  H_{S/J}(n) &< q \left( \frac{n + b - 2}{b - 2} \right) + \left( \frac{n + b_1'}{b_1'} \right) \\
  &\quad + \left( \frac{n + b_2' - 1}{b_2'} \right) + \cdots + \left( \frac{n + b_v' - (v - 1)}{b_v'} \right) = h_{S/J}(n),
\end{align*}
\]

the strict inequality being consequence of the fact \( c_v' = b_v' + 1 \geq 1 \) together with Lemma 2.1(ii).

Now to prove (ii) let us observe that by the definition of the \( c_i' \) we have for all \( i \geq 2 \) that \( p_i = v_{i-1} \). Let \( q > 0 \) (resp. \( q = 0 \)). From the local cohomology long exact sequence associated to (*) and the induction hypothesis we get for all \( i \geq 2 \) that \( H^i_{R^+}(R/I)_{n-1} \cong H^i_{R^+}(R/I)_n \) for all \( n \geq v_{i-1} - (i - 1) + 1 \) (resp. \( n \geq v_{i-1} - (i - 1) \)). Hence, \( H^i_{R^+}(R/I)_n = 0 \) for all \( n \geq v_{i-1} - i + 1 = p_i - i + 1 \) (resp. \( n \geq v_{i-1} - i = p_i - i \)).
The only thing left to complete the proof is to show that \( H^{d+}_{R/\mathfrak{m}}(R/\mathfrak{I})_n = 0 \) for all \( n \geq p \) (resp. \( n \geq p - 1 \)), and \( H^{R/\mathfrak{I}}_R(n) = h_{R/\mathfrak{I}}(n) \) for all \( n \geq p \) (resp. \( n \geq p - 1 \)).

Since \( p = p_1 \geq p_2 \geq \cdots \), by Grothendieck's formula we get \( H^{R/\mathfrak{I}}_R(n) - h_{R/\mathfrak{I}}(n) = -\lambda_{R_0}(H^{p-1}_{R_0}(R/\mathfrak{I})_n) \) for all \( n \geq p \) (resp. \( n > p - 1 \)). Assume that \( H^{p-1}_{R_0}(R/\mathfrak{I})_n \neq 0 \) for some \( n \geq p \), then we would have

\[
H^{R/\mathfrak{I}}_R(n) < h_{R/\mathfrak{I}}(n) = q \binom{n+b-1}{b-1} + \binom{n+c'_1}{n} + \binom{n+c'_2-1}{n-1} + \cdots + \binom{n+c'_{p-1} - (p-1)}{n-(p-1)}.
\]

Since this is a Euclidean division, repeatedly applying Theorem 2.9 we would get for all \( i \geq n \) \( H^{R/\mathfrak{I}}_R(i) < h_{R/\mathfrak{I}}(i) \). Thus, we get the result in the case \( q > 0 \), and in the case \( q = 0 \) it only remains to show that \( H^{p-1}_{R_0}(R/\mathfrak{I})_p = 0 \). If \( H^{p-1}_{R_0}(R/\mathfrak{I})_{p-1} \neq 0 \) we would have

\[
H^{R/\mathfrak{I}}_R(p-1) \leq \binom{p-1+c'_1}{p-1} + \binom{p-1+c'_2-1}{p-2} + \cdots + \binom{p-1+c'_{p-1} - (p-2)}{1}
\]

and this is a \((p-1)\)-binomial expansion; again this contradicts the definition of \( h_{R/\mathfrak{I}} \).

\[\square\]

Remark 3.4. Notice that in the case \( R/\mathfrak{I} = gr_a(A) \) with \( \text{depth}(A) \geq 1 \), by [7, Theorem 5.2], we have \( a_0(R/\mathfrak{I}) < a_1(R/\mathfrak{I}) \). Hence, we can assure that the maximum in Theorem 3.3(iii) is \( p-1 \) if \( q = 0 \) (resp. \( p \) if \( q > 0 \)).

As a corollary we obtain a result which was proved by Gotzmann and Green in the case \( R_0 \) is a field. The proof is analogous to those of Corollaries 2.5 and 2.11.

Corollary 3.5. Let \((R_0, \mathfrak{m})\) be an Artinian local equicharacteristic ring, \( R = R_0 [X_1, \ldots, X_p] \) and \( I \subseteq R_+ \) a homogeneous ideal such that \( \dim(R/I) \geq 1 \). Then:

(i) There exist integers \( c_1 \geq \cdots \geq c_s \geq 0 \) such that

\[
h_{R/\mathfrak{I}}(X) = \binom{X+c_1}{c_1} + \binom{X+c_2-1}{c_2} + \cdots + \binom{X+c_s - (s-1)}{c_s}.
\]

(ii) Let \( s_i = \#\{j \mid c_j \geq i-1\} \); then for \( i \geq 1 \) we have \( H^{R_0}_{R_0}(R/\mathfrak{I})_n = 0 \) for all \( n \geq s_i - 1 \).

(iii) \( i(H^{R/\mathfrak{I}}_R) \leq \max\{a_0(R/\mathfrak{I}) + 1, s - 1\} \).

Remark 3.6. In Theorem 3.3 we have in fact that \( q \neq 0 \) if and only if \( \text{height}(I) = 0 \) (that is, \( \dim(R/I) = \dim(R) = h \)). So the case where Theorem 3.3 and Corollary 3.5 are different is the case \( R_0 \) is not a field and \( I \subseteq \mathfrak{m}R \). In Section 4 we will check that in fact we always have \( p_i \leq s_i \), see Proposition 4.6.
Proposition 3.7. Let \( R_0 \) be an Artinian equicharacteristic local ring, \( b \geq 1 \) an integer, \( R = R_0[X_1, \ldots, X_b] \) and \( I \subseteq R_+ \) a homogeneous ideal, and consider the development

\[
h_{R/G}(X) = q\left(\frac{X + b - 1}{b - 1}\right)
+ \left(\frac{X + c'_1}{c'_1}\right) + \left(\frac{X + c'_2}{c'_2}\right) + \cdots + \left(\frac{X + c'_p - (p - 1)}{c'_p}\right)
\]

as in Theorem 3.3. Let \( m = \max\{i(H_{R/I}), p\} \) and for \( n \geq 0 \) let us consider the Euclidean division \( H(n) = (n+b-1)q(n) + r(n) \) as in Theorem 2.9. Then we have:

(i) For all \( n \geq m \) it holds \( q(n) = q \) and \( r(n + 1) = (r(n))_+ \).

(ii) \( I \) is generated in degrees at most \( m \) and in particular if \( H_{R/I}^0(R/I) = 0 \) then \( I \) is generated in degrees at most \( p \).

Proof. For all \( n \geq m \) we have

\[
H_{R/I}(n) = h_{R/I}(n) = q\left(\frac{n + b - 1}{b - 1}\right) + \left(\frac{n + c'_1}{c'_1}\right)
+ \left(\frac{n + c'_2 - 1}{c'_2}\right) + \cdots + \left(\frac{n + c'_p - (p - 1)}{c'_p}\right).
\]

Since this is a Euclidean division we get (i). To prove (ii), notice that since \( R_1I_n \subseteq I_{n+1} \) we have \( H_{R/I}(n+1) = \lambda_{R_{n+1}}(R_{n+1}/I_{n+1}) < \lambda_{R_n}(R_{n+1}/R_1I_n) \). As in the proof of Theorem 2.9 we get \( \lambda_{R_n}(R_{n+1}/R_1I_n) \leq (n+b-1)q(n) + (r(n))_+ \). If \( n \geq m \), then by (i) \( (n+b-1)q(n) + (r(n))_+ = H_{R/I}(n+1) \), therefore \( I_{n+1} = R_1I_n \) for all \( n \geq m \). Finally, if \( H_{R/I}^0(R/I) = 0 \) then \( a_0(R/I) = -\infty \) and hence by Theorem 3.3(iii) we have \( i(H_{R/I}) \leq p \). \( \square \)

Definition 3.8. Let \( P \in \mathbb{Q}[X] \); we will say that \( P \) admits a Gotzmann development if either \( P = 0 \) or there exist integers \( c_1 \geq c_2 \geq \cdots \geq c_s \geq 0 \) such that

\[
P(X) = \left(\frac{X + c_1}{c_1}\right) + \left(\frac{X + c_2 - 1}{c_2}\right) + \cdots + \left(\frac{X + c_s - (s - 1)}{c_s}\right).
\]

In this case we will call the expression above the Gotzmann development of \( P \). Notice that \( c_1, \ldots, c_s \) are uniquely determined by \( P \); they will be called the Gotzmann coefficients of \( P \). We define also \( s_q = \#\{i \mid c_i \geq q - 1\} \) for all \( q \geq 1 \).

The first fact to notice is that not all polynomials in \( \mathbb{Q}[X; \mathbb{N}] \) admit a Gotzmann development. For example, if \( 2X = (X + c_1) + (X + c_2 - 1) + \cdots + (X + c_s - (s - 1)) \), it should be \( c_1 = c_2 = 1 \) and \( c_3 = \cdots = c_s = 0 \). But then we would have \( 2X = (X + 1) + (X + 1 - 1) + (s - 2) = 2X + 1 + s - 2 = 2X + s - 1 \), which is absurd since \( s = s_1 \geq s_2 = 2 \).

Next we give a triangular equation system to compute the normalized Hilbert–Samuel coefficients from the Gotzmann coefficients and reciprocally.
Proposition 3.9. Let $P \in \mathbb{Q}[X; \mathbb{N}]$ a polynomial of degree $d - 1 \geq 0$ admitting a Gotzmann development. Denote by $c_1, \ldots, c_s$ the Gotzmann coefficients of $P$, by $s_j = \#\{i \mid c_i \geq j - 1\}$ and by $e_0, \ldots, e_{d-1}$ the normalized Hilbert–Samuel coefficients of $P$. Then for all $0 \leq i \leq d - 1$ we have
\[
e_i = (-1)^i s_{d-i} + \sum_{j=d-i+1}^{d} (-1)^{d-j} \binom{s_j + 1}{j}.
\]

Proof. By induction on $d$. If $d = 1$, $P(n) = e_0$ for all $n$, and its Gotzmann coefficients are $c_1 = \cdots = c_0 = 0$, so $s = s_1 = e_0$. In the case $d > 1$ we have
\[
\sum_{i=0}^{d-1} (-1)^i e_i \binom{X + d - 1 - i}{d - 1 - i} = P(X) = \sum_{i=1}^{s_1} \binom{X + c_i - (i - 1)}{c_i}.
\]
Given any function $f : \mathbb{Z} \to \mathbb{Z}$ we define $\Delta f(n) = f(n) - f(n - 1)$ for all $n \in \mathbb{Z}$. Notice that again $\Delta f : \mathbb{Z} \to \mathbb{Z}$, $\Delta$ is a $\mathbb{Z}$-linear operator and $\Delta \binom{X+1}{i} = \binom{X+i}{i-1}$. We have then
\[
\sum_{i=0}^{d-2} (-1)^i e_i \binom{X + d - 2 - i}{d - 2 - i} = \Delta P(X) = \sum_{i=1}^{n} \binom{X + b_i - (i - 1)}{b_i},
\]
where $t_1 = s_2$ and $b_i = c_i - 1$. If $t_q = \#\{i \mid b_i \geq q - 1\}$, by induction hypothesis we get for all $0 \leq i \leq d - 2$
\[
e_i = (-1)^i t_{d-i} + \sum_{j=d-i+1}^{d-1} (-1)^{d-1-j} \binom{t_j + 1}{j-d-1+i+1}.
\]
Since $t_q = s_{q+1}$ for all $q > 1$ it only remains to compute $e_{d-1}$. Let us give before an expression of $P$ which involves $s_q$:
\[
P(X) = \sum_{i=1}^{s} \binom{X + c_i - (i - 1)}{c_i} = \binom{X + d}{d} - \sum_{j=2}^{d} \binom{X + j - s_j - 1}{j} - \binom{X - s_1 + 1}{1},
\]
note that $s_{d+1} = 0$. Evaluating the two expressions of $P$ at $X = -1$ we get
\[
(-1)^{d-1} e_{d-1} = -\sum_{j=2}^{d} \binom{j - s_j - 2}{j} - \binom{-s_1}{1} = s_1 + \sum_{j=2}^{d} (-1)^{j+1} \binom{s_j + 1}{j}.
\]

Remark 3.10. We obtain polynomial bounds $a_{i+1}(R/I) \leq f_{i+1}(e_0, \ldots, e_{d-i-1})$ and $(-1)^i e_i \geq g_i(e_0, \ldots, e_{i-1})$ for all $0 \leq i \leq d - 1$. For example, $e_1 \leq \binom{e_0}{2}$, which answers a question in [8], $a_d(R/I) \leq e_0 - d - 1$ (see [7, Lemma 4.2]) and $a_{d-1}(R/I) \leq \binom{e_{d-1}}{2} - e_1 - d$.
4. Characterization of Hilbert polynomials

In this section we study when a polynomial \( P \in \mathbb{Q}[X; \mathbb{N}] \) can be the Hilbert polynomial of a standard algebra. We show in Theorem 4.4 that \( P \) is a Hilbert polynomial if and only if it admits a Gotzmann development. We also characterize the minimal number of variables for which \( P \) is admissible. The characterizations given are effective; see Section 5 for the algorithms.

**Definition 4.1.** Given an Artinian equicharacteristic local ring \((R_0, \mathfrak{m})\) and \( P \in \mathbb{Q}[X; \mathbb{N}] \), we will say that \( P \) is **admissible** if it is the Hilbert polynomial of a standard \( R_0 \)-algebra. If \( b \geq 1 \) is an integer, we will say that \( P \) is \( b \)-admissible if there exists a homogeneous ideal \( I \subseteq R_+ \), where \( R = R_0[X_1, \ldots, X_b] \), such that \( P = h_{R/I} \).

Notice that if \( P \) is \( b \)-admissible then \( P \) is \( b' \)-admissible for all \( b' \geq b \), see Corollary 2.12. In order to decide whether \( P \) is admissible it will suffice to study when \( P \) can be interpolated by an admissible function. Let us define a special admissible function that will do:

**Definition 4.2.** Let \( c_1 \geq c_2 \geq \cdots \geq c_s \geq 0 \) be integers. We define the **Gotzmann function** \( G[c_1, \ldots, c_s] \) associated to \( c_1, \ldots, c_s \), by

\[
G[c_1, \ldots, c_s](n) = \begin{cases} 
\binom{n+c_1}{c_1} + \binom{n+c_2-1}{c_2} + \cdots + \binom{n+c_{s-1}}{c_{s-1}} & \text{if } n \leq s-1, \\
\binom{n+c_1}{c_1} + \binom{n+c_2-1}{c_2} + \cdots + \binom{n+c_{s-1}}{c_{s-1}} & \text{if } n \geq s-1.
\end{cases}
\]

**Lemma 4.3.** For all \( c_1 \geq c_2 \geq \cdots \geq c_s \geq 0 \) integers, \( G[c_1, \ldots, c_s] \) is an admissible function such that \( G[c_1, \ldots, c_s](1) = c_1 + 2 \).

**Proof.** It follows from Lemma 2.1 and [11, Proposition 4.3]. \( \Box \)

**Theorem 4.4 (Characterization of Hilbert polynomials).** Let \( P \in \mathbb{Q}[X; \mathbb{N}] \) and \( R_0 \) an equicharacteristic Artinian local ring. Then the following conditions are equivalent:

(i) There exists a standard \( R_0 \)-algebra \( S \) such that \( h_S = P \).

(ii) \( P \) admits a Gotzmann development.

**Proof.** We have just seen that (i) implies (ii) in Corollary 3.5. Assume that \( P \) admits a Gotzmann development. Using Corollary 2.11, it is enough to find a function \( H : \mathbb{N} \rightarrow \mathbb{N} \) such that \( H(0) = \lambda_{R_0}(R_0) \), \( H(n+1) \leq (H(n))_n^+ \) and \( H(n) = P(n) \) for \( n \geq 0 \). The simplest is possibly \( H = G[c_1, \ldots, c_s] \) for \( n > 0 \) and \( H(0) = \lambda_{R_0}(R_0) \). \( \Box \)

Notice that the above criterion is effective. For instance, we see that \( P(X) = 2X \) is not the Hilbert polynomial of any standard \( R_0 \)-algebra.

The following theorem is a version of Theorem 4.4 which takes into account the number of variables. It decides whether \( P \) is \( b \)-admissible or not in terms of
combinatorial properties of $P$, and it will be used in Section 5 to compute the minimal $b$ for which a function $H$ is $b$-admissible.

**Theorem 4.5.** Let $(R_0, m)$ be an Artinian equicharacteristic local ring, $b \geq 1$ an integer and $P = \sum_{i=0}^{c} (-1)^i e_i \binom{x+c-i}{c-i} \in \mathbb{Q}[X; \mathbb{N}]$. Then we have:

(i) If $P$ is $b$-admissible then $b \geq c+1$ and $P$ admits a Gotzmann development.

(ii) If $b \geq c+2$ and $P$ admits a Gotzmann development then $P$ is $b$-admissible.

(iii) $P$ is $(c+1)$-admissible if and only if either of the following conditions hold:

(a) $0 < e_0 < \lambda_{R_0}(R_0)$ and $P(X) - e_0 \binom{x+c}{c}$ admits a Gotzmann development,

(b) $e_0 = \lambda_{R_0}(R_0)$ and $P(X) = e_0 \binom{x+c}{c}$.

**Proof.** Let $r = \lambda_{R_0}(R_0)$.

(i) Assume that $P$ is $b$-admissible. By Corollary 3.3, $P$ admits a Gotzmann development. Moreover, $c = \deg(P) = \dim(R/I) - 1 < \dim(R) - 1 = b - 1$.

(ii) By Corollary 2.12 it is enough to find an admissible function $H : \mathbb{N} \rightarrow \mathbb{N}$ such that $H(1) = c + 2$ and $H(n) = P(n)$ for all $n \gg 0$. Since this has been done in Lemma 4.3, (ii) is proved.

To prove (iii), assume in the first place that $P$ is $(c+1)$-admissible. Then by Theorem 3.3, there exist integers $c > c'_1 \geq \cdots \geq c'_p \geq 0$ and $0 \leq q \leq r$ such that

$$P(X) = q \binom{X+c}{c} + \binom{X+c'_1}{c'_1} + \binom{X+c'_2}{c'_2} + \cdots + \binom{X+c'_p - (p-1)}{c'_p}.$$ 

Since $c = \deg(P)$ it follows that $q > 0$. From the equality between the two expressions of $P$, by comparing degrees we get $q = e_0$ and $P(X) - e_0 \binom{x+c}{c} = \binom{X+c'_1}{c'_1} + \binom{X+c'_2 - 1}{c'_2} + \cdots + \binom{X+c'_p - (p-1)}{c'_p}$. Furthermore, for $n \gg 0$ we have an inequality of Euclidean divisions

$$e_0 \binom{n+c}{c} + \binom{n+c'_1}{c'_1} + \binom{n+c'_2 - 1}{c'_2} + \cdots + \binom{n+c'_p - (p-1)}{c'_p} = P(n) = H_{R/I}(n) \leq r \binom{n+c}{c},$$

hence we get (a) and (b). Reciprocally, assume that $P$ verifies the conditions in (a) or (b). Let $P(X) - e_0 \binom{x+c}{c} = \binom{X+c_1}{c_1} + \binom{X+c_2 - 1}{c_2} + \cdots + \binom{X+c_s - (s-1)}{c_s}$ be the Gotzmann development, put $s = 0$ if $P(X) - e_0 \binom{x+c}{c} = 0$. By comparing degrees we get $c_1 \leq c - 1$ and, therefore,

$$P(n) = e_0 \binom{n+c}{c} + \binom{n+c_1}{c_1} + \binom{n+c_2 - 1}{c_2} + \cdots + \binom{n+c_s - (s-1)}{c_s}$$

is a Euclidean division for all $n \geq s-1$. To show that $P$ is $(c+1)$-admissible it suffices to construct $H : \mathbb{N} \rightarrow \mathbb{N}$ verifying Theorem 2.9 (ii) and such that $H(n) = P(n)$ for all


\[ n \gg 0. \text{ It is now immediate that} \]
\[ H(n) = \begin{cases} 
\lambda_{R_0}(R_0) \binom{n+c}{c} & \text{if } n \leq s - 1 \\
P(n) & \text{if } n \geq s
\end{cases} \]

verifies the conditions required. \( \square \)

Gotzmann developments can be very complicated; for example, the Gotzmann development of \( 8(n+3) \) has 161427 terms (apply the formulas in Proposition 3.9). Notice that if \( R_0 \) is an Artinian ring with \( \lambda_{R_0}(R_0) = 8 \) and \( R = R_0[X_1, X_2, X_3, X_4] \), then these expressions are the ones obtained for \( h_R \) in parts (i) of Theorem 3.3 and Corollary 3.5, respectively. In fact, the expression of \( h_{R/I} \) given in Theorem 3.3 is always better than the one obtained in Corollary 3.5.

**Proposition 4.6.** Let \( R = R_0[X_1, \ldots, X_b] \), \( I \subseteq R_+ \) a homogeneous ideal and

\[
h_{R/I}(x) = q \left( \frac{x + b - 1}{b - 1} \right)
+ \left( \frac{x + c'_1}{c'_1} \right) + \left( \frac{x + c'_2 - 1}{c'_2} \right) + \cdots + \left( \frac{x + c'_p - (p - 1)}{c'_p} \right)
= \left( \frac{x + c_1}{c_1} \right) + \left( \frac{x + c_2 - 1}{c_2} \right) + \cdots + \left( \frac{x + c_s - (s - 1)}{c_s} \right)
\]

the expressions of \( h_{R/I} \) obtained in Theorem 3.3 and Corollary 3.5, respectively. Define \( p_i \) and \( s_i \) as in these two results; then for all \( i \geq 1 \) it holds \( p_i \leq s_i \).

**Proof.** We may assume \( q \neq 0 \), hence \( c_1 = b - 1 \). The result is obvious for \( b = 1 \). For \( b \geq 2 \) we have \( q \left( \binom{n+b-1}{b-1} \right) + G[c'_1, \ldots, c'_p](n) = G[c_1, \ldots, c_s](n) \) for all \( n \gg 0 \), hence applying the operator \( \Delta \) we get

\[ q \left( \frac{n + b - 2}{b - 2} \right) + G[c'_1 - 1, \ldots, c'_{p_2} - 1](n) = G[c_1 - 1, \ldots, c_{s_2} - 1](n). \]

So by induction on \( b \) it is enough to show that \( p \leq s \). We will proceed by induction on \( p \). If \( p = 0 \) the statement is clear, so we assume that \( p > 0 \). If \( c'_p = 0 \), then from Theorems 4.4 and 4.5 there exist \( b_1 \geq b_2 \geq \cdots \geq b_t \geq 0 \) such that for all \( n \gg 0 \) we have \( q \left( \binom{n+b-1}{b-1} \right) + G[c'_1, \ldots, c'_{p-1}](n) = G[b_1, \ldots, b_t](n) \), hence

\[
q \left( \frac{n + b - 1}{b - 1} \right) + G[c'_1, \ldots, c'_p](n) = q \left( \frac{n + b - 1}{b - 1} \right) + G[c'_1, \ldots, c'_{p-1}](n) + 1
= G[b_1, \ldots, b_t, 0](n).
\]

So we must have \( s = t + 1 \), and by induction hypothesis we have \( p - 1 \leq t \).
If \( c'_{p} > 0 \), let \( t = c'_{p} < b - 1 = c_{1} \). Then applying \( \Delta' \) we have for all \( n \gg 0 \)

\[
q \left( \frac{n + b - 1 - t}{b - 1 - t} \right) + G[c'_{1} - t, \ldots, c'_{p} - t](n) = G[c_{1} - t, \ldots, c_{s_{i+1}} - t](n)
\]

therefore by the case \( c'_{p} = 0 \) we have \( p \leq s_{i+1} \leq s \). \( \square \)

5. Admissibility of functions. Ideals with a given Hilbert function

Our aim in this section is to give an algorithm to decide whether an asymptotically polynomial function \( H \) is admissible. The natural way to encode \( H \) should be to give a finite number of values of \( H \), say \( H(0), H(1), \ldots, H(n_{0}) \), and \( h(X) \in \mathbb{Q}[X; \mathbb{N}] \) such that \( H(n) = h(n) \) for all \( n > n_{0} \). \( H \) is admissible if and only if it verifies the conditions in Theorem 2.1 (ii); the problem is to verify these conditions in a finite number of steps. The theory of Gotzmann developments provides us with a method to do so. Furthermore, if \( H \) is admissible we compute the minimal value \( b \) for which \( H \) is \( b \)-admissible. We also describe how to get a generating system, which will be a minimal in the case \( R_{0} \) is a field, for an ideal \( I \subseteq R_{0}[X_{1}, \ldots, X_{b}] \) such that \( H = H_{R_{0}/I} \). Let us begin by giving an algorithm to decide whether a polynomial \( P \in \mathbb{Q}[X] \) is an admissible polynomial.

5.1. Algorithm to compute the Gotzmann development

Here we give an algorithm to compute, if it exists, the Gotzmann development of a polynomial. The strategy is to compute first the normalized Hilbert–Samuel coefficients and then compute from them the Gotzmann coefficients using Proposition 3.9. The following proposition provides a triangular equation system in \( e_{1}, \ldots, e_{c} \) and a criterion to decide whether \( P \in \mathbb{Q}[X ; \mathbb{N}] \).

**Proposition 5.1.** Let \( h = a_{c}X^{c} + \cdots + a_{0} \in \mathbb{Q}[X] \) be a polynomial of degree \( c \) and \( e_{0}, \ldots, e_{c} \) its normalized Hilbert–Samuel coefficients. Then we have for all \( 0 \leq i \leq c - 1 \).

\[
\sum_{k=0}^{i} \binom{i}{k} e_{c-k} = (-1)^{\frac{c}{2}} h(-i - 1).
\]

In particular, \( h \in \mathbb{Q}[X ; \mathbb{N}] \) if and only if \( c!a_{c} \in \mathbb{N} \) and \( h(-1), \ldots, h(-c) \in \mathbb{Z} \).

**Proof.** We have \( h(X) = \sum_{k=0}^{c} (-1)^{c-k} e_{c-k} \binom{X+k}{k} \), hence for all \( 0 \leq i \leq c - 1 \) we get \( h(-i - 1) = \sum_{k=0}^{i} (-1)^{c-k} e_{c-k} \binom{k-i-1}{k} = \sum_{k=0}^{i} (-1)^{c} e_{c-k} \binom{i}{k} \) as we wanted to prove. The second part is consequence of the fact that the system matrix is unipotent upper triangular with coefficients in \( \mathbb{N} \). \( \square \)
Algorithm (GOTZMANN-DEVELOPMENT)

**INPUT:** \( P(X) = a_c X^c + \cdots + a_0 \in \mathbb{Q}[X] \).

**OUTPUT:** \( e_0(P), \ldots , e_c(P) \); the Gotzmann development of \( P \), if it exists, and its length \( s \).

*Step 1:* If \( e_0 = c!a_c \not\in \mathbb{N} \) or \( P(-1), \ldots , P(-c) \) are not all integers then NO-GOTZMANN-DEVELOPMENT (by Proposition 5.1).

*Step 2:* Obtain \( e_c, e_{c-1}, \ldots , e_1 \) by solving the triangular system of equations of Proposition 5.1.

*Step 3:* Obtain \( s_1, \ldots , s_c \) by solving the triangular system of equations of Proposition 3.9. If \( s_i < s_{i+1} \) for some \( c-1 \leq i \geq 1 \) then NO-GOTZMANN-DEVELOPMENT.

5.2 Algorithm to check \( b \)-admissibility

Fix an Artinian equicharacteristic local ring \( (R_0,m) \) and a set of data \((r,i_1,\ldots ,i_{n_0}, h(X))\) describing a function \( H : \mathbb{N} \to \mathbb{N} \). We assume that \( r = \lambda_{R_0}(R_0), \) \( i_n \in \mathbb{N} \) for all \( 1 \leq i \leq n_0, \) \( h(X) \in \mathbb{Q}[X] \) and \( h(n_0) \neq i_{n_0} \). Hence, we have \( H(0) = r, H(n) = i_n \) for all \( 1 \leq n \leq n_0 \) and \( i(H) = n_0 + 1 \). Notice that any admissible function can be encoded in this way. Then we are going to use Corollary 2.12, Theorem 4.5, Proposition 5.1, Theorem 4.4 and Proposition 3.7 in order to check whether \((r,i_1,\ldots ,i_{n_0}, h(X))\) describe an admissible function and, in such case, to compute the minimal \( b \) for which \( H \) is admissible. For any \( x \in \mathbb{R} \), let \( \lfloor x \rfloor \) denote the least integer greater or equal than \( x \).

Algorithm (\( b \)-ADMISSIBILITY)

**INPUT:** \( r = \lambda_{R_0}(R_0), n_0 \in \mathbb{N}, H = (i_1, \ldots , i_{n_0}, h(X) = a_c X^c + \cdots + a_0) \in \mathbb{N}^{n_0} \times \mathbb{Q}[X] \).

**OUTPUT:** \( b_{\text{min}} = \min\{-b' \in \mathbb{N} \mid H \text{ is } b'\text{-admissible}\}, \) if it exists.

*Step 1:* If \( H(1) < c + 1 \) then NO-ADMISSIBLE (by Corollary 2.12 and Theorem 4.5(i)).

*Step 2:* Perform Steps 1 and 2 of algorithm GOTZMANN-DEVELOPMENT. If NO-GOTZMANN-DEVELOPMENT then NO-ADMISSIBLE.

*Step 3:* Set \( b_{\text{min}} = \max\{\lfloor H(1)/r \rfloor, c + 1\} \) (by Corollary 2.12 (iii) and Theorem 4.5(i)). If \( b_{\text{min}} > c + 1 \) then skip to Step 6 (by Theorem 4.5(ii)).

*Step 4:* If \( e_0 > r \) then skip to Step 6 (by Theorem 4.5(iii)).

*Step 5:* Check whether \( h(X) - e_0 \binom{X+c}{c} \) verifies Theorem 4.5(iii):

5.1. If \( e_1 = \cdots = e_c = 0 \), set \( p = 0 \) and go to Step 7.

5.2. If the first nonzero \( e_i \) verifies \((-1)^i e_i < 0\), set \( b_{\text{min}} = b_{\text{min}} + 1 \) and skip to Step 6.

5.3. If \( e_0 = r \) set \( b_{\text{min}} = b_{\text{min}} + 1 \) and skip to Step 6. If \( e_0 < r \) compute the Gotzmann development of \( h(X) - e_0 \binom{X+c}{c} \). If NO-GOTZMANN-DEVELOPMENT, set
\[ b_{\text{min}} = b_{\text{min}} + 1 \] and skip to Step 6. Otherwise, let \( p = \) length of the Gotzmann development and skip to Step 7.

**Step 6:** Compute the Gotzmann development of \( h \). If NO-GOTZMANN-DEVELOPMENT then NO-ADMISSIBLE (by Theorem 4.4). Otherwise let \( p \) be its length.

**Step 7:** Define \( m = \max\{i(H), p\} \) (by Proposition 3.7). For \( 0 \leq n \leq m \) compute the Euclidean division \( H(n) = \left(\binom{n-h_{\text{min}}-1}{h_{\text{min}}-1}\right) q(n) + r(n) \). If \( (q(n), r(n)) \leq (q(n-1), (r(n-1)_{n-1})_{n-1}) \) for all \( 1 \leq n \leq m \) then \( b_{\text{min}} \)-ADMISSIBLE. Otherwise

1. If \( b_{\text{min}} = c + 1 \), set \( b_{\text{min}} = b_{\text{min}} + 1 \) and skip to Step 6.
2. If \( b_{\text{min}} = H(1) \) then NO-ADMISSIBLE (by Corollary 2.12 (ii))
3. Otherwise set \( b_{\text{min}} = b_{\text{min}} + 1 \) and skip to the head of Step 7.

Notice that in Step 5.3 the normalized Hilbert–Samuel coefficients of \( h(X) - e_0\left(\frac{X+c}{c}\right) \) are \( e_j^{i} = (-1)^i e_{i-j}, 0 \leq j \leq c - i \), so we do not need to perform Steps 1 and 2 in algorithm GOTZMANN-DEVELOPMENT. Also in Step 6 the normalized Hilbert Samuel coefficients of \( h \) have already been computed in Step 2, so we can go directly to Step 3 in algorithm GOTZMANN-DEVELOPMENT.

### 5.3. Algorithm to construct an ideal with a given Hilbert function

Given a \( b \)-admissible function \( H = (r, i_1, \ldots, i_n, h(X)) \) and a composition series \( \mathcal{J} = \{0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_r = R_0\} \) in \( R_0 \) we will construct the ideal \( I_{H, \mathcal{J}} \subseteq R_0[X_1, \ldots, X_b]_+ \).

**Proposition 5.2.** Let \( k \) be a field, \( I \subseteq k[X_1, \ldots, X_b] = R \) a segment ideal and \( H = H_{R/I} \). Then it holds:

1. For all \( n \geq 1 \), \( H(n + 1) = (H(n))_{+}^{n+1} \) if and only if \( I_{n+1} = R_{1}I_{n} \).
2. \( I_{n} \) is generated as a \( k \)-vector space by the last \( (r+1+b-1)^{n}_b - (H(n))_{+}^{n+1} \) monomials in \( R_{n} \).

**Proof.** (i) follows from [1, Proposition 4.2.8] and Macaulay’s theorem. (ii) follows from [1, Lemma 4.2.5], and the proof of Theorem 2.9. □

**Algorithm (H-IDEAL)**

**INPUT:** \( r = \lambda_{R_0}(R_0) \), a composition series \( \mathcal{J} = \{0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_r = R_0\} \), a function \( H \) encoded as in algorithm \( b \)-ADM and \( b \geq 1 \) an integer for which we know that \( H \) is admissible.

**OUTPUT:** The \( \mathcal{J} \)-segment ideal \( I_{H, \mathcal{J}} \subseteq R_0[X_1, \ldots, X_b]_+ \).

**Step 1:** Set \( p = \) length of the Gotzmann development of \( h(X) - e_0\left(\frac{X+c}{c}\right) \) if \( b = c + 1 \) and \( p = \) length of the Gotzmann development of \( h \) if \( b > c + 1 \).

**Step 2:** Let \( m = \max\{i(H), p\} \). For \( 0 \leq n \leq m \) compute the Euclidean division \( H(n) = Nq(n) + r(n), \) where \( N = \left(\binom{n-h_{\text{min}}-1}{h_{\text{min}}-1}\right) \), and the values \( (g_1(n), g_2(n)) = (q(n-1) - q(n), (r(n-1)_{n-1})_{n-1}^{+} - r(n)) \geq (0, 0) \) for \( 1 \leq n \leq m \). Set \( \gamma(n) = \min\{r(n), \)
(r(n - 1)_{n-1})^+}$, then the generators of $I_{H,J}$ in degree $n$ are

$$J_{r-q(n)-1}X^v, \ldots, J_{r-q(n)-1}X^{v_{\gamma(n)}}, J_{r-q(n)}X^{v_{\gamma(n)+1}}, \ldots, J_{r-q(n)}X^{v_N}$$

if $g_1(n) > 1$,

$$J_{r-q(n)}X^{v_{\gamma(n)+1}}, \ldots, J_{r-q(n)}X^{v_{\gamma(n)+\gamma_2(n)}}$$

if $g_1(n) = 0$.

Notice that the generators we have skipped are superfluous by Proposition 5.2. By Proposition 3.7, this set of $R_0$-submodules generates $I_{H,J}$.

Let us finally make some remarks about the case $R_0$ is a field:

**Remark 5.3.** If $R_0$ is a field, $H$ is an admissible function and $b = H(1)$, then the generating system obtained above is $\bigcup_{n=1}^{m-1} \{X^{v_{H(n+1)+1}}, \ldots, X^{v_{H(n)+1}}\}$, and it is a minimal generating system for $I_H$. Applying then [3, Corollary 2.7], we can compute for every homogeneous ideal $J \subseteq R = R_0[X_1, \ldots, X_k]$ a bound for its minimal number of generators that will depend only on $H = H_{R,J}$. Namely, if $m = \max\{i(H), s\}$, where $s$ is the length of the Gotzmann development of $h$, then

$$v(J) \leq \sum_{n=1}^{m} (H(n) - (H(n-1)_{n-1})^+).$$

In this case we can also compute $H^0_{R,J}(R/J)$: let $I \subseteq R$ be a homogeneous ideal and $J \subseteq R$ be the homogeneous ideal such that $H^0_{R,J}(R/I) = J/I$. We have $J_n = I_n$ for all $n > a_0(R/I)$. In other words, $J = I^{sat}$ is the saturation of $I$: it is the biggest homogeneous ideal containing $I$ and having the same Hilbert polynomial, and verifies depth$(R/J) \geq 1$.

**Lemma 5.4.** Let $s$ be the length of the Gotzmann development of $h_{R,I}$ and $m = \max\{i(R/I), s\}$. Then we have:

(i) If $I$ is a monomial ideal then $J$ is monomial too.

(ii) $J$ is generated in degrees at most $s$.

(iii) If $I$ is a segment ideal and $\alpha \in R$ is a monomial, then for all $n \geq 1$, $\alpha X_1^\alpha \subseteq I$ if and only if $\alpha X_1^n \in I$.

(iv) If $I$ is a segment ideal then $J = \bigcup_{n \geq 1} (I : X_1^n)$.

(v) If $I$ is a segment ideal, then $J$ is generated by $\bigcup_{n=0}^{m-n} (I_{n+i} : X_1^i)$.

**Proof.** (i) Let $\alpha \in J$ be an element which we may assume to be homogeneous. Write $\alpha = t_1 + \cdots + t_r$ as a sum of terms; then it is enough to show that every $t_i \in J$. Since $\alpha \in J$ there exists $n \in \mathbb{N}$ such that $\alpha X^v \in I$ for all multi-indices $v$ with $|v| = n$. That is to say $t_1 X^v + \cdots + t_r X^v \in I$ for all $|v| = n$. Notice that this is a sum of terms and $I$ is a monomial ideal, hence $t_i X^v \in I$ for all $|v| = n$ and for all $i$, i.e., $t_i \in J$ for all $i$. (ii) is Proposition 3.7. (iii) follows from the fact that if $\beta, \gamma \in R_n$ are monomials with
\[ \beta > \gamma, \text{ then } \alpha \beta > \alpha \gamma. \text{ Since } X^n_i \text{ is the greatest monomial in } R_n, \text{ we are done. (iv) and (v) follow from the first three claims.} \]

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**Note added in proof:** When this paper was in press, the author became aware of the reference [13, Theorem 5.7]. There, a characterization theorem for Hilbert polynomials, which in the case of a field is equivalent to our Theorem 4.4, is given and it is attributed to Macaulay.

**References**