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Commutativity preserving linear maps and Lie automorphisms of strictly triangular matrix space ☆

You'an Cao *, Zhonghu Chen, Chong Huang

Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, People's Republic of China

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Abstract

In this paper we classify linear maps preserving commutativity in both directions on the space $N(\mathbf{F})$ of strictly upper triangular $(n+1)\times(n+1)$ matrices over a field $\mathbf{F}$. We show that for $n \geqslant 3$ a linear map φ on $N(\mathbf{F})$ preserves commutativity in both directions if and only if $\varphi = \varphi' + f$ where φ' is a product of standard maps on $N(\mathbf{F})$ and f is a linear map of $N(\mathbf{F})$ into its center. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction and statement of results

Let M be a matrix space over a field $\mathbf{F}$. A linear map φ on M is said to be *commutativity preserving* if $\varphi(A)$ commutes with $\varphi(B)$ for every pair of commuting elements $A, B \in M$. It is said to be *commutativity preserving in both directions* when the condition $AB = BA$ holds if and only if $\varphi(A)\varphi(B) = \varphi(B)\varphi(A)$. It is one of the linear preserver problems to classify commutativity preserving linear maps on matrix spaces (see [10, Section 3.4], [11] and [14, Section 7.1]). Several authors

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* Corresponding author. Tel.: +86-0732-8292791; fax: +86-732-829-3000.
E-mail address: cya@xtu.edu.cn (Y. Cao).

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have classified commutativity preserving linear maps on a number of variations of matrix spaces, see [1,5,6,9,12,13,15–17].

In this paper, we will classify linear maps preserving commutativity in both directions on the space $N(\mathbf{F})$ of strictly upper triangular $(n+1)\times(n+1)$ matrices over an arbitrary field $\mathbf{F}$. Our main result is:

Theorem 1.1. *Let $n \geqslant 3$. A linear map φ on $N(\mathbf{F})$ preserves commutativity in both directions if and only if φ is of the form*

$$\varphi = \psi_c \omega \mu_{b_4}^{(n2)} \mu_{b_3}^{(12)} \mu_{b_2}^{(n1)} \mu_{b_1}^{(11)} \sigma_T + f, \tag{1.1}$$

where the factors in the first term on the right-hand side are standard maps on $N(\mathbf{F})$ and f is a linear map of $N(\mathbf{F})$ into its center.

We will give the definitions of standard maps on $N(\mathbf{F})$ in Section 2 and prove this theorem in Section 3. By Theorem 1.1, we can obtain the following theorems (Theorems 1.2 and 1.3).

Theorem 1.2. *Let $n \geqslant 3$. Regard $N(\mathbf{F})$ as an associative algebra over $\mathbf{F}$. A map φ on $N(\mathbf{F})$ is an $\mathbf{F}$-algebra automorphism if and only if φ is of the form*

$$\varphi = \sigma_T(1+f), \tag{1.2}$$

where both σ_T and $1+f$ are standard $\mathbf{F}$-algebra automorphisms of $N(\mathbf{F})$.

On the other hand, we define the bracket operation $[A, B] = AB - BA$ on $N(\mathbf{F})$. It is clear that φ is commutativity preserving in both directions if and only if it preserves zero brackets in both directions. In this paper, we will use this fact repeatedly.

The bracket operation defines a structure of Lie algebra on $N(\mathbf{F})$. Choi et al. [6] mentioned that the results on linear maps preserving commutativity can be viewed in the content of Lie algebra, where one assumes that the linear map preserves zero products and the conclusion is that the map “essentially” preserves all products. Marcoux and Sourour [12] also pointed out that the linear maps that preserve zero Lie brackets in both directions differ only slightly from those that preserve all Lie brackets. These assertions are also true for the strictly upper triangular matrix space $N(\mathbf{F})$ when $n \geqslant 3$. In fact, we have the following theorem.

Theorem 1.3. *Let $n \geqslant 3$. Regard $N(\mathbf{F})$ as a Lie algebra over $\mathbf{F}$. A map φ on $N(\mathbf{F})$ is a Lie automorphism if and only if φ is of the form*

$$\varphi = \omega \mu_{b_2}^{(n1)} \mu_{b_1}^{(11)} \sigma_T(1+f) \quad \textit{if} \operatorname{char} \mathbf{F} \neq 2 \tag{1.3}$$

or

$$\varphi = \omega \mu_{b_4}^{(n2)} \mu_{b_3}^{(12)} \mu_{b_2}^{(n1)} \mu_{b_1}^{(11)} \sigma_T(1+f) \quad \textit{if} \operatorname{char} \mathbf{F} = 2, \tag{1.4}$$

where the factors on the right-hand side are standard Lie automorphisms of $N(\mathbf{F})$.

We will define standard **F**-algebra automorphisms and standard Lie automorphisms in Section 2 and prove Theorems 1.2 and 1.3 in Section 3.

Remark 1. The result of Theorem 1.2 is covered with our earlier one in [4], where for $n \geqslant 1$ we characterize R-algebra automorphisms of strictly upper triangular matrices over an arbitrary commutative ring R.

Remark 2. In [3], for $n \geqslant 3$ we characterize Lie automorphisms of strictly upper triangular matrices over a local ring that contains 2 as a unit and an integral domain of characteristic other than 2. In [3], ω, $\mu_{b_2}^{(n1)}\mu_{b_1}^{(11)}$ and $1 + f$ are called graph, extremal and central automorphisms, respectively, and σ_T is called inner or diagonal automorphism according as T is an upper triangular matrix having entries 1's on the main diagonal or an invertible diagonal matrix. It follows from Theorem 1.3 that over a field **F** of characteristic 2 there exist Lie automorphisms of $N(\mathbf{F})$ which cannot be expressed as a product of standard automorphisms defined in [3]. So for $N(\mathbf{F})$ over a field **F** of characteristic 2 the result of Theorem 1.3 is new.

Remark 3. For descriptions for Lie automorphisms of the upper triangular matrices over a commutative ring, see [2,7].

2. Preliminaries and notations

Let **F** be an arbitrary field and $\mathbf{F}^*$ the group of non-zero elements of **F**. Let $M_{n+1}(\mathbf{F})$ be the full matrix space of $(n+1) \times (n+1)$ matrices over **F**, and $N(\mathbf{F})$ the subspace of strictly upper triangular matrices in $M_{n+1}(\mathbf{F})$. Let $T(\mathbf{F})$ be the group of invertible upper triangular matrices in $M_{n+1}(\mathbf{F})$ and $U(\mathbf{F})$ the group of upper triangular matrices having entries 1's on the main diagonal in $M_{n+1}(\mathbf{F})$. Denote by E the identity matrix in $M_{n+1}(\mathbf{F})$ and by E_{ij} the matrix with sole non-zero element 1 in the (i, j) position. Then, $\{E_{ij} \mid 1 \leqslant i < j \leqslant n+1\}$ is the canonical basis of $N(\mathbf{F})$. For a matrix X, we denote by X^{t} the transpose of X and use corresponding lower case with subscripts x_{ij} to denote the (i, j) entry of X. If X is invertible, denote by x_{ij}^* the (i, j) entry of X^{-1}. For convenience sake, in a matrix expression $X = \sum x_{ij}E_{ij}$ the subscript i can be less than 1 and the subscript j can be greater than $n+1$ and we use the convention that the coefficient x_{ij} is regarded as zero if $i < 1$ or $j > n+1$ in some term $x_{ij}E_{ij}$.

Clearly, the following sets

$$N_k = \left\{ X \in N(\mathbf{F}) \;\middle|\; X = \sum_{j-i \geqslant k} x_{ij}E_{ij} \right\}, \quad k = 1, 2, \ldots,$$

are ideals of the **F**-algebra $N(\mathbf{F})$ and N_n is the center of the **F**-algebra $N(\mathbf{F})$. We assume $N_k = \{0\}$ for $k > n$. It is easy to check that

$$N_k N_l = \{XY \mid X \in N_k, Y \in N_l\} \subseteq N_{k+l}.$$

Denote by $\mathscr{S}$ the set of all linear maps on $N(\mathbf{F})$ that preserve commutativity in both directions and by $\mathscr{S}'$ the set of all bijections in $\mathscr{S}$. Denote by 1 the identity map on $N(\mathbf{F})$.

Linear maps of $N(\mathbf{F})$ into its center are given by $X \mapsto f(X)E_{1,n+1}$ where $f : N(\mathbf{F}) \to \mathbf{F}$ is a linear functional. We will use the term "linear functional on $N(\mathbf{F})$" to denote both the linear functional of $N(\mathbf{F})$ as well as the corresponding linear map of $N(\mathbf{F})$ into its center. It is clear that for $\varphi \in \mathscr{S}$ and a linear functional f on $N(\mathbf{F})$, the map $\varphi + f : X \mapsto \varphi(X) + f(X)E_{1,n+1}$ is in $\mathscr{S}$.

It is easy to check that when $n \geqslant 3$ the following linear maps on $N(\mathbf{F})$ are all in $\mathscr{S}$:
(a) $\psi_c : X \mapsto cX$ where c is a constant in $\mathbf{F}^*$.
(b) $\sigma_T : X \mapsto T^{-1}XT$ where $T \in T(\mathbf{F})$.
(c) $\omega = 1$ or ω_0 where $\omega_0 : X \mapsto -RX^{\mathrm{t}}R$ with $R = E_{1,n+1} + E_{2n} + \cdots + E_{n2} + E_{n+1,1}$.
(d) $\mu_b^{(ij)}$ for $b \in \mathbf{F}$, $i = 1, n$ and $j = 1, 2$, are defined by

$$\mu_b^{(11)} : X \mapsto X + bx_{12}E_{2,n+1}, \quad \mu_b^{(n1)} : X \mapsto X + bx_{n,n+1}E_{1n},$$

$$\mu_b^{(12)} : X \mapsto X + bx_{12}E_{3,n+1} + bx_{13}E_{2,n+1},$$

and

$$\mu_b^{(n2)} : X \mapsto X + bx_{n,n+1}E_{1,n-1} + bx_{n-1,n+1}E_{1n}.$$

We call the linear maps of types (a)–(d) defined above *standard maps*.

It is clear that σ_T is an $\mathbf{F}$-algebra automorphism of $N(\mathbf{F})$ and if f is a linear functional satisfying the additional condition: $f(XY) = 0$ for any $X, Y \in N(\mathbf{F})$, then $1 + f$ is also an $\mathbf{F}$-algebra automorphism of $N(\mathbf{F})$. These automorphisms are called *standard* $\mathbf{F}$*-algebra automorphisms* of $N(\mathbf{F})$.

On the other hand, σ_T, ω and $\mu_b^{(i1)}$, $i = 1, n$, are all Lie automorphisms of $N(\mathbf{F})$, and if f is a linear functional satisfying the additional condition: $f([X, Y]) = 0$ for any $X, Y \in N(\mathbf{F})$, then $1 + f$ is also a Lie automorphism of $N(\mathbf{F})$. In addition, when $\operatorname{char}\mathbf{F} = 2$, both $\mu_b^{(12)}$ and $\mu_b^{(n2)}$ are also Lie automorphisms of $N(\mathbf{F})$. These automorphisms are called *standard Lie automorphisms* of $N(\mathbf{F})$. The automorphisms $\mu_b^{(11)}$, $\mu_b^{(n1)}$ are called extremal automorphisms in [3] and when $\operatorname{char}\mathbf{F} = 2$, the automorphisms $\mu_b^{(12)}$ and $\mu_b^{(n2)}$ are generalization of the extremal automorphisms in [3]. For the notion for extremal automorphisms of $N(\mathbf{F})$, we are motivated by Gibbs [8], where the automorphisms of certain unipotent subgroups of Chevalley groups and Steinberg groups over a field are discussed.

Lemma 2.1.
(i) *If* $\varphi \in \mathscr{S}$, *then* $\operatorname{Ker}\varphi \subseteq N_n$.
(ii) $\varphi \in \mathscr{S}'$ *if and only if* $\varphi(E_{1,n+1}) \neq 0$.
(iii) *If* $\varphi \in \mathscr{S}'$, *then* $\varphi(E_{1,n+1}) = cE_{1,n+1}$ *for some* $c \in \mathbf{F}^*$.
(iv) *If* $\varphi \in \mathscr{S}$ *and* f *is a linear functional on* $N(\mathbf{F})$, *then both* φf *and* $f\varphi$ *are linear functionals on* $N(\mathbf{F})$.

Proof. (i) If $X \in N(\mathbf{F})$ such that $\varphi(X) = 0$, then for any $Y \in N(\mathbf{F})$ we have $\varphi(X)\varphi(Y) = \varphi(Y)\varphi(X)$. So $XY = YX$, i.e., X is in the center N_n of the $\mathbf{F}$-algebra $N(\mathbf{F})$.

(ii) Clearly, if φ is bijective, we have $\varphi(E_{1,n+1}) \neq 0$. Conversely, if $\varphi(E_{1,n+1}) \neq 0$ and φ is not bijective, then there exists some non-zero $X \in N(\mathbf{F})$ such that $\varphi(X) = 0$. By (i) we have $X = cE_{1,n+1}$ with some $c \in \mathbf{F}^*$. It follows that $\varphi(E_{1,n+1}) = 0$, a contradiction.

(iii) The assertion follows from the fact that $\varphi(N_n) = N_n$ and (ii).

(iv) It is clear that for $f\varphi$ the assertion is true. It is easy to see that if $\varphi(E_{1,n+1}) = cE_{1,n+1}$, then $\varphi f = cf$. $\square$

Lemma 2.2.
(i) $\psi_{c'}\psi_c = \psi_{c'c}$ *for any* $c', c \in \mathbf{F}^*$.
(ii) $\sigma_{T'}\sigma_T = \sigma_{TT'}$ *for any* $T', T \in T(\mathbf{F})$.
(iii) $\omega_0^2 = 1$.
(iv) $\mu_{b'}^{(ij)}\mu_b^{(ij)} = \mu_{b'+b}^{(ij)}$ *for any* $b', b \in \mathbf{F}$, $i = 1, n$, *and* $j = 1, 2$.

Proof. The proof is trivial. $\square$

By Lemma 2.1(ii) all the standard maps are in $\mathscr{S}'$. By Lemma 2.2, it is easy to check that $\psi_c^{-1} = \psi_{c^{-1}}$, $\sigma_T^{-1} = \sigma_{T^{-1}}$, $\omega_0^{-1} = \omega_0$ and $(\mu_b^{(ij)})^{-1} = \mu_{-b}^{(ij)}$.

Lemma 2.3.
(i) ψ_c *commutes with every linear map on* $N(\mathbf{F})$. *In particular*, ψ_c *commutes with every standard map.*
(ii) $\omega_0^{-1}\mu_b^{(ij)}\omega_0 = \mu_b^{(kj)}$ *where* $k = n$ *or* 1 *according as* $i = 1$ *or* n.
(iii) $\omega_0^{-1}\sigma_T\omega_0 = \sigma_{\omega_0(T^{-1})}$.
(iv-1) $\sigma_T^{-1}\mu_b^{(i1)}\sigma_T = \mu_{b'}^{(i1)} + f$ *where* $b' \in \mathbf{F}$ *and* f *is a linear functional on* $N(\mathbf{F})$.
(iv-2) $\sigma_T^{-1}\mu_b^{(i2)}\sigma_T = \sigma_{T'}\mu_{b_2}^{(i2)}\mu_{b_1}^{(i1)} + f = \mu_{b_2}^{(i2)}\mu_{b_1}^{(i1)}\sigma_{T'} + f$ *where* $T' \in T(\mathbf{F})$, b_1, $b_2 \in \mathbf{F}$ *and* f *is a linear functional on* $N(\mathbf{F})$.
(v) *If* $n \geqslant 4$, *then any pair of maps of type* (d) *is commutative except the pairs consisting of* $\mu_{b_1}^{(12)}$ *and* $\mu_{b_2}^{(42)}$ *when* $n = 4$. *If* $n = 4$, *we have*

$$\mu_{b_1}^{(12)}\mu_{b_2}^{(42)} = \sigma_T\mu_{b_2}^{(42)}\mu_{b_1}^{(12)} = \mu_{b_2}^{(42)}\mu_{b_1}^{(12)}\sigma_T,$$

where $T = E - b_1 b_2 E_{24}$.

Proof. The proof is routine, but tedious. We give the proof only for (iv-2) with $i = 1$. For any $X \in N(\mathbf{F})$,

$$\begin{aligned}
&\sigma_T^{-1}\mu_b^{(12)}\sigma_T(X) \\
&\quad = \sigma_T^{-1}\mu_b^{(12)}(T^{-1}XT)
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_T^{-1}\big(T^{-1}XT + bt_{11}^* t_{22} x_{12} E_{3,n+1} \\
&\qquad + b\big(t_{11}^* t_{33} x_{13} + t_{11}^* t_{23} x_{12} + t_{12}^* t_{33} x_{23}\big) E_{2,n+1}\big) \\
&= X + bt_{11}^* t_{22} t_{n+1,n+1}^* x_{12} (t_{13} E_{1,n+1} + t_{23} E_{2,n+1} + t_{33} E_{3,n+1}) \\
&\quad + bt_{n+1,n+1}^* \big(t_{11}^* t_{33} x_{13} + t_{11}^* t_{23} x_{12} + t_{12}^* t_{33} x_{23}\big)\big(t_{22} E_{2,n+1} + t_{12} E_{1,n+1}\big) \\
&= X + b_2 x_{12} E_{3,n+1} + (a x_{23} + b_1 x_{12} + b_2 x_{13}) E_{2,n+1} \\
&\quad + (c_1 x_{12} + c_2 x_{23} + c_3 x_{13}) E_{1,n+1} \\
&= \big(\sigma_{T'} \mu_{b_2}^{(12)} \mu_{b_1}^{(11)} + f\big)(X) \\
&= \big(\mu_{b_2}^{(12)} \mu_{b_1}^{(11)} \sigma_{T'} + f\big)(X),
\end{aligned}
$$

where

$$T' = E + aE_{3,n+1}, \quad a = bt_{12}^* t_{33} t_{22} t_{n+1,n+1}^*,$$

$$b_1 = 2bt_{11}^* t_{22} t_{23} t_{n+1,n+1}^*, \quad b_2 = bt_{11}^* t_{22} t_{33} t_{n+1,n+1}^*,$$

and

$$f : X \mapsto c_1 x_{12} + c_2 x_{23} + (c_3 - a) x_{13},$$

with

$$
\begin{aligned}
c_1 &= bt_{11}^* t_{n+1,n+1}^* (t_{22} t_{13} + t_{12} t_{23}), \\
c_2 &= bt_{12}^* t_{12} t_{33} t_{n+1,n+1}^*, \\
c_3 &= bt_{11}^* t_{12} t_{33} t_{n+1,n+1}^*. \qquad \square
\end{aligned}
$$

If $X \in N(\mathbf{F})$, we denote by $C(X)$ the centralizer of X in $\mathbf{F}$-algebra $N(\mathbf{F})$, i.e., $C(X) = \{Y \in N(\mathbf{F}) \mid XY = YX\}$.

Lemma 2.4. *If $X \in N_k \backslash N_{k+1}$ for $1 \leqslant k \leqslant n$, then*

$$\dim C(X) \leqslant \tfrac{1}{2} n(n+1) - (n-k). \tag{2.1}$$

In particular, if $X = E_{m,m+k}$ for $1 \leqslant m \leqslant n+1-k$, then the equality holds.

Proof. For the sake of convenience, we first give some notations. For $1 \leqslant k \leqslant n$ and $1 \leqslant m \leqslant n+1-k$, set

$$\mathscr{U}_{m,k} = \left\{ Y \in N_k \;\middle|\; Y = \sum_{\substack{i<m \\ i+k \leqslant j < m+k}} y_{ij} E_{ij} \right\},$$

$$\mathscr{V}_{m,k} = \left\{ Y \in N_k \;\middle|\; Y = \sum_{\substack{i<m\\ j>m+k}} y_{ij} E_{ij} \right\}$$

and

$$\mathscr{W}_{m,k} = \left\{ Y \in N_k \;\middle|\; Y = \sum_{\substack{m<i\leqslant n+1-k\\ j\geqslant i+k}} y_{ij} E_{ij} \right\}.$$

The matrices in $\mathscr{U}_{m,k}$, $\mathscr{V}_{m,k}$ and $\mathscr{W}_{m,k}$ are of the forms

$$\begin{array}{c} 1\ \cdots\ \cdots\ \cdots\ k \quad k+1\ \cdots\ \cdots\ m-1+k\ \ m+k\ \ m+1+k\ \cdots\ n+1 \\ \left(\begin{array}{cccccccccc} 0 & \cdots & 0 & * & \cdots & * & 0 & 0 & \cdots & 0 \\ & \ddots & & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ & & \ddots & & \ddots & * & 0 & 0 & \cdots & 0 \\ & & & \ddots & & \ddots & 0 & 0 & \cdots & 0 \\ & & & & \ddots & & \ddots & 0 & \cdots & 0 \\ & & & & & \ddots & & \ddots & \ddots & \vdots \\ & & & & & & \ddots & & \ddots & 0 \\ & & & & & & & \ddots & & 0 \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & & & 0 \end{array}\right) \end{array} \begin{array}{l} 1 \\ \vdots \\ m-1 \\ m \\ m+1 \\ \vdots \\ n+1-k \\ n+2-k \\ \vdots \\ n+1 \end{array},$$

$$\begin{array}{c} 1\ \cdots\ \cdots\ \cdots\ k \quad k+1\ \cdots\ \cdots\ m-1+k\ \ m+k\ \ m+1+k\ \cdots\ n+1 \\ \left(\begin{array}{cccccccccc} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ & \ddots & & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ & & \ddots & & \ddots & 0 & 0 & * & \cdots & * \\ & & & \ddots & & \ddots & 0 & 0 & \cdots & 0 \\ & & & & \ddots & & \ddots & 0 & \cdots & 0 \\ & & & & & \ddots & & \ddots & \ddots & \vdots \\ & & & & & & \ddots & & \ddots & 0 \\ & & & & & & & \ddots & & 0 \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & & & 0 \end{array}\right) \end{array} \begin{array}{l} 1 \\ \vdots \\ m-1 \\ m \\ m+1 \\ \vdots \\ n+1-k \\ n+2-k \\ \vdots \\ n+1 \end{array}$$

and

$$
\begin{array}{cccccccccc|l}
1 & \cdots & \cdots & \cdots\ k & k{+}1 & \cdots\ \cdots & m{-}1{+}k & m{+}k & m{+}1{+}k & \cdots\ n{+}1 & \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\
 & \ddots & & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & & \ddots & & \ddots & 0 & 0 & 0 & \cdots & 0 & m-1 \\
 & & & \ddots & & \ddots & 0 & 0 & \cdots & 0 & m \\
 & & & & \ddots & & \ddots & * & \cdots & * & m+1 \\
 & & & & & \ddots & & \ddots & \ddots & \vdots & \vdots \\
 & & & & & & \ddots & & \ddots & * & n+1-k \\
 & & & & & & & \ddots & & 0 & n+2-k \\
 & & & & & & & & \ddots & \vdots & \vdots \\
 & & & & & & & & & 0 & n+1
\end{array},
$$

respectively.

It is clear that $\mathscr{U}_{m,k}$, $\mathscr{V}_{m,k}$ and $\mathscr{W}_{m,k}$ are left, two-sided and right ideals of the **F**-algebra $N(\mathbf{F})$, respectively. Moreover, we have

$$
\mathscr{U}_{m,k} \cdot \mathscr{W}_{m,k} \subseteq \mathscr{V}_{m,k}, \quad \mathscr{V}^2_{m,k} = 0, \quad \mathscr{W}_{m,k} \cdot \mathscr{U}_{m,k} = 0,
$$

$$
\mathscr{W}_{m,k} \cdot \mathscr{V}_{m,k} = 0 \quad \text{and} \quad \mathscr{V}_{m,k} \cdot \mathscr{U}_{m,k} = 0.
$$

Assume $X \in N_k \backslash N_{k+1}$ with some $x_{m,m+k} \neq 0$. We write X as

$$
X = \begin{pmatrix}
0 & \cdots & 0 & * & \cdots & * & x_{1,m+k} & * & \cdots & * \\
 & \ddots & & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & & \ddots & & \ddots & * & x_{m-1,m+k} & * & \cdots & * \\
 & & & \ddots & & \ddots & x_{m,m+k} & x_{m,m+1+k} & \cdots & x_{m,n+1} \\
 & & & & \ddots & & \ddots & * & \cdots & * \\
 & & & & & \ddots & & \ddots & \ddots & \vdots \\
 & & & & & & \ddots & & \ddots & * \\
 & & & & & & & \ddots & & 0 \\
 & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & 0
\end{pmatrix}
$$

$$
= x_{m,m+k} E_{m,m+k} + \sum_{i=1}^{m-1} x_{i,m+k} E_{i,m+k} + \sum_{j=m+k+1}^{n+1} x_{mj} E_{mj} + U + V + W
$$

with $U \in \mathscr{U}_{m,k}$, $V \in \mathscr{V}_{m,k}$ and $W \in \mathscr{W}_{m,k}$. Set

$$T_1 = E + \sum_{i=1}^{m-1} x_{i,m+k} x_{m,m+k}^{-1} E_{im},$$

$$T_2 = E - \sum_{j=m+k+1}^{n+1} x_{mj} x_{m,m+k}^{-1} E_{m+k,j}$$

and $T = T_1 T_2$. Then

$$\begin{aligned} T^{-1}XT &= (T_1T_2)^{-1}X(T_1T_2) \\ &= \begin{pmatrix} 0 & \cdots & 0 & * & \cdots & * & 0 & * & \cdots & * \\ & \ddots & & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ & & \ddots & & \ddots & * & 0 & * & \cdots & * \\ & & & \ddots & & \ddots & x_{m,m+k} & 0 & \cdots & 0 \\ & & & & \ddots & & \ddots & * & \cdots & * \\ & & & & & \ddots & & \ddots & \ddots & \vdots \\ & & & & & & \ddots & & \ddots & * \\ & & & & & & & \ddots & & 0 \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & & & 0 \end{pmatrix} \\ &= x_{m,m+k}E_{m,m+k} + U' + V' + W' \end{aligned}$$

with $U' \in \mathscr{U}_{m,k}$, $V' \in \mathscr{V}_{m,k}$ and $W' \in \mathscr{W}_{m,k}$. Since $\dim C(X) = \dim C(T^{-1}XT)$ for any $T \in T(\mathbf{F})$, we can assume without loss of generality that

$$X = x_{m,m+k}E_{m,m+k} + U + V + W,$$

where $x_{m,m+k} \neq 0$, $U \in \mathscr{U}_{m,k}$, $V \in \mathscr{V}_{m,k}$ and $W \in \mathscr{W}_{m,k}$.

Let $Z = \sum_{p<q} z_{pq}E_{pq} \in C(X)$. It follows from $ZX = XZ$ that

$$\begin{aligned} &\sum_{p=1}^{m-1} x_{m,m+k} z_{pm} E_{p,m+k} + ZU + ZV + ZW \\ &\quad = \sum_{q=m+k+1}^{n+1} x_{m,m+k} z_{m+k,q} E_{mq} + UZ + VZ + WZ, \end{aligned} \tag{2.2}$$

where $ZU \in \mathscr{U}_{m,k}$, $ZV, VZ \in \mathscr{V}_{m,k}$, $WZ \in \mathscr{W}_{m,k}$,

$$ZW = \left(\sum_{1 \leqslant p < q \leqslant n+1} z_{pq}E_{pq} \right) \cdot \left(\sum_{\substack{m<i\leqslant n+1-k \\ j \geqslant i+k}} x_{ij}E_{ij} \right)$$

$$= \sum_{j=m+k+1}^{n+1} \sum_{p=1}^{j-k-1} \left(\sum_{i=\max\{p,m\}+1}^{j-k} z_{pi} x_{ij} \right) E_{pj} \tag{2.3}$$

and

$$UZ = \left(\sum_{\substack{i<m \\ i+k \leqslant j<m+k}} x_{ij} E_{ij} \right) \cdot \left(\sum_{1 \leqslant p<q \leqslant n+1} z_{pq} E_{pq} \right)$$
$$= \sum_{i=1}^{m-1} \sum_{q=i+k+1}^{n+1} \left(\sum_{j=i+k}^{\min\{m+k,q\}-1} x_{ij} z_{jq} \right) E_{iq}. \tag{2.4}$$

If we regard the entries z_{pq} of Z as unknowns, then $ZX = XZ$ yields a system of $\frac{1}{2}n(n+1)$ homogeneous linear equations in $\frac{1}{2}n(n+1)$ unknowns. The dimension of $C(X)$ is equal to that of the solution space of the system of equations. Also, we know by (2.2)–(2.4) that the above system of equations contains the following $n-k$ homogeneous linear equations

$$\left.\begin{array}{ll} x_{m,m+k} z_{pm} = \sum_{j=p+k}^{m+k-1} x_{pj} z_{j,m+k}, & p = 1, \ldots, m-1, \\ x_{m,m+k} z_{m+k,q} = \sum_{i=m+1}^{q-k} x_{iq} z_{mi}, & q = m+k+1, \ldots, n+1. \end{array}\right\} \tag{2.5}$$

Since each of the $n-k$ unknowns $z_{1m}, z_{2m}, \ldots, z_{m-1,m}$ and $z_{m+k,m+k+1}, z_{m+k,m+k+2}, \ldots, z_{m+k,n+1}$ occurs once and only once in all the $n-k$ equations in (2.5), these $n-k$ homogeneous linear equations are linearly independent. Hence the dimension of the solution space of the system of equations given by $ZX = XZ$ is less than or equal to $\frac{1}{2}n(n+1) - (n-k)$ and so the desired inequality (2.1) holds.

If $X = E_{m,m+k}$, then Eq. (2.2) becomes

$$\sum_{p=1}^{m-1} z_{pm} E_{p,m+k} = \sum_{q=m+k+1}^{n+1} z_{m+k,q} E_{mq}.$$

It is easy to see that the dimension of $C(X)$ is equal to $\frac{1}{2}n(n+1) - (n-k)$. □

Lemma 2.5. *For any $\varphi \in \mathscr{S}'$, $\varphi(N_k) = N_k$ and φ induces a linear bijection of the quotient space N_k/N_{k+1} onto itself, $k = 1, 2, \ldots$*

Proof. If we can prove that $\varphi(N_k \backslash N_{k+1}) \subset N_k \backslash N_{k+1}$, then the assertions of the lemma follow from the bijectivity of φ. Now, we use induction on $l = n-k$ to prove that $\varphi(N_k \backslash N_{k+1}) \subset N_k \backslash N_{k+1}$. When $l = 0$, the assertion follows from Lemma 2.1(iii). Assume that for any $\varphi \in \mathscr{S}'$ the assertion holds for $n-k \leqslant l$ with $l \geqslant 0$. Let $X \in N_k \backslash N_{k+1}$ where $k = n-l-1$ and $\varphi \in \mathscr{S}'$. Assume $Y = \varphi(X) \in N_m \backslash N_{m+1}$. We need to prove $m = k$.

If $m > k$, it follows from $n - m \leqslant n - k - 1 = l$ and the induction hypothesis that $X = \varphi^{-1}(Y) \in \varphi^{-1}(N_m \backslash N_{m+1}) \subseteq N_m \backslash N_{m+1} \subseteq N_{k+1}$, a contradiction. Thus $m \leqslant k$. To prove $m = k$, we first consider the special case that $X = E_{i,i+k}$. It follows from Lemma 2.4 that $\dim C(\varphi(E_{i,i+k})) = \dim C(E_{i,i+k}) = \frac{1}{2}n(n+1) - (n-k)$. If $m < k$, again by Lemma 2.4, $\dim C(\varphi(E_{i,i+k})) \leqslant \frac{1}{2}n(n+1) - (n-m) < \frac{1}{2}n(n+1) - (n-k)$, a contradiction. So $m = k$ for $X = E_{i,i+k}$, i.e., $\varphi(E_{i,i+k}) \in N_k \backslash N_{k+1}$ for $1 \leqslant i \leqslant n+1-k$. Since any $X \in N_k \backslash N_{k+1}$ can be expressed as $X = \sum_{j-i \geqslant k} x_{ij} E_{ij}$ with $x_{ij} \in \mathbf{F}$, by the argument above and the induction hypothesis we get $\varphi(X) \in N_k$. From the induction hypothesis, it is easy to see that $\varphi(X) \in N_k \backslash N_{k+1}$. $\square$

For any $\varphi \in \mathscr{S}'$, assume

$$\varphi(E_{i,i+1}) = \sum_{j=1}^{n} a_{ji} E_{j,j+1} \bmod N_2 \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Then φ determines a matrix

$$A(\varphi) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

where the entries a_{ji} are dependent on φ.

Lemma 2.6. *For any $\varphi \in \mathscr{S}'$,* $\det A(\varphi) \neq 0$ *and the entries of the matrix $A(\varphi)$ satisfy the following relations*:

$$a_{r\alpha} a_{s\beta} = a_{r\beta} a_{s\alpha} \quad \textit{if}\ |r - s| = 1 \ \textit{and}\ |\alpha - \beta| > 1.$$

Proof. Since φ induces a linear bijection of the quotient space N_1/N_2 onto itself by Lemma 2.5, $\det A(\varphi) \neq 0$.

If $|r - s| = 1$ and $|\alpha - \beta| > 1$, then $[E_{\alpha,\alpha+1}, E_{\beta,\beta+1}] = 0$. So $[\varphi(E_{\alpha,\alpha+1}), \varphi(E_{\beta,\beta+1})] = 0$, on the left-hand side of which the coefficient of $E_{r,s+1}$ for $r < s$ or $E_{s,r+1}$ for $s < r$ is $\pm(a_{r\alpha} a_{s\beta} - a_{r\beta} a_{s\alpha})$. Hence $a_{r\alpha} a_{s\beta} - a_{r\beta} a_{s\alpha} = 0$, i.e., $a_{r\alpha} a_{s\beta} = a_{r\beta} a_{s\alpha}$. $\square$

Lemma 2.7. *Let $\varphi \in \mathscr{S}'$ such that $A(\varphi)$ is the identity matrix. Then*

$$\varphi(E_{i,i+k}) \equiv b_{ii}^{(k)} E_{i,i+k} \bmod N_{k+1} \quad \textit{for}\ 2 \leqslant k \leqslant n \ \textit{and}\ 1 \leqslant i \leqslant n+1-k, \tag{2.6}$$

where $b_{ii}^{(k)} \neq 0$.

Proof. Since $A(\varphi)$ is the identity matrix, we have

$$\varphi(E_{i,i+1}) \equiv E_{i,i+1} \bmod N_2 \quad \text{for } 1 \leqslant i \leqslant n. \tag{2.7}$$

By Lemma 2.5, for $1 < k \leqslant n$ and $1 \leqslant i \leqslant n+1-k$ we may assume

$$\varphi(E_{i,i+k}) \equiv b_{1i}^{(k)} E_{1,1+k} + \cdots + b_{n+1-k,i}^{(k)} E_{n+1-k,n+1} \bmod N_{k+1}. \tag{2.8}$$

If $k = n$, (2.6) holds by Lemma 2.1(iii). Assume $k < n$. We use a case-by-case analysis to prove $b_{si}^{(k)} = 0$ for $s \neq i$ in (2.8).

(A-1) $s \geqslant 2$ and $s-1 \neq i+k$. Since $[E_{s-1,s}, E_{i,i+k}] = 0$, we have $[\varphi(E_{s-1,s}), \varphi(E_{i,i+k})] = 0$. By (2.7) and (2.8) we obtain

$$b_{si}^{(k)} E_{s-1,s+k} - b_{s-k-1,i}^{(k)} E_{s-k-1,s} \equiv 0 \bmod N_{k+2},$$

which implies $b_{si}^{(k)} = 0$.

(A-2) $s \leqslant n-k$ and $i \neq s+k+1$. As in (A-1), it follows from $[E_{i,i+k}, E_{s+k,s+k+1}] = 0$ that $b_{si}^{(k)} = 0$.

Since $s-1 = i+k$ implies that $i \neq s+k+1$, it follows from (A-1) and (A-2) that $b_{si}^{(k)} = 0$ for $2 \leqslant s \leqslant n-k$ and $s \neq i$, and it remains only to consider the following two cases (B-1) and (B-2). In case (B-2), it follows from $s = n+1-k$ and $s-1 = i+k$ that $k < \frac{1}{2}n$. In case (B-1), noting that $i \leqslant n+1-k$, we also have $k < \frac{1}{2}n$. So if $k \geqslant \frac{1}{2}n$, then cases (B-1) and (B-2) do not occur and we always have $b_{si}^{(k)} = 0$ for any $s \neq i$.

(B-1) $s = 1, i = s+k+1$. By the argument above, we have $k < \frac{1}{2}n$ and so $n-k > \frac{1}{2}n$. By the comment above,

$$\varphi(E_{1+k,n+1}) \equiv b_{1+k,1+k}^{(n-k)} E_{1+k,n+1} \bmod N_{n+1-k},$$

where $b_{1+k,1+k}^{(n-k)} \neq 0$ by Lemma 2.5. Hence applying φ to $[E_{i,i+k}, E_{1+k,n+1}] = 0$, we obtain $b_{1i}^{(k)} = 0$.

(B-2) $s = n+1-k, s-1 = i+k$. As in (B-1), it follows from $[E_{1,n+1-k}, E_{i,i+k}] = 0$ that $b_{n+1-k,i}^{(k)} = 0$.

Thus we have proved that $b_{si}^{(k)} = 0$ in (2.8) for $s \neq i$. Finally, by Lemma 2.5 we have $b_{ii}^{(k)} \neq 0$. □

3. Proofs of Theorems 1.1–1.3

Throughout this section, we assume that $n \geqslant 3$ and assume without loss of generality that φ is bijective. In fact, if φ is not bijective, we have $\varphi(E_{1,n+1}) = 0$ by Lemma 2.1(ii). Let f be a linear functional on $N(\mathbf{F})$ such that $f(E_{1,n+1}) \neq 0$. Then $\varphi + f \in \mathscr{S}'$ again by Lemma 2.1(ii). Thus φ can be replaced with $\varphi + f$.

First, we will prove Theorem 1.1. The "if" part of the theorem is clear. For the "only if" part, we will prove it for the case $n \geqslant 4$ and the case $n = 3$, respectively.

First, we assume $n \geqslant 4$ and we will prove that any map φ in $\mathscr{S}'$ can be expressed as form (1.1) via Lemmas 3.1–3.8.

Lemma 3.1. *We can take $\omega = 1$ or ω_0 such that the matrix $A(\omega^{-1}\varphi)$ is diagonal.*

Proof. Let $A(\varphi) = (a_{ji})_{n\times n}$. We prove in turn the following statements:

(I) *If $a_{r\alpha} \neq 0$ and $|\alpha - \beta| > 2$, then $a_{r\beta} = 0$.*

Assume that $a_{r\beta} \neq 0$. Take s such that $|r - s| = 1$. By Lemma 2.6, we have $a_{s\alpha}/a_{r\alpha} = a_{s\beta}/a_{r\beta}$. Denote by p this ratio. For any γ with $1 \leqslant \gamma \leqslant n$, we have $|\alpha - \gamma| > 1$ or $|\beta - \gamma| > 1$. So by Lemma 2.6, $a_{r\alpha}a_{s\gamma} = a_{r\gamma}a_{s\alpha}$ or $a_{r\beta}a_{s\gamma} = a_{r\gamma}a_{s\beta}$. Hence, $a_{s\gamma} = p \cdot a_{r\gamma}, \gamma = 1, \ldots, n$. Therefore, rows r and s of the matrix $A(\varphi)$ are linearly dependent. This contradicts the non-singularity of $A(\varphi)$ and so (I) holds.

(II) *If $a_{r\alpha} \neq 0$, $|\alpha - \beta| = 2$ and $1 < r < n$, then $a_{r\beta} = 0$.*

Assume that $a_{r\beta} \neq 0$. Take γ such that $|\alpha - \gamma| = |\beta - \gamma| = 1$. If $\delta \neq \alpha, \beta$ and γ, then $a_{r\delta} = 0$ by (I). Moreover, $a_{r-1,\delta} = a_{r+1,\delta} = 0$. Otherwise, say $a_{r-1,\delta} \neq 0$. Since $|\alpha - \delta| > 2$ or $|\beta - \delta| > 2$, say $|\alpha - \delta| > 2$, by Lemma 2.6 we have $a_{r-1,\alpha}a_{r\delta} = a_{r-1,\delta}a_{r\alpha}$. The left-hand side of this equality is zero, but the right-hand side is not zero, a contradiction. Again by Lemma 2.6, we have $a_{r-1,\alpha}a_{r\beta} = a_{r-1,\beta}a_{r\alpha}$ and $a_{r+1,\alpha}a_{r\beta} = a_{r+1,\beta}a_{r\alpha}$. Hence,

$$(a_{r-1,\beta}, a_{r\beta}, a_{r+1,\beta}) = \frac{a_{r\beta}}{a_{r\alpha}}(a_{r-1,\alpha}, a_{r\alpha}, a_{r+1,\alpha}).$$

Thus the minor of the matrix $A(\varphi)$ consisting of rows $r-1, r, r+1$ and columns α, γ, β is zero and the other entries in these three rows are all zero, so these rows are linearly dependent. This contradicts the non-singularity of the matrix $A(\varphi)$. Hence, $a_{r\beta} = 0$ and (II) holds.

(III) *For $1 < r < n$, row r of the matrix $A(\varphi)$ has only one non-zero entry that is in some column except columns 1 and n and $a_{11} \neq 0$ or $a_{n1} \neq 0$.*

Assume $a_{r\alpha} \neq 0$ for some α. If we have $a_{r\beta} \neq 0$ with $\alpha \neq \beta$, then $|\alpha - \beta| = 1$ by (I) and (II). We assert that the non-zero entries in rows $r-1$ and $r+1$ of the matrix $A(\varphi)$ are all in columns α and β. Otherwise, say $a_{r-1,\gamma} \neq 0, \gamma \neq \alpha, \beta$. Then $|\alpha - \gamma| > 1$ or $|\beta - \gamma| > 1$. Say $|\alpha - \gamma| > 1$. By Lemma 2.6, $a_{r-1,\alpha}a_{r\gamma} = a_{r-1,\gamma}a_{r\alpha}$. But $a_{r\gamma} = 0$ by (I) or (II). This yields a contradiction. Furthermore, again by (I) and (II), the non-zero entries in row r of the matrix $A(\varphi)$ are also in columns α and β. Again, this contradicts the non-singularity of the matrix $A(\varphi)$.

Thus we have proved that row r of the matrix $A(\varphi)$ has only one non-zero entry. Assume $a_{r\alpha}$ is the sole non-zero entry in row r. We assert that $\alpha \neq 1, n$. Otherwise, take β such that $|\alpha - \beta| = 1$. Then for $1 < \gamma < n$ with $\gamma \neq \alpha, \beta$, we have $|\alpha - \gamma| > 1$. By Lemma 2.6, $a_{r\alpha}a_{r-1,\gamma} = a_{r\gamma}a_{r-1,\alpha}$. It follows from $a_{r\gamma} = 0$ that $a_{r-1,\gamma} = 0$. The same argument shows that $a_{r+1,\gamma} = 0$. Thus, the non-zero entries in rows $r-1, r$ and $r+1$ of the matrix $A(\varphi)$ are all in columns α and β. This contradicts the non-singularity of the matrix $A(\varphi)$. Hence, the sole non-zero entry in any row of $A(\varphi)$ except rows 1 and n is in some column except columns 1 and n.

Finally we have $a_{11} \neq 0$ or $a_{n1} \neq 0$. Otherwise, the entries in the first column of $A(\varphi)$ are all zero, a contradiction.

(IV) *Set* $\omega = 1$ *if* $a_{11} \neq 0$ *and* $\omega = \omega_0$ *otherwise. Then the diagonal entries of* $A = A(\omega^{-1}\varphi)$ *are non-zero.*

It is clear that for $A = A(\omega^{-1}\varphi)$, $a_{11} \neq 0$ and so $a_{1n} = 0$ by (I). It follows from the non-singularity of the matrix $A(\varphi)$ that $a_{nn} \neq 0$. Next, we have $a_{22} \neq 0$. In fact, if $a_{22} = 0$, assume $a_{2\alpha} \neq 0$ with $\alpha > 2$. By Lemma 2.6, $a_{11}a_{2\alpha} = a_{1\alpha}a_{21} = 0$ since $a_{21} = 0$, a contradiction. So $a_{22} \neq 0$. In the same way, we can prove that $a_{11} \neq 0, \dots, a_{ii} \neq 0$ imply $a_{i+1,i+1} \neq 0$, $i = 2, \dots, n-2$.

(V) *The matrix* $A = A(\omega^{-1}\varphi)$ *is diagonal.*

By (I)–(IV) above, for the matrix $A = A(\omega^{-1}\varphi)$ we have obtained the following results:
(i) In the first row, $a_{11} \neq 0$ and $a_{14} = \cdots = a_{1n} = 0$.
(ii) In the ith row for $1 < i < n$, the sole non-zero element is a_{ii}.
(iii) In the nth row, $a_{nn} \neq 0$ and $a_{n1} = \cdots = a_{n,n-3} = 0$.
Namely,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-3,n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{n-1,n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{pmatrix}$$

for $n > 5$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & a_{53} & a_{54} & a_{55} \end{pmatrix} \quad \text{for } n = 5$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad \text{for } n = 4.$$

Hence, for $n > 3$ we have

$$\omega^{-1}\varphi(E_{12}) \equiv a_{11}E_{12} \bmod N_2,$$
$$\omega^{-1}\varphi(E_{n,n+1}) \equiv a_{nn}E_{n,n+1} \bmod N_2.$$

Applying $\omega^{-1}\varphi$ to $[E_{12}, E_{1n}] = 0$ and $[E_{2,n+1}, E_{n,n+1}] = 0$, we obtain

$$\left.\begin{aligned} \omega^{-1}\varphi(E_{1n}) &\equiv b_{11}^{(n-1)}E_{1n} \bmod N_n,\\ \omega^{-1}\varphi(E_{2,n+1}) &\equiv b_{22}^{(n-1)}E_{2,n+1} \bmod N_n, \end{aligned}\right\} \tag{3.1}$$

where $b_{11}^{(n-1)} \neq 0$ and $b_{22}^{(n-1)} \neq 0$. When $n > 5$, we have

$$\left.\begin{aligned} \omega^{-1}\varphi(E_{23}) &\equiv a_{12}E_{12} + a_{22}E_{23} \bmod N_2,\\ \omega^{-1}\varphi(E_{34}) &\equiv a_{13}E_{12} + a_{33}E_{34} \bmod N_2. \end{aligned}\right\} \tag{3.2}$$

Applying $\omega^{-1}\varphi$ to $[E_{23}, E_{2,n+1}] = 0$ and $[E_{34}, E_{2,n+1}] = 0$, by (3.1) and (3.2) we obtain $a_{12}b_{22}^{(n-1)}E_{1,n+1} = 0$ and $a_{13}b_{22}^{(n-1)}E_{1,n+1} = 0$, which imply $a_{12} = 0$ and $a_{13} = 0$, respectively. Similarly, $a_{n,n-1} = 0$ and $a_{n,n-2} = 0$. Thus A is diagonal. When $n = 4, 5$, in the same way, we can also prove that A is diagonal. □

Lemma 3.2. *Let ω be as above such that $A(\omega^{-1}\varphi)$ is diagonal. Then there exists a diagonal matrix $D' \in T(\mathbf{F})$ such that $A(\sigma_{D'}^{-1}\omega^{-1}\varphi)$ is the identity matrix.*

Proof. By Lemma 3.1, we have

$$\omega^{-1}\varphi(E_{i,i+1}) \equiv a_{ii}E_{i,i+1} \bmod N_2 \quad \text{for } 1 \leqslant i \leqslant n.$$

Since $\det A(\omega^{-1}\varphi) \neq 0, a_{ii} \neq 0$. Set $D' = \operatorname{diag}\{1, a_{11}, (a_{11}a_{22}), \ldots, (a_{11}\cdots a_{nn})\}$. Then

$$\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{i,i+1}) \equiv E_{i,i+1} \bmod N_2 \quad \text{for } 1 \leqslant i \leqslant n.$$

This means that $A(\sigma_{D'}^{-1}\omega^{-1}\varphi)$ is the identity matrix. □

Lemma 3.3. *There exist a $T' \in U(\mathbf{F})$ and $b_3', b_4' \in \mathbf{F}$ such that*

$$\begin{aligned} &\left(\mu_{b_3'}^{(12)}\right)^{-1}\left(\mu_{b_4'}^{(n2)}\right)^{-1}\sigma_{T'}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{i,i+1})\\ &\quad\equiv E_{i.i+1} \bmod N_{n-1} \quad \textit{for } 1 \leqslant i \leqslant n. \end{aligned} \tag{3.3}$$

Proof. First, we will use induction on k to prove that there exist $T_k \in U(\mathbf{F}), k = 1, 2, \ldots, n-3$, such that

$$\sigma_{T_k}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{i,i+1}) \equiv E_{i,i+1} \bmod N_{k+1} \quad \text{for } 1 \leqslant i \leqslant n. \tag{3.4}$$

Let $T_1 = E$. Then Lemma 3.2 shows that (3.4) for $k = 1$ is true, and for (3.4) we may assume $n > 4$. Assume that there exists a $T_{k-1} \in U(\mathbf{F})$ with $1 \leqslant k-1 \leqslant n-4$

such that (3.4) for $k-1$ is true. Set $\theta=\sigma_{T_{k-1}}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi$. It is clear that $A(\theta)$ is still the identity matrix. Assume

$$\theta(E_{i,i+1})\equiv E_{i,i+1}+\sum_{j=1}^{n+1-k}a_{ji}^{(k)}E_{j,j+k} \bmod N_{k+1}\quad \text{for } 1\leqslant i\leqslant n. \tag{3.5}$$

For (3.5), we first prove the following claim:

Claim. $a_{si}^{(k)}=0$ *for* $s\neq i,i+1-k$.

We give a case-by-case analysis. In the following discussion, assume $s\neq i,i+1-k$.

(A-1) $s\leqslant n-k$ and $s\neq i-k,i-k-1$. Applying θ to $[E_{i,i+1},\ E_{s+k,s+k+1}]=0$, we have

$$\begin{aligned}&\left[E_{i,i+1}+\sum_{j=1}^{n+1-k}a_{ji}^{(k)}E_{j,j+k},\,E_{s+k,s+k+1}+\sum_{j=1}^{n+1-k}a_{j,s+k}^{(k)}E_{j,j+k}\right]\\&\quad\equiv a_{i+1,s+k}^{(k)}E_{i,i+1+k}+a_{si}^{(k)}E_{s,s+k+1}\\&\qquad -a_{s+k+1,i}^{(k)}E_{s+k,s+2k+1}-a_{i-k,s+k}^{(k)}E_{i-k,i+1}\\&\quad\equiv 0 \bmod N_{k+2}.\end{aligned}$$

Hence $a_{si}^{(k)}=0$.

(A-2) $s\geqslant 2$ and $s\neq i+1,i+2$. As in (A-1), applying θ to $[E_{s-1,s},E_{i,i+1}]=0$, we have $a_{si}^{(k)}=0$.

Since $s=i-k$ or $i-k-1$ implies that $s\neq i+1,i+2$, it follows from (A-1) and (A-2) that for $2\leqslant s\leqslant n-k$ the claim is true and it remains to consider the following cases (B-1), (B-2), (C-1) and (C-2).

(B-1) $s=1$ and $s=i-k$. By Lemma 2.7 we have

$$\theta(E_{1+k,3+k})\equiv b_{1+k,1+k}^{(2)}E_{1+k,3+k} \bmod N_3,$$

where $b_{1+k,1+k}^{(2)}\neq 0$. Applying θ to $[E_{1+k,2+k},E_{1+k,3+k}]=0$, we obtain

$$\left[E_{1+k,2+k}+\sum_{j=1}^{n+1-k}a_{j,1+k}^{(k)}E_{j,j+k}+\cdots,\ b_{1+k,1+k}^{(2)}E_{1+k,3+k}+\cdots\right]=0.$$

On the left-hand side of the above equality there is a term $a_{1,1+k}^{(k)}b_{1+k,1+k}^{(2)}E_{1,3+k}$ and the other terms do not contain the basis element $E_{1,3+k}$. So $a_{1,1+k}^{(k)}=0$, i.e., $a_{si}^{(k)}=0$.

(B-2) $s=1$ and $s=i-k-1$. As in (B-1), applying θ to $[E_{2+k,3+k},E_{1+k,4+k}]=0$, we obtain $a_{1,k+2}^{(k)}=0$, i.e., $a_{si}^{(k)}=0$.

(C-1) $s=n+1-k$ and $s=i+1$. As above, applying θ to $[E_{s-2,s},E_{s-1,s}]=0$, we obtain $a_{s,s-1}^{(k)}=0$, i.e., $a_{si}^{(k)}=0$.

(C-2) $s = n + 1 - k$ and $s = i + 2$. Applying θ to $[E_{s-3,s}, E_{s-2,s-1}] = 0$, we obtain $a^{(k)}_{s,s-2} = 0$, i.e., $a^{(k)}_{si} = 0$.

Thus, the claim is proved and (3.5) may be rewritten as

$$\theta(E_{i,i+1}) \equiv E_{i,i+1} + a^{(k)}_{i+1-k,i} E_{i+1-k,i+1} + a^{(k)}_{ii} E_{i,i+k} \bmod N_{k+1} \quad \text{for } 1 \leqslant i \leqslant n. \tag{3.6}$$

To complete the induction on k, we need again use induction on l to prove that there exist $S_l \in U(\mathbf{F})$, $l = 0, 1, \ldots, n$, such that for $1 \leqslant i \leqslant l$,

$$\sigma^{-1}_{S_l}\theta(E_{i,i+1}) \equiv E_{i,i+1} \bmod N_{k+1} \tag{3.7}$$

and for $l + 1 \leqslant i \leqslant n$,

$$\sigma^{-1}_{S_l}\theta(E_{i,i+1}) \equiv E_{i,i+1} + c^{(l)}_{i+1-k,i} E_{i+1-k,i+1} + c^{(l)}_{ii} E_{i,i+k} \bmod N_{k+1}. \tag{3.8}$$

Let $S_0 = E$. Then it follows from (3.6) that (3.8) with $c^{(0)}_{i+1-k,i} = a^{(k)}_{i+1-k,i}$ and $c^{(0)}_{ii} = a^{(k)}_{ii}$ is trivially true, and (3.7) does not occur. Assume that (3.7) and (3.8) hold for some $S_{l-1} \in U(\mathbf{F})$ with $0 \leqslant l - 1 \leqslant n - 1$. In particular,

$$\sigma^{-1}_{S_{l-1}}\theta(E_{l,l+1}) \equiv E_{l,l+1} + c^{(l-1)}_{l+1-k,l} E_{l+1-k,l+1} + c^{(l-1)}_{ll} E_{l,l+k} \bmod N_{k+1}.$$

Set $Z = E - c^{(l-1)}_{l+1-k,l} E_{l+1-k,l} + c^{(l-1)}_{ll} E_{l+1,l+k}$ and $S_l = Z S_{l-1}$. In fact, $c^{(l-1)}_{l+1-k,l} = 0$ if $l \neq k$. This is clear for $l < k$. And for $l > k$ applying $\sigma^{-1}_{S_{l-1}}\theta$ to $[E_{l-k,l-k+1}, E_{l,l+1}] = 0$, we have $c^{(l-1)}_{l+1-k,l} E_{l-k,l+1} \equiv 0 \bmod N_{k+2}$. Hence, $c^{(l-1)}_{l+1-k,l} = 0$. It is easy to check that (3.7) and (3.8) with

$$c^{(l)}_{i+1-k,i} = c^{(l-1)}_{i+1-k,i} + \delta_{l+k,i} c^{(l-1)}_{ll},$$

where δ_{ji} denotes the Kronecker delta, and $c^{(l)}_{ii} = c^{(l-1)}_{ii}$ for S_l hold. Thus, the induction on l is completed. Set $T_k = S_l T_{k-1}$. Then (3.4) for k is true, and the induction on k is completed. Hence we have proved that for $n \geqslant 4$, $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n - 3$, (3.4) is true. In particular, for $k = n - 3$, we have

$$\sigma^{-1}_{T_{n-3}}\sigma^{-1}_{D'}\omega^{-1}\varphi(E_{i,i+1}) \equiv E_{i,i+1} + \sum_{j=1}^{3} a^{(n-2)}_{ji} E_{j,j+n-2} \bmod N_{n-1} \quad \text{for } 1 \leqslant i \leqslant n. \tag{3.9}$$

For $k = n - 2$, repeating the arguments in (A-1), (A-2), (B-1) and (C-1) above, we obtain that in (3.9) $a^{(n-2)}_{si} = 0$ for $s \neq i, i + 3 - n$ except $a^{(n-2)}_{31}$ and $a^{(n-2)}_{1n}$. (*Note*. For $k = n - 2$, the arguments in (B-2) and (C-2) are invalid since $E_{1+k,4+k} = E_{n-1,n+2}$ and $E_{s-3,s} = E_{03}$ are not elements in $N(\mathbf{F})$.) Thus (3.9) may be rewritten as

$$\sigma_{T_{n-3}}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{12}) \equiv E_{12} + a_{11}^{(n-2)}E_{1,n-1} + a_{31}^{(n-2)}E_{3,n+1} \bmod N_{n-1},$$

$$\begin{aligned}&\sigma_{T_{n-3}}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{i,i+1})\\ &\quad\equiv E_{i,i+1} + a_{i+3-n,i}^{(n-2)}E_{i+3-n,i+1} + a_{ii}^{(n-2)}E_{i,i+n-2} \bmod N_{n-1}\\ &\quad\text{for } i = 2, \dots, n-1,\end{aligned}$$

and

$$\sigma_{T_{n-3}}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{n,n+1}) \equiv E_{n,n+1} + a_{1n}^{(n-2)}E_{1,n-1} + a_{3n}^{(n-2)}E_{3,n+1} \bmod N_{n-1}.$$

In the same argument as above, we can use induction to prove that there exists an $S \in U(\mathbf{F})$ such that $T' = ST_{n-3}$ satisfies

$$\left.\begin{aligned}&\sigma_{T'}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{12}) \equiv E_{12} + b_3'E_{3,n+1}\\ &\sigma_{T'}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{i,i+1}) \equiv E_{i,i+1}, \quad i = 2, \dots, n-1,\\ &\sigma_{T'}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi(E_{n,n+1}) \equiv E_{n,n+1} + b_4'E_{1,n-1}\end{aligned}\right\} \bmod N_{n-1}$$

with $b_3', b_4' \in \mathbf{F}$. It is easy to check that $\sigma_{T'}$, $\mu_{b_3'}^{(12)}$ and $\mu_{b_4'}^{(n2)}$ satisfy (3.3). The proof is completed. □

Lemma 3.4. *Let* $\theta_1 = (\mu_{b_3'}^{(12)})^{-1}(\mu_{b_4'}^{(n2)})^{-1}\sigma_{T'}^{-1}\sigma_{D'}^{-1}\omega^{-1}\varphi$. *Then there exist a* $T'' \in U(\mathbf{F})$ *and* $b_1', b_2' \in \mathbf{F}$ *such that*

$$\left(\mu_{b_1'}^{(11)}\right)^{-1}\left(\mu_{b_2'}^{(n1)}\right)^{-1}\sigma_{T''}^{-1}\theta_1(E_{i,i+1}) \equiv E_{i,i+1} \bmod N_n \quad \textit{for } 1 \leqslant i \leqslant n.$$

Proof. By Lemma 3.3, we have

$$\theta_1(E_{i,i+1}) \equiv E_{i,i+1} + a_{1i}^{(n-1)}E_{1n} + a_{2i}^{(n-1)}E_{2,n+1} \bmod N_n \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

For $2 < i < n$, applying θ_1 to $[E_{12}, E_{i,i+1}] = 0$, we have $a_{2i}^{(n-1)}E_{1,n+1} = 0$, from which it follows that $a_{2i}^{(n-1)} = 0$. Similarly, for $1 < i < n-1$, applying θ_1 to $[E_{i,i+1}, E_{n,n+1}] = 0$, we have $a_{1i}^{(n-1)} = 0$. Furthermore, $[E_{12}, E_{n,n+1}] = 0$ implies that $a_{2n}^{(n-1)} = -a_{11}^{(n-1)}$. Thus, we have

$$\left.\begin{aligned}&\theta_1(E_{12}) \equiv E_{12} + a_{11}^{(n-1)}E_{1n} + a_{21}^{(n-1)}E_{2,n+1}\\ &\theta_1(E_{23}) \equiv E_{23} + a_{22}^{(n-1)}E_{2,n+1}\\ &\theta_1(E_{i,i+1}) \equiv E_{i,i+1}, \quad i = 3, \dots, n-2,\\ &\theta_1(E_{n-1,n}) \equiv E_{n-1,n} + a_{1,n-1}^{(n-1)}E_{1n}\\ &\theta_1(E_{n,n+1}) \equiv E_{n,n+1} + a_{1n}^{(n-1)}E_{1n} - a_{11}^{(n-1)}E_{2,n+1}\end{aligned}\right\} \bmod N_n.$$

Set $T'' = E - a_{1,n-1}^{(n-1)}E_{1,n-1} + a_{11}^{(n-1)}E_{2n} + a_{22}^{(n-1)}E_{3,n+1}$. Then

$$\left.\begin{aligned} &\sigma_{T''}^{-1}\theta_1(E_{12}) \equiv E_{12} + a_{21}^{(n-1)} E_{2,n+1} \\ &\sigma_{T''}^{-1}\theta_1(E_{i,i+1}) \equiv E_{i,i+1}, \quad i = 2, \ldots, n-1, \\ &\sigma_{T''}^{-1}\theta_1(E_{n,n+1}) \equiv E_{n,n+1} + a_{1n}^{(n-1)} E_{1n} \end{aligned}\right\} \bmod N_n.$$

Set $b_1' = a_{21}^{(n-1)}$ and $b_2' = a_{1n}^{(n-1)}$. Then $(\mu_{b_1'}^{(11)})^{-1}(\mu_{b_2'}^{(n1)})^{-1}\sigma_{T''}^{-1}\theta_1$ acts trivially on $E_{i,i+1} \bmod N_n$ for $1 \leqslant i \leqslant n$. The proof is completed. □

Lemma 3.5. *Set* $\theta_2 = (\mu_{b_1'}^{(11)})^{-1}(\mu_{b_2'}^{(n1)})^{-1}\sigma_{T''}^{-1}\theta_1$. *Then*

$$\theta_2(E_{i,i+k}) \equiv \beta_k E_{i,i+k} \bmod N_{k+1} \quad \textit{for } 1 \leqslant k < n \textit{ and } 1 \leqslant i \leqslant n+1-k,$$

where $\beta_1 = 1$ *and* $\beta_k \in \mathbf{F}^*$, $k = 2, \ldots, n-1$.

Proof. By Lemma 3.4, the assertion for $k = 1$ is true. So we need only to consider the case of $2 \leqslant k \leqslant n-1$. Since the matrix $A(\theta_2)$ is the identity matrix, by Lemma 2.7 we have

$$\theta_2(E_{i,i+k}) \equiv b_{ii}^{(k)} E_{i,i+k} \bmod N_{k+1} \quad \text{for } 2 \leqslant k \leqslant n-1 \text{ and } 1 \leqslant i \leqslant n+1-k.$$

For $1 \leqslant i \leqslant n-k$, applying θ_2 to

$$\left[E_{i,i+k} + E_{i+1,i+1+k},\ E_{i,i+1} + E_{i+k,i+1+k}\right] = 0,$$

we obtain that

$$\left(b_{ii}^{(k)} - b_{i+1,i+1}^{(k)}\right) E_{i,i+1+k} \equiv 0 \bmod N_{k+2}.$$

So $b_{ii}^{(k)} = b_{i+1,i+1}^{(k)}$. Set $\beta_k = b_{ii}^{(k)}$ for $2 \leqslant k \leqslant n-1$. Then the proof is completed. □

Lemma 3.6. *Let* θ_2 *and* β_k *for* $1 \leqslant k < n$ *be as above. Then*

$$\theta_2(E_{i,i+k}) \equiv \beta_k E_{i,i+k} \bmod N_n \quad \textit{for } 1 \leqslant k < n \textit{ and } 1 \leqslant i \leqslant n+1-k. \tag{3.10}$$

Proof. It follows from Lemma 3.4 that (3.10) for $k = 1$ is true. We use induction on l to prove that for $2 \leqslant k \leqslant n-1$, $1 \leqslant i \leqslant n+1-k$ and $0 \leqslant l \leqslant n-k-1$,

$$\theta_2(E_{i,i+k}) \equiv \beta_k E_{i,i+k} \bmod N_{1+k+l}. \tag{3.11}$$

Then when $l = n-k-1$, (3.11) shows that this lemma holds. First, Lemma 3.5 shows that (3.11) for $l = 0$ is true. Next, assume that (3.11) for $l = p-1 < n-k-1$ is true. We need to prove that (3.11) for $l = p$ is true. In order to shorten the subscripts, we put $k + p = m$. Assume that for $2 \leqslant k \leqslant n-1$ and $1 \leqslant i \leqslant n+1-k$,

$$\theta_2(E_{i,i+k}) \equiv \beta_k E_{i,i+k} + \sum_{j=1}^{n+1-m} c_{ji}^{(k)} E_{j,j+m} \bmod N_{1+m}. \tag{3.12}$$

To prove that (3.11) for $l = p$ is true, we need to show that the coefficients $c_{ji}^{(k)} = 0$ in the above equation. We use a case-by-case analysis.

(A-1) $j \geqslant 2$ and $j \neq i, i+k+1$. Applying θ_2 to $[E_{j-1,j}, E_{i,i+k}] = 0$, we obtain

$$c_{ji}^{(k)} E_{j-1,j+m} - c_{j-1-m,i}^{(k)} E_{j-1-m,j} \equiv 0 \bmod N_{2+m},$$

which implies $c_{ji}^{(k)} = 0$.

(A-2) $j \leqslant n-m$ and $j \neq i+k-m, i-m-1$. As in (A-1), applying θ_2 to $[E_{i,i+k}, E_{m+j,m+j+1}] = 0$, we obtain $c_{ji}^{(k)} = 0$.

Since $j = i$ or $i+k+1$ implies $j \neq i+k-m, i-m-1$, it follows from (A-1) and (A-2) that $c_{ji}^{(k)} = 0$ for $2 \leqslant j \leqslant n-m$ and (3.12) can be rewritten as

$$\theta_2(E_{i,i+k}) \equiv \beta_k E_{i,i+k} + c_{1i}^{(k)} E_{1,1+m} + c_{n+1-m,i}^{(k)} E_{n+1-m,n+1} \bmod N_{1+m}$$
$$\text{for } 2 \leqslant k \leqslant n-1 \text{ and } 1 \leqslant i \leqslant n+1-k, \tag{3.13}$$

where $c_{1i}^{(k)} = 0$ for $1 \neq i+k-m, i-m-1$ and $c_{n+1-m,i}^{(k)} = 0$ for $n+1-m \neq i, i+k+1$. In particular, we have

$$\theta_2(E_{i,i+2}) \equiv \beta_2 E_{i,i+2} + c_{1i}^{(2)} E_{1,3+p} + c_{n-1-p,i}^{(2)} E_{n-1-p,n+1} \bmod N_{3+p}$$
$$\text{for } 1 \leqslant i \leqslant n-1, \tag{3.14}$$

Next, we consider the remaining cases.

(B-1) $j = 1$ and $j = i-m-1$. In this case we have $i = m+2$. Apply θ_2 to $[E_{m+2,m+2+k}, E_{m+1,m+3}] = 0$. By (3.13) and (3.14),

$$\delta_{m+2+k,n-1-p} \beta_k c_{n-1-p,m+1}^{(2)} E_{m+2,n+1} + \beta_2 c_{1,m+2}^{(k)} E_{1,m+3}$$
$$- \delta_{m+3,n+1-m} \beta_2 c_{n+1-m,m+2}^{(k)} E_{m+1,n+1} \equiv 0 \bmod N_{3+m},$$

which implies $c_{1,m+2}^{(k)} = 0$, i.e., $c_{ji}^{(k)} = 0$.

(B-2) $j = 1$ and $j = i-p$. In this case, $i = 1+p$. By (3.13) we have

$$\theta_2(E_{2+p,2+m}) \equiv \beta_k E_{2+p,2+m} + c_{n+1-m,2+p}^{(k)} E_{n+1-m,n+1} \bmod N_{1+m}.$$

Applying θ_2 to $[E_{1+p,1+m} + E_{2+p,2+m}, E_{1+p,2+p} + E_{1+m,2+m}] = 0$, we have

$$c_{1,1+p}^{(k)} E_{1,2+m} - \left(c_{n+1-m,1+p}^{(k)} + c_{n+1-m,2+p}^{(k)}\right)$$
$$\times (\delta_{2+p,n+1-m} E_{1+p,n+1} + \delta_{2+m,n+1-m} E_{1+m,n+1}) \equiv 0 \bmod N_{2+m}.$$

Hence $c_{1,1+p}^{(k)} = 0$, i.e., $c_{ji}^{(k)} = 0$.

In view of (B-1) and (B-2), we have $c_{1i}^{(k)} = 0$ in (3.13).

(C-1) $j = n+1-m$ and $j = i+k+1$. In this case, we have $i = n-m-k$. As in case (B-1), the equation $[E_{n-1-m,n+1-m}, E_{n-m-k,n-m}] = 0$ implies $c_{n+1-m,n-m-k}^{(k)} = 0$, i.e., $c_{ji}^{(k)} = 0$.

(C-2) $j = n + 1 - m$ and $j = i$. In this case, $i = n + 1 - m$. By the above arguments,

$$\theta_2(E_{n-m,n-p}) \equiv \beta_k E_{n-m,n-p} \bmod N_{1+m}.$$

As in (B-2), the equation $[E_{n-p,n-p+1}+E_{n-m,n-m+1}, E_{n-m,n-p}+E_{n+1-m,n+1-p}]=0$ implies $c^{(k)}_{n+1-m,n+1-m} = 0$, i.e., $c^{(k)}_{ji} = 0$.

The proof is completed. □

Lemma 3.7. *Let θ_2 and β_k for $1 \leqslant k < n$ be as above. Then there exist a diagonal matrix $D'' \in T(\mathbf{F})$ and a scalar $c \in \mathbf{F}^*$ such that for $1 \leqslant k < n$ and $1 \leqslant i \leqslant n + 1 - k$,*

$$\psi_c^{-1}\sigma_{D''}^{-1}\theta_2(E_{i,i+k}) \equiv E_{i,i+k} \bmod N_n. \tag{3.15}$$

Proof. Set $D'' = \mathrm{diag}\{d_1, \ldots, d_n, d_{n+1}\} \in T(\mathbf{F})$ with $d_k = \beta_k$ for $1 \leqslant k \leqslant n - 1$, $d_n = d_2 d_{n-1}$ and $d_{n+1} = d_2 d_n$. Keep in mind the fact that $d_1 = \beta_1 = 1$. We first show that

$$d_k d_l = d_p d_q \quad \text{for } 1 \leqslant k, l, p, q \leqslant n \text{ and } k + l = p + q \leqslant n + 2. \tag{3.16}$$

If $k = p$, then (3.16) is clear. Assume $k \neq p$. First consider the case of $k + l = p + q \leqslant n$. Applying θ_2 to $[E_{1,1+k} + E_{1,1+p},\ E_{1+k,1+k+l} - E_{1+p,1+p+q}] = 0$, we obtain that (3.16) is true. Next consider the case of $k + l = p + q = n + 1$. In this case, it is enough to prove

$$d_k d_l = d_n \quad \text{for} \quad 1 \leqslant k, l \leqslant n \quad \text{and} \quad k + l = n + 1. \tag{3.17}$$

When $k = 1, 2$, (3.17) is clear. Assume that for k with $2 \leqslant k \leqslant \frac{1}{2}(n - 1)$ (3.17) is true. Then $d_{k+1}d_{n-k} = d_k d_2 d_{n-k} = d_k d_{n+1-k} = d_n$. Thus (3.16) for $k + l = p + q = n + 1$ is true. Similarly, (3.16) for $k + l = p + q = n + 2$ holds. By (3.16) we have $d_2 d_k d_i = d_2 d_{i+k-1} = d_{i+k}$, which implies

$$d_k d_i d_{i+k}^{-1} = d_2^{-1} \quad \text{for } 1 \leqslant k \leqslant n \text{ and } 1 \leqslant i \leqslant n + 1 - k. \tag{3.18}$$

Then by Lemma 3.6 and (3.18), we have

$$\sigma_{D''}^{-1}\theta_2(E_{i,i+k}) \equiv d_k d_i d_{i+k}^{-1} E_{i,i+k} \bmod N_n \equiv d_2^{-1} E_{i,i+k} \bmod N_n$$
$$\text{for } 1 \leqslant k < n \text{ and } 1 \leqslant i \leqslant n + 1 - k.$$

Set $c = d_2^{-1}$. Then (3.15) holds. □

Lemma 3.8. *Set $\theta_3 = \psi_c^{-1}\sigma_{D''}^{-1}\theta_2$. Then there exists a linear functional f' on $N(\mathbf{F})$ such that $\theta_3 = 1 + f'$.*

Proof. By Lemmas 3.7 and 2.1(iii) we may assume that

$$\theta_3(E_{ij}) = E_{ij} + f_{ij} E_{1,n+1}, \quad f_{ij} \in \mathbf{F}, \ 1 \leqslant i < j \leqslant n + 1.$$

Let f' be the linear functional on $N(\mathbf{F})$ such that $f'(E_{ij}) = f_{ij}$ for $1 \leqslant i < j \leqslant n + 1$. Then f' satisfies the lemma. □

Proof of Theorem 1.1 (*for* $n \geqslant 4$). It follows from Lemmas 3.1–3.8 that

$$\begin{aligned}\varphi &= \omega\sigma_{D'}\sigma_{T'}\mu_{b'_4}^{(n2)}\mu_{b'_3}^{(12)}\theta_1\\ &= \omega\sigma_{D'}\sigma_{T'}\mu_{b'_4}^{(n2)}\mu_{b'_3}^{(12)}\sigma_{T''}\mu_{b'_2}^{(n1)}\mu_{b'_1}^{(11)}\theta_2\\ &= \omega\sigma_{D'}\sigma_{T'}\mu_{b'_4}^{(n2)}\mu_{b'_3}^{(12)}\sigma_{T''}\mu_{b'_2}^{(n1)}\mu_{b'_1}^{(11)}\sigma_{D''}\psi_c\theta_3\\ &= \varphi' + \varphi' f',\end{aligned}$$

where $\varphi' = \omega\sigma_{D'}\sigma_{T'}\mu_{b'_4}^{(n2)}\mu_{b'_3}^{(12)}\sigma_{T''}\mu_{b'_2}^{(n1)}\mu_{b'_1}^{(11)}\sigma_{D''}\psi_c$. By Lemmas 2.1(iv), 2.2 and 2.3 it is easy to show that φ is of form (1.1). □

Proof of Theorem 1.1 (*for* $n = 3$). First we show that in the matrix $A = A(\varphi) = (a_{ji})$ for any $\varphi \in \mathscr{S}'$ we have $a_{21} = a_{23} = 0$. Otherwise, say $a_{21} \neq 0$. By Lemma 2.6, we have $a_{13}a_{21} = a_{11}a_{23}$ and $a_{21}a_{33} = a_{23}a_{31}$. Thus $a_{i3} = (a_{23}/a_{21})a_{i1}, i = 1, 2, 3$. This contradicts the non-singularity of A. Now assume that φ is any map in $\mathscr{S}'$. We will show that φ can be expressed as form (1.1).

Since $a_{23} = 0$, we have $a_{13} \neq 0$ or $a_{33} \neq 0$. Set $\omega = 1$ if $a_{33} \neq 0$ and $\omega = \omega_0$ otherwise. Set $\theta_1 = \omega^{-1}\varphi$ and still denote $A(\theta_1)$ by $A = (a_{ji})$. Then we have $a_{21} = a_{23} = 0$ and $a_{33} \neq 0$. Set $b'_4 = a_{33}^{-1}a_{13}$ and $\theta_2 = (\mu_{b'_4}^{(32)})^{-1}\theta_1$ and denote $A(\theta_2)$ by $A = (a_{ji})$. Then we have $a_{21} = a_{23} = a_{13} = 0$ and $a_{11} \neq 0$. Set $b'_3 = a_{11}^{-1}a_{31}$ and $\theta_3 = (\mu_{b'_3}^{(12)})^{-1}\theta_2$ and still denote $A(\theta_3)$ by $A = (a_{ji})$. Then there is only one non-zero entry in each of the first and the third columns of A and the sole non-zero entry is a_{11} and a_{33}, respectively. Assume that

$$\theta_3(E_{i,i+2}) \equiv b_{1i}E_{13} + b_{2i}E_{24} \bmod N_3 \quad \text{for } i = 1, 2.$$

Applying θ_3 to $[E_{12}, E_{13}] = 0$, we obtain $a_{11}b_{21}E_{14} = 0$ and so $b_{21} = 0$. Furthermore, by Lemma 2.5 we have $b_{11} \neq 0$. Similarly, $b_{12} = 0$ and $b_{22} \neq 0$. Thus it follows from $[E_{13}, E_{23}] = 0$ that $b_{11}a_{32}E_{14} = 0$ and so $a_{32} = 0$. Similarly, we have $a_{12} = 0$. Therefore, the matrix A is diagonal. Set $D = \operatorname{diag}\{1, a_{11}, a_{11}a_{22}, a_{11}a_{22}a_{33}\}$ and $\theta_4 = \sigma_D^{-1}\theta_3$. Then $A(\theta_4)$ is the identity matrix. We may assume that

$$\theta_4(E_{i,i+1}) \equiv E_{i,i+1} + c_{1i}E_{13} + c_{2i}E_{24} \bmod N_3 \quad \text{for } i = 1, 2, 3.$$

Applying θ_4 to $[E_{12}, E_{34}] = 0$, we have $(c_{23} + c_{11})E_{14} = 0$ and so $c_{23} = -c_{11}$. Set $T' = E - c_{12}E_{12} + c_{11}E_{23} + c_{22}E_{34}$ and $\theta_5 = \sigma_{T'}^{-1}\theta_4$. Then

$$\left.\begin{aligned}\theta_5(E_{12}) &\equiv E_{12} + c_{21}E_{24}\\ \theta_5(E_{23}) &\equiv E_{23}\\ \theta_5(E_{34}) &\equiv E_{34} + c_{13}E_{13}\end{aligned}\right\} \bmod N_3.$$

Set $b'_1 = c_{21}, b'_2 = c_{13}$ and $\theta_6 = (\mu_{b'_1}^{(11)})^{-1}(\mu_{b'_2}^{(31)})^{-1}\theta_5$. Then

$$\theta_6(E_{i,i+1}) \equiv E_{i,i+1} \bmod N_3 \quad \text{for } i = 1, 2, 3.$$

In addition, by the argument above we may assume that

$$\theta_6(E_{i,i+2}) \equiv b'_{ii} E_{i,i+2} \bmod N_3 \quad \text{for } i = 1, 2,$$

where $b'_{ii} \neq 0$. It follows from $[E_{12} + E_{34}, E_{13} + E_{24}] = 0$ that $b'_{11} = b'_{22}$. Set $D' = \operatorname{diag}\{1, d, d^2, d^3\}$ with $d = b'_{11}$ and $\theta_7 = \sigma_{D'}^{-1}\theta_6$. Then

$$\theta_7(E_{ij}) \equiv d^{-1} E_{ij} \bmod N_3 \quad \text{for } 1 \leqslant i < j \leqslant 4 \text{ and } j - i < 3.$$

Set $\theta_8 = \psi_c^{-1}\theta_7$ with $c = d^{-1}$. Then

$$\theta_8(E_{ij}) \equiv E_{ij} \bmod N_3 \quad \text{for } 1 \leqslant i < j \leqslant 4 \text{ and } j - i < 3.$$

Thus we may assume that

$$\theta_8(E_{ij}) = E_{ij} + f_{ij} E_{14}, \quad f_{ij} \in \mathbf{F}, \ 1 \leqslant i < j \leqslant 4.$$

Hence $\theta_8 = 1 + f'$ where f' is a linear functional on $N(\mathbf{F})$ such that $f'(E_{ij}) = f_{ij}$ for $1 \leqslant i < j \leqslant 4$. It follows from the series of arguments above that

$$\varphi = \omega \mu_{b'_4}^{(32)} \mu_{b'_3}^{(12)} \sigma_D^{-1} \sigma_{T'}^{-1} \mu_{b'_2}^{(31)} \mu_{b'_1}^{(11)} \sigma_{D'} \psi_c (1 + f').$$

Finally, by Lemmas 2.1(iv), 2.2 and 2.3 we can obtain that φ is of form (1.1). □

Proof of Theorem 1.2. It is clear that a map on $N(\mathbf{F})$ of form (1.2) is an $\mathbf{F}$-algebra automorphism. Conversely, assume that φ is any $\mathbf{F}$-algebra automorphism on $N(\mathbf{F})$. By Theorem 1.1, we can assume

$$\varphi = \psi_c \omega \mu_{b_4}^{(n2)} \mu_{b_3}^{(12)} \mu_{b_2}^{(n1)} \mu_{b_1}^{(11)} \sigma_T + f.$$

Set $\theta = \psi_c \omega \mu_{b_4}^{(n2)} \mu_{b_3}^{(12)} \mu_{b_2}^{(n1)} \mu_{b_1}^{(11)} + f'$ where $f' = f \sigma_T^{-1}$ is a linear functional on $N(\mathbf{F})$. Then $\varphi = \theta \sigma_T$ and θ is an $\mathbf{F}$-algebra automorphism. First, we assert that $\omega = 1$. In fact, if $\omega = \omega_0$, then for any $c \in \mathbf{F}^*$ and any $b_i \in \mathbf{F}$, $i = 1, \ldots, 4$, by calculation we have

$$\left.\begin{aligned} \theta(E_{n,n+1}) &= -c(E_{12} + b_4 E_{3,n+1} + b_2 E_{2,n+1}) \\ &\quad + f'(E_{n,n+1}) E_{1,n+1} \quad \text{for } n > 3, \\ \theta(E_{34}) &= -c(E_{12} + b_4 E_{34} + b_2 b_3 E_{13} + b_2(1 + b_3 b_4) E_{24}) \\ &\quad + f'(E_{34}) E_{14} \quad \text{for } n = 3, \end{aligned}\right\} \tag{3.19}$$

$$\theta(E_{n-1,n}) = -cE_{23} + f'(E_{n-1,n}) E_{1,n+1} \quad \text{for } n \geqslant 3, \tag{3.20}$$

and

$$\theta(E_{n-1,n+1}) = -c(E_{13} + b_4E_{2,n+1}) + f'(E_{n-1,n+1})E_{1,n+1} \quad \text{for } n \geqslant 3. \tag{3.21}$$

Since $E_{n-1,n+1} = E_{n-1,n}E_{n,n+1}$, by (3.19) and (3.20) it follows that $\theta(E_{n-1,n+1}) = c^2b_4E_{2,n+1}$. This contradicts (3.21). Therefore, $\omega = 1$ and

$$\theta(E_{n,n+1}) = \begin{cases} c(E_{n,n+1} + b_4E_{1,n-1} + b_2E_{1n}) & \\ \quad + f'(E_{n,n+1})E_{1,n+1} & \text{for } n > 3, \\ c(E_{34} + b_4E_{12} + b_2b_3E_{24} & \\ \quad + b_2(1 + b_3b_4)E_{13}) + f'(E_{34})E_{14} & \text{for } n = 3 \end{cases}$$

and

$$\theta(E_{n-1,n}) = cE_{n-1,n} + f'(E_{n-1,n})E_{1,n+1} \quad \text{for } n \geqslant 3.$$

Since $E_{n,n+1}E_{n-1,n} = 0$, it follows that $c^2b_4E_{1n} = 0$, which implies $b_4 = 0$. Furthermore, by $E^2_{n,n+1} = 0$ we get $b_2 = 0$. Similarly, it follows from $E_{23}E_{12} = 0$ and $E^2_{12} = 0$ that $b_3 = 0$ and $b_1 = 0$. Thus $\theta = \psi_c + f'$. Since $\theta(E_{13}) = \theta(E_{12})\theta(E_{23})$, it follows that $cE_{13} + f'(E_{13})E_{1,n+1} = c^2E_{13}$ and so $c = 1$. Thus $\theta = 1 + f'$. Furthermore, for any $X, Y \in N(\mathbf{F})$, since $\theta(XY) = \theta(X)\theta(Y)$, it follows that $f'(XY) = 0$. By Lemma 2.1(iv), $\varphi = \sigma_T(1 + f'')$ where $f'' = \sigma_T^{-1}f$ is a linear functional on $N(\mathbf{F})$ with the property that $f''(XY) = 0$ for any $X, Y \in N(\mathbf{F})$. The proof is completed. □

Proof of Theorem 1.3. It is clear that a map on $N(\mathbf{F})$ of form (1.3) for char $\mathbf{F} \neq 2$ or of form (1.4) for char $\mathbf{F} = 2$ is a Lie automorphism. Conversely, assume that φ is any Lie automorphism on $N(\mathbf{F})$. As in the proof of Theorem 1.2, we can assume that $\varphi = \theta\sigma_T$ where $\theta = \psi_c\omega\mu^{(n2)}_{b_4}\mu^{(12)}_{b_3}\mu^{(n1)}_{b_2}\mu^{(11)}_{b_1} + f'$ with $f' = f\sigma_T^{-1}$. Clearly, θ is a Lie automorphism. First, consider the case $\omega = \omega_0$. Since $E_{n-1,n+1} = [E_{n-1,n}, E_{n,n+1}]$, it follows from (3.19) and (3.20) that $\theta(E_{n-1,n+1}) = c^2(b_4E_{2,n+1} - E_{13})$. Comparing this equation with (3.21), we have $c = 1$ and $b_4 = -b_4$. So $b_4 = 0$ when char $\mathbf{F} \neq 2$. Similarly, the equation $[E_{12}, E_{23}] = E_{13}$ implies that $b_3 = 0$ when char $\mathbf{F} \neq 2$. Using similar arguments, we can prove that if $\omega = 1$, we also have $c = 1$ for any field $\mathbf{F}$ and $b_3 = b_4 = 0$ when char $\mathbf{F} \neq 2$. Thus we have proved that $\theta = \eta + f'$ where $\eta = \omega\mu^{(n1)}_{b_2}\mu^{(11)}_{b_1}$ when char $\mathbf{F} \neq 2$ and $\eta = \omega\mu^{(n2)}_{b_4}\mu^{(12)}_{b_3}\mu^{(n1)}_{b_2}\mu^{(11)}_{b_1}$ when char $\mathbf{F} = 2$.

Finally, for any $X, Y \in N_n(\mathbf{F})$, the facts that η is a Lie automorphism of $N_n(\mathbf{F})$ and $\theta([X, Y]) = [\theta(X), \theta(Y)]$ imply that $f'([X, Y]) = 0$. By Lemma 2.1(iv), $\varphi = \eta\sigma_T(1 + f'')$ where $f'' = (\eta\sigma_T)^{-1}f$ is a linear functional on $N_{n+1}(\mathbf{F})$ with the property that $f''([X, Y]) = 0$ for any $X, Y \in N_{n+1}(\mathbf{F})$. The proof is completed. □

4. The exceptional case $n = 2$

The following theorem is the analogy of Proposition 8 in [12].

Theorem 4.1. *Let $n = 2$ and φ be a linear map on $N(\mathbf{F})$. Then:*
(i) *φ is commutativity preserving if and only if $\varphi(E_{13}) \in \mathbf{F}E_{13}$, or the range of φ is a commutative subspace of $N(\mathbf{F})$.*
(ii) *$\varphi \in \mathscr{S}$ if and only if $\varphi(E_{13}) \in \mathbf{F}E_{13}$ and the range of φ is non-commutative.*

Proof. (i) If the range of φ is commutative, then φ obviously preserves commutativity. If $\varphi(E_{13}) \in \mathbf{F}E_{13}$, for commuting matrices $X, Y \in N(\mathbf{F})$, $\{E_{13}, X, Y\}$ cannot be a basis of $N(\mathbf{F})$, and so $\{E_{13}, X, Y\}$ are linearly dependent. Hence $\{E_{13}, \varphi(X), \varphi(Y)\}$ are also linearly dependent and $\varphi(X)$ commutes with $\varphi(Y)$.

Conversely, if φ preserves commutativity and the range of φ contains two non-commuting matrices $\varphi(X)$ and $\varphi(Y)$, then $\{E_{13}, \varphi(X), \varphi(Y)\}$ are linearly independent and so span $N(\mathbf{F})$. Since $\varphi(E_{13})$ commutes with $\varphi(X)$ and $\varphi(Y)$, $\varphi(E_{13})$ is in the center of $N(\mathbf{F})$. Hence $\varphi(E_{13}) \in \mathbf{F}E_{13}$.

(ii) If $\varphi \in \mathscr{S}$, then the range of φ must be non-commutative and so $\varphi(E_{13}) \in \mathbf{F}E_{13}$ by (i). Conversely, assume that $\varphi(E_{13}) \in \mathbf{F}E_{13}$ and the range of φ is non-commutative. If $\varphi(E_{13}) = 0$, adding an appropriate linear functional on $N(\mathbf{F})$ to φ, we obtain a linear map φ_1 on $N(\mathbf{F})$ such that $\varphi_1(E_{13}) = E_{13}$ and the range of φ_1 is also non-commutative. It is easy to see that if a subspace of $N(\mathbf{F})$ of $\dim < 3$ contains E_{13}, then it must be commutative. Hence φ_1 is surjective and so bijective. From (i), we see that both φ_1 and φ_1^{-1} preserve commutativity. Thus φ_1 preserves commutativity in both directions, and so does φ. □

References

[1] L. Beasley, Linear transformations on matrices: the invariance of commuting pairs of matrices, Linear and Multilinear Algebra 6 (1978) 179–183.
[2] You'an Cao, Automorphisms of certain Lie algebras of upper triangular matrices over a commutative ring, J. Algebra 189 (1997) 506–513.
[3] You'an Cao, Automorphisms of the Lie algebras of strictly upper triangular matrices over certain commutative rings, Linear Algebra Appl. 329 (2001) 175–187.
[4] You'an Cao, Jingtong Wang, A note on algebra automorphisms of strictly upper triangular matrices over commutative rings, Linear Algebra Appl. 311 (2000) 187–193.
[5] G.H. Chan, M.H. Lim, Linear transformations on symmetric matrices that preserve commutativity, Linear Algebra Appl. 47 (1982) 11–22.
[6] M.D. Choi, A.A. Jafarian, H. Radjavi, Linear maps preserving commutativity, Linear Algebra Appl. 87 (1987) 227–241.
[7] D.Ž. Doković, Automorphisms of the Lie algebra of upper triangular matrices over a connected commutative ring, J. Algebra 170 (1994) 101–110.
[8] J.A. Gibbs, Automorphisms of certain unipotent groups, J. Algebra 14 (1970) 203–228.
[9] C. Kunicki, Commutativity preserving and normal preserving linear transformations on M_2, Linear and Multilinear Algebra 45 (1999) 341–347.

[10] C.-K. Li, S. Pierce, Linear preserver problems, Amer. Math. Monthly 108 (2001) 591–605.
[11] C.-K. Li, N.-K. Tsing, Linear preverser problems: a brief introduction and some special techniques, Linear Algebra Appl. 162–164 (1992) 217–235.
[12] L.W. Marcoux, A.R. Sourour, Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras, Linear Algebra Appl. 288 (1999) 89–104.
[13] M. Omladić, On operators preserving commutativity, J. Funct. Anal. 66 (1986) 105–122.
[14] S. Pierce et al., A survey of linear preserving problems, Linear and Multilinear Algebra, 42 (1992) 1–129.
[15] S. Pierce, W. Watkins, Invariants of linear maps on matrix algebras, Linear and Multilinear Algebra 6 (1978/79) 185–200.
[16] H. Radjavi, Commutativity-preserving operators on symmetric matrices, Linear Algebra Appl. 61 (1984) 219–224.
[17] W. Watkins, Linear maps that preserve commuting pairs of matrices, Linear Algebra Appl. 14 (1976) 29–35.