The algebra $IA^{fuz}$: a framework for qualitative fuzzy temporal reasoning

Silvana Badaloni\textsuperscript{a,*}, Massimiliano Giacomin\textsuperscript{b}

\textsuperscript{a} Dipartimento di Ingegneria dell’Informazione, Università di Padova, Via Gradenigo 6B, I-35131 Padova, Italy
\textsuperscript{b} Dipartimento di Elettronica per l’Automazione, Università di Brescia, Via Branze 38, I-25123 Brescia, Italy

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Abstract

The aim of this work is to integrate the ideas of flexibility and uncertainty into Allen’s interval-based temporal framework, defining a new formalism, called $IA^{fuz}$, which extends classical Interval Algebra (IA), in order to express qualitative fuzzy constraints between intervals. We generalize the classical operations between IA-relations to $IA^{fuz}$-relations, as well as the concepts of minimality and local consistency, referring to the framework of Fuzzy Constraint Satisfaction Problem. We analyze the most interesting reasoning tasks in our framework, which generalize the classical problems of checking consistency, finding a solution and computing the minimal network in the context of IA. In order to solve these tasks, we devise two constraint propagation algorithms and a Branch & Bound algorithm. Since these tasks are NP-difficult, we address the problem of finding tractable sub-algebras of $IA^{fuz}$, by extending to our fuzzy framework the classical pointizable sub-algebras $SA_c$ and $SA$, as well as the maximal tractable subalgebra $\mathcal{H}$ introduced by Nebel. In particular, we prove that the fuzzy extension of the latter, called $\mathcal{H}_c^{fuz}$, shares with its classical counterpart a maximality property, in that it is the unique maximal subalgebra of $IA^{fuz}$ which contains the fuzzy extensions of Allen’s atomic relations.

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1. Introduction

Representing and reasoning about time is an essential task in many areas of Artificial Intelligence, such as planning, scheduling, human–machine interaction, natural language understanding, diagnosis, temporal data-base management, and robotics. Several approaches for temporal reasoning have been proposed in the literature, which can be distinguished on the basis of whether they focus on the qualitative or quantitative aspect of temporal knowledge. Among the approaches of the first kind, the Interval Algebra (IA) [1] and the Point Algebra (PA) [2,3] deal with a qualitative representation of temporal knowledge relative to intervals and points respectively. On the other hand, quantitative approaches, such as [4–6], deal with metric temporal statements concerning points. Furthermore, hybrid approaches
have been proposed in order to combine the expressiveness capabilities of these formalisms [7,8], where qualitative and metric information are integrated in a single model.

All of these proposals rely on the framework of Constraint Satisfaction Problem (CSP), in that they approach the relevant reasoning tasks by representing the temporal objects as variables with temporal domains, and the available temporal knowledge as a set of constraints between these variables. Unfortunately, these temporal constraint-based reasoning approaches inherit from CSP a number of fundamental limitations, mainly related to a lack of flexibility and a limited representation of uncertainty [9,10]. First, while usually in real world problems constraints are satisfied to a degree, rather than satisfied or not satisfied, only hard temporal constraints can be represented in classical approaches, making it impossible to tolerate partial violation of constraints and to account for preferences among feasible solutions. Moreover, a related issue concerns the inability to associate different priorities to constraints, with the aim of satisfying as many as possible of the most important ones. Finally, classical approaches account for uncertainty, which pervades most practical problems, in a limited way. For instance, in Interval Algebra a constraint is expressed as a set of equally possible atomic relations which can hold between two intervals, so that the uncertainty relative to their mutual position is only related to the cardinality of this set. It is impossible to express more refined knowledge concerning the uncertainty affecting constraints, as in the case where the presence of a particular constraint is not certain, but we have some idea about its degree of “plausibility”.

In order to overcome these limitations, the CSP formalism has been extended in a fuzzy direction, by replacing classical crisp constraints with soft constraints modeled by fuzzy relations. Along the same line, a number of temporal reasoning approaches based on the Fuzzy Constraint Satisfaction Problem (FCSP) [9] has been devised, such as [11,12], in order to handle temporal information in terms of points (or times of events) and in terms of fuzzy metric constraints between them. A relevant application has been proposed in job-shop scheduling problems [9], where the temporal displacement of activities is represented in terms of their endpoints. For instance, a typical constraint in this context concerns the duration of a particular activity $A_1$, modeled as a distribution of values taken by the difference between its endpoints; this way, it is possible to state example that activity $A_1$ lasts between 5 and 10 minutes, but that its duration should be as short as possible.

While these proposals deal with soft temporal constraints extending classical quantitative point-based approaches, in this paper we focus our attention on representing the qualitative aspect of temporal knowledge, by extending the interval-based framework proposed by Allen [1]. The motivations underlying our proposal can be summarized as follows. From a human-computer interaction perspective, temporal intervals seem to be a natural means of human reference to time [13], as witnessed by the adoption of temporal specification about intervals in many applications of temporal reasoning, such as those proposed for the medical domain [14]. Furthermore, in many real world problems only qualitative temporal knowledge is needed or available; this is the case of planning systems that need a temporal world model in order to handle concurrency [15]. Of course, it is always possible to translate relations between intervals into relations between their endpoints, thus referring to a point-based representation of temporal knowledge. However, as in the classical case this conversion is not so convenient both for computational reasons [1] and mainly because not all binary relations between intervals can be represented as binary relations between endpoints [2]. Thus, our extension of Interval Algebra allows us to represent temporal relations that cannot be expressed in current fuzzy temporal reasoning formalisms. As an example, a constraint of the kind “activity $A_1$ must be disjoint from activity $A_2$, and it is better that $A_1$ is before $A_2$” may be required in a practical scheduling application: this constraint can be represented in our approach but it cannot in a point-based framework.

The aim of this work is to integrate the notions of flexibility and uncertainty into classical Interval Algebra (IA), defining a new formalism, that we call $IA^{flex}$, which includes different types of temporal constraints:

- soft constraints, that enable us to express preferences among solutions;
- prioritized constraints, the priority indicating how essential it is that a constraint be satisfied;
- uncertain constraints, expressed by means of prioritized constraints, their priority corresponding to their degree of plausibility.

Dealing with computational aspects, a major problem arises both with classical temporal CSPs as well as with FCSPs, namely the tractability of the reasoning task. It is well known that classical Allen’s Interval Algebra is non-tractable, since the main interesting problems in this framework, such as consistency checking, are NP-complete [3]. Since this fact seems to preclude general, large-scale applications, there has been a lot of work in identifying tractable
sub-classes of IA [3,16–18]. The problem of tractability also holds for all of the fuzzy extensions of temporal CSPs, including ours. In fact, since IA\(fuz\) is defined as a generalization of IA, if the complete expressive power of its language is maintained then the intractability of the reasoning problem affects IA\(fuz\) too. Thus, the second main aim of this paper is to investigate whether an extension of classical results about tractability of temporal sub-algebras is possible, by addressing the problem of finding tractable sub-algebras of IA\(fuz\). For this purpose, we first show how point-algebras PA\(_c\) and PA [3], and interval sub-algebras SA\(_c\) and SA [16] can be fuzzy extended. Then, we prove that the fuzzy extensions of SA\(_c\) and SA, called SA\(_c^{fuz}\) and SA\(_{fuz}\) respectively, are tractable sub-algebras of IA\(fuz\). Finally, we show that an analogous result holds for the fuzzy extension of the ORD-Horn maximal subclass [18], and that the latter is the unique maximal tractable subalgebra of IA\(fuz\) which contains atomic relations. The relevant proofs follow our general methodology devised in [19], which exploits the relationships between minimality and \(k\)-consistency of fuzzy networks and minimality and \(k\)-consistency of their \(\alpha\)-cuts.

The paper is organized as follows. After giving a complete formalization of the fuzzy interval algebra IA\(fuz\) in Section 2, we define in Section 3 the main reasoning tasks to be addressed in this framework. In Section 4, we describe how to deal with prioritized constraints and uncertainty. Section 5 deals with algorithmic issues, introducing the Path-Consistency, Branch & Bound and AAC algorithms in the context of IA\(fuz\)-networks, while the main tractability results are shown in Sections 6 and 7. Finally, Section 8 provides a comparison with related work and some conclusions, while Appendix A contains all the proofs of the results presented in the paper.

2. The algebra IA\(fuz\)

2.1. Basic definitions and semantics

In the temporal reasoning framework of classical Interval Algebra (IA), temporal knowledge concerns qualitative relations about intervals. More specifically, given a set of variables representing intervals and related by a set of qualitative constraints, the typical reasoning tasks consist in checking the consistency of the constraints, finding a solution, and inferring implicit constraints from the given ones. Each constraint is a binary relation between a pair of intervals, represented by a disjunction of atomic relations:

\[ I_1(\text{rel}_1, \ldots, \text{rel}_m)I_2 \]

where \(\text{rel}_p\) is an element of the set \(\mathcal{I}\), shown in Table 1, which includes the 13 mutually exclusive atomic relations that may exist between two intervals [1] (\(eq\) stands for \textit{equal}, \(b\) stands for \textit{before}, etc.). In order to solve the interesting tasks mentioned above, Interval Algebra can be viewed as an instance of the Constraint Satisfaction Problem (CSP), since an interval can be interpreted as an element of \(\mathbb{R}^2\) and a relation between a pair of intervals as a subset of \(\mathbb{R}^2 \times \mathbb{R}^2\). This way, general CSP techniques and algorithms can be specialized for the framework of qualitative temporal reasoning, often exploiting specific properties that follow from the particular shapes characterizing IA-constraints [16,20–22].

In order to integrate the ideas of flexibility and uncertainty into Allen’s framework, we relax the definition of IA-constraints, by assigning to every atomic relation \(\text{rel}_p\) a degree \(\alpha_p\), which indicates the preference degree of the corresponding assignment among the others [23]. Accordingly, we deal with relations between intervals \(I_1\) and \(I_2\) in the form

\[ I_1RI_2 \text{ with } R = (\text{rel}_1[\alpha_1], \ldots, \text{rel}_{13}[\alpha_{13}]) \]

where \(\alpha_p\) is the preference degree of \(\text{rel}_p\) (\(p = 1, \ldots, 13\)), denoted as \(d_{\text{er}}(\text{rel}_p)\). Preference degrees belong to a totally ordered set \(L\), which includes a top element, denoted as 1, and a bottom element denoted as 0: while in all the examples that will be made in the following we will assume \(L\) as the interval [0, 1], other choices are possible, such as a finite set of qualitative degrees.

<table>
<thead>
<tr>
<th>(\text{rel})</th>
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<tr>
<td>(\text{rel}^{-1})</td>
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Table 1

Allen’s atomic relations and their inverse
As for the semantics of these relations, which we call IA\textsuperscript{fuz}-relations, we rely on the Fuzzy Constraint Satisfaction Problem framework (FCSP) [9], which generalizes the CSP framework by replacing hard constraints with fuzzy relations [24]. In particular, we interpret the variables $I_i$, denoting intervals, as elements of $\mathbb{R}^2$, and a constraint $I_1 R I_2$, where $R$ is an IA\textsuperscript{fuz}-relation, as a fuzzy constraint between the two intervals $I_1$ and $I_2$ expressed by a fuzzy subset of $\mathbb{R}^2 \times \mathbb{R}^2$. The latter is defined by the following membership function $\mu_R$:

$$
\mu_R(I_1, I_2) = \deg_R(\text{rel}_p \in I: I_1 \text{rel}_p I_2) \quad (1)
$$

where $I_1$ and $I_2$ denote instances in $\mathbb{R}^2$ of the variables $I_1$ and $I_2$, respectively. In words, in case of ‘atomic’ IA\textsuperscript{fuz}-relations of the form $\text{rel}_q[\alpha]$, those pairs of intervals which satisfy “classically” $\text{rel}_q$ have membership degree $\alpha$, all of the others have membership degree 0; generic IA\textsuperscript{fuz}-relations represent the union of the fuzzy subsets corresponding to every $\text{rel}_p[\deg_R(\text{rel}_p)]$. This way, the membership function $\mu_R$ indicates to what extent each atomic relation satisfies the constraint represented by $R$.

As an example, the constraint concerning the disjointness of activities mentioned in the previous section may be expressed by means of the IA\textsuperscript{fuz}-constraint

$$I_1(b, a[0.7]) I_2$$

where $I_1$ and $I_2$ denote the temporal intervals during which $A_1$ and $A_2$ are executed, respectively. According to (1), this constraint is represented by the fuzzy relation shown in Fig. 1.

As in the classical case, we formalize our interval-based fuzzy framework in terms of algebra of relations:

**Definition 1.** IA\textsuperscript{fuz} is the algebra with the following underlying set

$$\mathcal{I}A_{\text{fuz}} = \{(b[\alpha_1], a[\alpha_2], m[\alpha_3], mi[\alpha_4], d[\alpha_5], di[\alpha_6], o[\alpha_7], oi[\alpha_8], s[\alpha_9], si[\alpha_{10}], f[\alpha_{11}], fi[\alpha_{12}], eq[\alpha_{13}])\}$$

where $\alpha_p \in L$, $p = 1, \ldots, 13$

and with the unary operation of inversion and binary operations of conjunction and composition (defined in the following).

In this framework, temporal knowledge can be represented by means of interval-based constraint networks, that we define in a similar way as IA-networks:

**Definition 2.** An IA\textsuperscript{fuz}-network $\mathcal{N}$ is a pair $\langle \mathcal{X}, \mathcal{C} \rangle$, where $\mathcal{X} = \{I_1, \ldots, I_n\}$ is a set of variables (representing intervals), and $\mathcal{C} = \{R_{ij} \in \mathcal{I}A_{\text{fuz}} \mid 0 < i, j \leq n, i \neq j\}$ is the set of IA\textsuperscript{fuz}-relations that constrain variables (without loss of generality, we assume that a relation is defined for every couple of intervals).

An IA\textsuperscript{fuz}-network $\mathcal{N} = \langle \mathcal{X}, \mathcal{C} \rangle$ can be graphically represented as a labeled directed graph, whose nodes represent variables and whose edges are labeled with the corresponding IA\textsuperscript{fuz}-relations. Accordingly, each element of the set $\{(I_i, I_j) \in \mathcal{X}^2 \mid I_i \neq I_j\}$, the latter being denoted as $\mathcal{E}(\mathcal{N})$, is called an edge of $\mathcal{N}$. Sometimes $(I_i, I_j)$ will be simply denoted as $(i, j)$.

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1 In order to make our language compatible with classical notation, if $\alpha_p = 0$ then $\text{rel}_p$ is omitted, while if $\alpha_p = 1$ then $\text{rel}_p[\alpha_p]$ is denoted as $\text{rel}_p$. 

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Fig. 1. Example of IA\textsuperscript{fuz}-relation.
Having defined the semantics of IA\textsuperscript{fuz}-relations in terms of fuzzy constraints, an IA\textsuperscript{fuz}-network \( (\mathcal{X}, C) \) is naturally interpreted as a FCSP whose variables have \( \mathbb{R}^2 \) as domain and whose constraints are the IA\textsuperscript{fuz}-relations of \( C \). As a consequence, we can define the usual concepts of the general FCSP framework in the context of our interval based framework: in the rest of this subsection, we investigate how the definitions of local consistency and solution are generalized to flexible notions in the IA\textsuperscript{fuz}-framework.

In classical IA-networks, an assignment is defined as a labeling of a subset of the edges with Allen’s atomic relations. Such an assignment is locally consistent if it satisfies all the constraints involved in the relative subnetwork and if it is consistent (i.e. it is possible to map each node involved by the assignment into an element of \( \mathbb{R}^2 \) and have the atomic relations of the assignment to hold). In our framework, local consistency is a gradual notion, i.e. it is expressed by a degree of satisfaction which denotes the acceptability of an assignment with respect to the soft constraints involved in the relative sub-network. In particular, according to the fuzzy conjunctive way of aggregating constraints \[24\], the degree of satisfaction corresponds to the least satisfied constraint.

**Definition 3.** Given an IA\textsuperscript{fuz}-network \( \mathcal{N} = (\mathcal{X}, C) \), a (partial) singleton labeling \( s \) of \( \mathcal{N} \) is a pair \( s = \langle N(s), \{s_{ij}\} \rangle \), where \( N(s) \subseteq \mathcal{X} \) identifies a subset of the nodes, and \( \{s_{ij}\}: \mathcal{E}(s) \to \mathcal{I} \), with \( \mathcal{E}(s) = \{(i, j) \in N(s) \times N(s) \mid i \neq j\} \), is a function that assigns an atomic relation \( s_{ij} \in \mathcal{I} \) to each of the edges \( (i, j) \in \mathcal{E}(s) \) involved in \( s \). If \( N(s) = \mathcal{X} \), and therefore \( \mathcal{E}(s) = \mathcal{E}(\mathcal{N}) \), then \( s \) is a complete singleton labeling. The set of all the complete singleton labelings of \( \mathcal{N} \) is denoted as \( SL_{\mathcal{N}} \).

**Definition 4.** A partial singleton labeling \( s \) is consistent iff
\[
\exists f : N(s) \to \mathbb{R}^2 \text{ such that } \forall (i, j) \in \mathcal{E}(s) f(i) s_{ij} f(j)
\]

**Definition 5.** Given an IA\textsuperscript{fuz}-network \( \mathcal{N} \), the degree of local consistency of a partial singleton labeling \( s \), denoted as \( \deg_{\mathcal{N}}(s) \), is defined as follows:
\[
\deg_{\mathcal{N}}(s) = \begin{cases} 
0 & \text{if } s \text{ is not consistent} \\
\min_{(i,j) \in \mathcal{E}(s)} \deg_{R_{ij}}(s_{ij}) & \text{otherwise}
\end{cases}
\]

As an example, the assignment \((I_1 m I_2, I_2 m I_3, I_1 b I_3)\) for the subnetwork of Fig. 2 has a degree of local consistency 0.5, corresponding to the least satisfied constraint \( R_{23} \), while the assignment \((I_1 m I_2, I_2 m I_3, I_1 m I_3)\) has a degree 0, since it is not possible to arrange the three intervals in the corresponding way.

Since the concept of local consistency is now graded, also the notion of solution of a given network becomes flexible: every solution satisfies network constraints to a degree, called degree of satisfaction, reflecting a trade-off among potentially conflicting constraints. Again, the degree of satisfaction corresponds to the least satisfied constraint.

**Definition 6.** Given an IA\textsuperscript{fuz}-network \( \mathcal{N} \), a solution \( s \) of \( \mathcal{N} \) is simply defined as a complete singleton labeling of \( \mathcal{N} \), and its degree of satisfaction is defined as \( \deg_{\mathcal{N}}(s) \). As a consequence, the set of the solutions of \( \mathcal{N} \), denoted as \( \text{SOL}(\mathcal{N}) \), is the fuzzy set defined by the following membership function:
\[
\mu_{\text{SOL}(\mathcal{N})}: SL_{\mathcal{N}} \to L, \quad \text{where } \mu_{\text{SOL}(\mathcal{N})}(s) = \deg_{\mathcal{N}}(s)
\]

![Fig. 2. Example of IA\textsuperscript{fuz}-network.](image)
2.2. Operations of the algebra $\text{IA}^{\text{fuz}}$

As it will be shown in the following, an important reasoning activity in our framework is rendering the constraints of a given $\text{IA}^{\text{fuz}}$-network more explicit, by means of constraint propagation algorithms that apply appropriate operations on the $\text{IA}^{\text{fuz}}$-relations associated to the network constraints. The three operations of $\text{IA}^{\text{fuz}}$-algebra correspond to the usual operations on fuzzy relations [9,25] adapted to our interval-based framework. More specifically, these operations are defined as follows (though not included in the definition of $\text{IA}^{\text{fuz}}$, we also define the operation of disjunction under which the set $\mathcal{IA}^{\text{fuz}}$ is closed).

**Definition 7.** Given the $\text{IA}^{\text{fuz}}$-relation $R = (\text{rel}[\alpha_1], \ldots, \text{rel}_{13}[\alpha_{13}])$, the unary operation of inversion $R^{-1}$ is defined as

$$R^{-1} = (\text{rel}_{13}^{-1}[\alpha_1], \ldots, \text{rel}_1^{-1}[\alpha_{13}])$$

where $\text{rel}_p^{-1}$ is defined as in Table 1.

**Definition 8.** Given two $\text{IA}^{\text{fuz}}$-relations $R' = (\text{rel}_1[\alpha_1'], \ldots, \text{rel}_{13}[\alpha_{13}'])$ and $R'' = (\text{rel}_1[\alpha_1''], \ldots, \text{rel}_{13}[\alpha_{13}''])$, the conjunction $R = R' \otimes R''$ is defined as

$$R = (\text{rel}_1[\alpha_1], \ldots, \text{rel}_{13}[\alpha_{13}])$$

$$\alpha_p = \min \{\alpha'_p, \alpha''_p\} \quad p \in \{1, \ldots, 13\}$$

**Definition 9.** Given two $\text{IA}^{\text{fuz}}$-relations $R' = (\text{rel}_1[\alpha_1'], \ldots, \text{rel}_{13}[\alpha_{13}'])$ and $R'' = (\text{rel}_1[\alpha_1''], \ldots, \text{rel}_{13}[\alpha_{13}''])$, the composition $R = R' \circ R''$ is defined as

$$R = (\text{rel}_1[\alpha_1], \ldots, \text{rel}_{13}[\alpha_{13}])$$

$$\alpha_p = \max_{q, r: \text{rel}_p \in \{R'_q \circ R''_r\}} \min \{\alpha'_q, \alpha''_r\} \quad p, q, r \in \{1, \ldots, 13\}$$

**Definition 10.** Given two $\text{IA}^{\text{fuz}}$-relations $R' = (\text{rel}_1[\alpha_1'], \ldots, \text{rel}_{13}[\alpha_{13}'])$ and $R'' = (\text{rel}_1[\alpha_1''], \ldots, \text{rel}_{13}[\alpha_{13}''])$, the disjunction $R = R' \oplus R''$ is defined as

$$R = (\text{rel}_1[\alpha_1], \ldots, \text{rel}_{13}[\alpha_{13}])$$

$$\alpha_p = \max \{\alpha'_p, \alpha''_p\} \quad p \in \{1, \ldots, 13\}$$

It is easy to see that the operations on $\text{IA}^{\text{fuz}}$-relations generalize those defined in the context of classical IA. In particular, it can be easily verified that, for all $\overline{T}_1, \overline{T}_2 \in \mathbb{R}^2$:

- $\mu_{R_j^{-1}}(\overline{T}_2, \overline{T}_1) = \mu_{R_j}(\overline{T}_1, \overline{T}_2)$
- $\mu_{R'_R \circ R''_R}(\overline{T}_1, \overline{T}_2) = \min \{\mu_{R'_R}(\overline{T}_1, \overline{T}_2), \mu_{R''_R}(\overline{T}_1, \overline{T}_2)\}$
- $\mu_{R'_R \oplus R''_R}(\overline{T}_1, \overline{T}_2) = \max \{\mu_{R'_R}(\overline{T}_1, \overline{T}_2), \mu_{R''_R}(\overline{T}_1, \overline{T}_2)\}$

Therefore the operation of inversion has the same meaning as the classical one, while conjunction and disjunction correspond to the intersection and the union of the involved relations, respectively, as in the case of classical operations between IA-relations.

As far as composition is concerned, we recall that, in the case of classical IA-networks, $\text{rel}_p \in (R_{ik} \circ R_{kj})$ iff $\exists \text{rel}_q \in R_{ik}, \text{rel}_r \in R_{kj}$ such that the singleton labeling $\langle \{i, k, j\}, \{s_{ik} = \text{rel}_q, s_{kj} = \text{rel}_r, s_{ij} = \text{rel}_p\} \rangle$ is consistent. Composition in $\text{IA}^{\text{fuz}}$-algebra generalizes this operation in that, in case $R_{ik}$ and $R_{kj}$ are $\text{IA}^{\text{fuz}}$-relations, $R_{ik} \circ R_{kj}$ assigns to every atomic relation $\text{rel}_p$ the degree of consistency through which it can be extended to a consistent labeling involving $R_{ik}$ and $R_{kj}$. This is expressed by the following proposition:
Proposition 11. Let us consider an IA\textsuperscript{fuz}-network $\mathcal{N} = \langle \mathcal{X}, \mathcal{C} \rangle$ such that $\mathcal{X} = \{I_i, I_k, I_j\}$ and $(i, j)$ is unconstrained, i.e. $R_{ij} = (rel_1[1], \ldots, rel_{13}[1])$. We have that, for every $rel_p \in \mathcal{I}$,
\[
\text{deg}_{R_{ik} \circ R_{ij}}(rel_p) = \max_{s \in SL_\mathcal{N} : s_{ij} = rel_p} \{\text{deg}_\mathcal{N}(s)\}
\]

As an example of composition, consider the constraints $I_i R_{ij} I_j$ and $I_j R_{jk} I_k$, where $R_{ij} = (a[0.5], m[0.7])$ and $R_{jk} = (b[0.7])$. As shown in Fig. 3, $I_i$ is before $I_k$, otherwise $R_{ij}$ or $R_{jk}$ would be completely violated. Moreover, the greatest degree to which both $R_{ij}$ and $R_{jk}$ can be satisfied if $I_i$ is before $I_k$ is 0.7. In fact, $R_{ij} \circ R_{jk} = (b[0.5] \oplus b[0.7]) = b[0.7]$.  

3. Interesting reasoning tasks in the IA\textsuperscript{fuz}-algebra

In the context of classical Interval Algebra, the most interesting reasoning tasks are determining the consistency of a given IA-network, finding a relevant solution and computing the so-called equivalent minimal network [26,27]. As far as computational complexity is concerned, it has been shown in [2] that these problems are equivalent under polynomial Turing-reduction, namely an algorithm that solves one of them is able to solve, by means of an appropriate polynomial mapping, also the other ones. In this section, we analyze how these problems are generalized to our fuzzy framework, and we show that their equivalence is maintained in the context of IA\textsuperscript{fuz}.

As far as consistency checking is concerned, in the classical case this task amounts to determining whether the constraints of a given IA-network admit at least a solution. Since the solutions of an IA\textsuperscript{fuz}-network form a fuzzy set, in our framework this problem corresponds to evaluate the best degree to which the constraints of a given IA\textsuperscript{fuz}-network can be satisfied (we call this problem FISAT):

Definition 12. Given an IA\textsuperscript{fuz}-network $\mathcal{N}$, its degree of consistency $\text{deg}(\mathcal{N})$ is defined as
\[
\text{deg}(\mathcal{N}) = \max_{s \in SL_\mathcal{N}} \text{deg}_\mathcal{N}(s)
\]

Along the same line, it is easy to see that the fuzzy counterpart of the problem of finding a solution, i.e. a complete assignment which satisfies all network constraints, is the computation of an optimal solution, i.e. a complete assignment that satisfies the fuzzy constraints of the network in the best way (we call this problem FOSOL):

Definition 13. Given an IA\textsuperscript{fuz}-network $\mathcal{N}$, a solution $s \in SL_\mathcal{N}$ is optimal iff $\text{deg}_\mathcal{N}(s) = \text{deg}(\mathcal{N})$.

Finally, the definition of minimal network can be generalized to the IA\textsuperscript{fuz}-framework, by defining an IA\textsuperscript{fuz}-network $\mathcal{N}$ as minimal if all of its constraints are completely explicit. Intuitively, an IA\textsuperscript{fuz}-constraint $R_{ij}$ is completely explicit if every atomic relation $rel_p \in \mathcal{I}$ can be extended to a solution of the network maintaining its degree of satisfaction $\text{deg}_{R_{ij}}(rel_p)$. This yields the following definition of minimality:

Definition 14. An IA\textsuperscript{fuz}-network $\mathcal{N} = \langle \mathcal{X}, \mathcal{C} \rangle$ is minimal iff $\forall R_{ij} \in \mathcal{C}$ we have that $\forall rel_p \in \mathcal{I} \exists s \in SL_\mathcal{N}$ such that $s_{ij} = rel_p \land \text{deg}_\mathcal{N}(s) = \text{deg}_{R_{ij}}(rel_p)$.

As in the classical case, given an IA\textsuperscript{fuz}-network we are interested in finding the minimal network among the equivalent ones, i.e. those that involve the same variables and preserve the fuzzy set of its solutions:
Definition 15. Two \(\mathcal{IA}^{fuz}\)-networks \(\mathcal{N}_1 = \langle \mathcal{X}_1, \mathcal{C}_1 \rangle\) and \(\mathcal{N}_2 = \langle \mathcal{X}_2, \mathcal{C}_2 \rangle\) are equivalent iff \(\mathcal{X}_1 = \mathcal{X}_2\) and \(\text{SOL}(\mathcal{N}_1) = \text{SOL}(\mathcal{N}_2)\).

Following the same lines as in [28], it can be proved that for every \(\mathcal{IA}^{fuz}\)-network \(\mathcal{N}\) a minimal network equivalent to \(\mathcal{N}\) exists and that it is unique. We call \(\text{FIMIN}\) the problem of computing this network.

In the following, we will also consider the three problems introduced above in a restricted form, where the relations used in \(\mathcal{IA}^{fuz}\)-networks belong to a given subclass \(\mathcal{S}\) of \(\mathcal{IA}^{fuz}\) (we will refer to this kind of relations and networks as \(\mathcal{S}\)-relations and \(\mathcal{S}\)-networks, respectively). In this case, the relevant problems will be denoted as \(\text{FISAT}(\mathcal{S})\), \(\text{FOSOL}(\mathcal{S})\) and \(\text{FIMIN}(\mathcal{S})\). As in the classical case [18], it turns out that these problems are equivalent with respect to polynomial Turing-reductions, provided that \(\mathcal{S}\) includes the set \(\mathcal{B}\) of the atomic relations of \(\mathcal{IA}^{fuz}\), which is defined as follows:

Definition 16. The set of atomic relations of \(\mathcal{IA}^{fuz}\) is defined as
\[
\mathcal{B} = \{ (\text{rel}_\mathcal{I}(\alpha)) | \text{rel}_\mathcal{I} \in \mathcal{I} \land \alpha \in L \}\]

More specifically, given an algorithm \(\text{ALG}\) for one problem, it is possible to solve another problem by means of a number of iterations needed in the different cases.

Proposition 17. If \(\mathcal{B} \subseteq \mathcal{S}\), then \(\text{FISAT}(\mathcal{S})\) is directly solved by an algorithm for \(\text{FIMIN}(\mathcal{S})\), while \(\text{FIMIN}(\mathcal{S})\) can be solved by means of \(O(n^2)\) iterations of an algorithm for \(\text{FISAT}(\mathcal{S})\), where \(n\) is the number of nodes of the network at hand.

Proposition 18. If \(\mathcal{B} \subseteq \mathcal{S}\), then \(\text{FISAT}(\mathcal{S})\) can be solved by means of an algorithm for \(\text{FOSOL}(\mathcal{S})\) and an \(O(n^2)\) check of network edges, and \(\text{FOSOL}(\mathcal{S})\) can be solved by means of \(O(n^2)\) iterations of an algorithm for \(\text{FISAT}(\mathcal{S})\), where \(n\) is the number of nodes of the network at hand.

Proposition 19. If \(\mathcal{B} \subseteq \mathcal{S}\), then \(\text{FIMIN}(\mathcal{S})\) can be solved by means of \(O(n^2)\) iterations of an algorithm for \(\text{FOSOL}(\mathcal{S})\) coupled with an \(O(n^2)\) check of network edges, and \(\text{FOSOL}(\mathcal{S})\) can be solved by means of \(O(n^2)\) iterations of an algorithm for \(\text{FIMIN}(\mathcal{S})\), where \(n\) is the number of nodes of the network at hand.

The results are summarized in Table 2, where the \(O(n^2)\) check along the edges of the network is not considered. Notice that we distinguish between \(\text{FOSOL}\) and \(\text{FISAT}\), since we suppose that solving \(\text{FOSOL}\) does not provide the degree of satisfaction of the optimal solution. If instead this is the case (as it is reasonable to suppose), then of course the \(O(n^2)\) check is not required.

An immediate consequence of these results is the equivalence of the reasoning tasks considered above:

**Theorem 20.** The problems \(\text{FISAT}(\mathcal{S})\), \(\text{FOSOL}(\mathcal{S})\) and \(\text{FIMIN}(\mathcal{S})\) are equivalent under polynomial Turing-reduction.

It is well-known [2,3] that the reasoning tasks in the framework of classical \(\mathcal{IA}\)-algebra are NP-complete. Since our framework represents a generalization of \(\mathcal{IA}\), if the complete expressive power of its language it maintained then the intractability of the reasoning problems affects \(\mathcal{IA}^{fuz}\) too:

**Theorem 21.** The problems \(\text{FISAT}(\mathcal{IA}^{fuz})\), \(\text{FOSOL}(\mathcal{IA}^{fuz})\) and \(\text{FIMIN}(\mathcal{IA}^{fuz})\) are NP-difficult.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Polynomial mappings between interesting problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>from(\mathcal{S})</td>
<td>FISAT</td>
</tr>
<tr>
<td>FISAT</td>
<td>(O(n^2))</td>
</tr>
<tr>
<td>FOSOL</td>
<td>const.</td>
</tr>
<tr>
<td>FIMIN</td>
<td>const.</td>
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</table>
4. Prioritized constraints and uncertainty

So far, we have defined \( \mathcal{IA}_{fuz} \)-algebra in terms of flexible constraints, while leaving out of consideration the case in which temporal knowledge is affected by uncertainty. In fact, according to the semantics defined in Section 2.1, \( \mathcal{IA}_{fuz} \)-relations express preference among possible temporal displacements of the intervals at hand, under the assumption that we have only to choose among such displacements in order to maximize the degree of satisfaction of a set of known constraints.

However, soft constraints expressed by fuzzy sets are also able to model a different kind of temporal knowledge, i.e. expressing the degree of possibility that a certain relation holds between two temporal entities. This is the way that soft constraints are often used in fuzzy scheduling systems (see e.g. [29,30]), where they model, among other things, the uncertainty affecting the duration of a given activity. Adopting a similar interpretation of soft constraints, Dutta has introduced in [31] an event-based temporal logic, where membership degrees are used to express the possibility that a given interval contains a particular event. Another use of fuzzy sets in the context of Allen’s framework has been proposed by Guesgen et al. [32], in order to model imprecise spatial descriptions. In this framework, variables represent spatial locations with respect to some horizontal reference axis. The idea is that, in many real situations, an observer cannot be sure that a given atomic relation holds between two objects, therefore acceptance grades are assigned to Allen’s atomic relations to reflect this imprecision. As it will be shown in the following, our framework is able to accommodate this proposal as a subcase, as well as the one introduced by Dutta.

The dual nature of fuzzy constraints, i.e. either flexible or uncertain constraints, comes from the interpretation of the corresponding membership functions as possibility distributions in the context of Possibility Theory, which can account for both preference and uncertainty. While introducing Possibility Theory is outside the scope of the present paper (see e.g. [33,34]), we remark that the possibilistic approach to handling uncertain knowledge is based on an ordinal model; it has been argued that this feature makes it particularly robust to imprecision of parameters with respect to probabilistic representations (which are intrinsically quantitative) and thus particularly suited to cases where there is a lack of statistical data [35].

Turning to the context of \( \mathcal{IA}_{fuz} \)-algebra, the interpretation of \( \mathcal{IA}_{fuz} \)-relations as uncertain constraints leads to consider e.g. the relation \( I_1(b, a[0.7])I_2 \) as assigning a maximal degree of possibility to the case that \( I_1 \) is before \( I_2 \), a possibility limited to 0.7 that \( I_1 \) is after \( I_2 \), and as stating that is definitely impossible for the two intervals to intersect in some way [36].

As it may be the case in real situations, we consider in the following the case in which the set of flexible temporal constraints to be satisfied is not known with certainty, depending for example on the environmental conditions, but we have some idea about their degree of “plausibility” that must be taken into account in order to appropriately choose a solution. As it is shown in [9,29], this kind of problem can be solved in the context of the FCSP by means of the same formalism devoted to handle flexible constraints. The key to integrate uncertainty in the FCSP framework is the definition of prioritized constraints, consisting of fuzzy constraints with a priority degree that indicates, using the language of Possibility Theory, the degree of necessity of their satisfaction. Here we don’t review the underlying formal basis that comes from Possibility Theory, referring the reader to [29,33,34], rather we introduce prioritized constraints in the \( \mathcal{IA}_{fuz} \)-framework, then we define the concept of uncertain \( \mathcal{IA}_{fuz} \)-problem and briefly explain how it can be equivalently stated by means of an \( \mathcal{IA}_{fuz} \)-network.

In order to introduce prioritized constraints, we have to assume the existence of a totally ordered scale \( P \) which includes the values taken by priority degrees. Such a scale is related to \( L \), in that an order-reversing bijection \( c \) from \( P \) to \( L \) is assumed to exist, such that \( c(P) = L \), \( \forall x_1, x_2 \in P: x_1 \geq x_2 \iff c(x_1) \leq c(x_2) \), and, denoting again the top and bottom element of \( P \) as 1 and 0 respectively, \( c(0) = 1 \) and \( c(1) = 0 \). In case \( L = P = [0, 1] \), a typical choice is \( c(x) = 1 - x \), as it will be assumed in the examples made in the following.

**Definition 22.** A prioritized \( \mathcal{IA}_{fuz} \)-constraint is a pair \( (R, \alpha) \), where \( R \in \mathcal{IA}_{fuz} \) and \( \alpha \in P \).

As it is proved in [9], interpreting priority degrees as necessity degrees of the associated constraints makes it possible to recast prioritized constraints as ordinary fuzzy constraints; applying the same proof in the context of our temporal framework, it turns out that a generic prioritized \( \mathcal{IA}_{fuz} \)-constraint can be represented by an ordinary \( \mathcal{IA}_{fuz} \)-relation, that we call associate \( \mathcal{IA}_{fuz} \)-relation:
Definition 23. Given a prioritized IA\textsuperscript{fuz}-constraint $C' = (R', \alpha)$, where $R' = (rel_1[\alpha'_1], \ldots, rel_{13}[\alpha'_{13}])$, the associate IA\textsuperscript{fuz}-relation $A(C')$ is defined as $A(C') = (rel_1[\alpha_1], \ldots, rel_{13}[\alpha_{13}])$, where $\alpha_p = \max(\alpha'_p, c(\alpha))$ for all $p = 1, \ldots, 13$.

The underlying idea is that if the priority of $C'$ is 1, i.e. $C'$ is mandatory, then the associate relation is equal to $R'$, otherwise the lower the priority, the higher the preference degrees assigned by $A(C')$ to atomic relations, independently of the original constraint. This way, even atomic relations completely excluded by $R'$ can be accepted as part of a solution, its degree of satisfaction being less constrained the lower the priority of $C'$ is. An example of associate relation is shown in Fig. 4.

We are now in a position to introduce the concept of uncertain IA\textsuperscript{fuz}-problem, modeled as a set of variables denoting intervals constrained by a set of IA\textsuperscript{fuz}-constraints whose presence is uncertain:

Definition 24. An uncertain IA\textsuperscript{fuz}-problem is a triple $(\mathcal{X}, C', \Pi)$, where:

- $\mathcal{X} = \{I_1, \ldots, I_n\}$ is a set of variables that denote intervals;
- $C' = \{C'_1, \ldots, C'_m\}$ is a set of possible IA\textsuperscript{fuz}-constraints between them, where each $C'_k$ is of the form $C'_k: I_1k(rel_1[\alpha'_{k1}], rel_2[\alpha'_{k2}], \ldots, rel_{13}[\alpha'_{k13}])I_2k$;
- $\Pi : C' \rightarrow P$ is a possibility distribution defined on the constraints of $C'$.

Intuitively, we are not sure about what constraints of $C'$ define the real problem, but we have a possibility distribution $\Pi$ expressing the “plausibility” that each constraint belongs to it. Then, the goal is to find a solution which maximizes the degree of necessity of satisfying the real problem, where the notion of necessity, which we leave here as an intuitive concept, can be formally defined in the context of Possibility Theory. As shown in [9], it can be proved that such a solution corresponds to an optimal solution of an equivalent FCSP, defined by the constraints of $C'$, with a priority degree attached to every constraint $C'_k \in C'$ equal to the corresponding plausibility degree $\Pi(C'_k)$: in a nutshell, more plausible constraints must be satisfied first. In the context of the IA\textsuperscript{fuz}-framework, given an uncertain IA\textsuperscript{fuz}-problem we define an associate IA\textsuperscript{fuz}-network $\mathcal{N}$ such that solving FOSOL for $\mathcal{N}$ amounts to solving the original uncertain IA\textsuperscript{fuz}-problem:

Definition 25. Given an uncertain IA\textsuperscript{fuz}-problem $\mathcal{P} = (\mathcal{X}, C', \Pi)$, whose elements are defined as in Definition 24, the associate IA\textsuperscript{fuz}-network $A(\mathcal{P})$ is the pair $(\mathcal{X}', \mathcal{C})$, where any $R_{ij} \in \mathcal{C}$ is defined as follows:

- if $\exists C'_k \in C'$ such that $I_{1k} = I_i$ and $I_{2k} = I_j$, then $R_{ij} = A(C'_k(\Pi(C'_k)))$, i.e. $\forall \mathcal{P} \in \{1, \ldots, 13\}$ $\deg_{R_{ij}}(rel_p) = \max(\alpha'_{p}, c(\Pi(C'_k)))$;
- otherwise, $R_{ij}$ denotes the absence of a constraint, i.e. $\forall rel_p \in \mathcal{I}$ $\deg_{R_{ij}}(rel_p) = 1$.

Proposition 26. Given an uncertain IA\textsuperscript{fuz}-problem $\mathcal{P}$, the solutions that have a maximal necessity of satisfying $\mathcal{P}$ are the optimal solutions of the IA\textsuperscript{fuz}-network $A(\mathcal{P})$.

Besides a ‘pure interpretation’ of soft constraints as either flexible or uncertain constraints, another issue is to encompass both notions of flexibility and uncertainty in the same framework, i.e. having a number of fuzzy relations whose instantiation can be controlled and other ones that are instead decided by Nature (and can therefore only be
observed. In this case, a pessimistic approach may be adopted, such as the one proposed in [37] to handle quantitative constraints for scheduling problems, where the basic idea is to find those choices of controllable relations that achieve the best satisfaction degree whatever choices are made by Nature. In a similar spirit, various notions of controllability have been defined in the context of classical quantitative point-based temporal reasoning [38], namely strong, weak and dynamic controllability. In particular, dynamic controllability relaxes the requirement of finding a solution which achieves consistency for any situation, by considering the possibility of partially instantiating controllable constraints on the basis of previous observations, in such a way that a complete consistent solution can always be constructed. Recently, an extension of this framework to handle preferences has been proposed in [39], where both free and contingent constraints are modeled by a preference function taking value in the fuzzy semiring. Extending the above notions to our qualitative framework represents an important issue, requiring additional constructions, that we leave as a topic for future work. Therefore, we will focus in the following on the IA \textit{fuz}-framework as it has been introduced in Section 2, with an underlying semantics enforcing the notion of flexibility.

5. Algorithms

In this section, we describe some of the algorithms that can be used to solve the reasoning tasks introduced in Section 3. In particular, we have developed two kinds of algorithms:

- two constraint propagation algorithms, called PC2\textit{fuz} and AAC\textit{fuz}, that are used to render a given IA\textit{fuz}-network more explicit;
- a Branch & Bound algorithm, which finds an optimal solution of a given IA\textit{fuz}-network.

While Branch & Bound is designed to tackle FOSOL, constraint propagation algorithms are mainly related to the FIMIN problem, since the equivalent network they compute represents an approximation of the minimal network, the latter being achieved for particular tractable fragments of IA\textit{fuz}-algebra (see Section 7). However, constraint-propagation algorithms are related to FISAT and FOSOL too: in fact, they typically return an upper bound to the degree of consistency of the input IA\textit{fuz}-network, and are used to speed up the execution of the Branch & Bound algorithm both as pre-processing algorithms and as a forward-checking technique.

5.1. Path-consistency algorithm

In the general FCSP framework, path-consistency algorithms make network constraints more explicit by enforcing on the input network a local consistency condition, which corresponds to a particular case of $k$-consistency. Adapting the general definition of $k$-consistency to our framework, an IA\textit{fuz}-network can be defined as $k$-consistent if and only if, for every set of $k - 1$ nodes, every assignment with a degree of local consistency $\alpha$ can be extended to any other $k$th variable maintaining the same degree $\alpha$.

Definition 27. Given two singleton labelings $s'$ and $s''$, $s'$ extends $s''$ iff $N(s'') \subseteq N(s')$ and $\forall (i, j) \in E(s'') s'_{ij} = s''_{ij}$.

Definition 28. An IA\textit{fuz}-network $N = (\mathcal{X}, \mathcal{C})$ is $k$-consistent iff $\forall \mathcal{Y} \subseteq \mathcal{X}$, $\forall \mathcal{Y} = \{I_1, \ldots, I_{k-1}\}$, $\forall I_k \in \mathcal{X}$ such that $I_k \not\in \mathcal{Y}$, $\forall s$ singleton labeling such that $N(s) = \mathcal{Y}$ $\exists$ $s'$ singleton labeling such that:

- $N(s') = (\mathcal{Y} \cup \{I_k\})$,
- $s'$ extends $s$,
- $\deg_N(s') = \deg_N(s)$.

Definition 29. An IA\textit{fuz}-network is path-consistent iff it is 3-consistent.

Taking into account Proposition 11, it is easy to see that path-consistency of an IA\textit{fuz}-network is equivalent to the condition $R_{ik} \leq (R_{ij} \circ R_{jk})$ for all triples $(i, j, k)$ of nodes, where the symbol ‘$\leq$’ denotes “inclusion” between IA\textit{fuz}-relations:
Definition 30. Given two IA-relations $R_1$ and $R_2$, $R_1$ is included in $R_2$, i.e. $R_1 \subseteq R_2$, iff all the atomic relations of $R_1$ belong to $R_2$:

$$\forall rel_p \in R_1 \quad rel_p \in R_2$$

In case $R_1$ and $R_2$ are $IA^{\text{fuz}}$-relations, $R_1 \subseteq R_2$ iff

$$\forall rel_p \in I \quad \deg_{R_1}(rel_p) \leq \deg_{R_2}(rel_p)$$

In devising our path-consistency algorithm for $IA^{\text{fuz}}$-networks, we start from the basic one developed for classical IA-networks, described e.g. in [40] and shown in Fig. 5. In a nutshell, the condition $R_{ik} \leq (R_{ij} \circ R_{jk})$ is enforced for all the triples of nodes $(i, j, k)$ (see lines 5–7 and 10–12), thus ensuring path-consistency of the network. In this process, a list $Q$ is used to maintain the edges that are modified and can further constrain other relations of the network (i.e. $(i, k)$ and $(k, j)$ in line 9 and 14 respectively). Taking into account that the operations on $IA^{\text{fuz}}$-relations generalize the classical ones preserving their main properties (see Section 2.2), it is easy to extend to our algorithm, that we call $PC1^{\text{fuz}}$, the correctness proof of the classical path-consistency algorithm, that can be found e.g. in [3].

In order to improve the efficiency of the algorithm $PC1^{\text{fuz}}$, we devise in the following some relevant refinements that are based, amongst other things, on a specific property of composition introduced below.

Definition 31. We indicate as $I[\alpha]$ the $IA^{\text{fuz}}$-relation made up of all atomic relations with a degree of preference $\alpha$, i.e. $I[\alpha] = \langle \text{rel}_1[\alpha], \text{rel}_2[\alpha], \ldots, \text{rel}_{13}[\alpha] \rangle$.

Definition 32. Given an $IA^{\text{fuz}}$-relation $R$, we denote the maximum among the preference degrees of $R$ as $\max_{R}$, and the minimum as $\min_{R}$:

$$\max_{R} = \max_{1 \leq p \leq 13} \{ \deg_{R}(rel_p) \}$$

$$\min_{R} = \min_{1 \leq p \leq 13} \{ \deg_{R}(rel_p) \}$$

Given an $IA^{\text{fuz}}$-network $N = \langle X, C \rangle$, and given $R_{ij} \in C$, we denote $\max_{R_{ij}}$ and $\min_{R_{ij}}$ as $\max_{ij}$ and $\min_{ij}$, respectively.

Lemma 33. For every $IA^{\text{fuz}}$-relation $R$, and for every $\alpha \in L$, we have that $R \circ I[\alpha] = I[\alpha] \circ R = I[\min \{\max_{R}, \alpha\}]$.

With reference to Fig. 6, the first refinement can be envisaged by considering the case in which an edge $(j, k)$ of the network is unconstrained, i.e. $R_{jk} = I[1]$. In case of classical IA-networks, the composition $R_{ij} \circ R_{jk}$ does not constrain $R_{ik}$ (see line 5), therefore it is possible to improve the efficiency of the algorithm by avoiding the
execution of lines 5–9 if \((j, k)\) is unconstrained (more generally, if either \((i, j)\) or \((j, k)\) is unconstrained). In our framework we cannot directly adopt the same technique, since Lemma 33 yields \(R_{ij} \circ R_{jk} = I[\max_{ij}]\). However, it is easy to see that the operation \(R_{ik} \leftarrow R_{ik} \otimes (R_{ij} \circ R_{jk})\) amounts to truncating the preference degrees of \(R_{ik}\) to \(\max_{ij}\); intuitively, this is due to the path consistency requirement that every assignment of \(R_{ik}\) could be extended to the 3-subnetwork maintaining its preference degree: since no solution of the subnetwork can have a degree of satisfaction greater than \(\max_{ij}\), the preference degrees of \((i, k)\) can be limited to \(\max_{ij}\) maintaining the equivalence of the subnetwork. A generalization of this operation to all edges \((i, j)\) of the network \(N\) at hand leads to limit the preference degrees to the following upper bound to \(\deg(N)\):

\[
\text{ConsSup} = \min_{(i,j) \in \mathcal{E}(N)} \{\max_{ij}\} \quad (2)
\]

In particular, the value of ConsSup is updated during constraint propagation, and, of course, it can only decrease: a first improvement to the algorithm \(\text{PC1}^\text{fuz}\) may be obtained by upper-bounding the preference degrees of all edges to ConsSup every time the latter decreases.

However, it is easy to notice that this operation can be limited to those edges involved in the current step of constraint propagation (i.e. \((i, j)\), \((j, k)\) and \((i, k)\) in lines 5–14), if a final truncation of all the preference degrees of the network is performed as the last step of the algorithm. Moreover, we prove that the network computed by the algorithm does not change if, during constraint propagation, the preference degrees greater than or equal to ConsSup are left out of account. In particular, this yields two further refinements to \(\text{PC1}^\text{fuz}\):

1. if either \(R_{ij}\) or \(R_{jk}\) in line 5 has all its preference degrees greater than or equal to ConsSup, then the steps of lines 6–9 can be avoided, since they would not modify preference degrees lower than ConsSup in \(R_{ik}\) (the same holds for \(R_{ki}\) and \(R_{ij}\) in line 10);
2. in line 9, \((i, k)\) can be inserted into \(Q\) only if a preference degree of \(R_{ik}\) which is strictly lower than ConsSup has been modified (the same holds for \((k, j)\) in line 14).

By enforcing these adjustments, we obtain the algorithm shown in Fig. 7, that we call \(\text{PC2}^\text{fuz}\), in which every edge is cut to ConsSup only in the final step (see line 20). Notice that, in particular, this new algorithm handles the aforementioned situation shown in Fig. 6 by avoiding the propagation of the constraints \(R_{ij}\) and \(R_{jk}\); this happens also in the case that \(R_{jk} \geq I[\text{ConsSup}]\), of which \((j, k)\) unconstrained is a subcase. In line 21, the final value of ConsSup is returned as an upper bound to the degree of consistency of the network, which can be exploited in the Branch & Bound algorithm (see Section 5.3).

To put the considerations made above in different terms, the algorithm \(\text{PC2}^\text{fuz}\) enforces for all the triples of nodes \((u, v, w)\) the condition

\[
(R_{uw} \otimes I[\text{ConsSup}]) \leq (R_{uv} \otimes I[\text{ConsSup}]) \circ (R_{vw} \otimes I[\text{ConsSup}]) \quad (3)
\]

which can be interpreted as a relaxed form of the path consistency condition \(R_{uw} \leq R_{uv} \circ R_{vw}\), in that only preference degrees strictly lower than ConsSup are affected. The list \(Q\) maintains all those edges that can constrain other edges; the following proposition ensures that, throughout the execution of the algorithm, all subnetworks \((u, v, w)\) whose edges \((u, v)\) and \((v, w)\) are not included in \(Q\) satisfy (3):

![Fig. 6. Constraint-propagation in case of an unconstrained edge.](image-url)
invoked, the algorithm PC2 holds in our context too, here we show how classical AAC can be generalized in order to handle IA of the input network belong to the SA-subalgebra of IA \[17\]. Since, as it will be shown in Section 7, a similar result computed by AAC better approximates the minimal one, and, in particular, that the latter is achieved if the relations stronger than path-consistency, which is equivalent to minimality of 3-subnetworks, it turns out that the network called AAC, which enforces on the input network minimality of all of its 4-subnetworks. Since this condition is stronger than path-consistency algorithm for classical IA-networks, the worst-case computational complexity is augmented by a factor equal to the number of levels of preference used to define the network, which is at most 13.

The resulting algorithm, called AAC, is shown in Fig. 8, where the following notation is used:

Fig. 7. The improved path-consistency algorithm PC2\(^{\text{fuz}}\) for IA\(^{\text{fuz}}\)-networks.

### Proposition 34.
Let us consider the algorithm PC2\(^{\text{fuz}}(\mathcal{N})\), with \(\mathcal{N} = (\mathcal{X}, \mathcal{C})\), and let us assume that, initially, ConsSup = min\((i,j)\in\mathcal{E}(\mathcal{N})\) max\(i,j\). Then, after the completion of every iteration of the while-loop (lines 2–19) we have that, for every ordered triple \((u, v, w)\) of distinct nodes of \(\mathcal{N}\), at least one of the following conditions holds:

1. \((u, v) \in Q\).
2. \((v, w) \in Q\).
3. \((R_{uw} \otimes I[\text{ConsSup}]) < (R_{uw} \otimes I[\text{ConsSup}]) \circ (R_{vw} \otimes I[\text{ConsSup}])\).

By relying on the proposition above, the correctness of PC2\(^{\text{fuz}}\) can be formally proved. With respect to the path-consistency algorithm for classical IA-networks, the worst-case computational complexity is augmented by a factor equal to the number of levels of preference used to define the network, which is at most 13.

### Theorem 35.
Let \(\mathcal{N}\) be an IA\(^{\text{fuz}}\)-network with \(n\) nodes, and let \(k\) be the number of levels of preference used in it. When invoked, the algorithm PC2\(^{\text{fuz}}\) runs to completion in \(O(kn^2)\) time and achieves a path-consistent equivalent network.

#### 5.2. The algorithm AAC\(^{\text{fuz}}\)

In the context of classical IA-networks, van Beek and Cohen propose in [16] a constraint propagation algorithm, called AAC, which enforces on the input network minimality of all of its 4-subnetworks. Since this condition is stronger than path-consistency, which is equivalent to minimality of 3-subnetworks, it turns out that the network computed by AAC better approximates the minimal one, and, in particular, that the latter is achieved if the relations of the input network belong to the SA-subalgebra of IA [17]. Since, as it will be shown in Section 7, a similar result holds in our context too, here we show how classical AAC can be generalized in order to handle IA\(^{\text{fuz}}\)-networks.

The resulting algorithm, called AAC\(^{\text{fuz}}\), is shown in Fig. 8, where the following notation is used:

\[
\text{RELATED-PATHS}(i, j) = \{ (k, i, j, l) \mid 1 \leq k < l \leq n, \ k, l \neq i, j \} \cup \\
\{ (i, j, l, k) \mid 1 \leq k, l \leq n, \ k \neq l, \ k, l \neq i, j \}
\]

and the composition operation between IA\(^{\text{fuz}}\)-relations is extended to triples of nodes in the following way:
1. $Q \leftarrow \{(i, j) \mid 1 \leq i < j \leq n\}$
2. while $(Q \neq \emptyset)$
3. select and delete $(u,v)$ from $Q$
4. for $(i,k,l,j) \in \text{RELATED-PATHS}(u,v)$
5. $t \leftarrow R_{ij} \otimes (\Delta_{ikl} \circ \Delta_{klj})$
6. if $(t \neq R_{ij})$
7. then $R_{ij} \leftarrow t$
8. $R_{ji} \leftarrow t^{-1}$
9. $Q \leftarrow Q \cup \{(i,j)\}$

Fig. 8. The AAC$_{\text{fuz}}$ algorithm for IA$_{\text{fuz}}$-networks.

Fig. 9. Enforcing minimality of 4-subnetworks.

$$\Delta_{ikl} \circ \Delta_{klj} = \bigoplus \{(\text{rel}_P[\alpha_P] \circ \text{rel}_S[\alpha_S]) \otimes (\text{rel}_Q[\alpha_Q] \circ \text{rel}_T[\alpha_T]) \mid$$
$$\alpha_P = \deg_{R_{ik}}(\text{rel}_P), \alpha_R = \deg_{R_{kl}}(\text{rel}_R)$$
$$\alpha_T = \deg_{R_{lj}}(\text{rel}_T), \alpha_P, \alpha_R, \alpha_T > 0$$
$$\alpha_S = \deg_{(\text{rel}_P[\alpha_P] \circ \text{rel}_T[\alpha_T]) \otimes R_{kj}}(\text{rel}_S), \alpha_S > 0$$
$$\alpha_Q = \deg_{(\text{rel}_P[\alpha_P] \circ \text{rel}_S[\alpha_S]) \otimes R_{li}}(\text{rel}_Q), \alpha_Q > 0\}$$

where the notation $\bigoplus S$ indicates the disjunction of the elements included in the set $S$.

The idea underlying the algorithm, similar to that of classical AAC, is to enforce minimality of every possible subgraph of four vertices $(i,k,l,j)$ (see Fig. 9), by updating $R_{ij}$ in such a way that every atomic relation $\text{rel}_p$ labeling $(i,j)$ can be extended to a singleton labeling of $(i,k,l,j)$ maintaining its degree of preference $\deg_{R_{ij}}(\text{rel}_p)$: this is enforced in line 7 by means of the assignment $R_{ij} \leftarrow R_{ij} \otimes (\Delta_{ikl} \circ \Delta_{klj})$. As in PC1$_{\text{fuz}}$, a list $Q$ is used to maintain all those edges that can further constrain other edges of the network: after the extraction of $(u,v)$ from $Q$ in line 3, RELATED-PATHS$(u,v)$ returns all the subgraphs of four vertices $(i,k,l,j)$ in which $(u,v)$ participates, and $R_{ij}$ is updated as described above.

Thus, the correctness proof of classical AAC can be extended to AAC$_{\text{fuz}}$. As with the path-consistency algorithm, the computational complexity is augmented with respect to the classical (crisp) case by a factor equal to the number of levels of preference used to define the input network.

**Theorem 36.** Let $\mathcal{N} = (\mathcal{X}, \mathcal{C})$ be an IA$_{\text{fuz}}$-network with $n$ nodes, and let $k$ be the number of levels of preference used in it. When invoked on $\mathcal{N}$, the algorithm AAC$_{\text{fuz}}$ runs to completion in $O(kn^4)$ time and computes a network equivalent to $\mathcal{N}$ such that all its subgraphs of four vertices are minimal.

5.3. Finding a solution: Branch & Bound algorithm

In this section, we explicitly consider the FOSOL problem, by proposing a general algorithm to find an optimal solution of a given IA$_{\text{fuz}}$-network. As it has been shown in Section 3, FOSOL is a generalization to IA$_{\text{fuz}}$ of the classical problem of finding a solution of an IA-network: accordingly, classical backtracking algorithm with incremental path-consistency [1,40] can be generalized to a Branch & Bound algorithm, which searches through the possible singleton labelings of the network until it finds one that is optimal.
B&B(\mathcal{N})
1. \alpha_{inf} \leftarrow 0
2. \alpha_{sup} \leftarrow \text{PC}_{2}^{\text{inc}}(\mathcal{N})
3. \text{if} (\alpha_{sup} = 0)
4. \text{then return} \text{ complete inconsistency}
5. order the edges of \mathcal{N} in a list \mathcal{E}
6. \textbf{bf} \leftarrow \emptyset
7. \text{RECB&B}(\mathcal{N}, \mathcal{E}, \alpha_{sup}, \alpha_{inf}, \textbf{bf})
8. \text{return} \textbf{bf}

RECB&B(\mathcal{N}, \mathcal{E}, \alpha_{sup}, \alpha_{inf}, \textbf{bf})
1. \textbf{extract} the first edge \((i,j)\) from \mathcal{E}
2. \textbf{R} \leftarrow \text{R}_{ij}
3. \textbf{for} \,(\text{rel}_q \in \mathcal{I} : \text{deg}_R(\text{rel}_q) > \alpha_{inf})
4. \quad \text{R}_{ij} \leftarrow \text{rel}_q[\text{deg}_R(\text{rel}_q)]
5. \quad \text{ConsSup}^{' \prime} \leftarrow \text{PC}_{2}^{\text{inc}}(\mathcal{N})
6. \quad \textbf{if} (\text{ConsSup}^{' \prime} > \alpha_{inf})
7. \quad \textbf{then if} (\mathcal{E} \neq \emptyset)
8. \quad \quad \text{RECB&B}(\mathcal{N}, \mathcal{E}, \alpha_{sup}, \alpha_{inf}, \textbf{bf})
9. \quad \textbf{else} \textbf{bf} \leftarrow \mathcal{N}
10. \quad \quad \alpha_{inf} \leftarrow \text{ConsSup}^{' \prime}
11. \quad \quad \textbf{if} (\alpha_{inf} = \alpha_{sup})
12. \quad \quad \textbf{then halt the searching process}

Fig. 10. The Branch & Bound algorithm to find an optimal solution.

The resulting algorithm is shown in Fig. 10, where it has been structured in two procedures: B&B(\mathcal{N}), which basically initializes the network and variables for the search, and a recursively called procedure RECB&B(\mathcal{N}, \mathcal{E}, \alpha_{sup}, \alpha_{inf}, \textbf{bf}), where \mathcal{N} and \mathcal{E} are passed by copying, while \alpha_{sup}, \alpha_{inf} and \textbf{bf} are passed by pointer.

Search is pruned by maintaining the degree of satisfaction of the current best solution \textbf{bf} in the variable \alpha_{inf}, and by using an upper bound to \text{deg}(\mathcal{N}), stored in \alpha_{sup}, which corresponds to the value of ConsSup returned from PC_{2}^{\text{inc}} (applied in line 2 of B&B(\mathcal{N}) as a pre-processing algorithm). Each time an edge is instantiated with an atomic relation, the incremental path-consistency algorithm PC_{2}^{\text{inc}} is applied to the resulting network (this algorithm is obtained from PC_{2}^{\text{inc}} by changing line 1 in such a way that \textbf{Q} is initialized to the single edge that has been instantiated). The returned value ConsSup^{' \prime} gives an upper bound to the maximum degree of satisfaction that can be reached extending the current partial instantiation, therefore if ConsSup^{' \prime} is not strictly greater than \alpha_{inf} then the algorithm backtracks. Once a complete labeling is obtained (see line 9 in RECB&B), its degree of satisfaction is equal to ConsSup^{' \prime}: this is due to the fact that atomic relations belong to the subclass \text{SA}_{c}^{\text{inc}} of \text{IA}_{c}^{\text{inc}}-relations, for which path-consistency entails minimality\textsuperscript{2} (see Proposition 63 in Section 7.2). As a consequence, ConsSup^{' \prime} is assigned to \alpha_{inf} in order to update the degree of satisfaction of the current best solution (see line 10 of RECB&B). Moreover, if ConsSup^{' \prime} reaches the upper-bound to the degree of consistency of the network, then the labeling is an optimal solution, so that the searching process can be halted (see line 12 of RECB&B).

6. From point-algebras to fuzzy point-algebras: PA_{c}^{\text{fuz}} and PA^{\text{fuz}}

As it has been pointed out in the introduction, current approaches to fuzzy temporal reasoning adopt time points as basic domain temporal entities, and handle binary metric constraints between them. Point-based formalisms have also been proposed in the context of classical non-fuzzy approaches to handle qualitative temporal knowledge, mainly because they play a particular role in the study of tractable classes of IA. In particular, classical Point Algebra PA [2] is defined in a similar way as IA-algebra, but considering points instead of intervals: since there are three basic relations that can hold between two points, namely <, = and >, eight possible relations between them can be expressed in

\textsuperscript{2} In particular, minimality ensures that the obtained labeling assigns the same preference degree to all atomic relations, and that the latter is equal to the degree of satisfaction of the corresponding solution.
PA, i.e. $\emptyset, <, \leq, =, \geq, \neq, ?$. A particular subset of PA is the PA$_c$-algebra [16,20], which includes all PA-relations without $\neq$.

**Proposition 37.** The subclass $\text{PA}_c \subseteq \text{PA}$ is an algebra, i.e. it is closed under the operations of inversion, conjunction and composition.

As it has been proved in [16,17], both PA and PA$_c$ have interesting computational properties, that make it possible to solve the relevant reasoning tasks in polynomial time. In particular, minimality of a given PA or PA$_c$-network is entailed by a local consistency condition, which can be achieved by means of a polynomial algorithm. This result can be exploited in the context of Interval Algebra as well, in order to define tractable subalgebras of IA. More specifically, in [16] two sub-algebras of IA have been defined, called SA$_c$ and SA, that include all those IA-relations that can be expressed by PA$_c$ and PA-relations between the endpoints of the involved intervals, respectively: both SA$_c$ and SA inherit the computational properties of the corresponding point-algebras, that are pointed out in the following propositions (the relevant proofs can be found in [16,17]).

**Proposition 38.** Let $N$ be a PA$_c$ or SA$_c$-network: if $N$ is path-consistent then it is also minimal.

**Corollary 39.** Given a PA$_c$-network $N$, its equivalent minimal network is a PA$_c$-network as well.

**Proposition 40.** Let $N$ be a PA or SA-network: if all of its 4-subnetworks are minimal then $N$ is minimal as well.

In order to investigate whether an analogous role can be played by point-algebras in our fuzzy framework too, in this subsection we introduce the fuzzy extensions of PA and PA$_c$, that we call PA$_{fuz}$ and PA$_{fuz}c$, respectively. Moreover, as it will be shown in the following, this can be exploited to identify the subclass of IA$_{fuz}$-relations that can be expressed by means of a point-based formalism and, on the other hand, those relations that cannot be captured by existing point-based approaches.

The algebra $\text{PA}_{fuz}$ is defined in the same way as $\text{IA}_{fuz}$, by considering points instead of intervals and PA-relations instead of Allen’s relations.

**Definition 41.** $\text{PA}_{fuz}$ is the algebra with the following underlying set

$$
\mathcal{I}_P = \{ (< [\alpha_1], = [\alpha_2], > [\alpha_3]) \mid \alpha_p \in L, \ p = 1, 2, 3 \}
$$

closed under the operations of inversion, conjunction and composition.

The operations of inversion, conjunction and composition are defined by means of a straightforward adaptation to $\text{PA}_{fuz}$ of Definitions 7, 8 and 9, respectively.

As far as the fuzzy extension of PA$_c$ is concerned, the idea is to exclude the fuzzy counterpart of the $\neq$ relation, which intuitively corresponds to the class of PA$_{fuz}$-relations $\{(< [\alpha_1], = [\alpha_2], > [\alpha_3]) : \alpha_2 < \alpha_1 \land \alpha_2 < \alpha_3\}$.

**Definition 42.** $\text{PA}_{c\ fuz}$ is the subclass of $\text{PA}_{fuz}$ made up of the following set

$$
\mathcal{I}_P_c = \{ (< [\alpha_1], = [\alpha_2], > [\alpha_3]) \mid \alpha_2 \geq \min \{\alpha_1, \alpha_3\} \}
$$

and the operations of inversion, conjunction and composition inherited from $\text{PA}_{fuz}$.

**Proposition 43.** The subclass $\text{PA}_{c\ fuz} \subseteq \text{PA}_{fuz}$ is an algebra, i.e. it is closed under the operations of inversion, conjunction and composition.

The relationship between $\text{PA}_{c\ fuz}$ and its classical counterpart $\text{PA}_c$ can be analyzed introducing the notion of $\alpha$-cut of a fuzzy relation, which corresponds to the crisp relation made up of those atomic relations that are assigned a preference degree greater than or equal to $\alpha$:
Definition 44. Given a value $\alpha \in L$ and a PA$_{\text{fuz}}$-relation $R$, its $\alpha$-cut $R_{\alpha}$ is the PA-relation

$$R_{\alpha} = \{ rel_p \in \{<, =, >\} | \deg_R(rel_p) \geq \alpha \}$$

If $R$ is an IA$_{\text{fuz}}$-relation, its $\alpha$-cut $R_{\alpha}$ is the IA-relation

$$R_{\alpha} = \{ rel_p \in I | \deg_R(rel_p) \geq \alpha \}$$

Proposition 45. Given a PA$_{\text{fuz}}$-relation $R$, $R \in$ PA$_{\text{fuz}}$ iff $\forall \alpha \in L: \alpha > 0$ $R_{\alpha} \in$ PA$_c$.

By means of Proposition 45, it is possible to prove that the results concerning tractability of classical PA$_c$ and PA hold in the context of the corresponding fuzzy algebras as well. While the relevant proofs are not reported here, they can be obtained along the same lines underlying the proofs of similar results that will be presented in the context of IA$_{\text{fuz}}$-subalgebras (see Propositions 63 and 64).

7. Tractable subalgebras of IA$_{\text{fuz}}$

In this section, we address the problem of finding tractable subalgebras of IA$_{\text{fuz}}$, by extending to our fuzzy framework the results holding in the context of classical Interval Algebra. To this purpose, we first analyze an important relationship holding between our fuzzy framework and the classical one, which is then exploited to extend SA$_c$ and SA to tractable subalgebras of IA$_{\text{fuz}}$, and finally we identify the unique maximal tractable subalgebra of IA$_{\text{fuz}}$ which includes all fuzzy atomic relations.

7.1. The role of the $\alpha$-cut in relating classical and fuzzy networks

The notion of $\alpha$-cut, introduced in Section 6 for PA$_{\text{fuz}}$ and IA$_{\text{fuz}}$-relations, plays an important role in relating the IA$_{\text{fuz}}$-algebra to its classical counterpart IA. In order to further characterize this relationship, obtaining a "bridge" between crisp and fuzzy constraint networks, we extend the definition of $\alpha$-cut to IA$_{\text{fuz}}$ and PA$_{\text{fuz}}$-networks. In particular, the $\alpha$-cut of a fuzzy constraint network is the crisp constraint network obtained from the original one by $\alpha$-cutting all of its constraints:

Definition 46. Given an IA$_{\text{fuz}}$ (PA$_{\text{fuz}}$) network $N = \langle X, C \rangle$, its $\alpha$-cut $N_{\alpha}$ is the IA (PA) network

$$N_{\alpha} = \langle X, C' \rangle, \quad \text{such that } R'_{ij} = (R_{ij})_{\alpha}$$

where $R_{ij}$ and $R'_{ij}$ denote a generic relation in $C$ and $C'$, respectively.

Recalling the definition of $\alpha$-cut of a fuzzy set [25], we are able to establish a first relationship between the fuzzy set of solutions of a fuzzy network and the crisp set of solutions of its $\alpha$-cut:

Lemma 47. Given a value $\alpha \in L: \alpha > 0$ and an IA$_{\text{fuz}}$-network (PA$_{\text{fuz}}$-network) $N$, we have that $[\text{SOL}(N)]_{\alpha} = \text{SOL}(N_{\alpha})$.

By exploiting Lemma 47, some important relationships can be established between the fuzzy and the classical framework. Such relationships will be essential to prove the main results concerning the tractability of the IA$_{\text{fuz}}$-subalgebras that will be introduced in the following subsections.

Proposition 48. An IA$_{\text{fuz}}$-network (PA$_{\text{fuz}}$-network) $N$ is minimal iff $\forall \alpha \in L: \alpha > 0$ $N_{\alpha}$ is minimal.

Proposition 49. An IA$_{\text{fuz}}$-network (PA$_{\text{fuz}}$-network) $N$ is $k$-consistent iff $\forall \alpha \in L: \alpha > 0$ $N_{\alpha}$ is $k$-consistent [19].

Finally, the following propositions establish the relationship between the operations of IA$_{\text{fuz}}$-algebra and the classical operations between IA-relations (only the proof concerning composition is reported, since the other ones are immediate).
Proposition 50. Given an IA$_{fuz}$-relation $R$, $\forall \alpha \in L$ we have that $(R^{-1})_\alpha = (R_\alpha)^{-1}$.

Proposition 51. Given two IA$_{fuz}$-relations $R_1$ and $R_2$, $\forall \alpha \in L$ we have that $(R_1 \otimes R_2)_\alpha = R_{1\alpha} \otimes R_{2\alpha}$.

Proposition 52. Given two IA$_{fuz}$-relations $R_1$ and $R_2$, $\forall \alpha \in L$ we have that $(R_1 \circ R_2)_\alpha = R_{1\alpha} \circ R_{2\alpha}$.

7.2. Fuzzy extension of pointizable algebras: SA$_{fuz}$ and SA$_{fuz}$c

In order to generalize the definition of SA to our fuzzy framework, we refer, as in the classical case, to the possibility of expressing an interval-based relation as a set of binary relations between points. In particular, since an IA$_{fuz}$-relation is interpreted as a fuzzy relation between two intervals $I_1$ and $I_2$, the idea is to check whether this relation can be translated into a PA$_{fuz}$-network involving their 4 endpoint-nodes $I_{1\lower}^-, I_{1\plus}^+, I_{2\lower}^-, I_{2\plus}^+$. We define the class of relevant PA$_{fuz}$-networks as follows (see Fig. 11):

Definition 53. Let us consider the class of PA$_{fuz}$ networks with 4 nodes, denoted as $\{I_{1\lower}^-, I_{1\plus}^+, I_{2\lower}^-, I_{2\plus}^+\}$, and let us indicate their binary relations as $R_{12}^-, R_{12}^+, R_{21}^-, R_{21}^+$. $\mathcal{S}_{PA}^{fuz}$ is the class of the networks such that $R_{11}^+ = \{< [\alpha_1] \}$ and $R_{22}^+ = \{< [\alpha_2] \}$, where $\alpha_1, \alpha_2 \in L$.

In the following, a network $N \in \mathcal{S}_{PA}^{fuz}$ will be denoted by the 6-tuple of its PA$_{fuz}$-relations. It is easy to see that each $N \in \mathcal{S}_{PA}^{fuz}$ has at most 13 distinct solutions with a degree of satisfaction strictly greater than 0, namely those which correspond to the possible relative dispositions of the intervals $I_1$ and $I_2$: we denote this set as SOL$_{IA}$, where SOL$_{IA} \subseteq \{<, =, >\} \times \cdots \times \{<, =, >\} = \{<, =, >\}^6$. Moreover, we denote as $f_{IA} : \text{SOL}_{IA} \rightarrow \mathcal{I}$ the function which associates to each element $s \in \text{SOL}_{IA}$ the corresponding Allen’s atomic relation, e.g. $f_{IA}(<, <, <, <, <, <) = b$.

The subclass $\text{SA}_c^{fuz} \subseteq \text{IA}_{fuz}$ is then defined as the set of IA$_{fuz}$-relations that can be translated into equivalent networks of $\mathcal{S}_{PA}^{fuz}$, where equivalence is expressed by means of the function $f_{IA}$:

Definition 54. Let $R$ be an IA$_{fuz}$-relation; $R \in \text{SA}_c^{fuz}$ iff $\exists N \in \mathcal{S}_{PA}^{fuz}$ such that $\forall s \in \text{SOL}_{IA}$ $\text{deg}_N(s) = \text{deg}_R(f_{IA}(s))$. We say that $N$ is a network equivalent to $R$.

Following the same line of reasoning, we define the subclass $\text{SA}_c^{fuz} \subseteq \text{SA}_{fuz}$ as the set of those relations that can be translated into PA$_c^{fuz}$-networks:

Definition 55. Let $R$ be an IA$_{fuz}$-relation; $R \in \text{SA}_c^{fuz}$ iff $\exists N \in \mathcal{S}_{PA}^{fuz}$ such that $N$ is a PA$_c^{fuz}$-network, and $\forall s \in \text{SOL}_{IA}$ $\text{deg}_N(s) = \text{deg}_R(f_{IA}(s))$. 

Fig. 11. The class $\mathcal{S}_{PA}^{fuz}$ of PA$_{fuz}$-networks.
It should be noticed that in this definition (as well as in Definition 54) we restrict our attention to the elements of SOL_{IA}, i.e. we don’t consider inconsistent assignments. This technical detail is due to the fact that the function f_{IA} is defined just in SOL_{IA}.

As an example of SA^{fuz}_c-relation, let us consider \( R = \{ b[0.7], m[0.3] \} \), which can be represented as shown in Fig. 12. It is easy to see that \( R \) can be translated into the PA^{fuz}_c-network \( N' \subseteq S^{fuz}_{PA} \) identified by \( R'_{12}^{-} = R'_{12}^{+} = R'_{12}^{-} = (\langle [0.7] \rangle, R'_{12}^{+} = (\langle [0.7] \rangle, [0.3]) \). In fact, \( N' \) has two solutions: if \( I_1^{-} < I_2^{-} \) the solution \( s_1 \) is \( I_1^{-} < I_1^{+} < I_2^{-} < I_2^{+} \) (corresponding to the atomic relation \( b \)), with \( \deg_N(s_1) = 0.7 \); if \( I_1^{+} = I_2^{-} \) the solution \( s_2 \) is \( I_1^{-} < I_1^{+} = I_2^{-} < I_2^{+} \) (corresponding to the atomic relation \( m \)), with \( \deg_N(s_2) = 0.3 \).

Since the definitions of SA^{fuz}_c and SA^{fuz}_c are parallel and extend the classical definitions of SA and SA_c, it is interesting to study the way in which these fuzzy subalgebras are related to their classical counterparts. In doing this, we will consider classical SA and SA_c algebras defined in the same way as SA^{fuz}_c and SA^{fuz}_c, with obvious adjustments. In particular, we will refer to the class of PA networks \( S_{PA} \), corresponding to \( S^{fuz}_{PA} \), defined as in Definition 53 but with \( R'_{12}^{+} = R'_{22}^{+} = (\langle \rangle) \). SA is defined according to Definition 54, where \( N' \subseteq S_{PA} \) must be such that \( s \in SOL(N) \) iff \( f_{IA}(s) \in R \), while SA_c is the subset of SA-relations \( N' \) for which \( N' \) is a PA_c-network.

The relevant proofs will make use of a number of definitions concerning networks of relations. In particular, we extend the notion of ‘inclusion’ introduced in Definition 30 to networks of crisp relations, then we introduce the notion of disjunction of networks, and finally we define the multiplication between a scalar and a crisp network.

**Definition 56.** Let \( N_1 = (\mathcal{X}', \mathcal{C}') \) and \( N_2 = (\mathcal{X}'', \mathcal{C}'') \) be two crisp networks (i.e. either PA or IA-networks). Then, \( N_1 \subseteq N_2 \) if \( \mathcal{X}' = \mathcal{X}'' \) (thus \( E(N_1) = E(N_2) \)) and \( \forall (i, j) \in E(N_1) \), \( R'_{ij} \subseteq R''_{ij} \), where \( R'_{ij} \subseteq \mathcal{C}' \) and \( R''_{ij} \subseteq \mathcal{C}'' \).

**Definition 57.** Let \( N_1 = (\mathcal{X}, \mathcal{C}') \) and \( N_2 = (\mathcal{X}, \mathcal{C}'') \) be two (either crisp or fuzzy) networks of relations. We define the disjunction \( (N_1 \cup N_2) \) as the network \( (\mathcal{X}, \mathcal{C}) \) such that \( \forall R_{ij} \in \mathcal{C} \), \( R_{ij} = (R'_{ij} \oplus R''_{ij}) \), where \( R'_{ij} \in \mathcal{C}' \) and \( R''_{ij} \in \mathcal{C}'' \).

**Definition 58.** Given an IA (PA) network \( N \) and a scalar \( \alpha \in L \), \( \alpha \ast N \) denotes the IA^{fuz} (PA^{fuz}) network whose edges are obtained from those of \( N \) by assigning to the corresponding atomic relations a preference degree equal to \( \alpha \). More formally, let \( N = (\mathcal{X}, \mathcal{C}) \), and let a generic \( R_{ij} \in \mathcal{C} \) be denoted as \( R_{ij} = \{ rel_{ij_1}, \ldots, rel_{ij_k} \} \); then, \( \alpha \ast N = (\mathcal{X}, \mathcal{C}') \), where for any \( R_{ij}^{fuz} \in \mathcal{C}' \) it holds that \( R_{ij}^{fuz} = \{ rel_{ij_1}[\alpha], \ldots, rel_{ij_k}[\alpha] \} \).

It turns out that SA^{fuz}_c and SA^{fuz}_c can be related to their classical counterparts by means of the notion of \( \alpha \)-cut; in particular, the same relation holds as the one between PA and PA^{fuz}_c, as well as the one between PA_c and PA^{fuz}_c (see Proposition 45).

**Lemma 59.** Let \( R_1 \) and \( R_2 \) be two SA-relations such that \( R_1 \subseteq R_2 \), and let \( N_1 \) and \( N_2 \) be two PA-networks of \( S_{PA} \) equivalent to \( R_1 \) and \( R_2 \), respectively. If \( N_1 \) is minimal, then \( N_1 \subseteq N_2 \).

**Proposition 60.** Given an IA^{fuz}-relation \( R, R \in SA^{fuz}_c \) iff \( \forall \alpha \in L: \alpha > 0 R_{\alpha} \in SA \).

**Proposition 61.** Given an IA^{fuz}-relation \( R, R \in SA^{fuz}_c \) iff \( \forall \alpha \in L: \alpha > 0 R_{\alpha} \in SA_c \).

Besides pointing out, from a theoretical point of view, the correspondence between our subclasses and the classical ones, these results make it possible to prove important properties characterizing SA^{fuz}_c and SA^{fuz}_c, that are essential in
Proposition 62. SA_{\text{fuz}} (SA_{c_{\text{fuz}}}) is an algebra with respect to the operations of inversion, conjunction and composition.

Furthermore, an important consequence of the results presented above is that SA_{c_{\text{fuz}}} and SA_{\text{fuz}} share with their classical counterparts SA_{c} and SA the computational properties described in Section 6. Following our general methodology introduced in [19], we don’t extend classical proofs to our fuzzy framework, but rather we generalize the results holding in the classical case by exploiting the relationships between minimality and $k$-consistency of fuzzy networks and minimality and $k$-consistency of their $\alpha$-cuts (see Propositions 48 and 49).

Proposition 63. If an SA_{c_{\text{fuz}}}-network is path-consistent, then it is also minimal.

Proposition 64. Let $N'$ be a SA_{\text{fuz}}-network. If all the 4-subnetworks of $N'$ are minimal, then $N'$ is minimal as well.

The above propositions show that the reasoning tasks considered in Section 3 are tractable for SA_{c_{\text{fuz}}} and SA_{\text{fuz}}.

Referring to FIMIN(SA_{\text{fuz}}), the minimal network equivalent to a given SA_{\text{fuz}}-network $N'$ can be computed by means of the $O(k \times n^3)$ path consistency algorithm PC2_{\text{fuz}}, which has been introduced in Section 5. In fact, since according to Proposition 62 SA_{c_{\text{fuz}}} is closed under the operations performed by the algorithm, PC2_{\text{fuz}} computes a path-consistent network equivalent to $N'$ which is still an SA_{c_{\text{fuz}}}-network, and therefore is minimal by Proposition 63. As for SA_{\text{fuz}}, similar considerations that rely on Proposition 64 make it possible to prove that FIMIN(SA_{\text{fuz}}) is solved by the algorithm AAC_{\text{fuz}}, with $O(k \times n^4)$ complexity.

Finally, the fact that FISAT and FOSOL problems are tractable as well turns out from Theorem 20. As it will be shown in the following subsection, an optimal solution, as well as its degree of satisfaction, can be computed by means of an efficient algorithm developed for classical IA-networks by Ligozat [42].

7.3. $\mathcal{H}_{\text{fuz}}$: the unique maximal tractable subalgebra of IA_{\text{fuz}}

In the previous subsections, we have extended to our fuzzy framework the so-called pointizable subalgebras of IA, for which efficient polynomial algorithms exist that compute the minimal network. An important result provided by Nebel and Bürckert in [18] is the introduction of a maximal tractable subclass of Interval Algebra, denoted as $\mathcal{H}$, which is a strict superset of SA (and therefore of SA_{c} as well). In particular, $\mathcal{H}$ includes all those Allen’s relations that can be represented as a conjunction of ORD-Horn clauses concerning endpoints, i.e. clauses containing at most one positive literal of the form $a = b$ or $a \leq b$. Tractability of $\mathcal{H}$ is proved by showing that path-consistency algorithm is sufficient for deciding consistency of $\mathcal{H}$-networks, so that the tasks of checking consistency and computing the minimal network take at most $O(n^5)$ and $O(n^5)$ time, respectively. An alternative characterization of $\mathcal{H}$-algebra is provided by Ligozat in [22], where it is defined as the set of pre-convex relations of IA. On the basis of this geometric characterization, a very simple method to test whether a given relation belongs to $\mathcal{H}$ is devised in [42], as well as an algorithm which computes without backtrack a solution of classical path-consistent $\mathcal{H}$-networks, taking $O(n^2)$-time: we denote this algorithm as LIG-ALG. The relationship between Horn representability as a syntactic concept and the geometric notion of pre-convexity has been deeply studied in [43] to characterize consistency of generalized interval algebra.

The results concerning tractability of $\mathcal{H}$ can be summarized by the following proposition.

Proposition 65. The subclass $\mathcal{H} \subseteq 2^I$ is a tractable subalgebra of IA. In particular, given a path-consistent $\mathcal{H}$-network $\mathcal{N} = (X, C)$, $\mathcal{N}$ is consistent (i.e. SOL($\mathcal{N}$) $\neq \emptyset$) if and only if $\forall R_{ij} \in C R_{ij} \neq \emptyset$. In this case, a solution of $\mathcal{N}$ can be obtained in $O(n^2)$-time by means of the algorithm LIG-ALG.

So, a first issue is to extend the classical algebra $\mathcal{H}$ to a tractable subalgebra of IA_{\text{fuz}}. In this respect, let us recall that the tractability properties of classical pointizable subalgebras have been extended to our fuzzy framework by exploiting the relationship between SA_{c_{\text{fuz}}} and SA_{c} and the relationship between SA_{\text{fuz}} and SA, expressed by
Propositions 61 and 60, respectively. This suggests to introduce the fuzzy extension of $H$, that we call $H^{\text{fuz}}$, by means of the following definition:

**Definition 66.** Let $R$ be an IA\textsuperscript{fuz}-relation; $R \in H^{\text{fuz}}$ iff $\forall \alpha \in L: \alpha > 0 \Rightarrow R_\alpha \in H$.

Along a similar line of reasoning as the one followed in the previous subsection (see Proposition 62), it is easy to see that an immediate consequence of this definition is the closure of $H^{\text{fuz}}$ under the operations of IA\textsuperscript{fuz}-algebra:

**Proposition 67.** $H^{\text{fuz}}$ is an algebra with respect to the operations of inversion, conjunction and composition.

Tractability of $H^{\text{fuz}}$ can be proved by referring to the problem FISAT, taking into account that, on the basis of Theorem 20, this suffices to ensure tractability of all of the other interesting reasoning tasks.

**Proposition 68.** Let $N = (X, C)$ be a path-consistent $H^{\text{fuz}}$-network. Then, it holds that $\forall R_{ij} \in C: \max_{ij} = \deg(N)$.

The proposition above shows that the path-consistency algorithm, which is polynomial in the number of nodes of the network, solves the FISAT problem, so that $H^{\text{fuz}}$ is tractable. As for FOSOL, this problem can be solved by exploiting the algorithm LIG-ALG, which computes a solution of classical path-consistent $H$-networks. In fact, by the proof of Proposition 68, it turns out that, given a path-consistent $H^{\text{fuz}}$-network $N$ and an arbitrary edge $(i, j) \in E(N)$, the network $N_{\max_{ij}}$ is a path consistent $H$-network whose solutions are optimal solutions of $N$. As a consequence, given an $H^{\text{fuz}}$-network $N$, an optimal solution can be computed as follows:

1. Run the path-consistency algorithm PC2\textsuperscript{fuz} on $N$;
2. Choose an edge $(i, j)$ and compute $\max_{ij}$;
3. Compute an optimal solution by applying Ligozat’s algorithm on $N_{\max_{ij}}$, i.e. considering only those atomic relations whose preference degrees are greater than or equal to $\max_{ij}$.

Since the complexity of PC2\textsuperscript{fuz} is $O(\kappa n^3)$ and that of LIG-ALG is $O(n^2)$, it turns out that the complexity of the overall algorithm for FOSOL is $O(\kappa n^3)$, where $\kappa$ is the number of preference degrees used in the network and $n$ is the number of nodes.

Turning to the classical subalgebra $H$, the importance of the relevant tractability results is in a sense strengthened by the maximality property which characterizes $H$. In particular, besides being a maximal subclass, it is proved in [18,22] that $H$ is the unique greatest tractable subclass among those that contain all atomic relations.

**Proposition 69.** All Allen’s atomic relations belong to $H$, i.e. $\forall \text{rel}_p \in \mathcal{I} (\text{rel}_p) \in H$. Moreover, for every subclass $A \subseteq 2^\mathcal{I}$ such that

- $A$ is an algebra,
- $A$ is tractable,
- $\forall \text{rel}_p \in \mathcal{I} (\text{rel}_p) \in A$,

it holds that $A \subseteq H$.

So, a second issue is to investigate whether $H^{\text{fuz}}$ has a maximality property corresponding to the one which characterizes its classical counterpart $H$, i.e. whether Proposition 69 can be extended to $H^{\text{fuz}}$ in some way. This leads us to check if $H^{\text{fuz}}$ contains each subclass $A \subseteq 2^\mathcal{I}$ such that $A$ is an algebra, is tractable and include the ‘fuzzy extension’ of Allen’s atomic relations. In this respect, let us recall that, in Section 3, the equivalence (under polynomial Turing-reduction) of interesting reasoning tasks has been proved for all those subsets of $\mathcal{I}A^{\text{fuz}}$ that include the set $B = \{ (\text{rel}_p \alpha) : \text{rel}_p \in \mathcal{I} \land \alpha \in L \}$, which therefore plays the role of classical Allen’s atomic relations in this case. Following this intuition, what should be proved is that $H^{\text{fuz}}$ is the unique maximal tractable subalgebra of $IA^{\text{fuz}}$ among those including the relations of $B$. As it will be shown in the remainder of this subsection, it turns out that this
Table 3
Complexity results for $\mathcal{I}A^{\text{fuz}}$-subalgebras

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{S}A^{\text{fuz}}$</th>
<th>$\mathcal{S}A^{\text{fuz}}$</th>
<th>$\mathcal{H}^{\text{fuz}}$</th>
<th>$\mathcal{I}A^{\text{fuz}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FISAT</td>
<td>$O(kn^3)$</td>
<td>$O(kn^3)$</td>
<td>$O(kn^3)$</td>
<td>NP-diff.</td>
</tr>
<tr>
<td>FOSOL</td>
<td>$O(kn^3)$</td>
<td>$O(kn^3)$</td>
<td>$O(kn^3)$</td>
<td>NP-diff.</td>
</tr>
<tr>
<td>FIMIN</td>
<td>$O(kn^3)$</td>
<td>$O(kn^4)$</td>
<td>$O(kn^3)$</td>
<td>NP-diff.</td>
</tr>
</tbody>
</table>

is the case for $\mathcal{H}^{\text{fuz}}$. In order to proceed with the relevant proof, we first introduce some relationships holding between a generic subset $\mathcal{A}$ of $\mathcal{I}A^{\text{fuz}}$ and that subset of $2^\mathcal{I}$ which is related to $\mathcal{A}$ by means of the notion of $\alpha$-cut.

**Definition 70.** Let $\mathcal{A}$ be a set of fuzzy relations (i.e. either $\mathcal{P}A^{\text{fuz}}$ or $\mathcal{I}A^{\text{fuz}}$-relations): the $\alpha$-cut of $\mathcal{A}$ is defined as $\mathcal{A}_\alpha = \bigcup_{R \in \mathcal{A}} \{ R_\alpha \}$.

**Lemma 71.** Let $\mathcal{A} \subseteq \mathcal{I}A^{\text{fuz}}$ be a sub-algebra of $\mathcal{I}A^{\text{fuz}}$ with respect to the operations of inversion, conjunction and composition. Then, $\forall \alpha \in L$ we have that $\mathcal{A}_\alpha$ is a sub-algebra of $\mathcal{I}A$ with respect to the classical operations of inversion, conjunction and composition.

**Lemma 72.** Let $\mathcal{A} \subseteq \mathcal{I}A^{\text{fuz}}$ be a tractable subclass of $\mathcal{I}A^{\text{fuz}}$ which is finite, i.e. $|\mathcal{A}| = N$ with $N$ finite. Then, $\forall \alpha \in L$ it turns out that $\mathcal{A}_\alpha$ is a tractable subclass of $\mathcal{I}A$.

We are now in a position to extend to $\mathcal{H}^{\text{fuz}}$ the maximality property of $\mathcal{H}$ stated in Proposition 69.

**Theorem 73.** $\mathcal{H}^{\text{fuz}}$ is the unique maximal tractable subalgebra of $\mathcal{I}A^{\text{fuz}}$ which includes all the relations of $\mathcal{B}$.

Table 3 summarizes the results concerning the complexity of the algorithms devised for the different algebras introduced throughout the paper. The result concerning FIMIN($\mathcal{H}^{\text{fuz}}$) has been obtained using the mapping from FOSOL to FIMIN shown in Table 2. It can be noticed that, with respect to their classical counterparts, the computational complexity of the algorithms is augmented by a factor equal to the number of preference degrees used in the network, which is at most $13\frac{n(n-1)}{2} = O(n^2)$.

8. Comparison with related work and conclusions

The idea that a fuzzy set is a natural way to model a soft constraint was first presented by Bellman and Zadeh [44]. In the context of temporal reasoning, fuzzy sets and possibility theory [25,33,34] have been used by several authors [9–12,45,46] as a rich and powerful setting to deal with approximate temporal reasoning, in order to provide the capability of expressing both flexible and uncertain constraints.

The aim of our proposal is to pursue this line of research in the context of qualitative temporal reasoning, by generalizing Allen’s Interval Algebra in a fuzzy direction: to this purpose, we attach preference degrees to atomic relations and we provide a relevant interpretation in terms of fuzzy sets. Thus, it is interesting to evaluate the expressive power of our framework compared to the classical IA-algebra, made up of 213 relations: in order to provide a quantitative comparison, we enumerate the set $\mathcal{I}A^{\text{fuz}}$ made up of all the relations that can be expressed in $\mathcal{I}A^{\text{fuz}}$-algebra. Of course, this set is infinite in as much as we choose $L$ as an infinite set, such as for instance the interval $[0, 1]$; however, according to the ordinal nature of our framework we can enumerate the possible ways of ordering the 13 preference degrees that characterize $\mathcal{I}A^{\text{fuz}}$-relations, where given two preference degrees $\alpha_i$ and $\alpha_j$ we count as distinct the three cases in which $\alpha_i < \alpha_j$, $\alpha_i = \alpha_j$ and $\alpha_i > \alpha_j$, respectively. Enumeration reveals that there are $5.269 \times 10^{11}$ orderings of this kind; since each one of them identifies the set of $\alpha$-cuts of the corresponding $\mathcal{I}A^{\text{fuz}}$-relation, it is also possible to evaluate in the same way the cardinality of the tractable subalgebras of $\mathcal{I}A^{\text{fuz}}$: the relevant results, shown in Table 4, point out that fuzzifying Allen’s relations provides a significant increase in the expressive power w.r.t. the classical framework, both for the full Interval Algebra [1] and for the tractable subalgebras that have been identified in the literature [3,16,18,20].
Besides the classical frameworks characterized by crisp constraints, several approaches to fuzzy temporal reasoning have been proposed in the literature. First, Dubois and Prade [46] introduced the notions of fuzzy date and fuzzy relation between fuzzy dates to represent ill-known dates, time intervals with fuzzy boundaries, fuzzy durations and uncertain precedence relations between events. All these concepts are formalized by relying on time points as the primitive notion, while intervals are defined as fuzzy sets of time-points between two dates. Starting from these definitions, Barro et al. [47] proposed a language for the representation of fuzzy temporal references, which is still based on the notion of fuzzy date and where intervals $I_{a,e,d}$ are defined as pairs of dates $(a,e)$ constrained by a temporal duration $d$: this way, all relations between temporal entities are interpreted as constraints on the distances between dates, which can then be projected into Fuzzy Temporal Constraint Satisfaction Networks. The latter were formalized by Godo and Vila in [11] and also underly their subsequent definition of a possibilistic temporal logic [10]. Basically, Fuzzy Temporal Constraint Networks (FTCN) generalize to the fuzzy framework the classical Temporal Constraint Satisfaction Problem (TCSP) [4]: they express temporal knowledge by means of a set of variables, which take value in a metric domain such as $\mathbb{R}$, and a set of fuzzy constraints between them, described by possibility distribution functions that constrain the temporal distance between couples of variables.

The main differences between these fuzzy temporal approaches and the approach developed here are twofold. First, we deal exclusively with the representation of qualitative temporal knowledge, while FTCN represent absolute locations of time points and their relative distances by means of quantitative fuzzy constraints. Then, we assume temporal intervals as a primitive notion, while FTCN variables represent time points: this makes it possible to express a much greater number of relations as far as the qualitative part of temporal knowledge is concerned. In fact, by means of FTCN it is well possible to express qualitative knowledge about time points; for instance, with reference to a pair of variables $(X_i, X_j)$, if the support of the possibility distribution constraining $X_j - X_i$ is included in the positive part of the real set, then we are sure that $X_i$ is before $X_j$, and we can also compute the relevant degree of possibility. However, the qualitative knowledge that can be expressed by means of a point-based approach corresponds to the point-algebra $PA^{fuz}$, and, on the basis of Definition 54, the subclass of $IA^{fuz}$ that can be translated into $PA^{fuz}$-networks is precisely the $SA^{fuz}$-algebra: taking into account the enumeration results of Table 4, we have that full $IA^{fuz}$-algebra makes it possible to express a number of relations which is more than five orders of magnitude greater.

A parallel result can be obtained by considering the tractable subclass of FTCN which has been identified in the literature. In this respect, both Marin et al. [48] and Vila and Godo [11] restrict the class of admissible possibility distributions to unimodal functions, i.e. such that all their $\alpha$-cuts are convex intervals. The resulting framework corresponds to a generalization of simple metric constraint networks (STP) [4], and inherits the relevant property that an $O(n^3)$ path-consistency algorithm suffices to compute minimal networks. Also the techniques for obtaining solutions of FTCN proposed in a more recent work [45] focus on this tractable class. As far as qualitative knowledge is concerned, it is easy to see that the fragment considered corresponds to $PA^{fuz}_{c}$-algebra: again, taking into account Definition 55 and the results shown in Table 4, we notice that our maximal tractable subalgebra $H^{fuz}$ includes a number of relations which is more than four orders of magnitude greater than the corresponding $SA^{fuz}_{c}$-algebra.

The results mentioned above, including the identification of a tractable subclass, have also been extended in [12,49] to the semiring-based soft constraint formalism [50], which includes the fuzzy framework as a subcase. As a matter of fact, the main results of this proposal rely on the idempotency of the aggregator operator $\times$ in the semiring, and this greatly restricts the choice of the specific instance of the framework: indeed, the authors recognize the fuzzy framework as the most significant one for temporal reasoning [49] and adopt it in their techniques to learn local preferences [49,51], thus relying again on the FTCN framework. On the other hand, since all the proofs that have been devised in the present paper rely on the properties of the ‘min’ operator that are common to those assumed in [50] for the aggregator operator, the $IA^{fuz}$-framework and the relevant results can easily be extended to the semiring-based soft constraint formalism, under the idempotency of the aggregator operator.

### Table 4

Enumeration of interval algebras in the classical and fuzzy settings

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Full IA</th>
<th>$H$</th>
<th>SA</th>
<th>$IA^{fuz}$</th>
<th>$H^{fuz}$</th>
<th>$SA^{fuz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA vs. $IA^{fuz}$</td>
<td>$8192$</td>
<td>$5.26858 \times 10^{11}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H$ vs. $H^{fuz}$</td>
<td>$868$</td>
<td>$1.27276 \times 10^{9}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SA vs. $SA^{fuz}$</td>
<td>$188$</td>
<td>$9.15564 \times 10^{5}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$SA_{c}$ vs. $SA^{fuz}_{c}$</td>
<td>$83$</td>
<td>$6.8216 \times 10^{4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Besides these point-based approaches to fuzzy temporal reasoning, there have only been two proposals, to the best of our knowledge, which assume time intervals as the primitive notion.

Dutta [31,52] proposed a first approach which models the lack of knowledge about events by means of fuzzy sets of time intervals. In particular, a set of precise and disjoint intervals is assumed as background, while the degree of membership \( \mu_i(e) \) of an event \( e \) in interval \( i \) represents the degree of possibility that interval \( i \) contains the event \( e \): the reasoning task is then to compute the possibility degree that a particular relation holds between two events, where such relation belongs to the set \{\(<, >, =\)\} corresponding to Allen’s atomic relations \{\(b, a, o\)\} re-defined in the coarse granularity model adopted in Dutta’s approach. Of course, such a problem can be codified in the \(IA^{fuz}\)-framework, by expressing both intervals and events as nodes of an \(IA^{fuz}\)-network, and fixing the relative disposition of disjoint intervals by means of classical atomic relations (which belong to \(IA^{fuz}\)). It turns out that Dutta’s approach correspond to a very limited fragment of \(IA^{fuz}\)-algebra, since only 7 kinds of \(IA^{fuz}\)-relations (enumerated as described above) suffice to construct the network, and, provided that the possibility distributions are normalized,\(^3\) the results prescribed in Dutta’s model can be obtained by computing the equivalent minimal network. On the other hand, several limitations affect Dutta’s proposal compared to our \(IA^{fuz}\)-algebra, which mainly concern the kind of reasoning activity allowed in the framework. Basically, information about the absolute location of events, in the form of the possibility distribution \(\mu_i()\), is required to start reasoning; information concerning the relative displacement of a pair of events cannot be specified, but can only be inferred, independently of information concerning other events, from the initial specification of \(\mu_i()\): actually, there is not a constraint propagation mechanism which infers, for instance, that if interval \(I_1\) is contained in \(I_2\) and the latter is before \(I_3\), then \(I_1\) is also before \(I_3\). Moreover, the granularity of the adopted temporal model limits the set of basic relations that can be assigned a degree of possibility to \{\(b, a, o, m, mi\)\}, while the other Allen’s relations do not have a counterpart in Dutta’s framework. A related issue is that granularity must be carefully chosen by the user, and this choice is critical for the results of the inference mechanism to be adequate.

An approach which is more directly related to ours is the one proposed by Guesgen et al. [32], which focuses on the so-called imprecise spatial descriptions. As mentioned in Section 4, the idea is to express the imprecision which characterizes the observation of a given atomic Allen’s relation \(rel_i\) by means of fuzzy values associated to atomic relations. In particular, greater values are assigned to those relations which are closer to \(rel_i\) according to the relation of conceptual neighborhood introduced by Freksa in [53]; actually, only 6 preference degrees are used, which is compatible with neighborhood of type B (see [53] for details about the different kinds of neighborhoods). The authors define operations of composition and intersection between fuzzy Allen relations in the same way as in the present paper, and introduce a constraint propagation algorithm which corresponds to the basic version of the path-consistency algorithm shown in Fig. 5 (i.e. PC1\(^{fuz}\)). Moreover, they suggest two strategies for the implementation of composition operation, called ‘Best First Relation Filling’ and ‘Minimizing Repeated Look-Ups’, that can be directly applied in our framework too. Enumeration reveals that, adopting neighborhood of type B, their framework covers 340 \(IA^{fuz}\)-relations; in contrast, we do not assume any restriction on the ordering of preference degrees, since the scope of our work is not limited to imprecise spatial reasoning as defined in [32]. On the other hand, all the concepts, algorithms, and the identification of tractable subclasses presented in the present paper can be exploited in the context of [32] too. In this respect, we have enumerated the \(IA^{fuz}\)-relations resulting from the adoption of the three kinds of neighborhood A, B, and C, and we have identified the relevant fragments contained in our tractable subalgebras. Table 5 shows the relevant results, which have been also extended to the union of neighborhood relations as indicated. It turns out that, in the case of neighborhood of type B, more than 90 per cent of relations belong to \(H^{fuz}\); in any case, if the relations of a given network turn out to belong to \(SA^{fuz}_c\), \(SA^{fuz}\) or \(H^{fuz}\), then the relevant properties can be exploited to solve the reasoning tasks that are of interest. Finally, besides exploiting the results provided by the present paper in the restricted framework of [32], one can envisage situations where full \(IA^{fuz}\)-algebra is needed in the context of imprecise spatial reasoning, e.g. by considering disjunctions of imprecise Allen’s relations.

Besides the motivations provided throughout the paper, the fact that these previous proposals can be grasped by our framework shows the importance of tackling the problem of reasoning with qualitative fuzzy temporal constraints in its full generality. We believe that our work provides a significant contribution in this respect, by placing Allen’s Interval Algebra in the general context of the FCSP framework, while most previous proposals have been concerned

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\(^3\) A possibility distribution \(\mu_i\) is normalized if there is a value \(e^*\) such that \(\mu_i(e^*) = 1\).
with quantitative temporal reasoning in point-based frameworks. On the other hand, this settlement also facilitates the integration of IA with quantitative approaches, in order to develop an integrated model capable of handling both quantitative and qualitative temporal information in a fuzzy setting, in a similar way as IA-algebra has been integrated with TCNs in [7]. The development of this model, which we began to explore in [54], is left as a topic for future work. Another issue is to refine the conditions of tractability explored in the paper, in order to design algorithms that, while being exponential in general, are able to find an optimal solution in polynomial time provided that its degree of satisfaction is greater than a certain preference level, defined on the basis of the IA-network at hand. Moreover, we will investigate the possibility of extending the notion of controllability to our framework, in order to handle uncertainty in presence of fuzzy constraints expressing preference, as well as the comparison of different algorithms on an empirical basis.

**Appendix A. Proofs**

**Proof of Proposition 11.** According to Definition 9, we have that
\[
\deg_{R_{ik} \circ R_{kj}}(rel_p) = \max_{q,r: rel_p \in \{rel_q \circ rel_r\}} \min \{\deg_{R_{ik}}(rel_q), \deg_{R_{kj}}(rel_r)\}
\]
(A.1)

Now, taking into account the definition of composition of classical IA-relations, we have that \(rel_p \in \{rel_q \circ rel_r\}\) iff the following singleton labeling \(s\) is consistent:

- \(N(s) = \{I_i, I_k, I_j\}\),
- \(s_{ik} = rel_q, s_{kj} = rel_r\) and \(s_{ij} = rel_p\).

As a consequence, (A.1) can be expressed as
\[
\deg_{R_{ik} \circ R_{kj}}(rel_p) = \max_{s \in SL_N, s \text{ consistent, } s_{ij} = rel_p} \min \{\deg_{R_{ik}}(s_{ik}), \deg_{R_{kj}}(s_{kj})\}
\]

Taking into account that \(\deg_{R_{ij}}(rel_p) = 1\), this yields \(\deg_{R_{ik} \circ R_{kj}}(rel_p)\) equal to
\[
\max_{s \in SL_N, s \text{ consistent, } s_{ij} = rel_p} \min \{\deg_{R_{ik}}(s_{ik}), \deg_{R_{kj}}(s_{kj}), \deg_{R_{ij}}(s_{ij})\}
\]

therefore, taking into account Definition 5, we get
\[
\deg_{R_{ik} \circ R_{kj}}(rel_p) = \max_{s \in SL_N: s_{ij} = rel_p} \{\deg_N(s)\}
\]
and we are done. \(\square\)

**Proof of Proposition 17.** It is easy to see that solving FIMIN(\(S\)) directly yields a solution to FISAT(\(S\)). In particular, given an \(S\)-network \(N\) we compute its equivalent minimal network \(N^{min}\), whose degree of consistency \(\deg(N^{min})\) is by Definition 15 equal to \(\deg(N)\). Taking into account Definition 14, it is easy to see that, for every IA\(^{fuz}\)-relation \(R_{ij}^{min}\) of \(N^{min}\), \(\deg(N^{min}) = \max_{rel_p \in \mathcal{I}} \deg_{R_{ij}^{min}}(rel_p)\). As a consequence, FISAT(\(S\)) can be directly solved by inspecting an arbitrary edge of \(N^{min}\).

To show that solving FIMIN(\(S\)) readily follows from solving FISAT(\(S\)), let us suppose to have an algorithm that, given an \(S\)-network \(N'\), computes its degree of consistency \(\deg(N')\). In order to determine the minimal network \(N^{min}\)
equivalent to a generic \( S \)-network \( N' \), we compute each of its \( O(n^2) \) IA\(^{fuz}\)-relations \( R_{ij}^{min} \) by means of 13 iterations of the algorithm for FISAT(\( S \)). In particular, at the \( p \)th iteration, with \( p \in \{1, \ldots, 13\} \), we compute \( \deg_{\min}(rel_p) \) by running the algorithm on a network \( N' \) which is obtained from \( N ' \) by letting \( R_{ij} \leftarrow rel_p[\deg_{R_{ij}}(rel_p)] \); since \( B \subseteq S, N' \) is an \( S \)-network, therefore the algorithm determines its degree of consistency \( \deg(N') \). Taking into account Definition 12, it is easy to see that \( \deg(N') = \max_{s \in SL_{N':s_{ij}}=rel_p} \deg_{N}(s) \), which according to Definitions 14 and 15 is equal to \( \deg_{\min}(rel_p) \). \( \square \)

**Proof of Proposition 18.** As for the first claim, it is easy to see that a solution to FOSOL(\( S \)) clearly yields a solution to FISAT(\( S \)): given an optimal solution \( s \) of a given \( S \)-network \( N' \), we have that \( \deg(N) = \deg_{N}(s) \), which can be computed as \( \min_{(i,j) \in E(N)} \deg_{R_{ij}}(s_{ij}) \) by checking the \( O(n^2) \) edges of the network \( N' \).

On the other hand, FOSOL(\( S \)) can be solved by \( O(n^2) \) iterations of the algorithm for FISAT(\( S \)) as follows. Given an \( S \)-network \( N' \), we choose an edge \((i,j) \in E(N)\), and for each \( rel_p \in I \) we run the algorithm for FISAT(\( S \)) on a network \( N' \) which is obtained from \( N' \) by letting \( R_{ij} \leftarrow rel_p[\deg_{R_{ij}}(rel_p)] \) (since \( B \subseteq S, N' \) is an \( S \)-network). For each \( rel_p \in I \), let \( \deg_{ij}'(rel_p) \) be the degree of consistency of \( N' \), and let \( \deg_{ij}'(rel_q) \) be a maximum among the computed values, i.e. \( \deg_{ij}'(rel_q) = \max_{rel_p \in I}(\deg_{ij}'(rel_p)) \); of course, \( \deg_{ij}'(rel_q) \) can be extended to an optimal solution of \( N' \). As a consequence, we can modify \( N' \) by letting \( R_{ij} \leftarrow rel_q[\deg_{ij}'(rel_q)] \), and repeat the process for all the \( O(n^2) \) edges of the network until the latter is labeled with atomic relations only: after \( O(n^2) \) iterations of the algorithm, the corresponding singleton labeling \( s \) is an optimal solution. \( \square \)

**Proof of Proposition 19.** The first claim directly follows from Proposition 17 (second claim) and Proposition 18 (first claim).

In a similar way, the second claim can be directly derived from Proposition 18 (second claim) and Proposition 17 (first claim). Alternatively, the proof can follow the same lines as that of the second claim of Proposition 18, but using the algorithm for FIMIN(\( S \)) instead of the one for FISAT(\( S \)). \( \square \)

**Proof of Lemma 33.** Let \( R = (rel_1[\alpha_1], \ldots, rel_{13}[\alpha_{13}]) \). We prove the lemma with reference to \( R \circ I[\alpha \circ R] \) (the proof is the same for \( I[\alpha \circ R] \)). Taking into account Definition 9, we have that \( R \circ I[\alpha] = (rel_1[\alpha_1] \circ I[\alpha]) \oplus \cdots \oplus (rel_{13}[\alpha_{13}] \circ I[\alpha]) \). Taking into account that in classical IA \( \forall rel_p \in I \) \( rel_p \circ I = I \), the latter turns out to be equivalent to \( I[\min(\alpha_1, \alpha)] \oplus \cdots \oplus I[\min(\alpha_{13}, \alpha)] = I[\max\{\min(\alpha_1, \alpha), \ldots, \min(\alpha_{13}, \alpha)\}] = I[\min(\max R, \alpha)] \). \( \square \)

**Proof of Proposition 34.** First, we prove the following lemmas (i.e. 74–78).

**Lemma 74.** Let \( R_{ij} \) and \( R_{jk} \) be two IA\(^{fuz}\)-relations, and \( \alpha \in L \). We have that

1. \( (R_{ij} \circ I[\alpha]) \circ R_{jk} = (R_{ij} \circ R_{jk}) \circ I[\alpha] \),
2. \( R_{ij} \circ (R_{jk} \circ I[\alpha]) = (R_{ij} \circ R_{jk}) \circ I[\alpha] \).

**Proof.** We prove the first equality (the other proof is similar). Let us denote the first member of the equation as \( R_1 \), i.e. \( R_1 = (R_{ij} \circ I[\alpha]) \circ R_{jk} \), and the second one as \( R_2 \), i.e. \( R_2 = (R_{ij} \circ R_{jk}) \circ I[\alpha] \). Taking into account the definitions of composition and conjunction, for every atomic relation \( rel_p \in I \) we have that

\[
\deg_{R_1}(rel_p) = \max_{u,v: rel_p \in [rel_u \circ rel_v]} \min \{ \deg_{R_{ij} \circ I[\alpha]}(rel_u), \deg_{R_{jk}}(rel_v) \} = \max_{u,v: rel_p \in [rel_u \circ rel_v]} \min \{ \min_{\alpha, \deg_{R_{ij}}(rel_u)}, \deg_{R_{jk}}(rel_v) \} = \max_{u,v: rel_p \in [rel_u \circ rel_v]} \min \{ \deg_{R_{ij}}(rel_u), \deg_{R_{jk}}(rel_v) \} = \deg_{R_2}(rel_p)
\]

and we are done. \( \square \)
Lemma 75. Given two IA\textsuperscript{fuz}-relations \(R_1\) and \(R_2\), \(R_1 \otimes R_2 = R_2\) if\(f\) \(R_2 \leq R_1\).

**Proof.** Taking into account the definition of conjunction, \(R_1 \otimes R_2 = R_2\) if\(f\) \(\forall rel_p \in I \min\{\deg_{R_1}(rel_p), \deg_{R_2}(rel_p)\} = \deg_{R_2}(rel_p)\). This holds if\(f\) \(\forall rel_p \in I\ deg_{R_2}(rel_p) \leq \deg_{R_1}(rel_p), i.e.\) if\(f\) \(R_2 \leq R_1\). □

Lemma 76. Let \(R_{uv}, R_{vw}, \text{ and } R_{uw}\) be three IA\textsuperscript{fuz}-relations, and let \(\alpha, \alpha' \in L\) such that \(\alpha > \alpha\). If \((R_{uw} \otimes I[\alpha]) \leq (R_{uv} \otimes I[\alpha]) \circ (R_{vw} \otimes I[\alpha]),\) then \((R_{uw} \otimes I[\alpha']) \leq (R_{uv} \otimes I[\alpha']) \circ (R_{vw} \otimes I[\alpha']).\)

**Proof.** Taking into account Lemma 74, the hypothesis can be expressed as \(R_{uw} \otimes I[\alpha] \leq (R_{uv} \circ R_{vw}) \otimes I[\alpha], i.e.\)

\[
\forall rel_p \in I \min\{\deg_{R_{uw}}(rel_p), \alpha\} \leq \min\{\deg_{R_{uv} \circ R_{vw}}(rel_p), \alpha\}
\]

Since \(\alpha > \alpha\), it is easy to see that this yields in turn

\[
\forall rel_p \in I \min\{\deg_{R_{uw}}(rel_p), \alpha'\} \leq \min\{\deg_{R_{uv} \circ R_{vw}}(rel_p), \alpha'\}
\]

which is equivalent to \((R_{uw} \otimes I[\alpha']) \leq (R_{uv} \circ R_{vw}) \otimes I[\alpha']\). Taking into account Lemma 74, this inequality can be expressed as \((R_{uw} \otimes I[\alpha']) \leq (R_{uv} \otimes I[\alpha']) \circ (R_{vw} \otimes I[\alpha']).\) □

Lemma 77. Let us consider the algorithm PC\textsuperscript{fuz}(\(N\)), with \(N = \langle X, C\rangle\), and let us assume that, initially,

\[
\text{ConsSup} = \min_{(i,j) \in E(\langle N\rangle)} \max_{ij}.
\]

Then, this condition remains true throughout the execution of the algorithm, with the exception of lines 9–12 and 16–19.

**Proof.** Within the body of the while-loop (lines 3–19), a modification of a network relation or of ConsSup, which might invalidate the condition on ConsSup, can occur just in lines 9, 10, 12 and 16, 17, 19. We prove that the condition remains true with reference to lines 9–12 (the proof is analogous for the other group). First, we notice that the assignment in line 9 is \(R_{ik} \leftarrow t = R_{ik} \otimes (R_{ij} \circ R_{jk})\), therefore the preference degrees of \(R_{ik}\) can only be diminished. As a consequence, also \(\min_{(i,j) \in E(\langle N\rangle)} \max_{ij}\) cannot increase: in order to maintain ConsSup updated, it is sufficient to check ConsSup against \(\max_{ij}\) after the modification of \(R_{ik}\), as it is done in line 12.

Finally, it is easy to see that the instruction in line 20 does not increase preference degrees, and potentially decreases only those preference degrees that are strictly greater than ConsSup: as a consequence, \(\min_{(i,j) \in E(\langle N\rangle)} \max_{ij}\) does not change, and the condition on ConsSup is maintained. □

Lemma 78. Let \(N\) be an IA\textsuperscript{fuz}-network, and let \(\text{ConsSup} = \min_{(i,j) \in E(\langle N\rangle)} \max_{ij}\). Given a triple \((i, j, w)\) of distinct nodes of \(N\), if it is the case that either

- \(\min_{ij} \geq \text{ConsSup} \text{ and } \max_{jw} \geq \text{ConsSup}; \text{ or}\)
- \(\max_{ij} \geq \text{ConsSup} \text{ and } \min_{jw} \geq \text{ConsSup}\)

then \((R_{uw} \otimes I[\text{ConsSup}]) \leq (R_{ij} \otimes I[\text{ConsSup}]) \circ (R_{jw} \otimes I[\text{ConsSup}]).\)

**Proof.** We prove the lemma in the case that \(\min_{ij} \geq \text{ConsSup} \text{ and } \max_{jw} \geq \text{ConsSup}\) (the other case is symmetric). Since \(\min_{ij} \geq \text{ConsSup}\), we have that \(R_{ij} \otimes I[\text{ConsSup}] = I[\text{ConsSup}]\). Moreover, since \(\max_{jw} \geq \text{ConsSup}\), \(\max_{rel_p \in I} \deg_{R_{uw} \otimes I[\text{ConsSup}]}(rel_p) = \text{ConsSup}\). As a consequence, taking into account Lemma 33 it turns out that \((R_{ij} \otimes I[\text{ConsSup}]) \circ (R_{jw} \otimes I[\text{ConsSup}]) = I[\text{ConsSup}]\). Since \(I[\text{ConsSup}] \geq (R_{uw} \otimes I[\text{ConsSup}])\), the lemma is proved. □

Now, let us turn to the proof of Proposition 34.

First, we prove that the thesis is verified after completion of the instruction of line 1. In particular, we prove that, given a generic triple of distinct nodes \((u, v, w)\), if \((u, v) \notin Q\) and \((v, w) \notin Q\) then

\[
(R_{uw} \otimes I[\text{ConsSup}]) \leq (R_{uv} \otimes I[\text{ConsSup}]) \circ (R_{vw} \otimes I[\text{ConsSup}])
\]

(A.2)
This easily follows from Lemma 78, taking into account that, if neither \((u, v)\) nor \((v, w)\) belongs to \(Q\), then it must be the case that \(\min_{vw} \geq \text{ConsSup}\) and \(\max_{vw} \geq \min_{uw} \geq \text{ConsSup}\).

Next, we reason by induction: assuming that the thesis is verified after line 1 or after the completion of the previous iteration of the while-loop (i.e. entering line 2) we prove that the thesis is verified after the completion of the current iteration.

First, let us examine lines 3 and 4, and let us first consider the case in which \(\min_{ij} \geq \text{ConsSup}\), so that the only effect of the current iteration is to update \(Q\) to \(Q \setminus (i, j)\): we prove the thesis by verifying that (A.2) holds for all triples of the kind \((i, j, w)\) and \((u, i, j)\). Again, this easily follows from Lemma 78, taking into account that, by Lemma 77, \(\max_{ji} \geq \text{ConsSup}\) and \(\max_{si} \geq \text{ConsSup}\).

Let us now turn to the case \(\min_{ij} < \text{ConsSup}\), so that the for-loop is entered (line 5): we have to prove that the thesis is verified after the last iteration of this loop. To this purpose we prove the following statement: after the \(k\)th iteration (starting with \(k = 1\)) the thesis is verified for all triples of distinct nodes with the exception of

- triples of the kind \((i, j, u)\) with \(u > k\),
- triples of the kind \((u, i, j)\) with \(u > k\),

where we recall that \(i\) and \(j\) are the nodes of the edge extracted from \(Q\). As a consequence, after the last iteration (with \(k = n\)), the set of ‘exceptions’ is empty, and the thesis is verified for all triples. We limit the proof to the first kind of triples, by considering lines 6–12 (the proof for the other kind of triples can be obtained by repeating the same steps of reasoning to lines 13–19).

First, let us consider in line 6 the case \(\min_{jk} \geq \text{ConsSup}\): as for the kind of triples we are considering, the iteration terminates without modifications of network edges. In this case, taking into account that \(\max_{ij} \geq \text{ConsSup}\) by Lemma 77, it turns out by Lemma 78 that \(\text{for} (i, j, k)\) satisfies (A.2), and we are done.

In the other case (\(\min_{jk} < \text{ConsSup}\)) line 7 is entered and variable \(t\) is assigned the relation \(R_{ik} \otimes (R_{ij} \circ R_{jk})\). Turning to line 8, we have again to consider two cases.

First, we consider the case in which it is not true that

\[
(\exists \text{rel}_p \in \mathcal{I}: [\deg_{t}(\text{rel}_p) < \text{ConsSup} \text{ and } \deg_{t}(\text{rel}_p) < \deg_{R_{ik}}(\text{rel}_p)])
\]

or, equivalently,

\[
\forall \text{rel}_m \in \mathcal{I} \quad [\deg_{t}(\text{rel}_m) \geq \text{ConsSup} \text{ or } \deg_{t}(\text{rel}_m) \geq \deg_{R_{ik}}(\text{rel}_m)]
\]

(A.3)

Since no modifications on network edges are done in this case, we have to verify that the triple \((i, j, k)\) satisfies (A.2). Taking into account (A.3), it can be easily shown that

\[
R_{ik} \otimes I[\text{ConsSup}] \leq t \otimes I[\text{ConsSup}]
\]

which substituting \(t\) with its right value (see line 7) becomes

\[
R_{ik} \otimes I[\text{ConsSup}] \leq (R_{ik} \otimes (R_{ij} \circ R_{jk})) \otimes I[\text{ConsSup}]
\]

Since the first member of this inequality appears in conjunction with another term in the second member, we must have in particular that

\[
R_{ik} \otimes I[\text{ConsSup}] = (R_{ik} \otimes (R_{ij} \circ R_{jk})) \otimes I[\text{ConsSup}]
\]

which according to Lemma 74 can also be expressed as

\[
R_{ik} \otimes I[\text{ConsSup}] = (R_{ik} \otimes I[\text{ConsSup}]) \otimes ((R_{ij} \otimes I[\text{ConsSup}]) \circ (R_{jk} \otimes I[\text{ConsSup}]))
\]

On the basis of Lemma 75, this can hold iff

\[
(R_{ik} \otimes I[\text{ConsSup}]) \leq (R_{ij} \otimes I[\text{ConsSup}]) \circ (R_{jk} \otimes I[\text{ConsSup}])
\]

which is (A.2) applied on \((i, j, k)\).

Finally, we have to consider the opposite case, in which the instructions of lines 9–12 are executed. In this case, after line 10 \(R_{ik}\) is equal to \(R_{ik} \otimes (R_{ij} \circ R_{jk})\), thus in particular \(R_{ik} \leq R_{ij} \circ R_{jk}\), which entails \(R_{ik} \otimes I[\text{ConsSup}] \leq (R_{ij} \circ R_{jk}) \otimes I[\text{ConsSup}]\): taking into account Lemma 74, it is easy to obtain (A.2) on \((i, j, k)\). Now, since the edge
(i, k) has been modified in lines 9–10, we have to prove that, at the end of the cycle, the thesis holds also for triples of the kind (i, k, u) and (u, i, k). This is evident by considering that, in line 11, (i, k) is included in Q, thus all of these triples satisfy the first or the second condition of the thesis. Finally, Lemma 76 entails that the updating of ConsSup in line 12 cannot invalidate condition (A.2), so that the thesis is maintained for all triples. □

**Proof of Theorem 35.** First, let us prove termination in O(kn^3) time. Line 1 and line 20 take O(n^2) time, since each of them performs an operation taking constant time on all of the O(n^2) edges of the network. As for the while-loop, we notice that an edge (i, k) is included in Q only if one of the preference degrees of R_{ik} is diminished (see lines 8 and 15): since an IA^{fus}-relation can be updated at most 13 * k times, no more than O(kn^3) pairs of intervals are ever entered into Q. Moreover, every time an edge (i, j) is removed from Q, at most O(n) iterations of the for-loop are executed, each of them taking constant time (see lines 6–19). This yields an overall complexity of O(kn^3).

Equivalence between the input and output network is preserved since the operations that modify network relations don’t alter SOL(\mathcal{N}). Considering the assignment of line 9, i.e. \( R_{ik} \leftarrow R_{ik} \otimes (R_{ij} \circ R_{jk}) \), this is evident taking into account the properties of composition operation (see Proposition 11), and the same holds for line 16. As for line 20, by Lemma 77 we have that

\[
\text{ConsSup} = \min_{(i,j) \in \mathcal{E}(\mathcal{N})} \max_{ij} \deg_{ij}
\]

which entails that \( \forall s \in SL_{\mathcal{N}} \deg_{\mathcal{N}}(s) \leq \text{ConsSup} \). As a consequence, the operations executed in line 20, that limit to ConsSup all the preference degrees that are greater than that, don’t alter SOL(\mathcal{N}).

Finally, we have to prove path-consistency of the output network. Since the exit condition of the while-loop is \( Q = \emptyset \) (see line 2), by Proposition 34 it must be the case that, entering line 20, for each triple (i, j, k)

\[
(R_{ik} \otimes I[\text{ConsSup}]) \leq (R_{ij} \otimes I[\text{ConsSup}]) \circ (R_{jk} \otimes I[\text{ConsSup}])
\]

After the assignments of line 20, i.e. \( \forall (i, j) R_{ij} \leftarrow R_{ij} \otimes I[\text{ConsSup}] \), we must have for each triple (i, j, k)

\[
R_{ik} \leq R_{ij} \circ R_{jk}
\]

that is, the network is path-consistent. □

**Proof of Theorem 36.** First, we prove the following lemmas (i.e. 79–80).

**Lemma 79.** Let \( \mathcal{N} = (X, \mathcal{C}) \) be a (classical) IA-network such that all of its relations are basic relations, i.e. \( \forall R_{ij} \in \mathcal{C} \exists rel_p \in \mathcal{I} \) such that \( R_{ij} = (rel_p) \). Then, \( \mathcal{N} \) is path-consistent if and only if it is consistent.

**Proof.** It is easy to see that, in the case of basic relations, consistency entails path-consistency. The reverse is ensured by the fact that atomic relations belong to a subalgebra of IA, called \( SA_c \), for which path-consistency entails minimality (see Proposition 38): obviously, a minimal network whose edges are non-empty is consistent. □

**Lemma 80.** Let \( \mathcal{N} \) be an IA^{fus}-network, and let \( R^* = R_{ij} \otimes (\Delta_{ikl} \circ \Delta_{klj}) \). For every \( rel_u \in \mathcal{I} \), we have that

\[
\deg_{R^*}(rel_u) = \max_{s: N(s)=[i,j,k,l] \wedge s_{ij}=rel_u} \deg_{\mathcal{N}}(s)
\]

**Proof.** Taking into account the definition of \( \Delta_{ikl} \circ \Delta_{klj} \), it is easy to see that the latter can be computed by means of the following procedure, where \( rel_u[\alpha_u] \in R \) simply denotes that \( \deg_{R}(rel_u) = \alpha_u \):

1. \( \Delta \leftarrow \emptyset \)
2. \( \text{for} \ (rel_p[\alpha_p] \in R_{ik}, rel_R[\alpha_R] \in R_{kl}, rel_T[\alpha_T] \in R_{lj}) \)
3. \( \text{for} \ (rel_S[\alpha_S] \in [(rel_R[\alpha_R] \circ rel_T[\alpha_T]) \otimes R_{lj}]) \)
4. \( \text{for} \ (rel_Q[\alpha_Q] \in [(rel_p[\alpha_p] \circ rel_R[\alpha_R]) \otimes R_{il}]) \)
5. \( \Delta \leftarrow \Delta \oplus [(rel_p[\alpha_p] \circ rel_S[\alpha_S]) \otimes (rel_Q[\alpha_Q] \circ rel_T[\alpha_T])]) \)

Introducing the notation \( \gamma \ast [rel_1, \ldots, rel_m] \) to indicate the IA^{fus}-relation \( (rel_1[\gamma], \ldots, rel_m[\gamma]) \), it is easy to see that the generic operation \( rel_u[\alpha_u] \circ rel_v[\alpha_v] \) can be computed as \( \min \{\alpha_u, \alpha_v\} \ast [rel_u, rel_v] \). Taking into account this, by means of some algebraic calculation it turns out that \( \Delta_{ikl} \circ \Delta_{klj} \) can be equivalently computed as follows:
1. \( \Delta \leftarrow \emptyset \)
2. \textbf{for} \(( \text{rel}_P, \text{rel}_R, \text{rel}_I \in \mathcal{I} )\)
3. \textbf{for} \(( \text{rel}_Q \in [\text{rel}_P \circ \text{rel}_R])\)
4. \textbf{for} \(( \text{rel}_S \in [\text{rel}_R \circ \text{rel}_I])\)
5. \( \Delta \leftarrow \Delta \oplus \{ \gamma \ast \{ (\text{rel}_P \circ \text{rel}_S) \otimes (\text{rel}_Q \circ \text{rel}_I) \} \}

where
\[
\gamma = \min \{ \deg_{\text{rel}_P}(\text{rel}_P), \deg_{\text{rel}_R}(\text{rel}_R), \deg_{\text{rel}_J}(\text{rel}_J), \deg_{\text{rel}_I}(\text{rel}_I), \deg_{\text{rel}_Q}(\text{rel}_Q) \}
\]

On the basis of this characterization of \( \Delta_{ikl} \circ \Delta_{klj} \), we have that \( \forall \text{rel}_u \in \mathcal{I} : \)
\[
\deg_{\text{rel}_u}(\text{rel}_u) = \max_{\mathcal{S}^{\ast}} \min_{s \in \mathcal{S}^{\ast}} \{ \deg_{\text{rel}_ik}(s_{ik}), \deg_{\text{rel}_jl}(s_{jl}), \ldots, \deg_{\text{rel}_ij}(s_{ij}) \}
\]
\[
= \max_{\mathcal{S}^{\ast}} \min_{s \in \mathcal{S}^{\ast}} \{ \deg_{\text{rel}_uw}(s_{uw}) \mid (v, w) \in \mathcal{E}(s) \}
\]

where \( \mathcal{S}^{\ast} \) denotes the set of singleton labelings \( s \) such that:
- \( N(s) = \{ i, j, k, l \} \); and
- \( s_{ij} = \text{rel}_u \); and
- \( s_{il} \in \{ s_{ik} \cup s_{kl} \} \land s_{kj} \in \{ s_{jk} \cup s_{ij} \} \land s_{ij} \in \{ s_{ik} \cup s_{jk} \} \land s_{ij} \in \{ s_{il} \cup s_{ij} \} \).

Now, it is easy to see that \( \mathcal{S}^{\ast} \) can be expressed as \( \{ s \mid N(s) = \{ i, j, k, l \} \land s_{ij} = \text{rel}_u \land s \text{ is path-consistent} \} \). According to Lemma 79, \( s \) is path-consistent iff it is consistent, therefore the above equality can be expressed as
\[
\deg_{\text{rel}_u}(\text{rel}_u) = \max_{s \text{ consistent, } N(s) = \{ i, j, k, l \}, s_{ij} = \text{rel}_u} \min_{s \in \mathcal{S}^{\ast}} \{ \deg_{\text{rel}_uw}(s_{uw}) \mid (v, w) \in \mathcal{E}(s) \}
\]
which directly yields the thesis. \( \square \)

Now, let us turn to the proof of Theorem 36, which is similar to that of Theorem 35.
First, let us notice that line 1 takes \( O(n^{2}) \) time. Next, by reasoning as in the aforementioned proof it turns out that no more than \( O(kn^{2}) \) edges are ever entered into \( Q \) : for each one of them, some operations taking constant time are performed on the \( O(n^{2}) \) related paths, yielding an overall complexity of \( O(kn^{4}) \). Equivalence between the input and output network is preserved, since the unique operation that modifies relations (see line 7) does not alter \( \text{SOL}(\mathcal{N}) \), as it can be easily shown taking into account Lemma 80. Finally, minimality of 4-subnetworks follows from the fact that, upon termination of the algorithm, \( Q = \emptyset \), therefore it must be the case that
\[
\forall R_{ij} \in \mathcal{C} \quad R_{ij} = R_{ij} \otimes (\Delta_{ikl} \circ \Delta_{klj})
\]
Taking into account Lemma 80, the thesis follows immediately. \( \square \)

**Proof of Proposition 43.** Given two \( \text{PA}^{\text{fuz}}_{\alpha} \)-relations \( R_1 \) and \( R_2 \), we have to show that \( R_1^{-1}, R_1 \otimes R_2 \) and \( R_1 \circ R_2 \) are \( \text{PA}^{\text{fuz}}_{\alpha} \)-relations. This can be directly proved by algebraic manipulation of these relations, i.e. by proving that the corresponding preference degrees satisfy Definition 42. Alternatively, a proof can be based on our general results introduced in [19], as it will be done for the subalgebras of \( \text{IA}^{\text{fuz}} \) (see the proof of Proposition 62 in the following). \( \square \)

**Proof of Proposition 45.** Let \( R = \{ < [\alpha_1], = [\alpha_2], > [\alpha_3] \} \) be a \( \text{PA}^{\text{fuz}}_{\alpha} \)-relation, and let us suppose by contradiction that \( \exists \alpha \in L : R_{\alpha} = \{ \neq \} \). In this case, we should have \( \alpha_2 < \alpha, \alpha_1 \geq \alpha \) and \( \alpha_3 \geq \alpha \), therefore in particular \( \alpha_2 < \min \{ \alpha_1, \alpha_3 \} \), but this contradicts Definition 42.
As for the other direction of the proof, suppose to the contrary that \( \alpha_2 < \min \{ \alpha_1, \alpha_3 \} \), and let \( \alpha = \min \{ \alpha_1, \alpha_3 \} \).
Since \( \alpha > \alpha_2 \), we have that \( \alpha > 0 \) and \( R_{\alpha} = \{ \neq \} \), thus contradicting the hypothesis. \( \square \)

**Proof of Lemma 47.** Given a complete singleton labeling \( s, s \in [\text{SOL}(\mathcal{N})]_{\alpha} \) iff \( \deg_{\mathcal{N}}(s) \geq \alpha \). Taking into account Definition 5 and the fact that \( \alpha > 0 \), this holds iff \( s \) is consistent and \( \forall (i, j) \in \mathcal{E}(\mathcal{N}) : s_{ij} \in (R_{ij})_{\alpha} \), i.e. iff \( s \in \text{SOL}(\mathcal{N}_{\alpha}) \). \( \square \)
Proof of Proposition 48. Since the two proofs are identical, we refer to IA_{fac}^-networks only.

First, let us suppose that \( \mathcal{N} \) is minimal, and let us consider a generic \( \alpha \in L \) such that \( \alpha > 0 \): we have to prove that \( \mathcal{N}_\alpha \) is minimal as well. Let \( rel_p \in \mathcal{I} \) be an atomic relation which satisfies a binary constraint \( (R_{ij})_\alpha \) in \( \mathcal{N}_\alpha \): by definition of \( \alpha \)-cut, we have that \( \deg_{R_{ij}}(rel_p) \geq \alpha \), therefore minimality of \( \mathcal{N} \) entails that \( rel_p \) can be extended to a solution \( s \) of \( \mathcal{N} \) such that \( s_{ij} = rel_p \) and \( \deg_{\mathcal{N}}(s) \geq \alpha \), i.e. \( s \in [\text{SOL}(\mathcal{N})]_\alpha \). Taking into account Lemma 47, we have that \( s \in [\text{SOL}(\mathcal{N}_\alpha)] \): in other words, \( rel_p \) has been extended to a solution of \( \mathcal{N}_\alpha \), thus proving that \( \mathcal{N}_\alpha \) is minimal.

As for the other direction of the proof, let us suppose that \( \forall \alpha < L \) such that \( \alpha > 0 \) \( \mathcal{N}_\alpha \) is minimal, and let \( rel_p \in \mathcal{I} \) be a generic atomic relation which satisfies a binary constraint \( R_{ij} \) of \( \mathcal{N} \) with a degree of preference equal to \( \beta \), i.e. \( \deg_{R_{ij}}(rel_p) = \beta \). In order to show that \( \mathcal{N} \) is minimal, we have to prove that \( rel_p \) can be extended to a solution \( s \) of \( \mathcal{N} \) such that \( \deg_{\mathcal{N}}(s) = \beta \). If \( \beta = 0 \), then the extension is trivial. In the other case, since \( \deg_{R_{ij}}(rel_p) = \beta \) then in particular \( rel_p \in (R_{ij})_\beta \), i.e. \( rel_p \) satisfies the constraint of the edge \((i,j)\) of \( \mathcal{N}_\beta \). Since the latter is minimal by the hypothesis, \( rel_p \) can be extended to a solution \( s \) of \( \mathcal{N}_\beta \), i.e. \( s \in [\text{SOL}(\mathcal{N}_\beta)] \) which by Lemma 47 entails that \( s \in [\text{SOL}(\mathcal{N})]_\beta \). As a consequence, \( \deg_{\mathcal{N}}(s) = \beta \), which taking into account that \( \deg_{R_{ij}}(rel_p) = \beta \) yields \( \deg_{\mathcal{N}}(s) = \beta \), and we are done. \( \square \)

Proof of Proposition 52. For every \( rel_p \in \mathcal{I} \), we have that \( rel_p \in (R_1 \circ R_2)_\alpha \) iff \( \deg_{R_1 \circ R_2}(rel_p) \geq \alpha \), and by the definition of composition this holds iff

\[
\max_{q,r} \min_{rel_q \circ rel_r} \{ \deg_{R_1}(rel_q), \deg_{R_2}(rel_r) \} \geq \alpha
\]

The above expression holds in turn iff \( \exists q,r : rel_p \in \{rel_q \circ rel_r\} \wedge \deg_{R_1}(rel_q) \geq \alpha \wedge \deg_{R_2}(rel_r) \geq \alpha \), i.e. iff \( \exists q,r : rel_p \in \{rel_q \circ rel_r\} \wedge rel_q \in R_{1\alpha} \wedge rel_r \in R_{2\alpha} \). This in turn is equivalent to \( rel_p \in R_{1\alpha} \circ R_{2\alpha} \), and we are done. \( \square \)

Proof of Lemma 59. Taking into account the definition of SA, \( \text{SOL}(N_1) = \{f_{IA}^{-1}(r) \mid r \in R_1\} \) and \( \text{SOL}(N_2) = \{f_{IA}^{-1}(r) \mid r \in R_2\} \): since \( R_1 \subseteq R_2 \), we have that \( \text{SOL}(N_1) \subseteq \text{SOL}(N_2) \). Since \( N_1 \) is minimal, any atomic relation of any of its edges can be extended to a solution, which is also a solution of \( N_2 \). As a consequence, the same atomic relation must be present in the same edge also in \( N_2 \). This means that \( N_1 \subseteq N_2 \). \( \square \)

Proof of Proposition 60. First, let us prove that if \( R \in \text{SA}_{fac}^- \), then all of its \( \alpha \)-cuts, with \( \alpha > 0 \), are SA-relations. Let \( \mathcal{N} \in \mathcal{S}_{\text{PA}_{fac}}^- \) be a PA_{fac}^-network equivalent to \( R \), and let us consider a generic \( \alpha \in L \) such that \( \alpha > 0 \). Since \( \mathcal{N} \in \mathcal{S}_{\text{PA}_{fac}}^- \), its \( \alpha \)-cut \( \mathcal{N}_\alpha \in \mathcal{S}_{\text{PA}}^- \): we show that \( R_\alpha \in \text{SA} \) by proving that \( \forall s \in \text{SOL}_{IA} \ s \in \text{SOL}(\mathcal{N}_\alpha) \) iff \( f_{IA}(s) \in R_\alpha \). Taking into account Lemma 47, we have that \( s \in [\text{SOL}(\mathcal{N}_\alpha)] \) iff \( s \in [\text{SOL}(\mathcal{N})]_\alpha \), i.e. iff \( \deg_{\mathcal{N}}(s) \geq \alpha \). According to the definition of \( \text{SA}_{fac}^- \)-relations (see Definition 54), this in turn holds iff \( \deg_{R}(f_{IA}(s)) \geq \alpha \), i.e. iff \( f_{IA}(s) \in R_\alpha \).

Let us now turn to the other part of the proof, i.e. let us suppose that \( \forall \alpha \in L : \alpha > 0 \) \( R_\alpha \in \text{SA} \): we prove that \( R \in \text{SA}_{fac}^- \) by constructing a network \( \mathcal{N} \in \mathcal{S}_{\text{PA}_{fac}}^- \) equivalent to \( R \). Let us denote as \( D_R \) the set of the preference degrees strictly greater than 0 used in \( R \), i.e.

\[
D_R = \bigcup_{rel_p \in \mathcal{I}} \{\deg_{R}(rel_p)\} \setminus \{0\}
\]

(A.4)

For each \( \alpha \in D_R \), let us consider the relation \( R_\alpha \). Since by the hypothesis \( R_\alpha \) is an SA-relation, there is a PA-network equivalent to \( R_\alpha \); let us consider in particular the unique minimal network equivalent to \( R_\alpha \), denoted as \( C(R_\alpha) \) (this is always possible since for any PA-network there is a unique equivalent minimal network). Now, we show that \( R \in \text{SA}_{fac}^- \) by proving that the network

\[
\mathcal{N} = \bigcup_{\beta \in D_R} \beta \ast C(R_\beta)
\]

(A.5)

which by construction belongs to \( \mathcal{S}_{\text{PA}_{fac}}^- \) is equivalent to \( R \). First, it is easy to see that (A.5) yields

\[
\forall \alpha \in D_R \quad \mathcal{N}_\alpha = \bigcup_{\beta \in D_R, \beta \geq \alpha} C(R_\beta)
\]

(A.6)
Now, in (A.6) the inequality $\beta \geq \alpha$ entails that $R_\beta \subseteq R_\alpha$, which taking into account Lemma 59, as well as the fact that $C(R_\beta)$ is minimal, yields $C(R_\beta) \subseteq C(R_\alpha)$; as a consequence, (A.6) can be expressed as

$$\forall \alpha \in D_R\ N_\alpha = C(R_\alpha)$$  \hspace{1cm} (A.7)

In order to prove that $N$ is equivalent to $R$, we have to show that $\forall s \in SOL_{IA}\ deg_N(s) = deg_R(f_{IA}(s))$. With reference to a generic element $s \in SOL_{IA}$, let $deg_N(s) = \alpha'$, where according to (A.4) and (A.5) it must be the case that $\alpha' \in (D_R \cup \{0\})$. Since $s \in [SOL(N)]_{\alpha'}$, in case $\alpha' > 0$ Lemma 47 yields $s \in SOL(N_{\alpha'}^c)$, which according to (A.7) yields in turn $s \in SOL(C(R_{\alpha'}))$: taking into account that $C(R_{\alpha'})$ is equivalent to $R_{\alpha'}$, we have that $f_{IA}(s) \in R_{\alpha'}$, i.e. $deg_R(f_{IA}(s)) \geq \alpha'$. Of course, the latter inequality is verified also in case $\alpha' = 0$, therefore we have proved that $deg_N(s) \leq deg_R(f_{IA}(s))$. In order to complete the proof, let us show that $deg_N(s) \geq deg_R(f_{IA}(s))$. Let $deg_R(f_{IA}(s)) = \beta'$, where according to (A.4) it must be the case that $\beta' \in (D_R \cup \{0\})$. Since $f_{IA}(s) \in R_{\beta'}$, in case $\beta' > 0$ we have that $s \in SOL(C(R_{\beta'}))$, which according to (A.7) yields $s \in SOL(N_{\beta'}^c)$. By Lemma 47, $s \in [SOL(N)]_{\beta'}$, i.e. $deg_N(s) \geq \beta'$. Of course, the latter inequality is verified also in case $\beta' = 0$, therefore $deg_N(s) \geq deg_R(f_{IA}(s))$, and we are done. \hfill \Box

**Proof of Proposition 61.** The proof is the same as that of Proposition 60, with the following adjustments.

In the ‘only if’ part of the proof, we notice that $\forall \alpha > 0\ N_\alpha$ is a $PA_\alpha$-network, therefore we can conclude that $R_\alpha \in SA_c$. In fact, since $R \in SA_c$ we have that $N$ is a $PA_c^{fuz}$-network, therefore Proposition 45 applied to the relations of $N$ entails that $\forall \alpha > 0\ N_\alpha$ is a $PA_c$-network.

In the ‘if’ part of the proof, we have to take into account Corollary 39 in order to choose $C(R_\alpha)$ as the minimal $PA_\alpha$-network equivalent to $R_\alpha$, and we use again Proposition 45 to prove that $N$ is a $PA_c^{fuz}$-network, so that $R$ turns out to be an $SA_c^{fuz}$-relation. In particular, taking into account (A.5) and (A.7) it is easy to see that $\forall \alpha \in L\ : \alpha > 0\ N_\alpha = C(R_\alpha)$. Since $R_\alpha \in SA_c$ by the hypothesis, $C(R_\alpha)$ is a $PA_c$-network: the fact that $N$ is a $PA_c^{fuz}$-network follows then from Proposition 45 applied to its relations. \hfill \Box

**Proof of Proposition 62.** First, let us prove closure under conjunction and composition, by considering two generic $SA_c^{fuz}$ ($SA_c^{fuz^+}$) relations $R_1$ and $R_2$. By Propositions 51 and 52 we have that, if $* \in \{\otimes, \circ\}$, then $\forall \alpha \in L\ : \alpha > 0\ (R_1 \ast R_2)_\alpha = R_{1\alpha} \ast R_{2\alpha}$. Since by Proposition 60 (Proposition 61) $R_1 \alpha \ast R_2 \alpha \ast R_{2\alpha}$ are $SA_c$ ($SA_c$) relations, taking into account that $SA_c$ is an algebra, we have that $\forall \alpha \in L\ : \alpha > 0\ (R_1 \ast R_2)_\alpha \in SA_c$: by Proposition 60 (Proposition 61) $(R_1 \ast R_2) \in SA_c^{fuz}$ ($SA_c^{fuz}$), and we are done.

As for the operation of inversion, the proof proceeds in a similar way, by exploiting Proposition 50 instead of Propositions 51 and 52. \hfill \Box

**Proof of Proposition 63.** Let $N$ be a path-consistent $SA_c^{fuz}$-network. By Propositions 61 and 49, we have that $\forall \alpha \in L\ : \alpha > 0\ N_\alpha$ is a path-consistent $SA_c$-network. Since, according to Proposition 38, classical path-consistent $SA_c$-networks are minimal, all $\alpha$-cuts of $N$ such that $\alpha > 0$ are minimal, therefore by Proposition 48 it turns out that $N$ is minimal as well. \hfill \Box

**Proof of Proposition 64.** It is easy to see that, by Proposition 48, for every $\alpha \in L\ : \alpha > 0$ all the 4-subnetworks of $N_\alpha$ are minimal as well. Moreover, by Proposition 60, we have that $\forall \alpha \in L\ : \alpha > 0\ N_\alpha$ is an $SA$-network. Taking into account the relevant result holding for classical $SA$-networks (see Proposition 40), it turns out that all the $\alpha$-cuts of $N$ such that $\alpha > 0$ are minimal, therefore by Proposition 48 we have that $N$ is minimal as well. \hfill \Box

**Proof of Proposition 68.** First, we prove the following lemma.

**Lemma 81.** Let $N = (\mathcal{X}, \mathcal{C})$ be a path-consistent $IA^{fuz}$-network: taken two arbitrary edges $(i, j), (k, m) \in \mathcal{E}(N)$, considering the relevant relations $R_{ij}, R_{km} \in \mathcal{C}$ we have that $\max_{ij} = \max_{km}$.

**Proof.** First, let us prove the lemma for two edges that have a node in common, e.g. $(i, j)$ and $(j, u_1)$ in Fig. 13. We reason by contradiction, by supposing without loss of generality that $\max_{ij} > \max_{ju_1}$: in this case, all the singleton labelings $s$ such that $N(s) = \{i, j, u_1\}$ have a degree local consistency $deg_N(s) \leq \max_{ju_1}$. As a consequence,
taken an atomic relation \( rel_p \) such that \( \deg_{R_{ij}}(rel_p) = \max_{ij} \), \( rel_p \) cannot be extended to the node \( u_1 \) maintaining its degree \( \max_{ij} \): but this entails that \( N \) is not path-consistent, against the hypothesis.

In the general case of two edges \((i, j)\) and \((k, m)\), the lemma can be proved by applying the above result along a path connecting \( j \) to \( k \), as shown in Fig. 13.  

Now, let us turn to the proof of Proposition 68.

Let us consider a generic edge \((i, j)\) \( \in \mathcal{E}(N) \). Of course, \( \deg(N) \leq \max_{ij} \): let us prove that \( \deg(N) \geq \max_{ij} \), so that the conclusion follows. If \( \max_{ij} = 0 \), then the inequality is trivially verified, so let us consider the case \( \max_{ij} > 0 \). Since \( N \) is path-consistent, by Lemma 81 we have that for all edges \((k, m) \in \mathcal{E}(N) \) \( \max_{km} = \max_{ij} \), therefore the \( \max_{ij} \)-cut \( N_{\max_{ij}} \) has no empty relations. Moreover, Proposition 49 entails that \( N_{\max_{ij}} \) is path-consistent, and according to Definition 66 \( N_{\max_{ij}} \) is an \( \mathcal{H} \)-network. Then, the classical property of \( \mathcal{H} \)-networks recalled in Proposition 65 entails that \( N_{\max_{ij}} \) is consistent, i.e. \( \text{SOL}(N_{\max_{ij}}) \neq \emptyset \), which taking into account Lemma 47 yields \( [\text{SOL}(N)]_{\max_{ij}} \neq \emptyset \). This in turns yields \( \deg(N) \geq \max_{ij} \), and we are done.

**Proof of Lemma 71.** Let us consider a generic \( \alpha \in L \), and two relations \( R_1, R_2 \in A_{\alpha} \): we have to prove that \( R_1^{-1}, R_1 \otimes R_2 \) and \( R_1 \circ R_2 \) belong to \( A_{\alpha} \). Since \( R_1, R_2 \in A_{\alpha} \), we have that \( \exists R_1^{\text{fac}}, R_2^{\text{fac}} \in A \) such that \( R_1 = (R_1^{\text{fac}})_{\alpha} \) and \( R_2 = (R_2^{\text{fac}})_{\alpha} \). Let us consider \( R_1 \circ R_2 \): it turns out that \( R_1 \circ R_2 = (R_1^{\text{fac}})_{\alpha} \circ (R_2^{\text{fac}})_{\alpha} \), which by Proposition 52 is in turn equal to \( (R_1^{\text{fac}} \circ R_2^{\text{fac}})_{\alpha} \). Now, since \( A \) is an algebra, \( (R_1^{\text{fac}} \circ R_2^{\text{fac}}) \in A \), therefore taking into account the definition of \( A_{\alpha} \) we have that \( (R_1^{\text{fac}} \circ R_2^{\text{fac}})_{\alpha} \in A_{\alpha} \).

As far as inversion and conjunction are concerned, the relevant proofs are similar and exploit Proposition 50 and 51 respectively.

**Proof of Lemma 72.** We prove tractability of \( A_{\alpha} \) with reference to the problem of determining consistency of a network. First, if \( \alpha = 0 \) then \( A_{\alpha} \) is made up of a unique IA-relation, i.e. that made up of all Allen’s atomic relations. Of course, \( A_{\alpha} \) is trivially tractable (i.e. all \( A_{\alpha} \)-networks are consistent). So, let us consider a generic \( \alpha \in L \) such that \( \alpha > 0 \).

First, given a generic \( A_{\alpha} \)-network \( N = \langle X, C \rangle \), according to Definition 70 we have that \( \forall R_{ij} \in C \exists R_{ij}^{\text{fac}} \in A \): \( (R_{ij}^{\text{fac}})_{\alpha} = R_{ij} \); as a consequence, the network \( N^{\text{fac}} = \langle X, C^{\text{fac}} \rangle \) with \( C^{\text{fac}} = \{ R_{ij}^{\text{fac}} : (i, j) \in \mathcal{E}(N) \} \) is an \( A \)-network such that \( (N^{\text{fac}})_{\alpha} = N \). Now, we notice that it is possible to devise a polynomial algorithm that, given a generic \( A_{\alpha} \)-network \( N \), computes a network \( N^{\text{fac}} \) such that \( (N^{\text{fac}})_{\alpha} = N \). In fact, for each \( R_{ij} \in C \) a corresponding \( R_{ij}^{\text{fac}} \) can be computed by enumerating all relations \( R \in A \) and checking whether \( R_{\alpha} = R_{ij} \): since there is only a finite number \( N \) of relations in \( A \), this operation can be performed in constant time, therefore the computation of \( N^{\text{fac}} \) takes \( O(n^2) \)-time, where \( n \) is the number of nodes of \( N \).

Now, a polynomial algorithm to check the consistency of a generic \( A_{\alpha} \)-network \( N \) can be arranged as follows. First, we compute in polynomial time an \( A \)-network \( N^{\text{fac}} \) such that \( (N^{\text{fac}})_{\alpha} = N \). Then, we check whether \( \deg(N^{\text{fac}}) \geq \alpha \) (since \( A \) is tractable, this can be done in polynomial time): since \( \text{SOL}(N) = \text{SOL}((N^{\text{fac}})_{\alpha}) \) which according to Lemma 47 is in turn equal to \( [\text{SOL}(N^{\text{fac}})]_{\alpha} \), this holds iff \( N \) is consistent.

**Proof of Theorem 73.** First, we prove the following lemmas (i.e. 82–83).
Lemma 82. The algebra $A$ includes all the relations of $B$, i.e. $B \subseteq A$.

Proof. We have to prove that, for all $rel_p \in I$ and for all $\alpha \in L$, $rel_p[\alpha] \in A$. Let us refer to a generic $rel_p \in I$ and a generic $\alpha \in L$. Then, $rel_p[\alpha] \in A$ for all $\alpha \in L$. Since classical $A$ includes all atomic relations, we have in particular that $rel_p \in A$, which can also be expressed as $(rel_p[\alpha])_{\alpha} \in A$. Taking into account Definition 66, it turns out that $rel_p[\alpha] \in A$. \hfill \Box

Lemma 83. Let $A \subseteq I, A$ be an algebra, and let $\alpha'$ be a non-empty subset of the preference degrees, i.e. $\alpha' \subseteq \alpha$, and for all $\alpha \in \alpha'$, $\exists \alpha \in \alpha$, $\forall \alpha \in \alpha$ such that $\alpha'$ is a finite algebra.

Proof. We reason by contradiction, by supposing that there is an IA-relational $R_{\alpha'}$, such that $R_{\alpha'} \notin A$. Let us consider the set $\alpha'$ made up of those relations of $A$ whose preference degrees are all in $\alpha'$, i.e. $\alpha' = \{R \in A | \forall \alpha \in \alpha' \}$. Since $\alpha'$ is a finite algebra, it must be the case that $\alpha' \subseteq K$ with $K \leq 13$. Now, let us consider the set $\alpha'$ made up of those relations of $A$ whose preference degrees are all in $\alpha'$, i.e. $\alpha' = \{R \in A | \forall \alpha \in \alpha' \}$. Since $\alpha' \subseteq K$, $\alpha'$ is a finite set, in particular it must be the case that $\alpha' \subseteq K$. According to Lemma 83, $\alpha'$ is a finite algebra, and is also tractable since it is contained in $A$, which is tractable by the hypothesis. Moreover, $R_{\alpha'} \in A$, and since $R_{\alpha'} \notin A$, we have that $\forall \alpha \in L: (\alpha^*) \notin A$. Since all the preference degrees of $\alpha'$ are included in $\alpha'$, it is easy to see that there is such an $\alpha$, say $\alpha^*$, which belongs to $\alpha'$, i.e. $\alpha^* \in \alpha' \cap (\alpha^*)^* \notin A$. Now, in order to get a contradiction, let us consider the set of IA-relations $A'_{\alpha^*}$. It is possible to prove the following properties for $A'_{\alpha^*}:

- $A'_{\alpha^*}$ is an algebra (taking into account Lemma 71 and the fact that $A'$ is an algebra);
- $A'_{\alpha^*}$ is tractable (taking into account Lemma 72 and the fact that $A'$ is a finite tractable subset of $I, A'$);
- $\forall \alpha \in I, \exists rel_p \in I, rel_p[\alpha^*] \in B \subseteq A$ and $\alpha^* \in A'$, we have that $rel_p[\alpha^*] \in A'$, therefore $rel_p[\alpha^*] = rel_p(\alpha^*)_{\alpha^*}$.

As a consequence, Proposition 69 can be applied to $A'_{\alpha^*}$, entailing that $A'_{\alpha^*} \subseteq A$. However, since $\alpha^* \in A'$ we have that $(\alpha^*)_{\alpha^*} \subseteq A'_{\alpha^*}$, while as mentioned before $(\alpha^*)_{\alpha^*} \notin A$. Therefore, we get a contradiction, and we are done. \hfill \Box

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