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Algebra Structure on the Hochschild Cohomology of the Ring of Invariants of a Weyl Algebra under a Finite Group[1]

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Let A_n be the nth Weyl algebra, and let $G \subset \mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{Aut}(A_n)$ be a finite group of linear automorphisms of A_n. In this paper, we compute the multiplicative structure on the Hochschild cohomology $HH^{\bullet}(A_n^G)$ of the algebra of invariants of G. We prove that, as a graded algebra, $HH^{\bullet}(A_n^G)$ is isomorphic to the graded algebra associated to the center of the group algebra $\mathbb{C}G$ with respect to a filtration defined in terms of the defining representation of G.

1. INTRODUCTION

1.1. Let us fix an algebraically closed ground field $\mathbb{C}$ of characteristic zero.

1.2. For $n \in \mathbb{N}$, the nth Weyl algebra A_n is the one freely generated by elements p_i and q_i, $1 \leq i \leq n$, subject to the commutation relations of Heisenberg,

$$\begin{aligned} &[p_i, p_j] = [q_i, q_j] = 0, && \forall i, j; \\ &[q_i, p_i] = 1, && \forall i; \\ &[q_j, p_i] = 0, && \forall i, j \text{ such that } i \neq j. \end{aligned}$$

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It can be realized either as the algebra of algebraic differential operators on the affine space $\mathbb{A}_n$ or as the Sridharan twisted enveloping algebra $\mathscr{U}_f\mathfrak{g}$ of an abelian Lie algebra $\mathfrak{g}$ of dimension $2n$ with respect to any non-degenerate Chevalley–Eilenberg 2-cocycle on $\mathfrak{g}$. It is a simple, left and right Noetherian algebra of Gabriel–Rentschler Krull dimension n, Gel'fand–Kirillov dimension $2n$, and global homological dimension n.

1.3. Sridharan [11] shows that the Hochschild cohomology $HH^\bullet(A_n) = H^\bullet(A_n, A_n) \cong \mathbb{C}$, concentrated in degree 0, and, in fact, that this characterizes the Weyl algebras among the twisted enveloping algebras of abelian Lie algebras. This result can be interpreted as the Poincaré lemma for quantum differential forms. The same methods can be used to show that, dually, $HH_\bullet(A_n) = H_\bullet(A_n, A_n) \cong \mathbb{C}$, concentrated in degree $2n$.

1.4. Consider a finite subgroup $G \subset \operatorname{Aut}(A_1)$ and the corresponding algebra of invariants A_1^G. As G varies, we obtain in this way a family of algebras, all of which are simple, left and right Noetherian, with Gel'fand–Kirillov dimension 2, Krull dimension 1, and global homological dimension 1; in particular, these numeric invariants do not allow us to separate them.

Alev and Lambre [2] compute the 0-degree Hochschild homology of these algebras: they show that $HH_0(A_1^G)$ is a vector space of dimension $s(G) - 1$, with $s(G)$ the number of irreducible representations of G. A theorem of Alev [1] which describes $\operatorname{Aut}(A_1)$ implies that each of its finite subgroups is conjugate to a subgroup in $\mathrm{SL}_2(\mathbb{C}) \subset \operatorname{Aut}(A_1)$, and the classification up to conjugation of these is classical; with this information one can compute $s(G)$ for each of the possible groups and conclude that the algebras under consideration are in fact non-isomorphic in pairs, apart from a few exceptions; for example, it is clear that there is a cyclic group with the same number of classes as the binary icosahedral group.

1.5. If one considers more generally the algebras A_n^G of invariants of A_n under the action of a finite subgroup $G \subset \mathrm{Sp}_{2n}(\mathbb{C}) \subset \operatorname{Aut}(A_n)$—we restrict our attention to linear automorphisms because we have no description of the whole automorphism group in this case—we again obtain a family of algebras indistinguishable, for fixed n, on the basis of the above numerical invariants alone. Alev *et al.* [4] obtain a generalization of the above formula for $HH_0(A_1^G)$: they show that $\nu_k = \dim_{\mathbb{C}} HH_k(A_n^G)$ is the number of conjugacy classes of G whose elements have unity as an eigenvalue with multiplicity exactly k.

We thus see that in general homology is not enough to separate this algebra, at least without further analysis: one can easily show that these numbers ν_k can be computed in terms of the character of the defining representation and the power maps of the group G, and it is known [7] that there are pairs of non-isomorphic finite groups for which these data coincide.

1.6. In part, the interest in these computations comes from the wish to understand the Poisson structures underlying the objects under consideration.

The algebra A_n has a natural filtration, that of Bernstein, such that the associated graded object $\operatorname{gr} A_n$ is a polynomial algebra on $2n$ variables, canonically endowed with a Poisson bracket deduced from the commutator of A_n. The action of $\mathrm{Sp}_{2n}(\mathbb{C})$ on $\operatorname{gr} A_n$ respects this structure, so that $(\operatorname{gr} A_n)^G$, for $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$, is naturally a Poisson algebra. Moreover, one can show that if $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$ is a finite subgroup, the graded algebra associated to A_n^G with respect to the restricted Bernstein filtration is exactly $(\operatorname{gr} A_n)^G$, with the same Poisson structure.

In particular, this means that we can regard the algebras A_n^G as quantizations of the Poisson algebras $(\operatorname{gr} A_n)^G$. This is the point of view of [3], where the authors show that, in a precise sense, the 0-degree Hochschild homology of A_1^G approximates the 0-degree Poisson homology of $(\operatorname{gr} A_1)^G$. This idea cannot be transferred to the general case because we do not have a definition of Poisson homology for non-smooth algebras directly amenable to calculations and because $(\operatorname{gr} A_n)^G$ is non-smooth for most finite subgroups of $\mathrm{Sp}_{2n}(\mathbb{C})$.

1.7. It is a result of [4] that there is a duality between the homology and the cohomology of the algebras at hand. In particular, we have $\dim_{\mathbb{C}} HH^k(A_n^G) = \nu_{2n-k}$ for each finite subgroup $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$. In this paper we complete the computation of the Hochschild cohomology making the algebra structure on $HH^\bullet(A_n^G)$ explicit. The final result is the following:

1.8. THEOREM. *Let $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$ be a finite subgroup. Let G act naturally on the nth Weyl algebra A_n. The subspace $V \subset A_n$ spanned by the standard generators is G-invariant for this action. Define $d : g \to \mathbb{N}_0$ by $d(g) = 2n - \dim_{\mathbb{C}} V^g$. For each $p \geq 0$, write $F_p\mathbb{C}G$ for the subspace of the group algebra $\mathbb{C}G$ spanned by the elements $g \in G$ such that $d(g) \leq p$.*

$F_\bullet\mathbb{C}G$ is an algebra filtration on $\mathbb{C}G$, so it restricts to an algebra filtration on the center $\mathscr{Z}G$ of $\mathbb{C}G$. There is a graded algebra isomorphism $HH^\bullet(A_n^G) \cong \operatorname{gr} \mathscr{Z}G$.

1.9. It is very easy to construct examples of the situation considered in the theorem. If G is a finite group and V is a faithful G-module of degree n, G acts faithfully on the algebra of algebraic differential operators on V, which is isomorphic to A_n, so we can regard $G \subset \operatorname{Aut}(A_n)$. One sees that —using the notation of the theorem—$d(g)$ is simply two times the codimension of the subspace of V fixed by g.

One particularly nice instance of this arises when we consider the canonical action of the Weyl group corresponding to a Cartan subalgebra

$\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{g}$ on the algebra of regular differential operators on $\mathfrak{h}^*$, the dual space of $\mathfrak{h}$.

1.10. In the next section, we recall the construction of the multiplicative structure on the Hochschild theory and indicate the reductions leading to its determination in our particular case. In Section 3 we carry out the various explicit computations needed for the proof of the theorem.

2. THE MULTIPLICATIVE STRUCTURE ON $HH^\bullet(A_n^G)$

2.1. Let us fix from now on $n \in \mathbb{N}$ and a finite subgroup $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$. We consider the natural action of G on the nth Weyl algebra A_n by linear automorphisms.

2.2. The computation of $HH^\bullet(A_n^G)$ presented in [4] is based on the fact that A_n^G and the crossed product $A_n \rtimes G$ are Morita equivalent; since Hochschild cohomology groups are invariant under this kind of equivalence, in order to determine $HH^\bullet(A_n^G)$ one can instead choose to compute $HH^\bullet(A_n \rtimes G)$. Now, the algebra structure on $HH^\bullet(A_n^G)$ can be defined in terms of the composition of iterated self-extensions of A_n^G in the category of A_n^G-bimodules; since this procedure is clearly invariant under equivalences, the *algebras* $HH^\bullet(A_n^G)$ and $HH^\bullet(A_n \rtimes G)$ coincide.

2.3. The next reduction depends on results of Stefan [12] and others, which show that, in our situation, for each $A_n \rtimes G$-bimodule M there is a natural spectral sequence with initial term $E_2^{p,q} \cong H^p(G, H^q(A_n, M))$ converging to $H^\bullet(A_n \rtimes G, M)$. Since our ground field has characteristic zero, group cohomology is trivial in positive degrees, and this spectral sequence immediately degenerates, giving us natural isomorphisms $H^\bullet(A_n \rtimes G, M) \cong H^\bullet(A_n, M)^G$.

We set $M = A_n \rtimes G$. It is easy to see that Stefan's spectral sequence is a spectral sequence of algebras in this case—for example, by using the resolutions given by the bar construction in order to compute the cohomologies of G and of A_n. The distribution of zeros in its initial term implies that there are no extension problems either in order to compute the cohomology groups or to compute the product maps. We thus conclude that the isomorphism between $HH^\bullet(A_n \rtimes G)$ and $H^\bullet(A_n, A_n \rtimes G)^G$ is multiplicative; we will determine this last algebra.

2.4. Let us recall the construction of the multiplicative structure on the functor $H^\bullet(A_n, -) = \mathrm{Ext}^\bullet_{A_n^e}(A_n, -)$. We fix a projective resolution $X^\bullet \twoheadrightarrow A_n$ of A_n as a A_n^e-module. Since plainly $\mathrm{Tor}_p^{A_n}(A_n, A_n) = 0$ for $p > 0$, $X^\bullet \otimes_{A_n} X^\bullet$ is an acyclic complex over $A_n \otimes_{A_n} A_n \cong A_n$. It follows from [5, Proposition IX.2.6] that it is a projective resultion of A_n as an A_n^e-module.

In particular, there is a morphism $\Delta\colon X^\bullet \to X^\bullet \otimes_{A_n} X^\bullet$ of resolutions lifting the identity map of A_n. If M and N are A_n^e-modules, the product

$$\smile\,:\mathrm{Ext}^\bullet_{A_n^e}(A_n, M) \otimes \mathrm{Ext}^\bullet_{A_n^e}(A_n, N) \to \mathrm{Ext}^\bullet_{A_n^e}\big(A_n, M \otimes_{A_n} N\big) \quad (1)$$

is induced by the composition

$$\begin{gathered} \hom_{A_n^e}(X^\bullet, M) \otimes \hom_{A_n^e}(X^\bullet, N) \\ \big\downarrow \psi \\ \hom_{A_n^e}\big(X^\bullet \otimes_{A_n} X^\bullet, M \otimes_{A_n} N\big) \\ \big\downarrow \Delta \\ \hom_{A_n^e}\big(X^\bullet, M \otimes_{A_n} N\big), \end{gathered}$$

with ψ standing for the evident hom-$\otimes$ "interchange map," up to the canonical isomorphisms

$$\begin{aligned} &H\big(\hom_{A_n^e}(X^\bullet, M)\big) \otimes H\big(\hom_{A_n^e}(X^\bullet, N)\big) \\ &\quad \cong H\big(\hom_{A_n^e}(X^\bullet, M) \otimes \hom_{A_n^e}(X^\bullet, N)\big). \end{aligned}$$

2.5. When $M = N = A_n \rtimes G$, we can compose the map (1) with the morphism induced by the product $\mu\colon (A_n \rtimes G) \otimes_{A_n} (A_n \rtimes G) \to A_n \rtimes G$. We obtain in this way the internal product of $H^\bullet(A_n, A_n \rtimes G)$.

The additivity of the functors involved and the decomposition $A_n \rtimes G \cong \bigoplus_{g \in G} A_n g$ of $A_n \rtimes G$ as an A_n^e-module—here and elsewhere $A_n g$ is the A_n^e-module obtained from A_n by twisting the right action by the automorphism g—have the consequence that this product is determined by its restrictions

$$\smile\,:\mathrm{Ext}^\bullet_{A_n^e}(A_n, A_n g) \otimes \mathrm{Ext}^\bullet_{A_n^e}(A_n, A_n h) \to \mathrm{Ext}^\bullet_{A_n^e}(A_n, A_n gh),$$

which we will compute in the next section.

3. EXPLICIT COMPUTATIONS

3.1. First of all, let us consider a filtered $\mathbb{C}$-algebra A with a positive ascending filtration such that the associated graded algebra $\operatorname{gr} A$ has no zero divisors. For each $x \in A$, we denote by $s(x)$ the principal symbol of x in $\operatorname{gr} A$.

Consider commuting elements $x_1, \ldots, x_n \in A$ and write, for each k such that $0 \leq k \leq n$, I_k and I'_k for the left ideals generated by $x_1, \ldots, x_k$ and $s(x_1), \ldots, s(x_k)$ in A and $\operatorname{gr} A$, respectively; in particular, $I_0 = I'_0 = 0$. Let us suppose that we have, for $1 \leq k \leq n$,

$$a \in \operatorname{gr} A,\ as(x_k) \in I'_{k-1} \Rightarrow a \in I'_{k-1}. \tag{2}$$

Let k be such that $1 \leq k \leq n$, $a \in F_m A$, and suppose that $ax_k \in I_{k-1}$, so that there are $a_i \in A$ for $i = 1, \ldots, k-1$ with $ax_k = \sum_{i=1}^{k-1} a_i x_i$. Then $s(ax_k) = s(\sum_{i=1}^{k-1} a_i x_i) \in I'_{k-1}$, and, as $s(ax_k) = s(a)s(x_k)$ because $\operatorname{gr} A$ is a domain, we see from (2) that $s(a) \in I'_{k-1}$. We conclude that there exist $b_i \in A$, for each $i = 1, \ldots, k-1$, and $a' \in F_{m-1}A$ such that $a = \sum_{i=1}^{k-1} b_i x_i + a'$. We have

$$a' x_k = ax_k - \sum_{i=1}^{k-1} b_i x_i x_k = ax_k - \sum_{i=1}^{k-1} b_i x_k x_i \in I_{k-1}.$$

By induction, this tells us that $a' \in I_{k-1}$ and, consequently, that $a \in I_{k-1}$.

We have shown that

$$a \in A,\ ax_k \in I_{k-1} \Rightarrow a \in I_{k-1}, \tag{3}$$

for each k such that $1 \leq k \leq n$, and we see that this is a condition that can be tested up to an appropriate filtration.

3.2. Let A_n be the nth Weyl algebra over $\mathbb{C}$, with generators p_i, q_i, for $i = 1, \ldots, n$. Let $\mu\colon A_n^e \to A_n$ be the canonical augmentation, and put $I = \ker \mu$. If $x \in A$, we will write $\partial x = 1 \otimes x - x \otimes 1$; plainly, $\partial x \in I$. A trivial computation shows that $[\partial x, \partial y] = 0$ whenever $x, y \in A_n$ are such that $[x, y] \in \mathbb{C}1$. In particular, the elements ∂p_i and ∂q_i, $i = 1, \ldots, n$, which generate I as a left A_n^e-module, commute.

There is an isomorphism of algebras $\phi\colon A_n^e \to A_{2n}$ uniquely determined by the conditions

$$\phi(p_i \otimes 1) = p_i,\ \phi(q_i \otimes 1) = q_i,\ \phi(1 \otimes p_i) = q_{i+n},$$
$$\text{and} \qquad \phi(1 \otimes q_i) = p_{i+n},$$

for each $i = 1, \ldots, n$. One has $\phi(\partial p_i) = -p_i + q_{i+n}$ and $\phi(\partial q_i) = -q_i + p_{i+n}$, and these elements obviously commute. When we consider the Bernstein filtration on A_{2n}, $\operatorname{gr} A_{2n}$ turns out to be a polynomial algebra on variables $x_i = s(p_i)$ and $y_i = s(q_i)$ for $i = 1, \ldots, 2n$; moreover, if $i = 1, \ldots, n$,

$$s(\phi(\partial p_i)) = -x_i + y_{i+n}, \qquad s(\phi(\partial q_i)) = -y_i + x_{i+n}.$$

It is clear that in this case (2) is satisfied, so that in turn the elements $\partial p_1, \ldots, \partial p_n$ and $\partial q_1, \ldots, \partial q_n$ satisfy (3).

We see that the left augmented algebra A^e with augmentation μ satisfies the hypotheses of Theorem VIII.4.2 of [5]; in particular, if we let $V = \bigoplus_{i=1}^n \mathbb{C}p_i \oplus \bigoplus_{i=1}^n \mathbb{C}q_i$, we have a projective resolution of A_n as a left A_n^e-module of the form $A_n^e \otimes \Lambda^\bullet V \twoheadrightarrow A_n$ with differentials $d\colon A_n^e \otimes \Lambda^p V \to A_n^e \otimes \Lambda^{p-1} V$ given by

$$d\big(a \otimes v_1 \wedge \cdots \wedge v_p\big) = \sum_{i=1}^{p} (-1)^{i+1} a\, \partial v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p.$$

There are, accordingly, natural isomorphisms

$$\operatorname{Ext}^\bullet_{A_n^e}(A_n, M) \cong H\big(\hom_{A_n^e}(A_n^e \otimes \Lambda^\bullet V, M)\big) \cong H(\hom(\Lambda^\bullet V, M))$$

for each left A_n^e-module M, where, in the last term, homology is computed with respect to differentials $d\colon \hom(\Lambda^d V, M) \to \hom(\Lambda^{d+1} V, M)$ such that

$$\begin{aligned} df\big(v_1 \wedge \cdots \wedge v_{p+1}\big) &= \sum_{i=1}^{p+1} (-1)^{i+1}\, \partial v_i f\big(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{p+1}\big) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \big[v_i, f\big(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{p+1}\big)\big] \end{aligned}$$

for each $f\colon \Lambda^d V \to M$.

3.3. Keeping the notations introduced in the previous paragraph, let $g \in \operatorname{Sp}(V)$, where we view V as a symplectic space in the usual way; there is a unique decomposition $V = V_1^g \oplus V_2^g$ preserved by g and such that $g|_{V_g^1} = Id_{V_1^g}$ and $Id_{V_2^g} - g|_{V_2^g} \in \mathrm{GL}(V_2^g)$. Let $d(g) = \dim V_2^g$. Whenever possible, we will suppress reference to the automorphism g in our notation.

Let $\omega \in (\Lambda^d V_2)^* \setminus 0$. The decomposition $V = V_1 \oplus V_2$ induces a decomposition

$$\Lambda^d V = \bigoplus_{p+q=d} \Lambda^p V_1 \otimes \Lambda^q V_2;$$

in particular, we see that $\Lambda^d V_2$ can be identified with a subspace of $\Lambda^d V$ and, in this identification, admits a natural complement; we extend ω to the whole of $\Lambda^d V$, prescribing it to be zero on this complement.

We define $\tilde{\omega}\colon \Lambda^d V \to A_n g$ by setting $\tilde{\omega}(v) = \omega(v) g$. We will show that $\tilde{\omega}$ represents a non-zero homology class of degree d in the complex $\hom(\Lambda^\bullet V, Ag)$ considered above.

Let us fix a basis $v_1, \ldots, v_{2n}$ of V in such a way that $v_1, \ldots, v_d \in V_2$ and $v_{d+1}, \ldots, v_{2n} \in V_1$ and choose indices $1 \leq i_1 < \cdots < i_{d+1} \leq 2n$; we have

$$\begin{aligned} d\tilde{\omega}&\left(v_{i_1} \wedge \cdots \wedge v_{i_{d+1}}\right) \\ &= \sum_{j=1}^{d+1} (-1)^{j+1}\left[v_{i_j}, \tilde{\omega}\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}\right)\right] \\ &= \sum_{j=1}^{d+1} (-1)^{j+1}\left[v_{i_j}, \omega\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}\right)\right]_g g. \end{aligned}$$

In the second equality, we see ω as taking values in A_n, and we write $[x, y]_g = xy - yg(x)$.

When $i_d > d$, $\omega(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}) = 0$ for every j such that $1 \leq j \leq d+1$, and this implies that, in this case, $d\tilde{\omega}(v_{i_1} \wedge \cdots \wedge v_{i_{d+1}}) = 0$. If, on the contrary, $i_d \leq d$, necessarily $i_j = j$ for each $1 \leq j \leq d$ and $\omega(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}) = 0$ if $1 \leq j \leq d$, so we have simply

$$d\tilde{\omega}\left(v_{i_1} \wedge \cdots \wedge v_{i_{d+1}}\right) = (-1)^d\left[v_{i_{d+1}}, \omega(v_1 \wedge \cdots \wedge v_d)\right]_g g.$$

As the values of ω are in the center of A_n and $v_{i_{d+1}} \in V_1$, this twisted commutator vanishes, and, again, we have $d\tilde{\omega}(v_{i_1} \wedge \cdots \wedge v_{i_{d+1}}) = 0$. Having considered each element in a basis of $\Lambda^{d+1}V$, we conclude that $d\tilde{\omega} = 0$; i.e., $\tilde{\omega}$ is a d-cocycle.

Suppose now that there is an $h \in \hom(\Lambda^{d-1}V, A)$ such that $d(hg) = \tilde{\omega}$. Then, writing $h_i = h(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_d) \in A_n$, we have

$$\sum_{i=1}^{d} (-1)^{i+1}[v_i, h_i]_g = \omega(v_1 \wedge \cdots \wedge v_d) \in \mathbb{C}1. \tag{4}$$

In particular, $[V_2, A_n]_g \cap \mathbb{C}1 \neq 0$.

For each $i = 1, \ldots, n$, let V^i be the subspace of V spanned by p_i and q_i, and let A^i be the subalgebra of A_n generated by p_i and q_i. Abusing a little of our notation, we see that $V = \bigoplus_{i=1}^n V^i$ and $A_n = \bigotimes_{i=1}^n A^i$. Suppose, without any loss of generality, that V_2 and V_1 are generated by p_i, q_i for $i = 1, \ldots, d$ and for $i = d+1, \ldots, n$, respectively, and that each V^i is preserved by g; let us write $g_i = g|_{V^i}$. We have

$$[V_2, A_n]_g = \sum_{i=1}^{d}\left[V^i, A_n\right]_g = \sum_{i=1}^{d} A^1 \otimes \cdots \otimes [V^i, A^i]_{g_i} \otimes \cdots \otimes A^n. \tag{5}$$

Theorem 4 in [2] states, among other things, that, for g an automorphism of A_1 different from the identity, $A_1 = \mathbb{C}1 \oplus [A_1, A_1]_g$. This implies that $A^i = \mathbb{C}1 \oplus [A^i, A^i]_{g_i}$ and, using (5), that $\mathbb{C}1$ and $[V_2, A_n]_g$ are transversal subspaces in A_n. This contradicts (4); we have thus proved that $\tilde{\omega}$ cannot be a coboundary in our complex.

3.4. In fact, this construction produces *all* d-cocycles with values in $\mathbb{C}g \subset A_n g$. To see this, let $\eta : \Lambda^d V \to A_n g$ be one such cocycle; let $v_1, \dots, v_{2n}$ be a basis of V as above with respect to which g acts diagonally. For each $1 \le i \le 2n$, let $\lambda_i \in \mathbb{C}$ be such that $gv_i = \lambda_i v_i$. If $1 \le i_1 < \cdots < i_{d+1} \le 2n$ and s satisfies $i_s \le d < i_{s+1}$, we have that

$$\begin{aligned}
0 &= d\eta\left(v_{i_1} \wedge \cdots \wedge v_{i_{d+1}}\right) \\
&= \sum_{j=1}^{d+1} (-1)^{j+1}\left[v_{i_j}, \eta\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}\right)\right]_g \\
&= \sum_{j=1}^{d+1} (-1)^{j+1}\left[v_{i_j}, \eta\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}\right)\right] \\
&\quad + \sum_{j=1}^{s} (-1)^{j+1}\left(1 - \lambda_{i_j}\right)\eta\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}\right)v_{i_j}.
\end{aligned}$$

Since the first sum in the last member is zero, we see that, for $1 \le j \le s$,

$$\eta\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_j} \wedge \cdots \wedge v_{i_{d+1}}\right) = 0.$$

If now $1 \le r_1 < \cdots < r_d \le 2n$ and $r_d > d$, there exists $t \in \{1, \dots, d\} \setminus \{r_1, \dots, r_d\}$. Let $1 \le i_1 < \cdots < i_{d+1} \le 2n$ be such that $\{i_1, \dots, i_{d+1}\} = \{r_1, \dots, r_d\} \cup \{t\}$ and suppose $i_f = t$. Our previous observation implies that

$$\eta\left(v_{r_1} \wedge \cdots \wedge v_{r_d}\right) = \eta\left(v_{i_1} \wedge \cdots \wedge \hat{v}_{i_f} \wedge \cdots \wedge v_{i_{d+1}}\right) = 0.$$

We see that $\eta(v_{r_1} \wedge \cdots \wedge v_{i_d})$ vanishes unless $r_d \le d$, and in this case $v_{r_j} = v_j$ if $1 \le j \le d$. It is clear now that η is one of the cocycles constructed the previous paragraph.

3.5. In our situation, and using the notation of the end of 3.3, we have an iterated product (cf. [5, XI.1])

$$\bigvee : \bigotimes_{i=1}^{n} \operatorname{Ext}^{\bullet}_{(A^i)^e}\left(A^i, A^i g_i\right) \to \operatorname{Ext}^{\bullet}_{A_n^e}(A_n, A_n g),$$

which is an isomorphism in view of Theorem XI.3.1 of [5]. On the other hand, we know from [4] that, for an algebra automorphism g of A_1,

$$\begin{aligned}
&\dim_{\mathbb{C}} \operatorname{Ext}^{\bullet}_{A_1^e}(A_1, A_1 g) \\
&\qquad = \begin{cases} 1, & \text{if } g = Id_{A_1} \text{ and } \bullet = 0 \text{ or if } g \ne Id_{A_1} \text{ and } \bullet = 2, \\ 0, & \text{in any other case.} \end{cases}
\end{aligned}$$

From these two facts, we easily deduce that $\operatorname{Ext}^{\bullet}_{A_n^e}(A_n, A_n g)$ is trivial except in degree d, where it is one-dimensional. Comparing dimensions, we see that the map $\omega \mapsto \tilde{\omega}$ is an isomorphism

$$\Lambda^d(V_2)^*[-d] \cong \operatorname{Ext}^{\bullet}_{A_n^e}(A_n, A_n g). \tag{6}$$

Here $M[-d]$ denotes the dth suspension of a graded space M.

3.6. Let $G \subset \operatorname{Sp}(V)$ be a finite subgroup. The natural action of G on V extends to a homogeneous action on the exterior algebra $\Lambda^{\bullet}V$, on one side, and, on the other, to an action by algebra automorphisms on A_n and, thence, on A_n^e. With respect to these actions, each module in the resolution $A_n^e \otimes \Lambda^{\bullet}V \twoheadrightarrow A_n$ is a G-module, and the differentials are G-equivariant.

If M is a left $A_n^e \rtimes G$-module, there is an homogeneous action of G on $\hom_{A_n^e}(A_n^e \otimes \Lambda^{\bullet}V, M)$, which is natural with respect to morphisms $M \to M'$ of $A_n^e \rtimes G$-modules and which, under the isomorphism of $\mathbb{C}$-spaces $\hom_{A_n^e}(A_n^e \otimes \Lambda^{\bullet}V, M) \cong \hom(\Lambda^{\bullet}V, M)$, corresponds to the usual diagonal action of G. Passing to homology, we obtain an action of G on $\operatorname{Ext}^{\bullet}_{A_n^e}(A_n, M) \cong H(\hom(\Lambda^{\bullet}V, M))$.

In particular, $\operatorname{Ext}^{\bullet}_{A_n^e}(A_n, A_n \rtimes G)$ is, in a natural way, a graded G-module. In view of the decomposition $A_n \rtimes G \cong \bigoplus_{g \in G} A_n g$ of $A_n \rtimes G$ as a left A_n^e-module and the considerations of the previous paragraph, we have an isomorphism

$$\operatorname{Ext}^{\bullet}_{A_n^e}(A_n, A_n \rtimes G) \cong \bigoplus_{g \in G} \left(\Lambda^{d(g)} V_2^g\right)^* g[-d(g)].$$

With respect to this isomorphism, the action of G can be described in the following way: let $g, h \in G$ and $\omega \in (\Lambda^{d(g)} V_2^g)^*$; left multiplication by h^{-1} induces an isomorphism $V_2^{hgh^{-1}} \to V_2^g$ which, in turn, determines an isomorphism $h^{\flat} : (\Lambda^{d(g)} V_2^g)^* \to (\Lambda^{d(hgh^{-1})} V_2^{hgh^{-1}})^*$. In this notation, we have

$$h(\omega g) = h^{\flat}(\omega) hgh^{-1}.$$

The verification of this claim reduces to a simple computation.

3.7. Since obviously $\operatorname{Tor}_+^{A_n}(A_n, A_n) = 0$, we know that

$$\begin{aligned} &A_n \otimes \Lambda^{\bullet}V \otimes A_n \otimes \Lambda^{\bullet}V \otimes A_n \\ &\qquad \cong \left(A_n^e \otimes \Lambda^{\bullet}V\right) \otimes_{A_n} \left(A_n^e \otimes \Lambda^{\bullet}V\right) \twoheadrightarrow A_n \otimes_{A_n} A_n \cong A_n \end{aligned}$$

is a projective resolution of A_n as a left A_n^e-module. There is a morphism of resolutions of A_n over 1_{A_n}, $\Delta : A_n \otimes \Lambda^{\bullet}V \otimes A_n \to A_n \otimes \Lambda^{\bullet}V \otimes A_n \otimes$

$\Lambda^\bullet V \otimes A_n$, given, in each degree d, by

$$\Delta(a \otimes v_1 \wedge \cdots \wedge v_d \otimes b) = \sum_{\substack{p+q=d \\ (i,j)\in S_{p,q}}} \varepsilon(i,j)\big(a \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \otimes 1 \otimes v_{j_1} \wedge \cdots \wedge v_{j_q} \otimes b\big),$$

if we let $S_{p,q}$ be the set of (p,q)-shuffles in the symmetric group S_{p+q} and if, for each such shuffle (i,j), $\varepsilon(i,j)$ is the signature of (i,j).

Given A_n^e-modules M and N, the product

$$\smile : \operatorname{Ext}^\bullet_{A_n^e}(A_n, M) \otimes \operatorname{Ext}^\bullet_{A_n^e}(A_n, N) \to \operatorname{Ext}^\bullet_{A_n^e}\big(A_n, M \otimes_{A_n} N\big) \quad (7)$$

is as explained in Section 2. Explicitly, under the usual identifications, if $\xi \in \hom(\Lambda^p V, M)$ and $\zeta \in \hom(\Lambda^q V, N)$, the product $\xi \smile \zeta \in \hom(\Lambda^{p+q} V, M \otimes_{A_n} N)$ is such that

$$(\xi \smile \zeta)\big(v_1 \wedge \cdots \wedge v_{p+q}\big) = \sum_{(i,j)\in S_{p,q}} \varepsilon(i,j)\,\xi\big(v_{i_1} \wedge \cdots \wedge v_{i_i}\big) \otimes \zeta\big(v_{j_1} \wedge \cdots \wedge v_{j_q}\big).$$

If there is a group G acting like in paragraph 3.6, from general principles or simply in view of this formula, we know that if M and N are $A_n^e \rtimes G$-modules, the product (7) is G-equivariant.

3.8. In the situation of paragraph 3.6, choose an arbitrary G-invariant inner product on V. It is easy to see that, for each $g \in G$, V_1^g and V_2^g are mutual orthogonal complements in V. If $g, h \in G$, we have

$$\big(V_2^g + V_2^h\big)^\perp = V_2^{g\,\perp} \cap V_2^{h\,\perp} = V_1^g \cap V_1^h \subset V_1^{gh}$$

so that

$$V_2^{gh} = V_1^{gh\,\perp} \subset \big(V_2^g + V_2^h\big)^{\perp\perp} = V_2^g + V_2^h. \quad (8)$$

3.9. There is an isomorphism $A_n g \otimes_{A_n} A_n h \cong A_n gh$ of A_n^e-modules, which we regard as an identification. Setting $M = A_n g$ and $N = A_n h$ in (7), we have a product map

$$\smile : \operatorname{Ext}^\bullet_{A_n^e}(A_n, A_n g) \otimes \operatorname{Ext}^\cdot_{A_n^e}(A_n, A_n h) \to \operatorname{Ext}^\bullet_{A_n^e}(A_n, A_n gh). \quad (9)$$

From degree considerations, we see that this is trivial unless $d(gh) = d(g) + d(h)$; if this is the case, (8) implies that $V_2^{gh} = V_2^g \oplus V_2^h$. Let $\omega \in (\Lambda^{d(g)} V_2^g)^*$ and $\phi \in (\Lambda^{d(h)} V_2^h)^*$ be nonzero forms, and consider a basis $v_1, \dots, v_{2n}$ of V such that $v_1, \dots, v_{d(g)}$ is a basis of V_2^g, $v_{d(g)+1}, \dots, v_{d(g)+d(h)}$

is a basis of V_2^h, and $v_{d(g)+d(h)+1}, \dots, v_{2n}$ is a basis of V_1^{gh}. Let $1 \le r_1 < \cdots < r_{d(g)+d(h)} \le 2n$ be arbitrary indices. If $r_{d(g)} > d(g)$ then

$$\begin{aligned}(\tilde{\omega} \smile \tilde{\phi})\big(v_{r_1} \wedge \cdots \wedge v_{r_{d(g)+d(h)}}\big) \\ = \sum_{(i,j)\in S_{d(g),d(h)}} \varepsilon(i,j)\,\tilde{\omega}\big(v_{r_{i_1}} \wedge \cdots \wedge v_{r_{i_{d(g)}}}\big)\tilde{\phi}\big(v_{r_{i_{d(g)+1}}} \wedge \cdots \wedge v_{r_{i_{d(g)+d(h)}}}\big)\end{aligned} \tag{10}$$

is zero because, for each $(i, j) \in S_{d(g),d(h)}$, $v_{r_{i_{d(g)}}} > d$, so the second factor in each term of the sum vanishes. If $r_{d(g)} \le d$ but $r_{d(g)+d(h)} > d(g) + d(h)$, similar reasoning shows that (10) is also zero.

We thus see that unless $r_i = i$ for each $1 \le i \le d(g) + d(h)$, $(\tilde{\omega} \smile \tilde{\phi})(v_{r_1} \wedge \cdots \wedge v_{r_{d(g)+d(h)}}) = 0$; that is, $\tilde{\omega} \smile \tilde{\phi}$ is one of the cocycles constructed in paragraph 3.3. It is not cohomologous to zero, because it is not zero on $\Lambda^{d(gh)} V_2^{gh}$, since

$$\begin{aligned}(\tilde{\omega} \smile \tilde{\phi})\big(v_1 \wedge \cdots \wedge v_{d(g)+d(h)}\big) \\ = \omega\big(v_1 \wedge \cdots \wedge v_{d(g)}\big)\phi\big(v_{d(g)+1} \wedge \cdots \wedge v_{d(g)+d(h)}\big) \neq 0.\end{aligned}$$

We conclude that (9) is either an isomorphism or zero, depending on whether $d(gh) = d(g) + d(h)$ or not.

3.10. We can choose a non-zero element $\omega_g \in (\Lambda^{d(g)} V_2^g)^*$ for each $g \in G$ in the following way. Let $\nu \in (\Lambda^2 V)^*$ be the symplectic form on V; since the action of G preserves ν, $\nu|_{\Lambda^2 V_2^g}$ is a symplectic form on V_2^g for each g. In particular, the $d(g)/2$th exterior power $\omega_g = (\nu|_{\Lambda^2 V_2^g})^{d(g)/2} \in (\Lambda^{d(g)} V_2^g)^* \setminus 0$—this makes sense because $d(g)$ is even because $g \in \mathrm{Sp}(V)$. It is clear that when $g, h \in G$ are such that $d(g) + d(h) = d(gh)$, $\tilde{\omega}_g \smile \tilde{\omega}_h = \tilde{\omega}_{gh}$ because $V_2^g \oplus V_2^h = V_2^{gh}$. Moreover, these elements are compatible with the action of G on $\mathrm{Ext}^\bullet_{A_n^e}(A_n, A_n \rtimes G)$, in the sense that $g\tilde{\omega}_h = \tilde{\omega}_{ghg^{-1}}$, because the action is symplectic.

We thus see that in terms of the basis $\{\omega_g\}_{g\in G}$ both the structure constants and the action of G become particularly pleasant.

3.11. Consider the filtration $F_\bullet \mathbb{C}G$ on $\mathbb{C}G$ such that $F_p \mathbb{C}G$ is spanned by the elements $g \in G$ such that $d(g) \le p$. Equation (8) implies that this is an algebra filtration on $\mathbb{C}G$.

It is clear from the previous paragraph that the map

$$\tilde{\omega}_g \in \mathrm{Ext}^\bullet_{A_n^e}(A_n, A_n \rtimes G) \mapsto s(g) \in \mathrm{gr}\,\mathbb{C}G$$

is an algebra isomorphism. The compatibility of the chosen basis of the domain of this map with the action of G tells us that this map is in fact

G-equivariant and, hence, that there is an isomorphism of graded algebras $\mathrm{Ext}^{\bullet}_{A_n^e}(A_n, A_n \rtimes G)^G \cong (\mathrm{gr}\,\mathbb{C}G)^G$.

3.12. Write $\mathscr{Z}G$ for the center of $\mathbb{C}G$, and consider for it the filtration induced by $F_{\bullet}\mathbb{C}G$. It is clear that $(\mathbb{C}G)^G = \mathscr{Z}G$, so that $\mathrm{gr}\,\mathscr{Z}G \subset (\mathrm{gr}\,\mathbb{C}G)^G$; and in fact this is an equality, because passing to the associated graded objects preserves the dimension. This proves Theorem 1.8.

4. SOME EXAMPLES

4.1. As mentioned in the Introduction, it is very easy to construct examples of the situation considered in our Theorem 1.8. Indeed, let G be a finite group and choose a faithful G-module V of degree n; G acts faithfully on the algebra of regular algebraic differential operators on V, which is isomorphic to A_n, so we can regard $G \subset \mathrm{Aut}(A_n)$. One sees that —in the notation of the theorem—$d(g)$ is simply two times the codimension of the subspace of V fixed by $g \in G$.

4.2. As we remarked in the Introduction, natural examples of this arise when one considers the action of the Weyl group corresponding to a Cartan subalgebra $\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{g}$ on the algebra of regular differential operators on the dual space $\mathfrak{h}^*$.

4.3. Let us write $C[G]$ for the algebra of $\mathbb{C}$-valued central functions on G. It is well-known (cf. [9]) that $C[G]$ is canonically endowed with the structure of a λ-ring with respect to which the Adams operations are given by $\psi^k(f)(g) = f(g^k)$ for $k \geq 0$, $f \in C[G]$, and $g \in G$.

Let t be a variable, and let $p \in C[G][\![t]\!]$ be the central function with polynomial values such that, for each $g \in G$, $p(g)$ is the characteristic polynomial of g in the representation V; define now $q \in C[G][\![t]\!]$ by setting, identically on G, $q(t) = t^n p(t^{-1})$. A simple computation shows that, if we let χ be the character of V, we have

$$\frac{d}{dt} \ln q(t) = -\sum_{k \geq 0} \psi^{k+1}(\chi) t^k.$$

It is clear that p and q have 1 as a zero of the same multiplicity, so that the function $d \in C[G]$ defined in Section 1.8 is given by

$$d = 2n + 2\,\mathrm{res}_1 \sum_{k \geq 0} \psi^{k+1}(\chi) t^k,$$

where we have written $\mathrm{res}_1 f$ for the residue at 1 of a function f meromorphic in a neighborhood of 1.

4.4. This equation implies that the numbers $\dim_{\mathbb{C}} HH^k(A_n \rtimes G)$ are determined by the character χ and the λ-ring structure on $C[G]$. Since we are working over a field of characteristic zero, this last structure is determined by the Adams operations. These, in turn, depend only on the power maps, i.e., the maps induced on the set of the conjugacy classes of G by exponentiation.

We thus see that if two groups G and G' are such that both their character tables and their power maps coincide (such pairs are shown to exist in [7]) and we choose corresponding faithful representations, which will of course have the same degree n, $HH^*(A_n^G)$ and $HH^*(A_n^{G'})$ will be isomorphic as graded vector spaces and, in fact, as algebras, since the matrix of structure constants of the center of a group algebra is determined by the character table.

4.5. We consider next in some detail the particular instance of Section 4.1 in which $G = S_n$, the symmetric group on n letters, acts on $V = \mathbb{C}^n$ by permutation of the canonical basis. This of course corresponds to the situation arising from the Weyl group action as in Section 4.2 in the case of Lie algebras of type A_n.

4.6. For each $n \geq 0$, let S_n be the symmetric group on $\{1, \dots, n\}$, and let $i_n \colon S_n \to S_{n+1}$ be the standard injection, under which S_n fixes $n+1$. Let $S_\infty = \lim S_n$ be the injective limit, the *restricted symmetric group* on an countable infinite number of letters.

4.7. A *partition* λ is a non-increasing sequence of non-negative integers $(\lambda_i)_{i \geq 1}$ which eventually vanish; let Π be the set of all partitions. If $\lambda \in \Pi$, let $l(\lambda)$ stand for the number of non-zero terms in λ and define the *weight* of λ to be $|\lambda| = \Sigma_{i \geq 1} \lambda_i$. Let Π_n be the set of partitions of weight n.

4.8. If $\pi \in S_n$, the *type* of π is the partition $\rho(\pi)$ listing the lengths of the cycles in a disjoint cycle decomposition of π; clearly, $|\rho(\pi)| = n$. If $\rho(\pi) = (r_1, \dots, r_l)$, $\rho(i_n(\pi)) = (r_1, \dots, r_l, 1)$, so that if we define the *stable type* of π to be the partition $\rho^\#(\pi) = (r_1 - 1, \dots, r_l - 1)$ we see that this is compatible with the injections i_n, and in consequence $\rho^\#$ is defined on S_∞. For $\lambda \in \Pi$, we set $C_\lambda = \{\pi \in S_\infty : \rho^\#(\pi) = \lambda\}$; it is easy to show that $\{C_\lambda\}_{\lambda \in \Pi}$ is precisely the decomposition of S_∞ into conjugacy classes.

4.9. For $n \geq 0$, let $\mathscr{Z}_n = \mathscr{Z}(\mathbb{Z}S_n)$, and if $\lambda \in \Pi$, $c_\lambda(n) = \Sigma_{g \in C_\lambda \cap S_n} g \in \mathscr{Z}_n$. Obviously we have $c_\lambda(n) = 0$ if $|\lambda| + l(\lambda) > n$.

4.10. If λ, $\mu \in \Pi$, there are integers $a_{\lambda\mu}^\nu(n)$, $\nu \in \Pi$, such that

$$c_\lambda(n) c_\mu(n) = \sum_{\nu \in \Pi} a_{\lambda\mu}^\nu(n) c_\nu(n),$$

with $a_{\lambda\mu}^\nu(n) = 0$ if $|\nu| > |\lambda| + |\mu|$. These numbers have great combinatorial interest, and explicit computations of specific cases can be found in the literature. In general, cf. [8], $a_{\lambda\mu}^\nu(n)$ depends polynomially on n and is actually independent of n when $|\nu| = |\lambda| + |\mu|$.

4.11. Let $\mathscr{B} \subset \mathbb{Q}[t]$ of polynomials which take integer values on integers, and let $\mathscr{Z}_\infty$ be the (possibly non-associative) $\mathscr{B}$-algebra which as a $\mathscr{B}$-module is free on the set $\{c_\lambda\}_{\lambda\in\Pi}$ and whose product is determined by

$$c_\lambda c_\mu = \sum_{\substack{\nu\in\Pi \\ |\nu|\le|\lambda|+|\mu|}} a^{\nu}_{\lambda\mu} c_\nu .$$

There are multiplicative $\mathbb{Z}$-linear morphisms $\mathscr{Z}_\infty \twoheadrightarrow \mathscr{Z}_n$ given by specialization $c_\lambda \mapsto c_\lambda(n)$ on the basis and by evaluation of polynomials at n on the coefficients, which collect to give a multiplicative map $\mathscr{Z}_\infty \to \Pi_{n\ge 0}\mathscr{Z}_n$, which turns out to be injective. This implies that $\mathscr{Z}_\infty$ is an associative algebra. Clearly, the kernel of $\mathscr{Z}_\infty \twoheadrightarrow \mathscr{Z}_n$ is generated by those c_λ such that $|\lambda| + l(\lambda) > n$ and the polynomials in $\mathscr{B}$ which vanish on n.

4.12. Let $F_n\mathscr{Z}_\infty$ be the $\mathscr{B}$-submodule spanned by the c_λ with $|\lambda| \le n$. We see at once that this defines an algebra filtration $F_\bullet\mathscr{Z}_\infty$ on $\mathscr{Z}_\infty$, and, in view of the last statement of paragraph 4.10, we can describe the associated graded algebra as follows: let $\mathscr{G}$ be the $\mathbb{Z}$-algebra with $\mathbb{Z}$-basis $\{c_\lambda\}_{\lambda\in\Pi}$ and product given by $c_\lambda c_\mu = \Sigma_{|\nu|=|\lambda|+|\mu|} a^{\nu}_{\lambda\mu} c_\nu$; then $\mathrm{gr}\mathscr{X}_\infty = \mathscr{B} \otimes_{\mathbb{Z}} \mathscr{G}$.

4.13. The epimorphisms $\mathscr{Z}_\infty \to \mathscr{Z}_n$ of paragraph 4.11 are compatible with the filtrations on the objects involved, so they give epimorphisms on associated graded objects; but it is easy to see that these are actually determined by epimorphisms $\mathscr{G} \twoheadrightarrow \mathrm{gr}\,\mathscr{Z}_n$, given by $c_\lambda \mapsto c_\lambda(n)$.

4.14. Let us write, for $\lambda = (r)$, $c_r = c_\lambda$. It is not difficult to show that the set $\{c_r\}_{r\ge 1}$ is algebraically independent in $\mathscr{G}$ and generates it rationally.

4.15. Let Λ be the ring of symmetric functions with integer coefficients on a countably infinite number of variables, and, for each $i \ge 0$, let h_i be the ith complete symmetric function, which is the sum of all monomials of total degree i. It turns out that $\Lambda = \mathbb{Z}[\{h_i\}_{i\ge 1}]$. If $\lambda = (r_1, \dots, r_l) \in \Pi$, we set $h_\lambda = h_{r_1} \cdots h_{r_l}$.

4.16. Define $u = \Sigma_{i\ge 0} h_i t^{i+1} \in \Lambda[[t]]$, and define elements $h_i^* \in \Lambda$ so that $t = \Sigma_{i\ge 0} h_i^* u^{i+1}$. Since the h_i freely generate Λ, we can define a ring morphism $\Psi : \Lambda \to \Lambda$ with $\Psi(h_i) = h_i^*$. Now one can verify from their definition that the h_i^* are also algebraically independent and generate Λ, so, in view of the symmetry of the construction, we see that Ψ is actually an involution. We extend the notation as in the previous paragraph to obtain a family $\{h_\lambda^*\}_{\lambda\in\Pi}$ indexed by all partitions which span Λ.

4.17. Let $\langle - | - \rangle$ be the bilinear form on Λ with values in $\mathbb{Z}$ such that $\langle h_\lambda | m_\mu \rangle = \delta_{\lambda\mu}$, where the m_μ are the monomial symmetric functions, obtained, for $\lambda = (r_1, \dots, r_l)$, by symmetrization from $x_1^{r_1} \cdots x_l^{r_l}$. Define functions $\{g_\lambda\}_{\lambda\in\Pi}$ such that $\langle g_\lambda, h_\mu^* \rangle = \delta_{\lambda\mu}$.

4.18. Now we can give a very concrete description of the ring $\mathscr{G}$ and, using the fact that the kernel of the epimorphisms $\mathscr{G} \twoheadrightarrow \mathrm{gr}\,\mathscr{Z}_n$ is easily identifiable, of the algebras $HH(A_n^{S_n})$. Indeed, it is a theorem proved in [10, Example I.7.25] that the map $\phi : \Lambda \to \mathscr{G}$ is such that $\phi(g_\lambda) = c_\lambda$ is a ring isomorphism. This allows us to do explicit computations in the algebras $HH(A_n^{S_n})$, by translating the problem into one involving symmetric polynomials.

4.19. For each $n \geq 0$, let RS_n be the representation ring of S_n; the sum $RS = \bigoplus_{n \geq 0} RS_n$ is a strictly commutative graded ring with product determined by its restrictions $RS_n \otimes RS_m \to RS_{n+m}$, given by $\chi \cdot \eta = \operatorname{ind}_{S_n \times S_m}^{S_{n+m}}(\chi \times \eta)$. There is a very natural isomorphism of rings Θ: $RS \to \Lambda$, essentially corresponding to taking the character of representations. It would be interesting to be able to explicitly relate elements and their products in $HH^{\bullet}(A_n^{S_n})$ to actual representations of the symmetric groups using the composition $\phi \circ \Theta$ of isomorphisms at hand. Unfortunately, one cannot expect to be able to restrict oneself to actual representations, since already $g_{(1)}$ corresponds under Θ to $-1 \in RS_1$, the opposite of the trivial representation of S_1, which, of course, is only a virtual representation.

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