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Combinatorial probability interpretation of certain modified orthogonal polynomials

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Abstract

A probabilistic interpretation of a modified Gegenbauer polynomial is supplied by its expression in terms of a combinatorial probability defined on a compound urn model. Also, a combinatorial interpretation of its coefficients is provided. In particular, probabilistic interpretations of a modified Chebyshev polynomial of the second kind and a modified Legendre polynomial together with combinatorial interpretations of their coefficients are deduced. Further, probabilistic interpretations of a modified Hermite and a modified Chebyshev polynomial of the first kind are supplied by their expressions in terms of combinatorial probability functions defined on two limiting forms of the compound urn model. Finally, combinatorial interpretations of their coefficients are obtained.

1. Introduction

In Combinatorics, orthogonal polynomials have been used to express enumerating functions of certain combinatorial configurations. Specifically, the mth Hermite polynomial may be viewed as the generating function of the number of fixed points over the set of involutions of $\{1, 2, \ldots, m\}$ (see, e.g. [6, p. 62]). Foata [2] used this interpretation to give a combinatorial proof of the Mehler formula. Also, Foata and Leroux [3] constructed a combinatorial model for the Jacobi polynomials and deduced their classical generating function. Extending slightly this model, Leroux and Strehl [9] obtained combinatorially many of the properties of the Jacobi polynomials. Labelle and Yeh [7,8], using appropriate combinatorial models derived combinatorial classical exact and asymptotic formulas for several orthogonal polynomials.

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In Probability, modified orthogonal polynomials have been used to express the probability function of certain compound (generalized) discrete distributions. Precisely, Kemp and Kemp [5] starting from the probability generating function of a compound Poisson distribution considered the case in which the probability generating function of the compounding distribution is a polynomial of order two. The probability function of this particular compound Poisson distribution were expressed in terms of modified Hermite polynomials. Plunkett and Jain [10] obtained the Hermite mixture, with mixing a gamma distribution, in terms of modified Gegenbauer polynomials. Kemp [4] studied it as a convolution of a binomial with a pseudo-binomial distribution. The Hermite and Gegenbauer distributions together with other distributions that are expressed in terms of orthogonal polynomials, were discussed as compound distributions in [1].

The Hermite, Legendre, Laguerre and Chebyshev polynomials were used by Watson [11] to introduce five distributions on the nonnegative integers. His approach was based on the fact that any nonnegative real function defined on a countable set with values summing to one represents a probability (mass) function of a random variable. Watson remarked that "it would be interesting to know classical probabilistic models which lead to such distributions" and added "although combinatorialists are now associating orthogonal polynomials with problems of enumeration, I have not been able to make a connection".

In the present paper, which is prompted by Watson's remarks, a combinatorial probability defined on a *compound urn model* is expressed in terms of a modified orthogonal polynomial. Note that this modification of an orthogonal polynomial is simply a transformation of it into a polynomial with positive coefficients and values for positive values of its argument. Then, a probabilistic interpretation of the modified orthogonal polynomial is supplied by its expression in terms of the combinatorial probability. Also, a combinatorial interpretation of its coefficients is provided. Section 2 is devoted to the Gegenbauer polynomials and their interesting particular cases of the Chebyshev polynomials of the second kind and the Legendre polynomials. The Hermite polynomials and the Chebyshev polynomials of the first kind are discussed in Sections 3 and 4, respectively.

2. Gegenbauer polynomials

The Gegenbauer polynomials $g_m^{(s)}(x)$, $m = 0, 1, \ldots,$ with $-1/2 < s < 0$ or $0 < s < \infty$, may be defined through their generating function by

$$\sum_{m=0}^{\infty} g_m^{(s)}(x) t^m = (1 - 2xt + t^2)^{-s}.$$

Expanding the generating function into powers of t and equating the coefficients of t^m on both sides of the resulting expression, we get

$$g_m^{(s)}(x) = \sum_{j=0}^{[m/2]} (-1)^j \binom{s+m-j-1}{m-j} \binom{m-j}{j} 2^{m-2j} x^{m-2j}, \quad m = 0, 1, \ldots,$$

where $[m/2]$ denotes the integral part of $m/2$. A modified Gegenbauer polynomial, with positive coefficients, that is positive for positive x is defined by

$$G_m^{(s)}(x) = \mathrm{i}^{-m} g_m^{(s)}(\mathrm{i}x), \quad m = 0, 1, \ldots, \mathrm{i} = \sqrt{-1},$$

so that

$$\sum_{m=0}^{\infty} G_m^{(s)}(x)t^m = (1-2xt-t^2)^{-s}$$

and

$$G_m^{(s)}(x) = \sum_{j=0}^{[m/2]} \binom{s+m-j-1}{m-j}\binom{m-j}{j} 2^{m-2j}x^{m-2j}, \quad m=0,1,\ldots. \tag{2.1}$$

A combinatorial probability interpretation of the modified Gegenbauer polynomial can be given through the following stochastic model.

Compound urn model. *Consider a finite or infinite set W of target and control urns, with each target urn divided into two cells (compartments) of capacity limited to one ball.* (a) *Assume that urns are randomly selected from W one after the other until s control urns are chosen, with probability θ of selecting a target urn at any trial. In the case of a finite set W, the sequential selection of urns is made with replacement so that the proportion θ of target urns in W remains unchanged for all trials.* (b) *Further, assume that two balls are distributed into each selected target urn, placing one ball in each cell, and that a ball has probability p of staying in it* (*and probability $q = 1-p$ of falling through*).

An urn containing at least one ball is referred to as *occupied urn*, while an urn containing two balls is referred to as *fully occupied urn*.

Note that assumption (a) implies that the distribution of the number N of target urns selected from W is negative binomial with the probability function

$$c_n = P(N=n) = \binom{s+n-1}{n}(1-\theta)^s\theta^n, \quad n=0,1,\ldots, 0<\theta<1.$$

Further, assumption (b) implies that the distribution of the number X of balls that stay in any specific target urn is binomial with the probability function

$$q_x = P(X=x) = \binom{2}{x}p^x q^{2-x}, \quad x=0,1,2, q=1-p, 0<p<1.$$

Also, the distribution of the number S_n of balls that stay in n target urns, which are selected from W, is again binomial with the probability function

$$q_m(n) = P(S_n=m) = \binom{2n}{m}p^m q^{2n-m}, \quad m=0,1,\ldots,2n$$

and

$$q_{n,m} = P(N=n, S_n=m) = c_n q_m(n), \quad m=0,1,\ldots,2n, n=0,1,\ldots.$$

Then, the distribution of the number S_N of balls that stay in the target urns that are selected from W, which is a compound negative binomial, with compounding a binomial, has probability function

$$p_m = P(S_N=m) = \sum_{n=0}^{\infty} c_n q_m(n), \quad m=0,1,\ldots,$$

and so

$$p_m = (1-\theta)^s(p/q)^m \sum_{n=[m/2]}^{\infty}\binom{s+n-1}{n}\binom{2n}{m}(\theta q^2)^n, \quad m=0,1,\ldots.$$

The probability p_m, $m = 0, 1, \ldots$, may be expressed in terms of a modified Gegenbauer polynomial as follows. Suppose that n target urns are selected from W and $2n$ balls are distributed into them according to the compound urn model. Then, the probability that k_0 urns are empty, k_1 urns contain one ball each and k_2 urns contain two balls each, with $k_0 + k_1 + k_2 = n$, is given by the trinomial probability

$$\frac{n!}{k_0!k_1!k_2!} q_0^{k_0} q_1^{k_1} q_2^{k_2} = \frac{n!2^{k_1}}{k_0!k_1!k_2!} p^{k_1+2k_2} q^{2k_0+k_1}.$$

Introducing the number of occupied urns $k = n - k_0$ and the number of balls that stay in them $m = k_1 + 2k_2$, it follows that $k_0 = n - k$, $k_1 = 2k - m$ and $k_2 = m - k$. Therefore the probability that k of n selected target urns are occupied and m balls stay in them is given by

$$p_{k,m}(n) = \binom{n}{k} \binom{k}{m-k} 2^{2k-m} p^m q^{2n-m}, \tag{2.2}$$

for $m = k, k+1, \ldots, 2k$ and $k = 0, 1, \ldots, n$. Multiplying it by the probability c_n, that n target urns are selected from W, and summing for $n = 0, 1, \ldots$, it follows that, the probability that k of the selected target urns are occupied and m balls stay in them is given by

$$p_{k,m} = \binom{k}{m-k} 2^{2k-m} p^m q^{2k-m} (1-\theta)^s \theta^k \sum_{n=k}^{\infty} \binom{n}{k} \binom{s+n-1}{n} (\theta q^2)^{n-k},$$

for $m = k, k+1, \ldots, 2k$ and $k = 0, 1, \ldots$. Since

$$\begin{aligned} \sum_{n=k}^{\infty} \binom{n}{k} \binom{s+n-1}{n} (\theta q^2)^{n-k} &= \binom{s+k-1}{k} \sum_{n=k}^{\infty} \binom{s+n-1}{n-k} (\theta q^2)^{n-k} \\ &= \binom{s+k-1}{k} \frac{1}{(1-\theta q^2)^{s+k}}, \end{aligned}$$

it reduces to

$$p_{k,m} = \binom{s+k-1}{k} (1-\lambda)^s \lambda^k \cdot \binom{k}{m-k} 2^{2k-m} \frac{p^m q^{2k-m}}{(1-q^2)^k}, \tag{2.3}$$

for $m = k, k+1, \ldots, 2k$ and $k = 0, 1, \ldots$, with $\lambda = \theta(1-q^2)/(1-\theta q^2)$.

A probability function interpretation of the indicated factors of $p_{k,m}$, analogous to that of the factors of $q_{n,m} = c_n q_m(n)$ can be given through the following stochastic model.

Modified compound urn model. *Consider the same set W of target and control urns as in the compound urn model and modify the assumptions as follows.* (a) *Assume that two balls are distributed into each target urn of W by placing one ball into each cell. Let U be the subset of W that contains the occupied and control urns.* (b) *Further, assume that urns are sequentially selected from U one after the other until s control urns are chosen. In the case of a finite set W, in which U is also finite, the sequential selection of urns is made with replacement so that the proportion of target urns in U remains unchanged for all trials.*

The probability λ of selecting an occupied target urn from U at any trial may be expressed in terms of the probabilities θ and q of the initial compound urn model as follows. Let A be the event of selecting a target urn from W and B, the event that a target urn remains empty after two balls are distributed in it by placing one ball in each cell. Clearly, $\lambda = P(A|A' \cup B')$ and so

$$\lambda = \frac{P(A \cap (A' \cup B'))}{P(A' \cup B')} = \frac{P(A \cap B')}{1 - P(A \cap B)} = \frac{P(A)P(B')}{1 - P(A)P(B)} = \frac{\theta(1-q^2)}{1-\theta q^2}.$$

Then, the distribution of the number K of occupied target urns selected from U is a negative binomial with probability function

$$c_k(\lambda) = P(K = k) = \binom{s+k-1}{k}(1-\lambda)^s \lambda^k, \quad k = 0, 1, \ldots,$$

where $\lambda = \theta(1-q^2)/(1-\theta q^2)$. Also, the probability $p_m(k)$ that m balls stay in k occupied target urns, which are selected from U, equals the conditional probability $p_{k,m}(k)/(1-q^2)$, that m balls stay in k target urns, which are selected from W, given that all k target urns are occupied, and so by (2.2),

$$p_m(k) = \binom{k}{m-k} 2^{2k-m} \frac{p^m q^{2k-m}}{(1-q^2)^k}, \quad m = k, k+1, \ldots, 2k, k = 0, 1, \ldots.$$

Consequently (2.3) is written as

$$p_{k,m} = c_k(\lambda) p_m(k), \quad m = k, k+1, \ldots, 2k, k = 0, 1, \ldots.$$

A reparametrization transforms the probability $p_{k,m}$ to the general term of the modified Gegenbauer polynomial. Specifically, set

$$\alpha = p\sqrt{\theta/(1-\theta q^2)}, \qquad \beta = q\sqrt{\theta/(1-\theta q^2)}.$$

Then

$$p = \frac{\alpha}{\alpha+\beta}, \qquad q = \frac{\beta}{\alpha+\beta}, \qquad \lambda = 2\alpha\beta + \alpha^2$$

and (2.3) is transformed into

$$p_{k,m} = (1 - 2\alpha\beta - \alpha^2)^s \binom{s+k-1}{k}\binom{k}{m-k} 2^{2k-m}\alpha^m \beta^{2k-m}, \tag{2.4}$$

for $k = [m/2], [m/2]+1, \ldots, m$ and $m = 0, 1, \ldots$. Note that the probability function of the number Y of balls that stay in any specific occupied target urn is

$$p_y = P(Y = y) = \begin{cases} 2\alpha\beta/(2\alpha\beta + \alpha^2), & y = 1, \\ \alpha^2/(2\alpha\beta + \alpha^2), & y = 2. \end{cases}$$

Also, since $0 < 2\alpha\beta + \alpha^2 < 1$, with $\alpha > 0$, $\beta > 0$, the new parameter space is

$$0 < \alpha < 1, \qquad 0 < \beta < \frac{1-\alpha^2}{2\alpha}.$$

Further, replacing in (2.4) the number k of occupied target urns by the number $j = m - k$ of fully occupied target urns it follows that

$$p_{m-j,m} = (1 - 2\alpha\beta - \alpha^2)^s \binom{s+m-j-1}{m-j}\binom{m-j}{j} 2^{m-2j}\alpha^m \beta^{m-2j}, \tag{2.5}$$

for $j = 0, 1, \ldots, [m/2], m = 0, 1, \ldots,$ with $0 < \alpha < 1$, $0 < \beta < (1-\alpha^2)/(2\alpha)$. Summing these probabilities for $j = 0, 1, \ldots, [m/2]$ and using (2.1), the probability p_m, $m = 0, 1, \ldots,$ that m balls stay in the target urns that are selected from W or U is expressed as

$$p_m = (1 - 2\alpha\beta - \alpha^2)^s G_m^{(s)}(\beta)\alpha^m, \quad m = 0, 1, \ldots, \tag{2.6}$$

with $0 < \alpha < 1$ and $0 < \beta < (1-\alpha^2)/(2\alpha)$.

A probabilistic interpretation of the mth modified Gegenbauer polynomial (2.1) is given by its expression,

$$G_m^{(s)}(\beta) = \frac{p_m}{p_0\alpha^m},$$

in terms of the combinatorial probability p_m and the probability $\alpha^2 = \lambda p_2$ of choosing a fully occupied target urn from U. Further, the jth coefficient of the mth modified Gegenbauer polynomial,

$$G_{m,j}^{(s)} = \binom{s+m-j-1}{m-j}\binom{m-j}{j}2^{m-2j}, \quad j = 0, 1, \ldots, [m/2], m = 0, 1, \ldots,$$

is the number of different selections of $m - j$ occupied target urns from U that include j fully occupied urns and $m - 2j$ urns each with one of its two cells occupied.

A generalization of the compound urn model is obtained by replacing assumption (a) by the assumption that the distribution of the number N of target urns selected from W is a negative binomial with parameters $\theta \in (0, 1)$ and s, not necessarily a positive integer but $s \in (0, \infty)$. The corresponding generalization of the modified compound urn model is obtained by replacing assumption (b) by the assumption that the distribution of the number K of occupied target urns selected from U is a negative binomial with parameters $\lambda \in (0, 1)$ and $s \in (0, \infty)$. The probabilistic interpretation of the modified Gegenbauer polynomial in terms of the combinatorial probability p_m is not affected by this generalization. As regards the combinatorial interpretation of the jth coefficient of the mth modified Gegenbauer polynomial, $G_{m,j}^{(s)}$, the factor $\binom{s+m-j-1}{m-j}$ does not express any more the different selections of $m-j$ occupied target urns from U; it can be considered as a selection weight of the $m - j$ occupied target urns. In this case the combinatorial interpretation of $G_{m,j}^{(s)}$ is modified as follows. The jth coefficient of the mth modified Gegenbauer polynomial, $G_{m,j}^{(s)}$, is the number of different weighted selections of $m - j$ occupied target urns from U, with weight $\binom{s+m-j-1}{m-j}$, that include j fully occupied urns and $m - 2j$ urns each with one of its two cells occupied.

Furthermore, the number of different selections (or the selection weight) of $m - j$ occupied target urns from U satisfies the recurrence relation

$$(m-j)\binom{s+m-j-1}{m-j} = (s+m-j-1)\binom{s+m-j-2}{m-j-1},$$

for $j = 1, 2, \ldots, m-1$, $m = 2, 3, \ldots$. Also, the number of distributions of the m balls stayed in the $m - j$ occupied target urns that include j fully occupied urns and $m - 2j$ urns each with one of its two cells occupied satisfies the recurrence relation

$$\binom{m-j}{j}2^{m-2j} = 2\binom{m-1-j}{j}2^{m-1-2j} + \binom{m-2-(j-1)}{j-1}2^{m-2-2(j-1)}, \tag{2.7}$$

for $j = 1, 2, \ldots, [(m-1)/2]$, $m = 2, 3, \ldots$. Therefore

$$\begin{aligned}(m-j)&\binom{s+m-j-1}{m-j}\binom{m-j}{j}2^{m-2j}\\ &= 2(s+m-j-1)\binom{s+m-j-2}{m-j-1}\binom{m-1-j}{j}2^{m-1-2j}\\ &\quad+(s+m-j-1)\binom{s+m-j-2}{m-j-1}\binom{m-2-(j-1)}{j-1}2^{m-2-2(j-1)},\end{aligned}$$

for $j = 1, 2, \ldots, [m/2]$, $m = 2, 3, \ldots$. After some algebraic manipulations, we get for the coefficients of the modified Gegenbauer polynomials the recurrence relation

$$mG^{(s)}_{m,j} = 2(s+m-1)G^{(s)}_{m-1,j} + (2s+m-2)G^{(s)}_{m-2,j-1},$$

for $j = 1, 2, \ldots, [m/2]$, $m = 2, 3, \ldots$, which implies for the modified Gegenbauer polynomials the recurrence relation

$$mG^{(s)}_m(\beta) = 2(s+m-1)\beta G^{(s)}_{m-1}(\beta) + (2s+m-2)G^{(s)}_{m-2}(\beta), \tag{2.8}$$

for $m = 2, 3, \ldots$, with $G^{(s)}_0(\beta) = 1$ and $G^{(s)}_1(\beta) = 2s\beta$.

In the particular case $s = 1$, the distribution of the number N of target urns selected from W is geometric with the probability function

$$c_n = P(N = n) = (1-\theta)\theta^n, \quad n = 0, 1, \ldots, 0 < \theta < 1.$$

The distribution of the number K of occupied target urns selected from U is again geometric with the probability function

$$c_k(\lambda) = P(K = k) = (1-\lambda)\lambda^k, \quad k = 0, 1, \ldots, \lambda = \theta(1-q^2)/(1-\theta q^2).$$

Also, the modified Gegenbauer polynomial reduces to the modified Chebyshev polynomial of the second kind

$$U_m(x) = \sum_{j=0}^{[m/2]}\binom{m-j}{j}2^{m-2j}x^{m-2j}, \quad m = 0, 1, \ldots. \tag{2.9}$$

Further, the probability p_m, $m = 0, 1, \ldots$, that m balls stay in the occupied target urns selected from U, until a control urn is chosen, is given by

$$p_m = (1-2\alpha\beta-\alpha^2)U_m(\beta)\alpha^m, \quad m = 0, 1, \ldots, \tag{2.10}$$

with $0 < \alpha < 1$, $0 < \beta < (1-\alpha^2)/(2\alpha)$, and a probabilistic interpretation of the mth modified Chebyshev polynomial of second kind is given by its expression,

$$U_m(\beta) = \frac{p_m}{p_0\alpha^m},$$

in terms of the combinatorial probability p_m and the probability $\alpha^2 = \lambda p_2$ of choosing a fully occupied target urn from U.

In the other particular case $s = 1/2$, the modified Gegenbauer polynomial reduces to the modified Legendre polynomial

$$P_m(x) = \sum_{j=0}^{[m/2]} \binom{1/2+m-j-1}{m-j} \binom{m-j}{j} 2^{m-2j} x^{m-2j}, \quad m = 0, 1, \ldots, \tag{2.11}$$

with the generating function

$$\sum_{m=0}^{\infty} P_m(x) t^m = (1 - 2xt - t^2)^{-1/2}.$$

The probability $p_m, m = 0, 1, \ldots$, that m balls stay in the occupied target urns selected from U, is given by

$$p_m = (1 - 2\alpha\beta - \alpha^2)^{1/2} P_m(\beta)\alpha^m, \quad m = 0, 1, \ldots, \tag{2.12}$$

with $0 < \alpha < 1, 0 < \beta < (1 - \alpha^2)/(2\alpha)$, and a probabilistic interpretation of the mth modified Legendre polynomial is given by its expression,

$$P_m(\beta) = \frac{p_m}{p_0 \alpha^m},$$

in terms of the combinatorial probability p_m and the probability $\alpha^2 = \lambda p_2$ of choosing a fully occupied target urn from U.

3. Hermite polynomials

The Hermite polynomials $h_m(x)$, $m = 0, 1, \ldots$, can be defined through their generating function by

$$\sum_{m=0}^{\infty} h_m(x) \frac{t^m}{m!} = \mathrm{e}^{2xt - t^2}$$

and so

$$h_m(x) = m! \sum_{j=0}^{[m/2]} (-1)^j \frac{1}{(m-j)!} \binom{m-j}{j} 2^{m-2j} x^{m-2j}, \quad m = 0, 1, \ldots.$$

A modified Hermite polynomial, with positive coefficients, that is positive for all positive x is defined by

$$H_m(x) = \mathrm{i}^{-m} h_m(\mathrm{i}x), \quad m = 0, 1, \ldots, \mathrm{i} = \sqrt{-1},$$

so that

$$\sum_{m=0}^{\infty} H_m(x) \frac{t^m}{m!} = \mathrm{e}^{2xt + t^2}$$

and

$$H_m(x) = m! \sum_{j=0}^{[m/2]} \frac{1}{(m-j)!} \binom{m-j}{j} 2^{m-2j} x^{m-2j}, \quad m = 0, 1, \ldots. \tag{3.1}$$

Consider the following limiting form of the compound urn model. Assume that $s \to \infty$ and $\theta \to 0$ so that $s\theta \to \mu$. Then, the distribution of the number N of occupied urns that are randomly selected from W is Poisson with the probability function

$$c_n = P(N = n) = \mathrm{e}^{-\theta} \frac{\theta^n}{n!}, \quad n = 0, 1, \ldots, 0 < \theta < \infty, \tag{3.2}$$

where the parameter μ is replaced by θ. The probability that k of the selected urns from W are occupied and m balls stay in them, on using (2.2) and (3.2), is obtained as

$$p_{k,m} = \binom{k}{m-k} 2^{2k-m} p^m q^{2k-m} \mathrm{e}^{-\theta} \frac{\theta^k}{k!} \sum_{n=k}^{\infty} \frac{(\theta q^2)^{n-k}}{(n-k)!}$$

and so

$$p_{k,m} = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} \cdot \binom{k}{m-k} 2^{2k-m} \frac{p^m q^{2k-m}}{(1-q^2)^k}, \tag{3.3}$$

for $m = k, k+1, \ldots, 2k$ and $k = 0, 1, \ldots,$ with $\lambda = \theta(1-q^2)$. Clearly, the distribution of the number K of occupied target urns selected from U, according to the modified compound urn model, is Poisson with the probability function

$$c_k(\lambda) = P(K = k) = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots, \lambda = \theta(1-q^2).$$

Also, the probability $p_m(k)$ that m balls stay in k occupied target urns, which are selected from U, is given by

$$p_m(k) = \binom{k}{m-k} 2^{2k-m} \frac{p^m q^{2k-m}}{(1-q^2)^k}, \quad m = k, k+1, \ldots, 2k, k = 0, 1, \ldots.$$

Thus

$$p_{k,m} = c_k(\lambda) p_m(k), \quad m = k, k+1, \ldots, 2k, k = 0, 1, \ldots.$$

Setting

$$\alpha = p\sqrt{\theta}, \qquad \beta = q\sqrt{\theta},$$

we get

$$p = \frac{\alpha}{\alpha+\beta}, \qquad q = \frac{\beta}{\alpha+\beta}, \qquad \lambda = 2\alpha\beta + \alpha^2$$

and (3.3) is transformed into

$$p_{k,m} = \mathrm{e}^{-2\alpha\beta-\alpha^2} \frac{1}{k!} \binom{k}{m-k} 2^{2k-m} \alpha^m \beta^{2k-m},$$

for $k = [m/2], [m/2]+1, \ldots, m, m = 0, 1, \ldots,$ with $\alpha > 0, \beta > 0$. Replacing the number k of occupied target urns by the number $j = m-k$ of fully occupied target urns it follows that

$$p_{m-j,m} = \mathrm{e}^{-2\alpha\beta-\alpha^2} \frac{1}{(m-j)!} \binom{m-j}{j} 2^{m-2j} \alpha^m \beta^{m-2j}, \tag{3.4}$$

for $j = 0, 1, \ldots, [m/2]$, $m = 0, 1, \ldots$, with $\alpha > 0$, $\beta > 0$. Summing these probabilities for $j = 0, 1, \ldots, [m/2]$ and using (3.1), the probability p_m, $m = 0, 1, \ldots$, that m balls stay in the target urns that are selected from U is expressed as

$$p_m = \mathrm{e}^{-2\alpha\beta-\alpha^2} H_m(\beta)\frac{\alpha^m}{m!}, \quad m = 0, 1, \ldots. \tag{3.5}$$

A probabilistic interpretation of the mth modified Hermite polynomial (3.1) is furnished by its expression,

$$H_m(\beta) = \frac{p_m}{p_0\alpha^m/m!},$$

in terms of the combinatorial probability p_m. Also, the jth coefficient of the mth modified Hermite polynomial

$$H_{m,j} = \frac{m!}{(m-j)!}\binom{m-j}{j}2^{m-2j}, \quad j = 0, 1, \ldots, [m/2], m = 0, 1, \ldots,$$

is the number of different weighted selections of $m-j$ occupied target urns from U, with weight $(m)_j = m!/(m-j)!$, that include j fully occupied urns and $m-2j$ urns with one of its two cells occupied. Further, multiplying recurrence relation (2.7) by the weight $(m)_j$, we get for the coefficients of the modified Hermite polynomials the recurrence relation

$$H_{m,j} = 2H_{m-1,j} + 2(m-1)H_{m-2,j-1},$$

for $j = 1, 2, \ldots, [m/2]$, $m = 2, 3, \ldots$, which implies for the modified Hermite polynomials the recurrence relation

$$H_m(\beta) = 2\beta H_{m-1}(\beta) + 2(m-1)H_{m-2}(\beta), \tag{3.6}$$

for $m = 2, 3, \ldots$, with $H_0(\beta) = 1$ and $H_1(\beta) = 2\beta$.

4. Chebyshev polynomials of the first kind

The Chebyshev polynomials of the first kind $t_m(x)$, $m = 1, 2, \ldots$, may be defined through their generating function by

$$\sum_{m=1}^{\infty} t_m(x)u^m = -\log(1 - 2xu + u^2)$$

and so

$$t_m(x) = \sum_{j=0}^{[m/2]}(-1)^j\frac{1}{m-j}\binom{m-j}{j}2^{m-2j}x^{m-2j}, \quad m = 1, 2, \ldots.$$

Note that $t_m(x)$, $m = 1, 2, \ldots$, are closely connected to the classical Chebyshev polynomials of the first kind $C_m(x)$, $m = 0, 1, \ldots$, defined by

$$\sum_{m=0}^{\infty} C_m(x)u^m = \frac{1-xu}{1-2xu+u^2},$$

with $C_0(x) = 1$. Indeed, differentiating the generating function of $t_m(x)$, $m = 1, 2, \ldots$, it readily follows that $t_m(x) = (2/m)C_m(x)$. A modified Chebyshev polynomial, with positive coefficients, that is positive for all positive x is defined by

$$T_m(x) = \mathrm{i}^{-m} t_m(\mathrm{i}x), \quad m = 1, 2, \ldots, \mathrm{i} = \sqrt{-1},$$

so that

$$\sum_{m=1}^{\infty} T_m(x) u^m = -\log(1 - 2xu - u^2)$$

and

$$T_m(x) = \sum_{j=0}^{[m/2]} \frac{1}{m-j} \binom{m-j}{j} 2^{m-2j} x^{m-2j}, \quad m = 1, 2, \ldots. \tag{4.1}$$

Consider the following limiting form of the compound urn model. Assume that the number of occupied urns that are randomly selected from W follows a zero truncated negative binomial distribution, with

$$P(N = n) = [(1-\theta)^{-s} - 1]^{-1} \binom{s+n-1}{n} \theta^n, \quad n = 1, 2, \ldots,$$

where $0 < \theta < 1$, $s > 0$, and let $s \to 0$. Clearly, the limiting distribution is a logarithmic with the probability function

$$c_n = [-\log(1-\theta)]^{-1} \frac{\theta^n}{n}, \quad n = 1, 2, \ldots, 0 < \theta < 1. \tag{4.2}$$

The probability that k of the selected urns from W are occupied and m balls stay in them, on using (2.2) and (4.2), is obtained as

$$p_{k,m} = \binom{k}{m-k} 2^{2k-m} p^m q^{2k-m} [-\log(1-\theta)]^{-1} \frac{\theta^k}{k} \sum_{n=k}^{\infty} \binom{n-1}{k-1} (\theta q^2)^{n-k}$$

and since

$$\sum_{n=k}^{\infty} \binom{n-1}{k-1} (\theta q^2)^{n-k} = \frac{1}{(1-\theta q^2)^k},$$

it reduces to

$$p_{k,m} = [-\log(1-\theta)]^{-1} \frac{\lambda^k}{k} \binom{k}{m-k} 2^{2k-m} \frac{p^m q^{2k-m}}{(1-q^2)^k}, \tag{4.3}$$

for $m = k, k+1, \ldots, 2k$ and $k = 1, 2, \ldots$, with $\lambda = \theta(1-q^2)/(1-\theta q^2)$. Also,

$$p_{k,0} = [-\log(1-\theta)]^{-1} \frac{(\theta q^2)^k}{k}, \quad k = 1, 2, \ldots. \tag{4.4}$$

Further, the distribution of the number K of occupied target urns selected from U, according to the modified compound urn model, is a modified logarithmic with the probability function

$$
\begin{aligned}
c_k(\lambda) = P(K=k) &= \sum_{n=k}^{\infty} P(N=n)P(K=k|N=n) \\
&= [-\log(1-\theta)]^{-1} \frac{[\theta(1-q^2)]^k}{k} \sum_{n=k}^{\infty} \binom{n-1}{k-1} (\theta q^2)^{n-k} \\
&= [-\log(1-\theta)]^{-1} \frac{[\theta(1-q^2)/(1-\theta q^2)]^k}{k},
\end{aligned}
$$

for $k = 1, 2, \ldots$, and

$$
\begin{aligned}
c_0(\lambda) = P(K=0) &= \sum_{n=1}^{\infty} P(N=n)P(K=0|N=n) \\
&= [-\log(1-\theta)]^{-1} \sum_{n=1}^{\infty} \frac{(\theta q^2)^n}{n} \\
&= [-\log(1-\theta)]^{-1} [-\log(1-\theta q^2)].
\end{aligned}
$$

Setting

$$
\alpha = p\sqrt{\theta/(1-\theta q^2)}, \qquad \beta = q\sqrt{\theta/(1-\theta q^2)},
$$

we get

$$
p = \frac{\alpha}{\alpha+\beta}, \qquad q = \frac{\beta}{\alpha+\beta}, \qquad \theta = \frac{(\alpha+\beta)^2}{1+\beta^2}, \qquad \lambda = 2\alpha\beta + \alpha^2
$$

and (4.3) and (4.4) are transformed into

$$
p_{k,m} = \left[-\log\left(1 - \frac{(\alpha+\beta)^2}{1+\beta^2}\right)\right]^{-1} \frac{1}{k} \binom{k}{m-k} 2^{2k-m} \alpha^m \beta^{2k-m},
$$

for $m = k, k+1, \ldots, 2k$, $k = 1, 2, \ldots$, and

$$
p_{k,0} = \left[-\log\left(1 - \frac{(\alpha+\beta)^2}{1+\beta^2}\right)\right]^{-1} \frac{[\beta^2/(1+\beta^2)]^k}{k}, \quad k = 1, 2, \ldots.
$$

Replacing the number k of occupied target urns by the number $j = m - k$ of fully occupied target urns it follows that

$$
p_{m-j,m} = \left[-\log\left(1 - \frac{(\alpha+\beta)^2}{1+\beta^2}\right)\right]^{-1} \frac{1}{m-j} \binom{m-j}{j} 2^{m-2j} \alpha^m \beta^{m-2j}, \tag{4.5}
$$

for $j = 0, 1, \ldots, [m/2]$, $m = 1, 2, \ldots$, with $0 < \alpha < 1$, $0 < \beta < (1-\alpha^2)/(2\alpha)$. Then, the probability p_m, $m = 0, 1, \ldots$, that m balls stay in the target urns that are selected from U, on using (4.1), is expressed as

$$
p_m = \left[-\log\left(1 - \frac{(\alpha+\beta)^2}{1+\beta^2}\right)\right]^{-1} T_m(\beta)\alpha^m, \quad m = 1, 2, \ldots, \tag{4.6}
$$

with $0 < \alpha < 1, 0 < \beta < (1-\alpha^2)/(2\alpha)$, and

$$p_0 = \left[-\log\left(1 - \frac{(\alpha+\beta)^2}{1+\beta^2}\right)\right]^{-1}\left[-\log\left(\frac{1}{1+\beta^2}\right)\right]. \tag{4.7}$$

A probabilistic interpretation of the mth modified Chebyshev polynomial of the first kind (4.1) is given by its expression,

$$T_m(\beta) = \frac{2\beta p_m}{p_1\alpha^m}, \quad m = 1, 2, \ldots,$$

in terms of the combinatorial probability p_m. Also, the jth coefficient of the mth modified Chebyshev polynomial of the first kind,

$$T_{m,j} = \frac{1}{m-j}\binom{m-j}{j}2^{m-2j}, \quad j = 0, 1, \ldots, [m/2], m = 1, 2, \ldots,$$

is the number of different weighted selections of $m-j$ occupied target urns from U, with weight $1/(m-j)$, that includes j fully occupied urns and $m-2j$ urns with one of its two cells occupied. Further, multiplying recurrence relation (2.7) by the weight $1/(m-j)$, we get, after a little algebra, the following recurrence relation for the coefficients of the modified Chebyshev polynomials of the first kind

$$mT_{m,j} = 2(m-1)T_{m-1,j} + (m-2)T_{m-2,j-1},$$

for $j = 1, 2, \ldots, [(m-1)/2]$, $m = 3, 4, \ldots$, which implies for the modified Chebyshev polynomials of the first kind the recurrence relation

$$mT_m(\beta) = 2(m-1)\beta T_{m-1}(\beta) + (m-2)T_{m-2}(\beta), \tag{4.8}$$

for $m = 3, 4, \ldots$, with $T_1(\beta) = 2\beta$ and $T_2(\beta) = 2\beta^2 + 1$.

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