Some notes on the Erdős–Szekeres theorem

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Received 25 October 1988
Revised 19 December 1989

Abstract

In the spirit of the Erdős–Szekeres theorem of 1935 we prove some canonical Ramsey Theorems in the plane. In Theorem 1 we are concerned with the number of points in the interior of a convex $n$-gon. In Theorem 2 we consider a set of points in general position along with a function from the set of points into the plane and prove the existence of a certain canonical configuration. A one-dimensional analog of Theorem 2 is a reformulation of a known theorem concerning intervals on the real line.

1. Introduction

In their classic paper [5] Erdős and Szekeres proved the following theorem.

**Theorem 0.** For every integer $n$, $n \geq 3$, there exists an integer $f(n)$ such that if $S$ is any set of $f(n)$ points in the plane, no three collinear, then there are $n$ points in $S$ which are the vertices of a convex $n$-gon.

The determination of $f(n)$ for $n \geq 6$ is still an open problem. Following the Erdős–Szekeres theorem it was conjectured that there exists a $g(n)$ such that if $S$ is any set of $g(n)$ points in the plane, no three collinear, then there are $n$ points of $S$ which are the vertices of a convex $n$-gon which does not contain any other points of the set in its interior. The conjecture was proved in [7] for $n = 3, 4, 5$, and was disproved in [8] for $n \geq 7$.

It is natural to look for a weaker conjecture. We conjecture the following.

**Conjecture.** For every two natural numbers $q$ and $n$, $n \geq 3$, there is a natural number $C(n, q)$ satisfying the following: Let $S$ be any set of points in the plane such that no three points are collinear. If $|S| \geq C(n, q)$, then there are points of $S$
which are the vertices of a convex $n$-gon for which the number of points of $S$ in its interior is divisible by $q$.

In this paper we follow the line of investigation initiated by Erdős and Szekeres and prove three theorems.

In Theorem 1 we prove the conjecture above for every $n$ and $q$ such that $n \equiv 2 \pmod{q}$ or $n \geq q + 3$.

In Theorem 2 we consider a set of points $S = \{x_1, x_2, \ldots, x_m\}$ in general position in the plane, $E^2$, along with a mapping $\alpha : S \to E^2$ such that $\alpha(x_j)$ for $j = 1, 2, \ldots, m$ does not lie on a line passing through two points of $S$. We prove the existence of a canonical subconfiguration of $S$, provided that the cardinality of $S$ is large enough. The idea of searching for canonical subconfigurations in combinatorial structures on which a function is imposed goes back to problems in combinatorial set theory considered by Erdős and Hajnal [3], see also [4]. However, we are motivated by more recent results of Alon and Caro [1] and Caro [2].

While presenting Theorem 2 in a seminar, we were asked by Prof. J. Schönheim about the one-dimensional analogue of Theorem 2. It turns out that the one-dimensional case of Theorem 2 is a reformulation of the following: Given $(n-1)^2 + 1$ intervals on a line, there are either $n$ pairwise disjoint or $n$ intersecting intervals.

2. Preliminaries

Notations and definitions. (1) Let $k$, $s$, $n_1, \ldots, n_s$ be positive integers. The Ramsey number $R = R_k(n_1, n_2, \ldots, n_s)$ denotes the minimum integer $R$ for which the following holds: Let $S$ be a set of cardinality $m$ each of whose $k$-subsets is colored by one of the $s$ colors $1, 2, \ldots, s$. If $m \geq R$, then there exists a color $i$, $1 \leq i \leq s$, and an $n_i$-subset of $S$ all of whose $k$-subsets are colored $i$.

(2) A set of points of $E^2$ is in general position if no three points of the set are collinear.

(3) Let $C$ be a convex polygon whose vertices are $x_1, x_2, \ldots, x_n$. The (?) line segments joining all pairs of vertices of $C$ determine a cell decomposition of $C$. Two convex polygons are said to be isomorphic if there is an incidence preserving one to one correspondence between the zero dimensional (points), one-dimensional (line segments), and two-dimensional cells. Note that this definition is a modification of the definition of isomorphic arrangements of lines given in [6, p. 4].

Observations. (1) Let $x_0, x_1, x_2$ be three points in general position. Up to isomorphism, in the sense of [6, p. 4], there is only one arrangement of the three lines determined by $x_0, x_1$ and $x_2$. This arrangement partitions into seven parts as shown in Fig. 1.
Some notes on the Erdős–Szekeres theorem

(2) Let $x_0, x_1, x_2, x_3$ be the vertices of a convex quadrilateral. If no two of the lines determined by $x_0, x_1, x_2, x_3$ are parallel, up to isomorphism, in the sense of [6, p. 4], then there is only one arrangement of these six lines as shown in Fig. 2.

(3) Up to isomorphism of convex polygons there is only one convex 5-gon and two convex 6-gons, one of which is the regular hexagon.

3. Results and proofs

In view of [7] the conjecture presented in the introduction is true for $n = 3, 4, 5$ and every $q$. The following theorem resolves the conjecture for certain $n$ and $q$. 
Theorem 1. Let $q$ and $n$ be two natural numbers such that either

1. $n \equiv 2 \pmod{q}$
2. $n \geq q + 3$

then the following holds. There is a number $C(n, q)$ such that if $S$ is any set of points of $E^2$ in general position and $|S| \geq C(n, q)$ then there are $n$ points of $S$ that are the vertices of a convex $n$-gon such that the number of points of $S$ contained in the convex $n$-gon is $0 \pmod{q}$.

Proof. (1) $n \equiv 2 \pmod{q}$. Let

$$C(n, q) = f(R_3(n, n, \ldots, n))^{q \text{ times}}$$

where $f$ is the Erdős–Szekeres function from Theorem 0. Let $S$ be a set of points in the plane as described in Theorem 1. By Theorem 0 there are $R_3(n, n, \ldots, n)$ points of $S$ which are the vertices of a convex $R_3(n, n, \ldots, n)$-gon $P$. Color the 3-subsets of the vertices of $P$ by the $q$ colors $0, 1, \ldots, q - 1$ corresponding to the number of points of $S$ modulo $q$ contained in the corresponding triangle. By the Ramsey Theorem there exists an $n$-subset $Q$ of the vertices of $P$ and an integer $k$, $0 \leq k \leq q - 1$, such that the number of points of $S$ in each triangle with vertices from $Q$ is $k \pmod{q}$. Since every $n$-gon can be partitioned into $n - 2$ triangles and $n - 2 = 0 \pmod{q}$ the result follows.

(2) $n \equiv q + 3$. Let $n_1$ and $n_2$ be integers such that $2 < n_1 < n < n_2$ and $n_i = 2 \pmod{q}$ for $i = 1, 2$. Let

$$C(n, q) = f(R_3(n_2, n_2, \ldots, n_2))^{q \text{ times}}$$

where $f$ is the Erdős–Szekeres function from Theorem 0. Let $S$ be a set of points in the plane as described in Theorem 1. As in the proof of part (1) we can assume that there exists an $n_2$-gon with vertices $x_1, x_2, \ldots, x_{n_2}$ (listed in a clockwise direction) and an integer $k$, $0 \leq k \leq q - 1$, such that the number of points of $S$ in each triangle $x_i x_j x_k$ where $1 \leq i, j, k \leq n_2$ is $k \pmod{q}$. If there is a triangle with vertices $x_i, x_j, x_k$ where $1 \leq i, j, k \leq n_2$ containing no points of $S$ then $k = 0$ and the theorem follows easily. Therefore we may assume that each such triangle contains at least one point of $S$. Let $n - n_1 = s$ and consider the $n_1$-gon with vertices $x_1, x_3, \ldots, x_2s-1, x_2s+1, x_2s+2, x_2s+3, \ldots, x_{s+n_1-1}, x_{s+n_1}$. (From $x_1$ to $x_{2s+1}$ every other vertex appears; thereafter every vertex appears.) Note that since $n_1 = 2 \pmod{q}$, the number of points of $S$ in the $n_1$-gon is $0 \pmod{q}$. Consider the following $s$ points $y_1, y_2, \ldots, y_s$ where $y_i$ is in the triangle $x_{2i-1} x_{2i} x_{2i+1}$ and has the property that the triangle $x_{2i-1} y_i x_{2i+1}$ contains no points of $S$. (The point $y_i$ may be selected as the point of $S$ in the triangle $x_{2i-1} x_{2i} x_{2i+1}$ closest to the line segment $x_{2i-1} x_{2i+1}$.) Now the $n$-gon

$$x_1 y_1 x_3 y_2 x_5 y_3 x_7 \cdots x_{2s-1} y_s x_{2s+1} x_{2s+2} \cdots x_{s+n_1}$$

is convex and contains a number of points of $S$ congruent to zero modulo $q$. 
Remarks. (1) For the case $n = 2 \pmod{q}$ we have proved a somewhat stronger result: If $n$ and $q$ are as above and $S$ is any set of points of $E^2$ in general position and if $T$ is any subset of $S$ such that $T$ is the set of vertices of a convex polygon then, provided $|T|$ is sufficiently large, there exists a convex $n$-gon $P$ with vertices from $T$ such that the number of points of $S$ in $P$ is $0 \pmod{q}$.

(2) For every integer $t$, $t \geq 3$ it is possible to locate $t - 2$ points in a convex $t$-gon in such a way that each triangle formed on the vertices of the $t$-gon contains exactly one point. Hence every $n$-gon formed on the vertices of the $t$-gon contains exactly $n - 2$ points. Consequently the stronger result mentioned in Remark (1) cannot be extended for $n \equiv 2 \pmod{q}$. The example alluded to is implicit in Theorem 2 and its proof.

(3) It was verified by computer that if $S$ is any set of points of $E^2$ in general position and if $T$ is any $6$-subset of $S$ such that $T$ is the set of vertices of a convex hexagon then there is a convex quadrilateral, say $P$, with vertices from $T$, such that the number of points of $S$ in $P$ is even. Thus the number $R_3(4, 4)$ in the proof of Theorem 1 can be replaced by $6$ for the case $n = 4$, $q = 2$.

Theorem 2. For every natural number $n$, $n \geq 5$, there is a natural number $A(n)$ satisfying the following property: Let $S = \{x_1, x_2, \ldots, x_m\}$ be a set of points of $E^2$ in general position and let $\alpha : S \to E^2$ be a mapping such that $\alpha(x_j)$ for $j = 1, 2, \ldots, m$, does not lie on a line passing through any two points of $S$. If $m \geq A(n)$, then there are $n$ points of the set $S$, say $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ which are the vertices of a convex $n$-gon $C$ with edges $x_{i_k}x_{i_{k+1}}$ ($x_{i_{n+1}} = x_{i_1}$). Moreover, one of the following holds:

(i) For $j = 1, 2, \ldots, n$ $\alpha(x_j)$ lies in the exterior of $C$.

(ii) There exists an integer $j$, $1 \leq j \leq n$, such that $\alpha(x_j)$ and $\alpha(x_{j+1})$ lie in the exterior of $C$ and $\alpha(x_k)$ for $k \neq j, j + 1$, lies in the triangle determined by the three lines $x_{i_k}x_{i_k+1}$, $x_{i_k}x_{i_j}$, $x_{i_{j+1}}x_{i_{j+1}}$.

Proof. Let $A(n) = f(R(n))$ where

$$R(n) = R_3(n, n, 5, 5, 5, 4, \ldots, 4)$$

and $f$ is the Erdős–Szekeres function from Theorem 0. Let $S = \{x_1, x_2, \ldots, x_m\}$ and let $\alpha : S \to E^2$ be as described in Theorem 2. If $m \geq f(R(n))$, then by Theorem 0 there are $R(n)$ points, say $y_1, y_2, \ldots, y_{R(n)}$, which are the vertices of a convex $R(n)$-gon. Color all the 3-subsets of $\{y_1, y_2, \ldots, y_{R(n)}\}$ by 39 colors named $0, 1(1,1), 1(1,2), \ldots, 1(6,6), 2, 3$, as follows. The set $\{y_r, y_s, y_t\}$ is colored by 0, 2, or 3 corresponding to how many of the points $\alpha(y_r), \alpha(y_s), \alpha(y_t)$ lie in the interior of the triangle $y_r y_s y_t$. If exactly one of the points $\alpha(y_r), \alpha(y_s), \alpha(y_t)$ lies in the interior of the triangle $y_r y_s y_t$, then according to Fig. 1 there are 36 possibilities for mapping the remaining two points into the six exterior regions determined by the three lines $y_r y_s, y_s y_t$ and $y_t y_r$. Let $y_r, y_s, y_t$ be a clockwise
ordering of the vertices of the triangle $y_r y_s y_t$. Suppose that $\alpha(y_r)$ lies in the interior of triangle $y_s y_t y_r$ then let $y_r$ correspond to $x_0$ of Fig. 1 and color \{y_r, y_s, y_t\} by $1_{(u,v)}$, $1 \leq u$, $v \leq 6$, if $\alpha(y_r)$ is in region $u$ and $\alpha(y_t)$ is in region $v$.

Because a convex 4-gon can be partitioned by a diagonal into two disjoint triangles, it is impossible that the interior of each triangle will contain the images of three vertices of the 4-gon. Therefore there is no 4-subset of \{y_1, y_2, \ldots, y_{R(n)}\} all whose 3-subsets are color 3. Similarly, since a convex 5-gon can be partitioned by two diagonals into three disjoint triangles it is impossible that the interior of each triangle will contain the images of two vertices of the 5-gon. Thus, there is no 5-subset all whose 3-subsets are color 2.

In Fig. 2 we illustrate the 18 possible regions into which the plane is divided by the lines arising from the set of four vertices of a convex quadrilateral. A computer was used to find all mappings of the vertices of a convex quadrilateral into the 18 regions which result in a monochromatic coloring of all 3-subsets of the vertices using one of the colors $1_{(u,v)}$, $1 \leq u$, $v \leq 6$. The results obtained are presented in Table 1.

A straightforward manual check eliminates the possibility of having a 5-set all of whose 3-subsets are colored by one of the colors $1_{(1,1)}$, $1_{(4,1)}$, $1_{(5,1)}$, $1_{(3,3)}$, $1_{(5,5)}$, $1_{(3,6)}$ leaving only the four mappings associated with the color $1_{(5,5)}$. It turns out that only the eleven interior regions determined by the diagonals of the convex 5-gon are needed in the above elimination. A further manual check shows that if all 3-subsets of the vertices of a convex 5-gon are colored $1_{(5,5)}$ then the vertices of the 5-gon can be relabeled so that the 5-gon along with the images of its vertices under $\alpha$ is as shown in Fig. 3.

Let $n \geq 5$. It follows that there is a subset of \{y_1, y_2, \ldots, y_{R(n)}\}, say \{x_1, x_2, \ldots, x_n\} all of whose 3-subsets are colored either 0 or $1_{(5,5)}$. If all the 3-subsets are colored 0, then $\alpha(x_i)$ for $j = 1, 2, \ldots, n$ lies in the exterior of the

### Table 1

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polygon determined by $x_i, \ldots, x_k$. Otherwise all the 3-subsets are colored $l_{(5,5)}$ and hence all 5-subsets and their images under $\alpha$ are as shown in Fig. 3. Now consider a convex $n$-gon all of whose 3-subsets are colored $l_{(5,5)}$. Since a convex $n$-gon can be partitioned into $n - 2$ disjoint triangles and since each triangle contains exactly one image point of a vertex, there are at least $n - 2$ image points of vertices in the interior of the convex $n$-gon. Furthermore, if a triangle $z_1z_2z_3$ contains the images $\alpha(z_1)$ and $\alpha(z_3)$, where $z_1$, $z_2$, $z_3$, and $z_4$ are vertices of the convex 4-gon, then consideration of the triangle $z_2z_3z_4$ leads easily to a contradiction. Hence exactly two vertices of the convex $n$-gon are mapped to the exterior of the $n$-gon. In view of the configuration illustrated in Fig. 3, the two points mapped to the exterior of the convex $n$-gon must form an edge, say $x_{ik}x_{j}$. Considering all the 5-gons $x_i x_{j} \ldots x_k x_{i}$, where $j = k + 2, \ldots, k - 3$ (addition is mod $n$), we get the configuration described in part (ii) of the conclusion of Theorem 2 and the proof of the theorem is complete.

Remark. Since every convex $n$-gon with a mapping satisfying (1) or (2) contains a $k$-gon satisfying (1) or (2) respectively for every $k$, $3 \leq k \leq n$ the assumption $n \geq 5$ in the statement of the theorem can be relaxed to $n \geq 3$.

The one-dimensional analog of Theorem 2 is the following.

Observation. Let $x_1 < x_2 < \cdots < x_s$ be $s$ real numbers and let $f : \{x_1, x_2, \ldots, x_s\} \to \mathbb{R} \setminus \{x_1, x_2, \ldots, x_s\}$. If $s \geq 2(n - 1)^2 + 1$, then there are $n$ points among $x_1, x_2, \ldots, x_s$, say, $x_{i_1} = y_1 < x_{i_2} = y_2 < \cdots < x_{i_n} = y_n$ such that one
of the following holds:

(i) \( y_i < f(y_i) \) for \( i = 1, 2, \ldots, n \) and \( f(y_i) < y_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \),

(ii) \( f(y_i) > y_i \) for \( i = 1, 2, \ldots, n \),

(iii) \( y_i > f(y_i) \) for \( i = 1, 2, \ldots, n \) and \( f(y_i) > y_{i-1} \) for \( i = 2, 3, \ldots, n \),

(iv) \( f(y_i) < y_i \) for \( i = 1, 2, \ldots, n \).

This observation is almost a reformulation of the following theorem.

**Theorem 3.** Given \((n - 1)^2 + 1\) intervals on a line, there are either \(n\) pairwise disjoint or \(n\) intersecting intervals.

In turn, Theorem 3 is an easy consequence of Dilworth's theorem along with Helly's theorem in dimension one.

**References**


