A theorem on upper–lower solutions for nonlinear elliptic systems and its applications

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Abstract

We report on a result of upper–lower solutions for nonlinear elliptic systems without the assumption of quasi-monotonicity. An application is described involving the existence of positive steady states of a certain interaction system arising in biology and medical sciences.

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1. Introduction

The method of upper and lower solutions has been used as one of the effective mathematical tools in studies of nonlinear partial differential equations (PDEs) under various boundary conditions. Such technique has been applied to many kind of nonlinear PDEs including coupled systems of parabolic and elliptic boundary-value problems by suitable construction of upper–lower solutions.

Today, the demand for more research studies addressing the general PDE problems with strong nonlinearities, such as nonlinear non-homogeneous systems under nonlinear boundary conditions without the quasi-monotone property, is increasing with the increase in mathematical models that are arising in various applied sciences. These models have strong nonlinearity such as non-quasi-monotonicities.

However, the classical upper–lower solution method is not applicable to such nonlinear PDE problems since the method does depend on a monotone iteration argument \([6,11,12]\). For parabolic systems with homogeneous boundary conditions, we refere readers to \([9]\).

In this article, we extend the classical result of an upper–lower solution technique to nonlinear non-homogeneous elliptic systems without the assumption of quasi-monotonicity under nonlinear boundary conditions. The method
employed to prove our main result is Schauder’s fixed point theorem. In addition, we introduce one of the models
emerges in biology and medical science. We obtain the result for the positive coexistence of a given model using the
extended upper–lower solution technique we developed and describe in this paper.

This paper is organized as follows. We provide the main theorem for nonlinear elliptic PDEs without quasi-
monotonicity in Section 2. In Section 3, we describe a model that deals with interaction among immune cells and
a virus. In Section 3, we present our investigation of the sufficient conditions for the existence of positive solutions of
time-independent systems. We also give a brief biological interpretation for our result.

2. A theorem on upper–lower solutions

In this section, we extend the result for classical upper–lower solutions to nonlinear non-homogeneous elliptic
systems without the assumption of quasi-monotonicity under nonlinear boundary conditions.

For the convenience of the reader, we state below a definition of upper and lower solutions of PDE systems.

The set of functions \( u_1 \geq u_j \geq u_m \) in the ball of radius \( K \) centered at origin in \( \mathbb{R}^m \)

Let \( \Omega \) be a smooth and bounded domain in \( \mathbb{R}^n \) and \( x = (x_1, x_2, \ldots, x_n) \in \Omega \). A function \( f(\xi_1, \ldots, \xi_m, x) \) is said
to be uniformly Lipschitzian independent of \( x \) for bounded \( (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \) if \( f \) is uniformly Lipschitzian in any bounded subset of \( (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \) and the Lipschitz constant \( P \) is independent of \( x \in \partial \Omega \). Let us consider the elliptic system

\[
\begin{align*}
-d_i \Delta u_i &= f_i(u_1, \ldots, u_m, x) \quad \text{in } \Omega, \quad i = 1, \ldots, m, \\
\alpha_i u_i + \beta_i \frac{\partial u_i}{\partial n} &= \psi_i(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m, x) \quad \text{on } \partial \Omega,
\end{align*}
\]

where the functions \( f_i \) are uniformly Lipschitzian independent of \( x \in \overline{\Omega} \), \( \psi_i \in C^2(\mathbb{R}^{m-1} \oplus \mathbb{R}^n) \), and \( \alpha_i, \beta_i \) in the
Hölder space \( C^{1+\alpha}(\overline{\Omega}) \) with \( \alpha_i + \beta_i > 0 \).

**Definition 1.** The set of functions \( \overline{u}_i \geq u_j \in C^{2+\alpha}(\overline{\Omega}) \), where \( \alpha \in (0, 1) \) are said to be coupled upper–lower solutions to
the elliptic system (1) if

\[
\begin{align*}
-d_i \Delta \overline{u}_i(x) &\geq \sup_{u_j \leq \xi_j \leq \overline{u}_j, j \neq i} f_i(\xi_1, \ldots, \xi_{i-1}, \overline{u}_i, \xi_{i+1}, \ldots, \xi_m, x) := \overline{f}_i \quad \text{in } \Omega, \\
\alpha_i \overline{u}_i + \beta_i \frac{\partial \overline{u}_i}{\partial n} &\geq \sup_{u_j \leq \xi_j \leq \overline{u}_j, j \neq i} \psi_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m, x) \quad \text{on } \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
-d_i \Delta u_j(x) &\leq \inf_{u_j \leq \xi_j \leq \overline{u}_j, j \neq i} f_i(\xi_1, \ldots, \xi_{i-1}, u_j, \xi_{i+1}, \ldots, \xi_m, x) := \underline{f}_i \quad \text{in } \Omega, \\
\alpha_i u_j + \beta_i \frac{\partial u_j}{\partial n} &\leq \inf_{u_j \leq \xi_j \leq \overline{u}_j, j \neq i} \psi_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m, x) \quad \text{on } \partial \Omega.
\end{align*}
\]

In the following theorem, no semi-quasimonotonicity is assumed, and the boundary interactions can be nonlinear. This
theorem thus generalizes the results of [6,11,12]. We apply Schauder’s fixed point theorem to prove this result.

**Theorem 1.** Let \( f_i \) in system (1) be uniformly Lipschitzian independent of \( x \), \( \psi_i \in C^2(\mathbb{R}^{m-1}) \) as above. If \( u_j \leq \overline{u}_i \) \((i = 1, \ldots, m)\), are coupled upper–lower solutions of the elliptic system (1), then there exists a solution \( u = (u_1, \ldots, u_m) \) to the system (1) such that \( u_i \leq u_j \leq \overline{u}_i \).

**Proof.** Let \( M \geq \max(\|u_i\|_\infty, \|\overline{u}_i\|_\infty, i = 1, \ldots, m) \).

Let \( P_i > 0 \) be the Lipschitz constant of the function \( f_i \). Denote \( \mathcal{K} = \{ \theta = (\theta_1, \ldots, \theta_m) \in \bigoplus_{i=1}^m C^1(\overline{\Omega}) : \theta_i \leq \theta_i \leq \overline{\theta}_i, \alpha_i u_j + \beta_i \frac{\partial u_j}{\partial n} \leq \alpha_i \theta_i + \beta_i \frac{\partial \theta_i}{\partial n}, \|\theta_i\|_\infty \leq M, \|\theta_i\|_{C^1(\overline{\Omega})} \leq M_i, i = 1, \ldots, m \} \) where the constant \( M \) will be specified below. Since \( (u_1, \ldots, u_m) \) and \( (\overline{u}_1, \ldots, \overline{u}_m) \) are in \( \mathcal{K} \), \( \mathcal{K} \neq \emptyset \). As usual, the norm in the direct sum space is defined as \( \|\theta\| := \sum_{i=1}^m \|\theta_i\| \).
For a given $\theta = (\theta_1, \ldots, \theta_m) \in \bigoplus_{i=1}^m C^{1+\alpha}(\overline{\Omega})$, consider the following equations for $i = 1, 2, \ldots, m$:

\[-d_i \Delta v_i + P_i v_i = P_i \theta_i + f_i(\theta_1, \ldots, \theta_m, x) \text{ in } \Omega,\]

\[\alpha_i v_i + \beta_i \frac{\partial v_i}{\partial n} = \psi_i(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_m, x) \text{ on } \partial \Omega.\]  

(2)

There exist [5, §2, Ch. 10] solutions $v_i \in C^{1+\alpha}(\overline{\Omega})$ and a constant $C_1 > 0$, which is independent of $(\theta_1, \ldots, \theta_m, x)$, such that

\[\|v_i\|_{C^{1,\alpha}} \leq C_1 \|\theta\|_{\infty} = C_1 \sum_{i=1}^m \|\theta_i\|_{\infty} \leq C_1 m M.\]  

(3)

Let $M = \max(C_1 m M, \|u\|_{C^{1,\alpha}}, \|u\|_{C^{1,\alpha}}, i = 1, \ldots, m)$.

Claim. The solution $v = (v_1, \ldots, v_m) \in K$. It suffices to show that $u_i \leq v_i \leq \bar{u}_i$. To see this, let $(\theta_1, \ldots, \theta_m) \in K$ be given, by Definition 1 it implies

\[-d_i \Delta \bar{u}_i + P_i \bar{u}_i \geq P_i v_i + f_i(\theta_1, \ldots, \theta_{i-1}, \bar{u}_i, \ldots, \theta_m, x) \text{ in } \Omega,\]

\[\alpha_i \bar{u}_i + \beta_i \frac{\partial \bar{u}_i}{\partial n} \geq \psi_i(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_m, x) \text{ on } \partial \Omega.\]

Subtracting Eq. (2) from this inequality, it follows that

\[-d_i \Delta (\bar{u}_i - v_i) + P_i (\bar{u}_i - v_i) \geq P_i (\bar{u}_i - v_i) + \left[f_i(\theta_1, \ldots, \theta_{i-1}, \bar{u}_i, \ldots, \theta_m, x) - f_i(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_m, x)\right] \geq 0 \text{ in } \Omega,\]

\[\alpha_i (\bar{u}_i - v_i) + \beta_i \frac{\partial (\bar{u}_i - v_i)}{\partial n} \geq 0 \text{ on } \partial \Omega.\]

In the above inequality we have made use of the facts that $(\theta_1, \ldots, \theta_m) \in K$ and that $P_i$ is the Lipschitzian constant of the function $f_i$. From the strong maximum principle together with the Hopf’s lemma, we then concludes $\bar{u}_i \geq v_i$.

Similarly, $u_i \leq v_i$ and $\alpha_i u_i + \beta_i \frac{\partial u_i}{\partial n} \leq \alpha_i v_i + \beta_i \frac{\partial v_i}{\partial n} \leq \alpha_i \bar{u}_i + \beta_i \frac{\partial \bar{u}_i}{\partial n}$, thus, $v = (v_1, \ldots, v_m) \in K$.

Since $\theta_i \in C^{1+\alpha}(\overline{\Omega})$, $P_i > 0$, the usual regularity of elliptic equations (Theorem 6.31 and its remarks in [4]) tells us that there exist solutions $v_i$ to Eq. (2), $v_i \in C^{2+\alpha}(\overline{\Omega})$ and that there exists a constant $C_2 > 0$ such that $\|v_i\|_{C^{2,\alpha}} \leq C_2 M$. We have thus shown that the solutions $v = (v_1, \ldots, v_m)$ satisfy following properties:

\[V(1) \quad v = (v_1, \ldots, v_m) \in K \cap \bigoplus_{i=1}^m C^{2+\alpha}(\overline{\Omega}),\]

\[V(2) \quad \|v_i\|_{C^{1,\alpha}} \leq C_1 m M \leq \overline{M},\]

\[V(3) \quad \|v\|_{C^{2,\alpha}} \leq C_2 \overline{M}.\]

It is clear that $K$ is convex and bounded. Consider now $K \subset \mathcal{X} := \bigoplus_{i=1}^m C(\overline{\Omega})$. Let $\overline{K}$ be the closure of $K$ in $\mathcal{X}$. Then $\overline{K}$ is bounded, closed and convex in $\mathcal{X}$. For a given $\theta := (\theta_1, \ldots, \theta_m) \in K$, denote the solution $v = (v_1, \ldots, v_m)$ of Eq. (2) with $v = T\theta$. We proved in above that $T(K) \subset K$.

By the properties $V(1)$, $V(3)$ and Arzelà-Ascoli theorem, the operator $TK \to K$ is compact. Let $\theta = (\theta_1, \ldots, \theta_m) \in K$ and $\theta^{(n)} = (\theta^{(n)}_1, \ldots, \theta^{(n)}_m) \in K$ that are converging to $\theta$. Then $\|\theta^{(n)} - \theta\|_{\infty} \to 0$. Let $v^{(n)} = T\theta^{(n)} \in K$. By the compactness of $T$, $v^{(n)} \in K$ has a convergent (sub)sequence in $\bigoplus_{i=1}^m C^{1+\alpha}(\overline{\Omega})$, $v^{(n)} \to v = (v_1, \ldots, v_n) \in \bigoplus_{i=1}^m C^{1+\alpha}(\overline{\Omega})$. One can show, in the same way as above, that $v \in K$. Therefore, we have extended the mapping $T$ to $\overline{K}$ by letting $T\theta = v$, for a given $\theta \in \overline{K}$. By the estimates (3), it is routine to verify that $v$ does not depend on the choice of the particular Cauchy sequence $\theta^{(n)}$ for $\theta \in \overline{K}$. We thus have

\[V(4) \quad v = T(\theta) \in K \quad \text{for } \theta \in \overline{K}.\]

Using the estimates (3) it is also routine to check that $T$ is continuous in $\overline{K} \subset \bigoplus_{i=1}^m C(\overline{\Omega})$. Let $\theta^{(n)} \in \overline{K}$, $j = 1, 2, \ldots$, be given and $v^{(n)} = T\theta^{(j)} \in K$. The direct sum norm $\|v^{(n)}\|_{C^{1,\alpha}} \leq m \overline{M}$ by the definition of $K$. Hence $\{v^{(n)}\}$ has
a Cauchy subsequence in $K \subset \bigoplus_{i=1}^{m} C(\Omega)$ again by Arzelá–Ascoli theorem. Therefore, $T : K \rightarrow K$ is a compact mapping. It follows from the Schauder fixed point theorem that there exists a function $u \in K$ such that $u = Tu$. By the regularity and $V(1), u \in \bigoplus_{i=1}^{m} C^{2+\alpha}(\Omega)$. Then it is easy to see that $u$ is the desired solution. \hfill \Box

3. An application

The interaction between HIV and the human immune system can be briefly described as follows. The HIV virus can deplete and cripple the immune T4 cells (CD4$^+$T cells) which perform the job of organizing some of the other immune cells to fight against the virus. These cells, especially, activate macrophages and T8 cells which can kill and suppress the virus as well as infected cells. The population of macrophages, in one respect, contributes to activation and proliferation of T cells. On the other hand, this population is susceptible to HIV infection and, therefore, as long-lived cells, serves as a major reservoir of HIV. The interactions among immune cells with HIV is thus formed by a set of immunology chains. We refer readers with interest in more details to [2,3,7,10,13]. We quote following citation from [13]: “Achieving immune control without the need for chronic anti-HIV medications is clearly a goal of enormous value.” Based on the immunological facts and results given by these papers, Based on the facts and results given by these papers’ research studies, the following six by six elliptic system is proposed as a model of human immune cells fighting with the HIV on a domain $\Omega$, without the administration of antiviral medications.

\[-d_i \Delta u_i = f_i(u_1,u_2,\ldots,u_6) \text{ in } \Omega, \quad i = 1, 2, \ldots, 6, \quad \text{(4)}\]

where $u_i := u_i(x)$ denote the densities represented, respectively,

\[u_1: \text{ uninfected CD4}^+\text{T cells,} \quad u_2: \text{cytotoxic T8 cells},\]

\[u_3: \text{uninfected macrophages,} \quad u_4: \text{(HIV) virus},\]

\[u_5: \text{uninfected stem cells,} \quad u_6: \text{infected stem cells}.\]

Here $\Omega$ is a non-specified domain which can be lymph nodes or certain organ tissues where T cells are undergoing a so-called homing, that is, roughly, the inward pointing flux of T cells across the boundary outpaces the outward flux. The stem cells cited above serve as the source that generates T cells and macrophages.

The diffusion rates $d_i > 0$ are constants and the reaction functions $f_i$ ($i = 1, 2, \ldots, 6$), which represent the immune interactions are

\[f_1(u_1,u_2,u_3,u_4,u_5,u_6) = a_1 u_5 - a_2 u_1 - a_3 \min(u_1,u_4) + a_4 \frac{u_1 u_3}{1 + u_1 u_3},\]

\[f_2(u_1,u_2,u_3,u_4,u_5,u_6) = a_5 u_5 - a_6 u_2 + a_7 \phi_1(u_1) \frac{u_2 u_3}{1 + u_2 u_3},\]

\[f_3(u_1,u_2,u_3,u_4,u_5,u_6) = a_8 u_5 - a_9 \min(u_3,u_4) + a_{10} \phi_1(u_1) u_3 - a_{11} u_3,\]

\[f_4(u_1,u_2,u_3,u_4,u_5,u_6) = \phi_4(u_1,u_3,u_4) + a_{12} u_6 - a_{13} \phi_2(u_1) u_2 u_4 - a_{14} u_4,\]

\[f_5(u_1,u_2,u_3,u_4,u_5,u_6) = a_{15} u_5 (c - u_5) - a_{16} \phi_3(u_1,u_4) u_5 + a_{17} \frac{u_1 u_5}{1 + u_1 u_5},\]

\[f_6(u_1,u_2,u_3,u_4,u_5,u_6) = -a_{18} u_6 + a_{16} \phi_3(u_1,u_4) u_5 - a_{19} u_2 u_6, \quad \text{(5)}\]

where the immune activation threshold functions

\[\phi_1(u_1) = \begin{cases} 
\frac{u_1}{a}, & \text{for } 0 \leq u_1 < a, \\
1, & \text{for } u_1 \geq a,
\end{cases}\]

\[\phi_2(u_1) = \begin{cases} 
\frac{u_1}{b}, & \text{for } 0 \leq u_1 < b, \\
1, & \text{for } u_1 \geq b,
\end{cases}\]

\[\phi_3(u_1,u_4) = \begin{cases} 
\frac{e^{u_1} - u_4}{e - 1} \frac{u_4}{1 + u_4}, & \text{for } 0 \leq u_1 \leq e, \\
0, & \text{for } u_1 > e,
\end{cases}\]

\[\phi_4(u_1,u_3,u_4) := a_{20} u_4 \min(u_4,u_1 + u_3) = \begin{cases} 
a_{20} u_4^2, & \text{for } 0 \leq u_4 < (u_1 + u_3), \\
a_{20} u_4 (u_1 + u_3), & \text{for } u_4 \geq (u_1 + u_3).\end{cases} \quad \text{(6)}\]
In above expressions the rates \( a_4, a_7, a_{10}, a_{12} \geq 0 \) while the remainder rates \( a_i \) and parameters \( a, b, c, e \) are all positive constants.

The boundary operator \( \mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_6) \) is defined as follows:

\[
\begin{align*}
(\mathcal{B}_6)u_6 &:= \frac{\partial u_6}{\partial n} = 0, \\
(\mathcal{B}_i)u_i &:= \theta_j(x)u_j + \eta_j(x)\frac{\partial u_j}{\partial n} = 0, \quad j = 1, 2, 3, 5, \\
(\mathcal{B}_4)u_4 &:= \frac{\partial u_4}{\partial n} + \alpha(x)u_4 = \beta(x)(u_1 + u_3), \quad \alpha(x) \geq 0, \beta(x) \geq 0 \text{ on } \partial \Omega.
\end{align*}
\]

We assume the functions \( \theta_j, \eta_j, \alpha, \beta \in C^{1+\alpha}(\overline{\Omega}) \). We also assume that \( \theta_j(x) \geq 0, \eta_j(x) \geq 0, \theta_j + \eta_j > 0 \) \( (j = 1, 2, 3, 5) \), and that the functions \( \alpha, \beta \) can be extended to the \( \overline{\Omega} \) such that \( \alpha, \beta \in C^2(\overline{\Omega}) \). The \( \vec{n} \) denotes outward-pointing normal direction along \( \partial \Omega \). The elliptic system under consideration is not smooth nor quasimonotone. Indeed, the function \( f_4 \) is neither non-increasing in nor non-decreasing in \( u_4 \) due to the threshold functions \( \phi_1, \phi_2, \phi_3 \).

A lemma for positive solutions to an elliptic equation is in order [8, Lemma 2].

**Lemma 1.** Let \( p(x, u) \) be a \( C^1 \) function and decreasing in \( u \) with \( \lim_{\xi \to \infty} p(x, \xi) \leq \lambda_1(-\Delta) \), the principal eigenvalue of \( -\Delta \) under Robin–Dirichlet boundary condition \( a(x)u + b(x)\frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \), where \( a, b \geq 0, a + b > 0 \) are smooth functions.

Then the equation \( -\Delta u = up(x, u) \) in \( \Omega \), \( a(x)u + b(x)\frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \) has a unique strictly positive solution if and only if the principal eigenvalue \( \lambda_1(\Delta + p(x, 0)) > 0 \) under the same boundary condition.

In what follows we denote \( \tau = \frac{1}{\lambda_1}\left(c + \sqrt{c^2 + 4a_17/a_{15}} \right) \), where the constant \( c \) is the carrying capacity of the logistic growth for stem cells \( u_5 \) in the fifth equation of system (5).

**Theorem 2.** Assume

\[
\begin{align*}
(i) & \quad a_{15}c - a_{16} > d_5\lambda_1(-\Delta), \text{ where } \lambda_1(-\Delta) \text{ is the principal eigenvalue of } -\Delta \text{ under the boundary condition } \theta_5 + \eta_5\frac{\partial}{\partial n} = 0, \\
(ii) & \quad a_{11} - a_{10} > 0, \\
(iii) & \quad ea_2 > (a_1\tau + a_4), \\
(iv) & \quad a_{20}(a_1\tau + a_4)a_2^{-1} + a_8\tau(a_{11} - a_{10})^{-1} - a_{14} < 0.
\end{align*}
\]

Then the system (4)–(5) possesses positive solutions \( u_1(x), \ldots, u_{6}(x) \), \( x \in \Omega \), such that the immune cells \( u_i(x) \geq m_i > 0 \), \( i = 1, 2, 3, 5 \), on \( \overline{\Omega} \) where \( m_i \) are certain positive constants.

**Note.** It can be seen from the proof below that the assumptions (iii) and (iv) are for positive steady-state of infected stem cells and virus; while assumptions (i) and (ii) are for that of immune cells.

**The Physical meaning.** We give, here, a brief description of the biological meaning in the field of immunology of above conditions stated in this theorem. The condition (i) demands that the growth rate \( a_{15} \) and the death rates \( a_{16} \) of stem cells satisfy \( a_{15}c - a_{16} > d_5\lambda_1(-\Delta) > 0 \). This indicates that a well renewable population of stem cells will play a vital role in stabilizing the infection. In order to have a positive steady-state, assumption (ii) demands that \( a_{10} < a_{11} \), where \( a_{11} \) denotes the natural death rate of macrophages \( u_3 \) while \( a_{10} \) represents the rate of immune stimulation mediated by T4 cells. Since a secondary infection can drive up \( a_{10} \), drastic disturbances can have a vital impact on the disruption to the formation of the steady-state in an infected individual as condition \( a_{10} < a_{11} \) fails to hold on. We know that a secondary infection with other viruses/bacteria may, in some cases, cause lethal results in the HIV infected individual. This disaster is usually and commonly attributed to the impairment of a weakened immune system. In view of Theorem 2, we can now understand mathematically that the breakup of the fabric of a steady-state, which is physiologically caused by a result of over-stimulating immune responses, can introduce a detrimental effect. This mathematical implication matches the clinical result [14].
Proof. We shall apply the result of Theorem 1.

The first part of the proof is to construct a set of functions $\bar{u}_i \geq u_i > 0$, $i = 1, \ldots, 6$. We then show in the second part that these functions meet with the definition of coupled upper–lower solutions. The stem cells $u_5, \bar{u}_5$ will perform crucial rules in the proof.

Let $\tau = \tau$ where $\tau$ is defined before the statement of Theorem 1. To define the lower solution $u_5$, we consider following equation:

$$-d_5 \Delta u_5 = a_{15} u_5 \left( c - \frac{a_{16}}{a_{15}} - u_5 \right),$$

with boundary condition $B_5 \cdot \theta_5 + \eta_5 \frac{\partial}{\partial n} = 0$. By Lemma 1 this equation has a unique strictly positive solution because $(a_{15} c - a_{16}) d_5^{-1} > \lambda_1(-\Delta)$ by assumption (i). Due to the boundary condition, we cannot apply the usual general maximal principle to claim that $\max u_5(x) \leq c - \frac{a_{16}}{a_{15}}$ because the maximum could occur on the boundary. However, this can be proved using upper–lower–solution technique as follows. Let $\phi > 0$ be the principal eigenfunction of $-d_5 \Delta$ with above boundary operator $B_5$. Then $\phi(x) > 0$ on $\Omega$. It is easy to check that $\varepsilon \phi$ is a lower solution for a small positive $\varepsilon$, while $\phi \equiv c - \frac{a_{16}}{a_{15}}$ is an upper solution, where $\varepsilon$ can be chosen so small that $\varepsilon \phi < \phi$. The uniqueness of positive solution to Eq. (8) then implies that $u_5 \leq \phi = c - \frac{a_{16}}{a_{15}}$. Therefore $u_5 = \tau > c > u_5$. The strong maximum principle along with Hopf’s lemma implies, from the above equation that $m_5 := \min x u_5(x) > 0$ for $x \in \Omega$.

Consider next following equation for $u_1(x)$:

$$-d_1 \Delta u_1 = a_1 u_1 - (a_2 + a_3) u_1 \text{ in } \Omega, \quad \theta_1(x) u_1 + \eta_1(x) \frac{\partial u_1}{\partial n} = 0 \text{ on } \partial \Omega. \quad (9)$$

Since, under the boundary condition in (9), the principal eigenvalue $\lambda_1(d_1 \Delta - (a_2 + a_3) I) < 0$ and the Green’s function $g(x, y)$ of the operator $d_1 \Delta - (a_2 + a_3) I$ is negative under the given boundary condition. These results are from Aronszajn–Smith theorem [1, §5]. This theorem says that for a self-adjoint operator, $A = -\sum_{i,j=1}^n a_{i,j}(x) D_i D_j + (\text{lower order terms})$ with Robin boundary condition $B u = 0$, if the operator $A$ is positive (negative) definite, then the Green’s function of $A$ must be positive (negative). In our case, $A = -d_1 \Delta + (\text{lower order terms})$. Now $-d_1 \Delta (a_2 + a_3) I) u_1 = a_1 u_1 > 0$ implies $u_1 > 0$ because the Green’s function $g(x, y) > 0$ and $u_1$ is the convolution $u_1 = g(x, y) * (a_1 u_1) = (-d_1 \Delta + (a_2 + a_3) I)^{-1}(a_1 u_1)$. Again the strong maximum principle along with Hopf’s lemma implies that $m_1 := \min u_1(x) > 0$ in $\Omega$. The same argument can be used for the existence of a unique positive solution $\bar{u}_1(x) > 0$ in $\Omega$ to the equation $-d_1 \Delta \bar{u}_1 = (a_1 \bar{u}_1 + a_4) - a_2 \bar{u}_1 = (a_1 \tau + a_4) - a_2 \bar{u}_1$ and $\theta_1(x) \bar{u}_1 + \eta_1(x) \frac{\partial \bar{u}_1}{\partial n} = 0$ on $\partial \Omega$. Comparing this Eq. (9), we, then, find $-d_1 \Delta (\bar{u}_1 - u_1) + a_2 (\bar{u}_1 - u_1) > 0$ in $\Omega$ and $\theta(\bar{u}_1 - u_1) + \eta_1 \frac{\partial (\bar{u}_1 - u_1)}{\partial n} = 0$ on $\Omega$.

Similarly, other functions $u_2, \bar{u}_2$ can be constructed as follows. Let $\bar{u}_2(x) \equiv (a_5 \tau + a_7) \eta_2$. Since the principal eigenvalue of the operator $-d_2 \Delta + a_6$, under boundary condition $\theta_2 + \eta_2 \frac{\partial}{\partial n} = 0$ is positive and $u_5 > 0$, the function $u_2$ can be defined as the unique solution to the equation

$$-d_2 \Delta u_2 = a_1 u_2 - a_6 u_2 \text{ in } \Omega, \quad \theta_2 + \eta_2 \frac{\partial}{\partial n} = 0 \text{ on } \partial \Omega. \quad (10)$$

We can similarly prove that there exists a unique positive solution $u_3(x) > 0$ in $\Omega$ satisfying following equation:

$$-d_3 \Delta u_3 = a_3 u_3 - (a_1 + a_9) u_3 \text{ in } \Omega \text{ and } \theta_3(x) u_3 + \eta_3(x) \frac{\partial u_3}{\partial n} = 0 \text{ on } \partial \Omega. \quad (11)$$

The function $\bar{u}_3(x)$ is the unique positive solution to the equation

$$-d_3 \Delta \bar{u}_3 = a_8 \bar{u}_3 + a_{10} \bar{u}_3 - a_{11} \bar{u}_3 \text{ in } \Omega, \quad \theta_3 \bar{u}_3 + \eta_3 \frac{\partial \bar{u}_3}{\partial n} = 0 \text{ on } \partial \Omega. \quad (12)$$

Since $a_{11} - a_{10} > 0$ by assumption (ii), following the same line of reasoning used above, we conclude that $\bar{u}_3(x) > 0$ and that $\bar{u}_3 > u_3$. Moreover, $\|\bar{u}_3\|_\infty \leq a_8 \bar{u}_3 / (a_{11} - a_{10}) = a_8 \tau (a_{11} - a_{10})$. 

Define \( \bar{u}_6(x) \equiv (a_{16}/a_{18}) \tau \). The lower solution function \( u_6(x) \) will be defined after the functions \( \bar{u}_4(x), u_4(x) \) are defined.

Notice that \( \bar{u}_2 = (a_5 + a_7)/a_6 > 0 \). Let \( g(x, y) \) be the Green’s function of the operator \( \mathcal{L} = -d_4 \Delta + (a_{14} + a_{13}) \bar{u}_2 \) under the boundary condition \( \frac{\partial u}{\partial n} + \alpha \cdot u = 0 \). Then, \( g(x, y) > 0 \) by Aronszajn–Smith theorem [1]. The function \( u_4(x) = \int_{\partial \Omega} g(x, y) \beta(y)(u_1 + u_3) \) gives a unique solution \( u_4(x) \) to the equation

\[
-d_4 \Delta u_4 = -(a_{14} + a_{13} \bar{u}_2) u_4 \quad \text{in} \ \Omega, \quad \frac{\partial u_4}{\partial n} + \alpha(x) u_4 = \beta(x)(u_1 + u_3) \quad \text{on} \ \partial \Omega, \tag{13}
\]

and it is clear that \( u_4(x) > 0 \) in \( \bar{\Omega} \).

In order to define the function \( \bar{u}_4(x) \), let us consider following equation:

\[
-d_4 \Delta u = a_20 u(\bar{u}_1 + \bar{u}_3) + a_{12} \bar{u}_6 - a_{14} u \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial n} + \alpha(x) u = \beta(x)(\bar{u}_1 + \bar{u}_3) \quad \text{on} \ \partial \Omega. \tag{14}
\]

As we have shown above, note that \( \| \bar{u} \|_\infty \leq (a_1 + a_4) a_2^{-1} \) and \( \| \bar{u}_3 \|_\infty \leq a_8 \tau (a_{11} - a_{10})^{-1} \). Under the Robin boundary condition \( \frac{\partial u}{\partial n} + \alpha(x) u = 0 \), the principal eigenvalue \( \lambda_1(-d_4 \Delta - a_{20}(\bar{u}_1 + \bar{u}_3) + a_{14}) > 0 \) by assumption (iv) and by the estimates on \( \bar{u}_1, \bar{u}_3 \) given above.

Notice that \( \bar{u}_6 > 0 \) and \( 0 \neq \beta(x)(u_1 + u_3) \geq 0 \), the Aronszajn–Smith theorem [1] ensures a positive solution \( \bar{u}_4 \) to Eq. (14).

Because \( -d_4 \Delta u_4 \leq -a_{14} u_4 \) by Eq. (13) and \( -d_4 \Delta \bar{u}_4 \geq -a_{14} \bar{u}_4 \) by Eq. (14), we have \( -d_4 \Delta (\bar{u}_4 - u_4) + a_{14}(\bar{u}_4 - u_4) \geq 0 \) and \( \frac{\partial (\bar{u}_4 - u_4)}{\partial n} + \alpha(x)(\bar{u}_4 - u_4) \geq 0 \).

The strong maximum principle along with Hopf’s lemma, then concludes that \( \bar{u}_4 > u_4 \). The strong maximum principle claims also that \( m_i := \min_i u_i(x) > 0 \) for \( x \in \bar{\Omega}, i = 2, 3, 4 \), in addition to the case of \( i = 1 \).

Let \( u_6(x) \) be the constant

\[
a_{16} e^{-1} [(e - a_{18}^{-1}(a_1 + a_4)) m_4 (c - a_{16}/a_{15}) (1 + m_4)^{-1} [a_{18} + a_{19}(a_5 + a_7)a_6^{-1}]^{-1}].
\]

By assumption (iii), \( u_6(x) > 0 \). Since \( \bar{u}_6(x) \equiv (a_{16}/a_{18}) \tau \), it can be verified that \( \bar{u}_6(x) \geq u_6(x) \).

We have proved that

\[
\bar{u}_i(x) \geq u_i(x), \quad i = 1, \ldots, 6. \tag{15}
\]

The final step is to verify that \( \bar{u}_i, u_i \) do form a coupled upper–lower–solution to the elliptic steady-state of system (4)–(7).

We list below an outline of the proof. Let

\[
\bar{f}_i := \sup_{y_i \leq \xi_i \leq y_j, i \neq j} f_i(\xi_1, \ldots, \xi_{i-1}, \bar{u}_i, \xi_{i+1}, \ldots, \xi_6),
\]

\[
f_i := \inf_{y_i \leq \xi_i \leq y_j, i \neq j} f_i(\xi_1, \ldots, \xi_{i-1}, u_i, \xi_{i+1}, \ldots, \xi_6),
\]

where functions \( u_i, \bar{u}_i \) are defined as above. Using the resulting estimates of their bounds, via a tedious computation, we, then, verify following equations and inequalities:

\[
-d_1 \Delta u_1 = a_{11} u_5 - a_{12} u_1 - a_{13} u_1 \leq a_{11} u_5 - a_{2} u_1 - a_{3} \min(u_1, \bar{u}_4) \leq \bar{f}_1, \quad \text{with} \ \theta_1 u_1 + \eta_1 \frac{\partial u_1}{\partial n} = 0 \quad \text{on} \ \partial \Omega;
\]

\[
-d_2 \Delta u_2 = a_{2} u_5 - a_{6} u_2 \leq \bar{f}_2, \quad \theta_2 u_2 + \eta_2 \frac{\partial u_2}{\partial n} = 0 \quad \text{on} \ \partial \Omega;
\]

\[
-d_3 \Delta u_3 = a_{6} u_5 - a_{11} u_3 \leq a_{8} u_5 - a_{11} u_3 - a_{9} \min(u_3, \bar{u}_4) \leq \bar{f}_3, \quad \text{with} \ \theta_3 u_3 + \eta_3 \frac{\partial u_3}{\partial n} = 0 \quad \text{on} \ \partial \Omega;
\]

\[
-d_4 \Delta u_4 = -(a_{14} + a_{13} \bar{u}_2) u_4 \leq \bar{f}_4, \quad \frac{\partial u_4}{\partial n} + \alpha(x) u_4 = \beta(x)(u_1 + u_3);
\]
\[-d_5 \Delta u_5 = a_1 u_5 (c - u_5) - a_1 u_5 \leq f_5, \quad \theta_5 u_5 + \eta_5 \frac{\partial u_5}{\partial n} = 0 \quad \text{on } \partial \Omega;\]
\[-d_6 \Delta u_6 = 0 \leq -a_1 u_6 + a_1 \frac{e - \max \alpha_i}{e} \min \frac{u_4}{1 + \min \alpha_i} u_5 - a_1 u_6 \left( \frac{a_5 \tau + a_7}{a_6} \right) \leq f_6,\]
with \(\frac{\partial u_6}{\partial n} = 0 \quad \text{on } \partial \Omega.\)

In the last inequality, we have used the definition of \(u_6\) and the fact of \(\alpha_i \leq a_i^{-1}(a_1 \tau + a_4).\)

Moreover,
\[-d_1 \Delta \bar{u}_1 = a_1 \bar{u}_5 - a_2 \bar{u}_1 + a_4 \geq \bar{f}_1, \quad \theta_1 \bar{u}_1 + \eta_1 \frac{\partial \bar{u}_1}{\partial n} = 0 \quad \text{on } \partial \Omega,\]
\[-d_2 \Delta \bar{u}_2 = a_5 \bar{u}_5 - a_6 \bar{u}_2 + a_7 \geq \bar{f}_2, \quad \theta_2 \bar{u}_2 + \eta_2 \frac{\partial \bar{u}_2}{\partial n} \geq 0 \quad \text{on } \partial \Omega,\]
\[-d_3 \Delta \bar{u}_3 = a_8 \bar{u}_5 + a_{10} \bar{u}_3 - a_{11} \bar{u}_3 \geq \bar{f}_3, \quad \theta_3 \bar{u}_3 + \eta_3 \frac{\partial \bar{u}_3}{\partial n} \geq 0 \quad \text{on } \partial \Omega.\]

Here we have made use of assumption (ii): \(a_{10} > a_{11}.\)

Also
\[-d_4 \Delta \bar{u}_4 = a_2 \bar{u}_4 (\bar{u}_1 + \bar{u}_3) - a_4 \bar{u}_6 - a_{14} \bar{u}_4 \geq \bar{f}_4,\]
\[\frac{\partial \bar{u}_4}{\partial n} + \alpha(x) \bar{u}_4 = \beta(x) (\bar{u}_1 + \bar{u}_3) \quad \text{on } \partial \Omega.\]

We have used in above the definition of \(\phi_4(u_1, u_3, u_4) = a_2 u_4 \min(u_1 + u_3, u_4).\)

By definition of \(\tau, a_{15} \tau (c - \tau) + a_{17} = 0,\) it follows from \(\bar{u}_5 \equiv \tau\) that
\[-d_5 \Delta \bar{u}_5 = a_{15} \bar{u}_5 (c - \bar{u}_5) + a_{17} = 0 \geq \bar{f}_5, \quad \theta_5 \bar{u}_5 + \eta_5 \frac{\partial \bar{u}_5}{\partial n} \geq 0 \quad \text{on } \partial \Omega.\]

Finally,
\[-d_6 \Delta \bar{u}_6 = -a_{18} \bar{u}_6 + a_{16} \bar{u}_5 = -a_{18} \bar{u}_6 + a_{16} \tau = 0 \geq \bar{f}_6 \quad \text{and} \quad \frac{\partial \bar{u}_6}{\partial n} = 0 \quad \text{on } \partial \Omega.\]

The above inequalities along with (15)–(17) justify that \(u_i, \bar{u}_i \ (i = 1, \ldots, 6),\) from a set of coupled upper–lower solutions. \(\square\)

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References