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A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo-contractions ☆

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ABSTRACT

In this paper, we propose an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions in the setting of real Hilbert spaces. By using our proposed scheme, we get a weak convergence theorem for a finite family of asymptotically k-strict pseudo-contractions and then we modify these algorithm to have strong convergence theorem by using the two hybrid methods in the mathematical programming. Our results improve and extend the recent ones announced by Ceng, et al.'s result [L.C. Ceng, Al-Homidan, Q.H. Ansari and J.C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Comput. Appl. Math. 223 (2009) 967–974] Qin, Cho, Kang, and Shang, [X. Qin, Y. J. Cho, S. M. Kang, and M. Shang, A hybrid iterative scheme for asymptotically k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 70 (2009) 1902–1911] and other authors.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. Let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : C \times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$\phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(\phi)$. Numerous problems in physics, optimization, and economics are reduced to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see [1–3,24–26]). In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(\phi)$ is nonempty and they also proved a strong convergence theorem.

Recall that a mapping $T : C \longrightarrow C$ is said to be asymptotically k-strictly pseudo-contractive (the class of asymptotically k-strictly pseudo-contractive maps was first introduced in Hilbert spaces by Qihou [5]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$ such that there exists $k \in [0, 1)$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Note that the class of asymptotically k-strict pseudo-contractions strictly includes the class of asymptotically nonexpansive mappings [6] which are mappings T on C such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|, \quad \forall x, y \in C, \tag{1.3}$$

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where the sequence $\{k_n\} \subset [1, \infty)$ is such that $\lim_{n\longrightarrow\infty} k_n = 1$. That is, T is asymptotically nonexpansive if and only if T is asymptotically 0-strictly pseudo-contractive.

Recall that a mapping $T : C \longrightarrow C$ is called a k-strict pseudo-contraction mapping if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. Note that the class of strict pseudo-contraction mappings strictly includes the class of nonexpansive mappings. Clearly, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. Construction of fixed points of nonexpansive mappings via Mann's algorithm [7] has extensively been investigated in the literature; See, for example [8,7,9–12] and references therein. If T is a nonexpansive self-mapping of C, then Mann's algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a real control sequence in the interval (0, 1).

If $T : C \longrightarrow C$ is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T. Reich [13] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a Fréchet differentiable norm. It is well known that Reich's result is one of the fundamental convergence results. Very recently, Marino and Xu [14] extended Reich's result [13] to strict pseudo-contraction mappings in the setting of Hilbert spaces.

Very recently, Motivated and inspired by the research work of Marino and Xu [14] and Takahashi and Takahashi [15], Ceng, Homidan, Ansari and Yao [16], introduced a new implicit iterative scheme for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a strict pseudo-contraction mapping defined in the setting of real Hilbert spaces. They gave some weak and strong convergence theorems for such iterative scheme. More precisely, they proved the following theorems.

Theorem 1.1 (*Ceng, Homidan, Ansari and Yao [16]*). *Let C be a closed convex subset of a Hilbert space H, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $T : C \longrightarrow C$ be a k-strict pseudo-contraction mapping for some $0 \leq k < 1$ such that $F(T) \cap EP(\phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_0 \in H$ and then by*

$$\begin{cases} \phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, & \forall y \in C \\ x_n = \alpha_{n-1}u_{n-1} + (1 - \alpha_{n-1})Tu_{n-1}; & \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$, *for some* $\alpha, \beta \in (k, 1)$, *and*
(2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(T) \cap EP(\phi)$.

Theorem 1.2 (*Ceng, Homidan, Ansari and Yao [16]*). *Let $C, H, T, \phi, \{x_n\}, \{u_n\}$ and $\{\alpha_n\}, \{r_n\}$ be as in Theorem 1.1. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of F if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F) = 0$, where $d(x_n, F)$ denotes the metric distance from the point x_n to F.*

On the other hand, motivate and inspired in [17,14], very recently, Qin, Cho, Kang and Shang [18] introduced the following algorithm for a finite family of asymptotically k-strict pseudo-contractions. Let $x_0 \in C$ and $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in (0, 1). The sequence $\{x_n\}$ generated by the following way:

$$\begin{aligned}
&x_1 = \alpha_0 x_0 + (1 - \alpha_0)T_1 x_0, \\
&x_2 = \alpha_1 x_1 + (1 - \alpha_1)T_2 x_1 \\
&\cdots \\
&x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1})T_N x_{N-1} \\
&x_{N+1} = \alpha_N x_N + (1 - \alpha_N)T_1^2 x_N \\
&\cdots \\
&x_{2N} = \alpha_{2N-1} x_{2N-1} + (1 - \alpha_{2N-1})T_N^2 x_{2N-1} \\
&x_{2N+1} = \alpha_{2N} x_{2N} + (1 - \alpha_{2N})T_1^3 x_{2N} \\
&\cdots
\end{aligned}$$

is called the explicit iterative sequence of a finite family of asymptotically k-strict pseudo-contractions $\{T_1, T_2, \dots, T_N\}$. Since, for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \dots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence the above table can be written in the following form:

$$x_n = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \quad \forall n \geq 1. \tag{1.7}$$

Then they proved some weak convergence theorems for a finite family of asymptotically k-strict pseudo-contractions by algorithm (1.7). More precisely, they proved the following theorem.

Theorem 1.3. *Let C be nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \leq k_i \leq 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \leq i \leq N\}$ and $k_n = max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $\cap_{i=1}^{N} F(T_i)$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (1.7). Assume that the control sequence $\{\alpha_n\}$ is chosen such that $k + \varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 0$ and some $\varepsilon \in (0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.*

On the other hand, recently Takahashi, Takeuchi and Kubota [19] introduced the new iterative methods for approximating the common fixed point of a family of nonexpansive mappings $\{T_n : C \longrightarrow C\}$ by using the hybrid method as follows: Let $x \in H$, for $C_0 = C$ and $x_0 = P_{C_0}x$, they define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_n x_{n-1}, \\ C_n = \{z \in C_{n-1} : \|y_{n-1} - z\| \leq \|x_{n-1} - z\|\}, \\ x_n = P_{C_n}x, \quad \forall n \geq 1 \end{cases} \tag{1.8}$$

where $0 \leq \alpha_n < \alpha < 1$ for all $n \geq 0$. Then, under appropriate conditions on $\{T_n\}$, they obtained a strong convergence theorem for the iterative scheme (1.8) in the setting of a real Hilbert space.

In this paper, inspired and motivated by the above researches, we suggest and analyze an iterative scheme for finding a common element of the set of common fixed points of a finite family of asymptotically k-strict pseudo-contraction and the set of solutions of an equilibrium problem in the framework of Hilbert spaces. Then we modify our iterative scheme to get strong convergence theorems by the hybrid algorithms. Our results extend and improve the corresponding recent results of Ceng, Homidan, Ansari and Yao [16] and Qin, Cho, Kang and Shang [18] and some others.

2. Preliminaries

Throughout the paper, we write $x_n \longrightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x, and $\omega_w(x_n) = \{x : x_{n_j} \rightharpoonup x$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 2.1 (*[14, Lemma 1.1]*). *Let H be a real Hilbert space. There hold the following identities*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, $\forall x, y \in H$,
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall t \in [0, 1], \forall x, y \in H$,
(iii) *If $\{x_n\}$ is a sequence in H weakly converging to z, then*

$$\limsup_{n\longrightarrow\infty} \|x_n - y\|^2 = \limsup_{n\longrightarrow\infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping.

It is also known that H satisfies Opial's condition [20], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| \leq \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.2 (*[21]*). *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and point $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$ the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\},$$

is convex and closed.

Lemma 2.3 (*[14, Lemma 1.3]*). *Let C be a closed convex subset of H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 (*[21, Lemma 1.5]*). *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \leq \|u - q\|$ for all n. Then $x_n \longrightarrow q$.*

Lemma 2.5 (*Kim and Xu [22]*). *Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and $T : C \longrightarrow C$ be an asymptotically k-strictly pseudo-contractive mapping for some $0 \leq k < 1$ with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and the fixed point set of T is nonempty. Then $(I - T)$ is demiclosed at zero.*

Lemma 2.6 (*[23, Lemma 2]*). *Let the sequences of numbers $\{a_n\}$ and $\{b_n\}$ be satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n, \quad a_n \geq 0, b_n \geq 0, \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty, \quad \forall n \geq 1.$$

If $\liminf_{n\longrightarrow\infty} a_n = 0$, then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.7 (*[9]*). *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be the three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\longrightarrow\infty} r_n$ exists.

Lemma 2.8 (*Kim and Xu [22]*). *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \longrightarrow C$ be an asymptotically k-strictly pseudo-contractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.9 (*Qin, Cho, Kang, and Shang [18]*). *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \longrightarrow C$ be a k-strictly asymptotically pseudo-contractive mapping. Then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

For solving the equilibrium problem, let us assume that the bifunction ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;
(A2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for any $x, y \in C$;
(A3) ϕ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\longrightarrow 0^+} \phi(tz + (1 - t)x, y) \leq \phi(x, y);$$

(A4) $\phi(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [1].

Lemma 2.10 (*[1]*). *Let C be a nonempty closed convex subset of H and let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The following lemma was also given in [4].

Lemma 2.11 (*[4]*). *Assume that $\phi : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \longrightarrow C$ as follows:*

$$S_r(x) = \left\{ z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\},$$

for all $z \in H$. Then, the following hold:

(i) *S_r is single-valued;*
(ii) *S_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle$;*
(iii) *$F(S_r) = EP(\phi)$;*
(iv) *$EP(\phi)$ is closed and convex.*

3. Weak convergence theorems

We are now in a position to prove some weak convergence theorems.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive* *mapping for some* $0 \le k_i \le 1$ *and a sequence* $\{k_{n,i}\}$ *such that* $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. *Let* $k = max\{k_i : 1 \le i \le N\}$ *and* $k_n = max\{k_{n,i} : 1 \le i \le N\}$. *Assume that* $F := \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$ *is nonempty. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated initially by arbitrary element* $x_0 \in C$ *and then by*

$$\begin{cases} \phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0; & \forall y \in C, \\ x_n = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}; & \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset [\alpha, \beta]$, *for some* $\alpha, \beta \in (k, 1)$, *and*
(C2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to an element of* F.

Proof. We divide the proof into five steps.

Step 1. We claim that $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists, $\forall q \in F$.
Indeed, Let $q \in F := \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$. Thus from the definition of S_r in Lemma 2.11, we have $u_{n-1} = S_{r_{n-1}}x_{n-1}$ and therefore

$$\begin{aligned} \|u_{n-1} - q\| &= \|S_{r_{n-1}}x_{n-1} - S_{r_{n-1}}q\| \\ &\le \|x_{n-1} - q\|, \quad \text{for all } n \ge 1. \end{aligned}$$

Since each $i \in \{1, 2, \ldots, N\}$, $T_i : C \longrightarrow C$ is an asymptotically k_i-strictly pseudo-contractive mapping, we have

$$\begin{aligned} \|x_n - q\|^2 &= \|\alpha_{n-1}(u_{n-1} - q) + (1-\alpha_{n-1})(T_{i(n)}^{h(n)}u_{n-1} - q)\|^2 \\ &= \alpha_{n-1}\|u_{n-1} - q\|^2 + (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n+1} - q\|^2 - \alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le \alpha_{n-1}\|u_{n-1} - q\|^2 - \alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\quad + (1-\alpha_{n-1})[k_{h(n)}^2\|u_{n-1} - q\|^2 + k\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2] \\ &\le k_{h(n)}^2\|u_{n-1} - q\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le k_{h(n)}^2\|x_{n-1} - q\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le (1 + (k_{h(n)}^2 - 1))\|x_{n-1} - q\|^2. \end{aligned} \tag{3.2}$$

It follows from Lemma 2.7 that $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists.
Step 2. We claim that $\lim_{n\longrightarrow\infty}\|u_n - u_{n+j}\| = 0$; $\forall j = 1, 2, \ldots, N$. Observing (3.2) again, we have

$$(1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - q\|^2 - \|x_n - q\|^2.$$

It follows from our assumptions that

$$(1-\beta)(\alpha - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - q\|^2 - \|x_n - q\|^2.$$

Taking the limit as $n \longrightarrow \infty$ yields that

$$\lim_{n\longrightarrow\infty}\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\| = 0. \tag{3.3}$$

This implies that

$$\|x_n - u_{n-1}\| = (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.4}$$

Let $q \in F$. Thus as above $u_{n-1} = S_{r_{n-1}}x_{n-1}$ and we have

$$\begin{aligned} \|u_{n-1} - q\|^2 &= \|S_{r_{n-1}}x_{n-1} - S_{r_{n-1}}q\|^2 \\ &\le \langle S_{r_{n-1}}x_n - S_{r_{n-1}}q, x_{n-1} - q\rangle \\ &= \langle u_{n-1} - q, x_{n-1} - q\rangle \\ &= \frac{1}{2}(\|u_{n-1} - q\|^2 + \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2), \end{aligned}$$

and hence

$$\|u_{n-1} - q\|^2 \le \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2.$$

Using (3.2), (C1), and the last inequality, we have

$$\begin{aligned}\|x_n - q\|^2 &= \|\alpha_{n-1}(u_{n-1} - q) + (1-\alpha_{n-1})(T_{i(n)}^{h(n)}u_{n-1} - q)\|^2\\ &\le k_{h(n)}^2\|u_{n-1} - q\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2\\ &\le k_{h(n)}^2\|u_{n-1} - q\|^2\\ &= k_{h(n)}^2(\|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2)\end{aligned}$$

and hence

$$k_{h(n)}^2\|x_{n-1} - u_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - q\|^2 - \|x_n - q\|^2.$$

The existence of $\lim_{n\longrightarrow\infty}\|x_n - q\|$ and $\lim_{n\longrightarrow\infty} k_{h(n)} = 1$ imply that

$$\lim_{n\longrightarrow\infty}\|x_{n-1} - u_{n-1}\| = 0. \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\|u_n - u_{n-1}\| \le \|u_n - x_n\| + \|x_n - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.6}$$

It follows that

$$\lim_{n\longrightarrow\infty}\|u_n - u_{n+j}\| = 0, \quad \forall j = 1, 2, \dots, N. \tag{3.7}$$

Applying (3.5) and (3.6), we obtain that

$$\|x_n - x_{n-1}\| \le \|x_n - u_n\| + \|u_n - u_{n-1}\| + \|u_{n-1} - x_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This also implies that $\lim_{n\longrightarrow\infty}\|x_n - x_{n+j}\| = 0; \forall j = 1, 2, \dots, N$.
Step 3. We claim that $\lim_{n\longrightarrow\infty}\|u_n - T_l u_n\| = 0$, $\lim_{n\longrightarrow\infty}\|x_n - T_l x_n\| = 0$ for all $l = 1, 2, \dots, N$. Since, for any positive integer $n > N$, it can be written as $n = (k(n)-1)N + i(n)$, where $i(n) \in \{1, 2, \dots, N\}$. Observe that

$$\begin{aligned}\|u_{n-1} - T_n u_{n-1}\| &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + \|T_{i(n)}^{h(n)}u_{n-1} - T_n u_{n-1}\|\\ &= \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + \|T_{i(n)}^{h(n)}u_{n-1} - T_{i(n)}u_{n-1}\|\\ &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + L\|T_{i(n)}^{h(n)-1}u_{n-1} - u_{n-1}\|\\ &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + L[\|T_{i(n)}^{h(n)-1}u_{n-1} - T_{i(n-N)}^{h(n)-1}u_{n-N}\|\\ &\quad + \|T_{i(n-N)}^{h(n)-1}u_{n-N} - u_{(n-N)-1}\| + \|u_{(n-N)-1} - u_{n-1}\|].\end{aligned} \tag{3.8}$$

Since, for each $n > N$, $n = (n-N)(\operatorname{mod} N)$ and $n = (k(n)-1)N + i(n)$, we have

$$n - N = (k(n)-1)N + i(n) = (k(n-N)-1)N + i(n-N).$$

That is

$$k(n-N) = k(n) - 1, \qquad i(n-N) = i(n).$$

Observe that

$$\begin{aligned}\|T_{i(n)}^{h(n)-1}u_{n-1} - T_{i(n-N)}^{h(n)-1}u_{n-N}\| &= \|T_{i(n)}^{h(n)-1}u_{n-1} - T_{i(n)}^{h(n)-1}u_{n-N}\|\\ &\le L\|u_{n-1} - u_{n-N}\|,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\|T_{i(n-N)}^{h(n)-1}u_{n-N} - u_{(n-N)-1}\| &= \|T_{i(n-N)}^{h(n-N)}u_{n-N} - u_{(n-N)-1}\|\\ &\le \|T_{i(n-N)}^{h(n-N)}u_{n-N} - T_{i(n-N)}^{h(n)-N}u_{(n-N)-1}\| + \|T_{i(n-N)}^{h(n-N)}u_{(n-N)-1} - u_{n-N-1}\|\\ &\le L\|u_{(n-N)-1} - u_{n-N}\| + \|T_{i(n-N)}^{h(n-N)}u_{(n-N)-1} - u_{n-N-1}\|.\end{aligned} \tag{3.10}$$

It follows from (3.8)–(3.10) that

$$\begin{aligned}\|u_{n-1} - T_n u_{n-1}\| &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + L(L\|u_{n-1} - u_{n-N}\| + L\|u_{(n-N)-1} - u_{n-N}\|\\ &\quad + \|T_{i(n-N)}^{h(n-N)}u_{(n-N)-1} - u_{n-N-1}\| + \|u_{(n-N)-1} - u_{n-1}\|).\end{aligned} \tag{3.11}$$

Applying (3.7) and (3.3) to (3.11), we obtain

$$\lim_{n\longrightarrow\infty} \|u_{n-1} - T_n u_{n-1}\| = 0. \tag{3.12}$$

Notice that

$$\begin{aligned}\|u_n - T_n u_n\| &\le \|u_n - u_{n-1}\| + \|u_{n-1} - T_n u_{n-1}\| + \|T_n u_{n-1} - T_n u_n\|.\\ &\le (1+L)\|u_n - u_{n-1}\| + \|u_{n-1} - T_n u_{n-1}\|.\end{aligned}$$

From (3.6) and (3.12), one can easily see that

$$\lim_{n\longrightarrow\infty} \|u_n - T_n u_n\| = 0.$$

We also have

$$\begin{aligned}\|u_n - T_{n+j} u_n\| &\le \|u_n - u_{n+j}\| + \|u_{n+j} - T_{n+j} u_{n+j}\| + \|T_{n+j} u_{n+j} - T_{n+j} u_n\|\\ &\le (1+L)\|u_n - u_{n+j}\| + \|u_{n+j} - T_{n+j} u_{n+j}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned}$$

for any $j = 1, 2, \ldots, N$, which give that

$$\lim_{n\longrightarrow\infty} \|u_n - T_l u_n\| = 0; \quad \forall l = 1, 2, \ldots, N.$$

Moreover, for each $l \in \{1, 2, \ldots, N\}$, we obtain that

$$\begin{aligned}\|x_n - T_l x_n\| &\le \|x_n - u_n\| + \|u_n - T_l u_n\| + \|T_l u_n - T_l x_n\|\\ &\le (1+L)\|x_n - u_n\| + \|u_n - T_l u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.13}$$

Step 4. We claim that

$$\omega_w(x_n) \subset F := \cap_{i=1}^N F(T_i) \cap EP(\phi), \tag{3.14}$$

where $\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}$.
Indeed, since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $w \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Applying (3.5), we can obtain that $u_{n_i} \rightharpoonup w$ as $i \longrightarrow \infty$. It follows from $\|u_n - T_l u_n\| \longrightarrow 0$ that

$$T_l u_{n_i} \rightharpoonup w, \quad \text{for all } l = 1, 2, \ldots, N.$$

Let us show $w \in EP(\phi)$. Since $u_n = T_{r_n} u_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge \phi(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge \phi(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup w$, from (A4) have

$$0 \ge \phi(y, w), \quad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $\phi(y_t, w) \le 0$. So, from (A1) and (A4) we have

$$0 = \phi(y_t, y_t) \le t\phi(y_t, y) + (1-t)\phi(y_t, w) \le t\phi(y_t, y)$$

and hence $0 \le \phi(y_t, y)$. From (A3), we have

$$0 \le \phi(w, y), \quad \forall y \in C$$

and hence $w \in EP(\phi)$.

Next, we prove that $w \in \cap_{i=1}^{N} F(T_i)$. Assume that $w \notin \cap_{i=1}^{N} F(T_i)$. Thus there exists $l \in \{1, \ldots, N\}$ such that $w \notin F(T_l)$. From (3.13) and Opial's condition,

$$\begin{aligned}\liminf_{i\longrightarrow\infty} \|x_{n_i} - w\| &< \liminf_{i\longrightarrow\infty} \|x_{n_i} - T_l w\| \\ &\le \liminf_{i\longrightarrow\infty}\{\|x_{n_i} - T_l x_{n_i}\| + \|T_l x_{n_i} - T_l w\|\} \\ &\le \lim_{i\longrightarrow\infty} L\|x_{n_i} - w\|,\end{aligned}$$

which derives a contradiction. This implies that $w \in \cap_{i=1}^{N} F(T_i)$.

Step 5. We show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\cap_{i=1}^{N} F(T_i) \cap EP(\phi)$.

Indeed, to verify that the assertion is valid, it is sufficient to show that $\omega_w(x_n)$ is a single-point set. We take w_1, $w_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup w_1$ and $x_{m_j} \rightharpoonup w_2$, respectively. Since $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists for each $q \in \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$ and $w_1, w_2 \in \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$, by Lemma 2.1(iii), we obtain

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - w_1\|^2 &= \lim_{j\longrightarrow\infty} \|x_{m_j} - w_1\|^2 \\ &= \lim_{j\longrightarrow\infty} \|x_{m_j} - w_2\|^2 + \|w_2 - w_1\|^2 \\ &= \lim_{j\longrightarrow\infty} \|x_{k_i} - w_2\|^2 + \|w_2 - w_1\|^2 \\ &= \lim_{j\longrightarrow\infty} \|x_{k_i} - w_1\|^2 + 2\|w_2 - w_1\|^2 \\ &= \lim_{n\longrightarrow\infty} \|x_n - w_1\|^2 + 2\|w_2 - w_1\|^2.\end{aligned}$$

Hence $w_1 = w_2$. This shows that $\omega_w(x_n)$ is a single-point set. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improve Theorem 3.1 of Ceng, Homidan, Ansari and Yao [16], from a k-strictly pseudo-contractive mapping to an finite family of the asymptotically k_i-strictly pseudo-contractive mappings.

A direct consequence of Theorem 3.1, we derive the following theorem of Qin, Cho, Kang and Shang [18].

Theorem 3.3 (*[18, Theorem 2.1]*). *Let C be nonempty closed convex subset of a real Hilbert space H and $N \ge 1$ be an integer. Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \le k_i \le 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \le i \le N\}$ and $k_n = max\{k_{n,i} : 1 \le i \le N\}$. Assume that $\cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated initially by arbitrary element $x_0 \in C$ and then by*

$$x_n = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}; \quad \forall n \ge 1,$$

where $\{\alpha_n\}$ is a sequence in $[\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$. Then, the sequences $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.

Proof. Put $\phi(x, y) = 0$ for all $x, y \in C$, $r_n = 1$ for all $n \ge 0$ in Theorem 3.1. Thus, we have $u_n = x_n$. Then the sequence $\{x_n\}$ generated in Theorem 3.3 converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$. □

4. Strong convergence theorems

Theorem 4.1. *Let C be nonempty closed convex subset of a real Hilbert space H and $N \ge 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \le k_i \le 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \le i \le N\}$ and $k_n = max\{k_{n,i} : 1 \le i \le N\}$. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$ is nonempty. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_0 \in C$ and*

$$\begin{cases}\phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, & \forall y \in C \\ x_n = \alpha_{n-1}u_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)} u_{n-1}, & \forall n \ge 1,\end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$, and
(2) $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of F if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F) = 0$, where $d(x_n, F)$ denotes the metric distance from the point x_n to F.

Proof. From the proof of Theorem 3.1, we know that $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists for each $q \in F$ and $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$. Hence $\{x_n\}$ is bounded. The necessity is apparent. We show the sufficiency. Indeed we suppose that $\liminf_{n\longrightarrow\infty} d(x_n, F) = 0$.

Since $x_n = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}$, from (3.2), we have

$$\|x_n - q\|^2 \le (1 + (k_{h(n)}^2 - 1))\|x_{n-1} - q\|^2.$$

From the fact that $x \ge 0$, $1 + x \le \mathrm{e}^x$, we can calculate

$$\begin{aligned}\|x_{n+m} - q\|^2 &= (1 + (k_{h(n+m)}^2 - 1))\|x_{(n+m)-1} - q\|^2 \\ &\le \mathrm{e}^{(k_{h(n+m)}^2 - 1)}\|x_{(n+m)-1} - q\|^2 \\ &\vdots \\ &\le \mathrm{e}^{\sum_{j=n}^{n+m}(k_{h(n)}^2 - 1)}\|x_n - q\|^2 \\ &\le \mathrm{e}^{\sum_{j=1}^{\infty}(k_{h(j)}^2 - 1)}\|x_n - q\|^2, \quad \text{for all } n, m \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Let $\mathrm{e}^{\sum_{j=1}^{\infty}(k_{h(j)}^2 - 1)} = M$, for some nonnegative number M. Thus,

$$\|x_{n+m} - q\|^2 \le M\|x_n - q\|^2, \quad \text{for all } n, m \in \mathbb{N}.$$

This gives that

$$\|x_{n+m} - q\| \le \sqrt{M}\|x_n - q\|, \quad \text{for all } n, m \in \mathbb{N}. \tag{4.3}$$

From Lemma 2.6, we have

$$\lim_{n\longrightarrow\infty} d(x_n, F) = 0.$$

Here after, we will prove that $\{x_n\}$ is a Cauchy sequence. For any $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that,

$$d(x_n, F) \le \frac{\varepsilon}{3\sqrt{M}}, \quad \forall n \ge N_1.$$

In particular, we obtain that $d(x_{N_1}, F) \le \frac{\varepsilon}{3\sqrt{M}}$. This implies that there exists $q_1 \in F$ such that

$$\|x_{N_1} - q_1\| = d(x_{N_1}, F) \le \frac{\varepsilon}{3\sqrt{M}}.$$

It follows from (4.3) that $n > N_1$,

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &= \|x_{n+m} - q_1\| + \|x_n - q_1\| \\ &\le \sqrt{M}\|x_{N_1} - q_1\| + \sqrt{M}\|x_{N_1} - q_1\| \\ &\le \sqrt{M}\frac{\varepsilon}{2\sqrt{M}} + \sqrt{M}\frac{\varepsilon}{2\sqrt{M}} = \varepsilon. \end{aligned} \tag{4.4}$$

Thus $\{x_n\}$ is a Cauchy sequence. Suppose that $x_n \longrightarrow x^* \in H$. Then

$$d(x^*, F) = \lim_{n\longrightarrow\infty} d(x_n, F) = 0.$$

Since for each $i = 1, 2, \ldots, N$, T_i is an asymptotically k_i-strictly pseudo-contractive mapping, we know form Lemma 2.9 that $\cap_{i=1}^N F(T_i)$ is closed and convex. Note that $EP(\phi)$ is closed and convex according to Lemma 2.11. Thus $F := \cap_{i=1}^N F(T_i) \cap EP(\phi)$ is closed and convex. Consequently, $x^* \in F$. By using $\|u_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we conclude that both $\{x_n\}$ and $\{u_n\}$ converge strongly to an element x^* of F. □

Remark 4.2. Theorem 4.1 mainly improve Theorem 3.2 of Ceng, Homidan, Ansari and Yao [16], from a k-strictly pseudo-contractive mapping to an finite family of the asymptotically k_i-strictly pseudo-contractive mappings.

Theorem 4.3. *Let C be nonempty closed convex subset of a real Hilbert space H and $N \ge 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \le k_i \le 1$ and a sequence $\{k_{n,i}\}$. Let $k = max\{k_i : 1 \le i \le N\}$ and $k_n = max\{k_{n,i} : 1 \le i \le N\}$. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi)$ is nonempty and let $x \in H$. For $C_0 = C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{cases} x_0 = P_{C_0}x, \\ u_{n-1} \in C \text{ such that } \phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_{n-1} = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}, \\ C_n = \{v \in C_{n-1} : \|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 + \theta_{n-1}\}, \\ x_n = P_{C_n}x, \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1-\alpha_{n-1})\rho_{n-1}^2 \longrightarrow \infty$ *as* $n \longrightarrow \infty$, *where* $\rho_{n-1} = \sup\{\|x_{n-1} - v\| : v \in F\} < \infty$, *and* $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$ *and* $\{r_n\} \subset (0, \infty)$ *such that* $\liminf_{n\longrightarrow\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_F x$.

Proof. We first show by induction that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C_n$ for all $n \geq 0$. It is obvious that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C = C_0$. Suppose that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C_{j-1}$ for some $j \in \mathbb{N}$. Hence, for any $q \in F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C_{j-1}$ and by $u_{j-1} = S_{r_{j-1}}x_{j-1}$, we have

$$\begin{aligned} \|u_{j-1} - q\| &= \|S_{r_{j-1}}x_{n-1} - S_{r_{j-1}}q\| \\ &\leq \|x_{j-1} - q\|. \end{aligned}$$

Hence

$$\begin{aligned} \|y_{j-1} - q\|^2 &= \|\alpha_{j-1}(u_{j-1} - q) + (1-\alpha_{j-1})(T_{i(j)}^{h(j)}u_{j-1} - q)\|^2 \\ &= \alpha_{j-1}\|u_{j-1} - q\|^2 + (1-\alpha_{j-1})\|T_{i(j)}^{h(j)}u_{j-1} - q\|^2 \\ &\quad - \alpha_{j-1}(1-\alpha_{j-1})\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2 \\ &\leq \alpha_{j-1}\|u_{j-1} - q\|^2 - \alpha_{j-1}(1-\alpha_{j-1})\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2 \\ &\quad + (1-\alpha_{j-1})[k_{h(j)}^2\|u_{j-1} - q\|^2 + k\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2] \\ &\leq \|x_{j-1} - q\|^2 + \theta_{j-1} - (1-\alpha_{j-1})(\alpha_{j-1} - k)\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2 \\ &\leq \|x_{j-1} - q\|^2 + \theta_{j-1}. \end{aligned} \tag{4.6}$$

Therefore, $q \in C_j$. This implies that

$$F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C = C_n, \quad \text{for all } n \geq 0.$$

Next, we prove that C_n is closed and convex for all $n \geq 0$. It is obvious that $C_0 = C$ is closed and convex. Suppose that C_{k-1} is closed and convex for some $k \in \mathbb{N}$. For $v \in C_{k-1}$, we know that $\|y_{k-1} - v\|^2 \leq \|x_{k-1} - v\|^2 + \theta_{k-1}$ is equivalent to

$$2\langle x_{k-1} - y_{k-1}, v\rangle \leq \|x_{k-1}\|^2 - \|y_{k-1}\|^2 + \theta_{k-1}.$$

So, C_k is closed and convex. Then, for any $n \geq 0$, C_n is closed and convex. This implies that $\{x_n\}$ is well defined. From Lemma 2.10, the sequence $\{u_n\}$ is also well defined. From $x_n = P_{C_n}x$, we have

$$\langle x - x_n, x_n - y\rangle \geq 0,$$

for each $y \in C_n$. Using $F \subset C_n$, we also have

$$\langle x - x_n, x_n - p\rangle \geq 0 \quad \text{for each } p \in F \text{ and } n \in \mathbb{N}.$$

Hence, for $p \in F$, we have

$$\begin{aligned} 0 &\leq \langle x - x_n, x_n - p\rangle \\ &= \langle x - x_n, x_n - x + x - p\rangle \\ &= -\langle x_n - x, x_n - x\rangle + \langle x - x_n, x - p\rangle \\ &= -\|x_n - x\|^2 + \|x - x_n\|\|x - p\|. \end{aligned}$$

This implies that

$$\|x - x_n\| \leq \|x - p\|, \quad \text{for all } p \in F \text{ and } n \in \mathbb{N}. \tag{4.7}$$

Hence $\{x_n\}$ is bounded. It follows that $\{y_n\}$ is also bounded. From $x_{n-1} = P_{C_{n-1}}x$ and $x_n = P_{C_n}x \in C_n \subset C_{n-1}$, we obtain

$$\langle x - x_{n-1}, x_{n-1} - x_n\rangle \geq 0. \tag{4.8}$$

It follow that, for $n \in \mathbb{N}$,

$$0 \leq \langle x - x_{n-1}, x_{n-1} - x_n\rangle$$

$$
\begin{aligned}
&= \langle x - x_{n-1}, x_{n-1} - x + x - x_n \rangle \\
&= -\|x - x_{n-1}\|^2 + \langle x - x_{n-1}, x - x_n \rangle \\
&\le -\|x - x_{n-1}\|^2 + \|x - x_{n-1}\|\|x - x_n\|,
\end{aligned}
$$

and hence

$$\|x - x_{n-1}\| \le \|x - x_n\|.$$

Hence $\{\|x_n - x\|\}$ is nondecreasing, so $\lim_{n\longrightarrow\infty} \|x_n - x\|$ exists.

Next we can show that $\lim_{n\longrightarrow\infty} \|x_n - x_{n-1}\| = 0$. Indeed, from (4.8) we get

$$
\begin{aligned}
\|x_{n-1} - x_n\|^2 &= \|x_{n-1} - x + x - x_n\|^2 \\
&= \|x_{n-1} - x\|^2 + 2\langle x_{n-1} - x, x - x_n\rangle + \|x - x_n\|^2 \\
&= -\|x_{n-1} - x\|^2 + 2\langle x_{n-1} - x, x - x_{n-1} + x_{n-1} + x_n\rangle + \|x - x_n\|^2 \\
&\le -\|x_{n-1} - x\|^2 + \|x - x_n\|^2.
\end{aligned}
$$

Since $\lim_{n\longrightarrow\infty} \|x - x_n\|$ exists, we have

$$\lim_{n\longrightarrow\infty} \|x_{n-1} - x_n\| = 0. \tag{4.9}$$

On the other hand, $x_n \in C_n$, we have

$$\|y_{n-1} - x_n\|^2 \le \|x_{n-1} - x_n\|^2 + \theta_{n-1}. \tag{4.10}$$

So, we have $\lim_{n\longrightarrow\infty} \|y_{n-1} - x_n\| = 0$. It follows that

$$\|y_{n-1} - x_{n-1}\| \le \|y_{n-1} - x_n\| + \|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.11}$$

Next, we claim that $\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0$. Indeed, let $q \in F$. Thus as above $u_{n-1} = S_{r_{n-1}} x_{n-1}$ and we have

$$
\begin{aligned}
\|u_{n-1} - q\|^2 &= \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} q\|^2 \\
&\le \langle S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} q, x_{n-1} - q\rangle \\
&= \langle u_{n-1} - q, x_{n-1} - q\rangle \\
&= \frac{1}{2}(\|u_{n-1} - q\|^2 + \|\alpha_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2).
\end{aligned}
$$

and hence

$$\|u_{n-1} - q\|^2 \le \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2.$$

This implies that

$$
\begin{aligned}
\|y_{n-1} - q\|^2 &= \|\alpha_{n-1}(u_{n-1} - q) + (1 - \alpha_{n-1})(T_{i(n)}^{h(n)} u_{n-1} - q)\|^2 \\
&= \alpha_{n-1}\|u_{n-1} - q\|^2 + (1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} u_{n-1} - q\|^2 - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2 \\
&\le \alpha_{n-1}\|u_{n-1} - q\|^2 - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2 \\
&\quad + (1 - \alpha_{n-1})[k_{h(n)}^2\|u_{n-1} - q\|^2 + k\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2] \\
&\le \|u_{n-1} - q\|^2 + \theta_{n-1} - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2 \\
&\le \|u_{n-1} - q\|^2 + \theta_{n-1} \\
&\le \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2 + \theta_{n-1}.
\end{aligned} \tag{4.12}
$$

Hence

$$
\begin{aligned}
\|x_{n-1} - u_{n-1}\|^2 &\le \|x_{n-1} - q\|^2 - \|y_{n-1} - q\|^2 + \theta_{n-1}. \\
&\le \|x_{n-1} - y_{n-1}\|\{\|x_{n-1} - q\| - \|y_{n-1} - q\|\} + \theta_{n-1}.
\end{aligned} \tag{4.13}
$$

It follows from (4.11) and the boundedness of the sequences $\{x_n\}$ and $\{y_n\}$ that

$$\lim_{n\longrightarrow\infty} \|x_{n-1} - u_{n-1}\| = 0. \tag{4.14}$$

From (4.9) and (4.14), we have

$$\|u_n - u_{n-1}\| \le \|u_n - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This implies that

$$\lim_{n\longrightarrow\infty} \|u_n - u_{n+j}\| = 0, \quad \text{for all } j \in \{1, 2, \ldots, N\}.$$

Form $y_{n-1} = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}$, $\forall n \geq 1$, we have

$$\begin{aligned}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - x_{n-1}\| &= \|y_{n-1} - \alpha_{n-1}u_{n-1} - (1-\alpha_{n-1})x_{n-1}\| \\ &\leq \|y_{n-1} - x_{n-1}\| + \alpha_{n-1}\|x_{n-1} - u_{n-1}\|.\end{aligned}$$

Applying (4.11) and (4.14) to the last inequality, we obtain

$$\lim_{n\longrightarrow\infty} \|T_{i(n)}^{h(n)}u_{n-1} - x_{n-1}\| = 0. \tag{4.15}$$

From (4.14) and (4.15), we have

$$\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\| \leq \|T_{i(n)}^{h(n)}u_{n-1} - x_{n-1}\| + \|x_{n-1} - u_{n-1}\| \longrightarrow 0. \tag{4.16}$$

By using the same method as in the proof of the Theorem 3.1, we easily obtain

$$\lim_{n\longrightarrow\infty} \|T_l u_n - u_n\| = 0 \quad \text{for all } l \in \{1, 2, \ldots, N\} \tag{4.17}$$

and

$$\omega_w(x_n) \subset F := \cap_{i=1}^N F(T_i) \cap EP(\phi). \tag{4.18}$$

This, together with (4.7) and Lemma 2.4 guarantees the strong convergence of $\{x_n\}$ to $p = P_F x$. From (4.14), we also have the strong convergence of $\{u_n\}$ to $p = P_F x$. This completes the proof. □

5. The CQ method for asymptotically *k*-strictly pseudo-contractive mappings

Theorem 5.1. *Let C be nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \leq k_i \leq 1$ and a sequence $\{k_{n,i}\}$. Let $k = max\{k_i : 1 \leq i \leq N\}$ and $k_n = max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi)$ is nonempty. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{cases} x_0 = u \in C \text{ chosen arbitrarily}, \\ u_{n-1} \in C \text{ such that } \phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0; \quad \forall y \in C, \\ y_{n-1} = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}, \\ C_{n-1} = \{v \in C : \|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 + \theta_{n-1}\}, \\ Q_{n-1} = \{v \in C : \langle x_0 - x_{n-1}, x_{n-1} - v\rangle \geq 0\}, \\ x_n = P_{C_{n-1}\cap Q_{n-1}}x_0, \quad \forall n \geq 1, \end{cases} \tag{5.1}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1-\alpha_{n-1})\rho_{n-1}^2 \longrightarrow \infty$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - v\| : v \in F\} < \infty$, and $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n\longrightarrow\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. We show first that the sequence $\{x_n\}$ is well defined. From the definition of C_{n-1} and Q_{n-1}, it is obvious that C_{n-1} is closed and Q_{n-1} is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. We prove that C_{n-1} is convex. For any $v_1, v_2 \in C_{n-1}$ and $t \in (0, 1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_{n-1}$. Since

$$\|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 + \theta_{n-1}$$

is equivalent to

$$2\langle x_{n-1} - y_{n-1}, v\rangle \leq \|x_{n-1}\|^2 - \|y_{n-1}\|^2 + \theta_{n-1}.$$

One can easily see that $v \in C_{n-1}$. Therefore we can obtain that C_{n-1} is convex. So, $C_{n-1} \cap Q_{n-1}$ is a closed convex subset of H for any $n \in \mathbb{N}$.

Next, we show that $\cap_{i=1}^N F(T_i) \cap EP(\phi) \subseteq C_{n-1}$. Indeed, let $q \in \cap_{i=1}^N F(S_i) \cap EP(\phi)$ and let $\{S_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.11. Then $q = S_{r_n}q$. From $u_{n-1} = S_{r_{n-1}}x_{n-1}$, we have

$$\begin{aligned}\|u_{n-1} - q\| &= \|S_{r_{n-1}}x_{n-1} - S_{r_{n-1}}q\| \\ &\leq \|x_{n-1} - q\|.\end{aligned} \tag{5.2}$$

By our assumptions, we have

$$\begin{aligned}\|y_{n-1}-q\|^2 &= \|\alpha_{n-1}(u_{n-1}-q)+(1-\alpha_{n-1})(T_{i(n)}^{h(n)}u_{n-1}-q)\|^2\\ &= \alpha_{n-1}\|u_{n-1}-q\|^2+(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n+1}-q\|^2\\ &\quad -\alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2\\ &\le \alpha_{n-1}\|u_{n-1}-q\|^2-\alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2\\ &\quad +(1-\alpha_{n-1})[k_{h(n)}^2\|u_{n-1}-q\|^2+k\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2]\\ &\le \|x_{n-1}-q\|^2+\theta_{n-1}-(1-\alpha_{n-1})(\alpha_{n-1}-k)\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2\\ &\le \|x_{n-1}-q\|^2+\theta_{n-1} \qquad (5.3)\end{aligned}$$

Therefore, $q \in C_{n-1}$ for all $n \ge 1$.

Next, we show that

$$\cap_{i=1}^N F(T_i)\cap EP(\phi)\subseteq Q_{n-1}, \quad \forall n\ge 1. \qquad (5.4)$$

We prove this by induction. For $n = 1$, we have $\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset C = Q_0$. Assume that $\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset Q_{n-1}$. Since x_n is the projection of x_0 onto $C_{n-1}\cap Q_{n-1}$, by Lemma 2.3, we have

$$\langle x_0-x_n, x_n-v\rangle \ge 0, \quad \forall v\in C_{n-1}\cap Q_{n-1}.$$

In particular, we have

$$\langle x_0-x_n, x_n-q\rangle \ge 0$$

for each $q\in\cap_{i=1}^N F(T_i)\cap EP(\phi)$ and hence $q\in Q_n$. Hence (5.4) holds for all $n \ge 1$. Therefore, we obtain that

$$\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset C_{n-1}\cap Q_{n-1}, \quad \forall n\ge 1.$$

Next, we show that $\lim_{n\longrightarrow\infty}\|x_n-x_{n-1}\|=0$.

Since $\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is a nonempty closed convex subset of H, there exists a unique $z'\in\cap_{i=1}^N F(T_i)\cap EP(\phi)$ such that

$$z' = P_{\cap_{i=1}^N F(T_i)\cap EP(\phi)}x_0.$$

From $x_n = P_{C_{n-1}\cap Q_{n-1}}x_0$, we have

$$\|x_n-x_0\|\le\|z-x_0\| \quad \text{for all } z\in C_{n-1}\cap Q_{n-1} \quad \text{and all } n\in\mathbb{N}.$$

Since $z'\in\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset C_{n-1}\cap Q_{n-1}$ we have

$$\|x_n-x_0\|\le\|z'-x_0\| \text{ all } n\in\mathbb{N}. \qquad (5.5)$$

Therefore, $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. From the definition of Q_{n-1}, we have $x_{n-1} = P_{Q_{n-1}}x_0$, which together with the fact that $x_n\in C_{n-1}\cap Q_{n-1}\subset Q_{n-1}$ implies that

$$\|x_0-x_{n-1}\|\le\|x_0-x_n\|. \qquad (5.6)$$

This show that the sequence $\{x_n-x_0\}$ is nondecreasing. So, we have $\lim_{n\longrightarrow\infty}\|x_n-x_0\|$ exists. Notice again that $x_{n-1} = P_{Q_{n-1}}x_0$ and $x_n\in Q_{n-1}$, which give that $\langle x_n-x_{n-1}, x_{n-1}-x_0\rangle\ge 0$. Therefore, we have

$$\begin{aligned}\|x_n-x_{n-1}\|^2 &= \|(x_n-x_0)-(x_{n-1}-x_0)\|^2\\ &= \|x_n-x_0\|^2-\|x_{n-1}-x_0\|^2-2\langle x_n-x_{n-1}, x_{n-1}-x_0\rangle\\ &\le \|x_n-x_0\|^2-\|x_{n-1}-x_0\|^2. \qquad (5.7)\end{aligned}$$

This together with the existence of $\lim_{n\longrightarrow\infty}\|x_n-x_0\|$ implies that $\lim_{n\longrightarrow\infty}\|x_n-x_{n-1}\|=0$. Since $x_n\in C_{n-1}$, we have

$$\|y_{n-1}-x_n\|^2\le\|x_{n-1}-x_n\|^2+\theta_{n-1}. \qquad (5.8)$$

So, we have $\lim_{n\longrightarrow\infty}\|y_{n-1}-x_n\|=0$. It follows that

$$\|y_{n-1}-x_{n-1}\|\le\|y_{n-1}-x_n\|+\|x_n-x_{n-1}\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \qquad (5.9)$$

Similar to the proof of Theorem 4.1, we have

$$\lim_{n\longrightarrow\infty}\|x_{n-1}-u_{n-1}\|=0 \qquad (5.10)$$

and $\omega_w(x_n)\subset F$. This, together with (5.5) and Lemma 2.4 guarantees the strong convergence of $\{x_n\}$ to $P = P_F x$. From (5.10), we also have the strong convergence of $\{u_n\}$ to $p = P_F x$. This completes the proof. □

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