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Multiple orthogonal polynomials of mixed type: Gauss–Borel factorization and the multi-component 2D Toda hierarchy

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Abstract

Multiple orthogonality is considered in the realm of a Gauss–Borel factorization problem for a semi-infinite moment matrix. Perfect combinations of weights and a finite Borel measure are constructed in terms of M-Nikishin systems. These perfect combinations ensure that the problem of mixed multiple orthogonality has a unique solution, that can be obtained from the solution of a Gauss–Borel factorization problem for a semi-infinite matrix, which plays the role of a moment matrix. This leads to sequences of multiple orthogonal polynomials, their duals and second kind functions. It also gives the corresponding linear forms that are bi-orthogonal to the dual linear forms. Expressions for these objects in terms of determinants from the moment matrix are given, recursion relations are found, which imply a multi-diagonal Jacobi type matrix with *snake* shape, and results like the ABC theorem or the Christoffel–Darboux formula are re-derived in this context (using the factorization problem and the generalized Hankel symmetry of the moment matrix). The connection between this description of multiple orthogonality and the multi-component 2D Toda hierarchy, which can be also understood and studied through a Gauss–Borel factorization problem, is discussed. Deformations of the weights, natural for M-Nikishin systems, are considered and the correspondence with solutions to the integrable hierarchy, represented as a collection of Lax equations, is explored. Corresponding Lax and Zakharov–Shabat matrices as well as wave functions and their adjoints are determined. The construction of discrete flows is discussed in terms of Miwa transformations which involve Darboux trans-

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formations for the multiple orthogonality conditions. The bilinear equations are derived and the τ-function representation of the multiple orthogonality is given.

Keywords: Multiple orthogonal polynomials; Integrable systems; Factorization problems; Nikishin systems; Recursion relations; Christoffel–Darboux formula; Tau-functions

Contents

1. Introduction

The topic of multiple orthogonality of polynomials is very close to that of simultaneous rational approximation (simultaneous Padé approximants) of systems of Cauchy transforms of measures. The history of simultaneous rational approximation starts in 1873 with the well-known article [21] in which Ch. Hermite proved the transcendence of the Euler number e. Later, around the years 1934–1935, K. Mahler delivered at the University of Groningen several lectures [26] where he settled down the foundations of this theory. Meanwhile, two of Malher's students, J. Coates and H. Jager, made important contributions in this respect (see [12] and [22]). In the case of Cauchy transforms, the simultaneous rational approximation definition may be written in terms of multiple orthogonality of polynomials as follows. Given an interval $\Delta \subset \mathbb{R}$ of the real line, let $\mathcal{M}(\Delta)$ denote all the finite Borel measures which have support, $\operatorname{supp}(\cdot)$ with infinitely many points in Δ, where they do not change sign. Fix $\mu \in \mathcal{M}(\Delta)$, and let us consider a system of weights $\vec{w} = (w_1, \dots, w_p)$ on Δ, with $p \in \mathbb{N}$. (In this paper a "weight" on an interval Δ is meant to be a real integrable function defined on Δ which does not change its sign on Δ.) Fix a multi-index $\vec{\nu} = (\nu_1, \dots, \nu_p) \in \mathbb{Z}_+^p$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, and denote $|\vec{\nu}| = \nu_1 + \dots + \nu_p$. There exist polynomials, $A_1, \dots, A_p$, not all identically equal to zero which satisfy the following orthogonality relations

$$\int_\Delta x^j \sum_{a=1}^{p} A_a(x) w_a(x)\, \mathrm{d}\mu(x) = 0, \qquad \deg A_a \leqslant \nu_a - 1, \quad j = 0, \dots, |\vec{\nu}| - 2. \tag{1}$$

Analogously, there exists a polynomial B not identically equal to zero, such that

$$\int_\Delta x^j B(x) w_b(x)\, \mathrm{d}\mu(x) = 0, \qquad \deg B \leqslant |\vec{\nu}|, \quad j = 0, \dots, \nu_b - 1,\ b = 1, \dots, p. \tag{2}$$

The resulting polynomials are said to be of type I and type II, respectively, with respect to the combination $(\mu, \vec{w}, \vec{\nu})$ of the measure μ, the systems of weights $\vec{w}$ and the multi-index $\vec{\nu}$. When $p = 1$ both definitions coincide with that of the standard orthogonal polynomials on the real line. The existence of a system of polynomials $(A_1, \dots, A_p)$ and a polynomial B defined from (1) and (2) respectively, are ensured because in both cases finding the coefficients of the polynomials is equivalent to solving a system of $|\vec{\nu}|$ linear homogeneous equations with $|\vec{\nu}| + 1$ unknown coefficients. From the theory of orthogonal polynomials we know that when $p = 1$ each polynomial $A_1 \equiv B$ has exactly degree $|\vec{\nu}| = \nu_1$; unfortunately if $p > 1$ that is not true in general. For instance, let us take a system of weights $\vec{w} = (w_1, w_1, \dots, w_1)$, in this case the solution vector space has dimension bigger than one, and we can find two solutions which are linearly independent. Hence, there is at least an $a \in \{1, \dots, p\}$ such that $\deg A_a < \nu_a - 1$ and $\deg B < |\vec{\nu}|$. Given a measure $\mu \in \mathcal{M}(\Delta)$ and a system of weights $\vec{w}$ on Δ a multi-index $\vec{\nu}$ is called type I or type II normal if $\deg A_a$ must equal to $\nu_a - 1$, $a = 1, \dots, p$, or $\deg B$ must equal to $|\vec{\nu}| - 1$, respectively. When for a pair $(\mu, \vec{w})$ all the multi-indices are type I or type II normal, then the pair is called type I perfect or type II perfect, respectively. The concepts of normality and perfectness were introduced by Malher (see Malher's, Coates' and Jager's articles cited above).

Multiple orthogonal of polynomials have been employed in several proofs of irrationality of numbers. For example, in [10], F. Beukers shows that Apery's proof (see [8]) of the irrationality of $\zeta(3)$ can be placed in the context of a combination of type I and type II multiple orthogonality, which is called mixed type multiple orthogonality of polynomials. More recently, mixed type approximation has appeared in random matrix and non-intersecting Brownian motion theories (see, for example, [11,14,24]). A formalization of this kind of orthogonality was initiated by V.N. Sorokin [35]. He studied a simultaneous rational approximation construction which is closely connected with multiple orthogonal polynomials of mixed type. Surprisingly, in [20] a Riemann–Hilbert problem was found for the theory of orthogonal polynomials, and later [38] this result was largely extended to type I and II multiple orthogonality. In [14] mixed type multiple orthogonality was analyzed from this perspective.

In order to introduce multiple orthogonal polynomials of mixed type we need two systems of weights $\vec{w}_1 = (w_{1,1}, \dots, w_{1,p_1})$ and $\vec{w}_2 = (w_{2,1}, \dots, w_{2,p_2})$ where $p_1, p_2 \in \mathbb{N}$ (as we said a set of functions which do not change their sign in Δ), and two multi-indices $\vec{\nu}_1 = (\nu_{1,1}, \dots, \nu_{1,p_1}) \in \mathbb{Z}_+^{p_1}$ and $\vec{\nu}_2 = (\nu_{2,1}, \dots, \nu_{2,p_2}) \in \mathbb{Z}_+^{p_2}$ with $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$. There exist polynomials $A_1, \dots, A_{p_1}$, not all identically zero, such that $\deg A_s < \nu_{1,s}$ which satisfy the following relations

$$\int\limits_{\Delta} \sum_{a=1}^{p_1} A_a(x) w_{1,a}(x) w_{2,b}(x) x^j \, \mathrm{d}\mu(x) = 0, \quad j = 0, \dots, \nu_{2,b} - 1, \ b = 1, \dots, p_2. \tag{3}$$

They are called mixed multiple-orthogonal polynomials with respect to the combination $(\mu, \vec{w}_1, \vec{w}_2, \vec{\nu}_1, \vec{\nu}_2)$ of the measure μ, the systems of weights $\vec{w}_1$ and $\vec{w}_2$ and the multi-indices $\vec{\nu}_1$ and $\vec{\nu}_2$. It is easy to show that finding the polynomials $A_1, \dots, A_{p_1}$ is equivalent to solving a system of $|\vec{\nu}_2|$ homogeneous linear equations for the $|\vec{\nu}_1|$ unknown coefficients of the polynomials. Since $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$ the system always has a non-trivial solution. The matrix of this system of equations is the so called moment matrix, and the study of its Gauss–Borel factorization will be the cornerstone of this paper. Observe that when $p_1 = 1$ we are in the type II case and if $p_2 = 1$ in type I case. Hence in general we can find a solution of (3) where there is an $a \in \{1, \dots, p_1\}$ such that $\deg A_a < \nu_{1,a} - 1$. When given a combination $(\mu, \vec{w}_1, \vec{w}_2)$ of a measure $\mu \in \mathcal{M}(\Delta)$ and systems of weights $\vec{w}_1$ and $\vec{w}_2$ on Δ if for each pair of multi-indices $(\vec{\nu}_1, \vec{\nu}_2)$ the conditions (3) determine that $\deg A_a = \nu_{1,a} - 1$, $a = 1, \dots, p_1$, then we say that the combination $(\mu, \vec{w}_1, \vec{w}_2)$ is perfect. The concept of *perfectness* will be rigorously introduced in Definition 2.

The seminal paper of M. Sato [32], and further developments performed by the Kyoto school through the use of the bilinear equation and the τ-function formalism [15–17], settled the basis for the Lie group theoretical description of integrable hierarchies, in this direction we have the relevant contribution by M. Mulase [29] in which the factorization problems, dressing procedure, and linear systems were the key for integrability. In this dressing setting the multicomponent integrable hierarchies of Toda type were analyzed in depth by K. Ueno and T. Takasaki [37]. See also the papers [9] and [23] on the multi-component KP hierarchy and [28] on the multi-component Toda lattice hierarchy. In a series of papers M. Adler and P. van Moerbeke showed how the Gauss–Borel factorization problem appears in the theory of the 2D Toda hierarchy and what they called the discrete KP hierarchy [1–5]. In these papers it becomes clear—from a group-theoretical setup—why standard orthogonality of polynomials and integrability of nonlinear equations of Toda type where so close. In fact, the Gauss–Borel factorization of the moment matrix may be understood as the Gauss–Borel factorization of the initial condition for the integrable hierarchy. To see the connection between the work of Mulase and that of Adler and van Moerbeke see [18]. Later on, in the recent paper [6], it is shown that the multiple orthogonal construction described in previous paragraphs was linked with the multi-component KP hierarchy. In fact, for a given set of weights $(\vec{w}_1, \vec{w}_2)$ and degrees $(\vec{\nu}_1, \vec{\nu}_2)$ the authors constructed a finite matrix that plays the role of the moment matrix and, using the Riemann–Hilbert problem of [14], where able to show that determinants constructed from the moment matrix were τ-functions solving the bilinear equation for the multi-component KP hierarchy. However, there is no mention in that paper to any Gauss–Borel factorization in spite of being the multicomponent integrable hierarchies connected with different factorization problems of these type. For further developments on the Gauss–Borel factorization and multi-component 2D Toda hierarchy see [7] and [27].

This motivated our initial research in relation with this paper; i.e., the construction of an appropriate Gauss–Borel factorization in the group of semi-infinite matrices leading to multiple orthogonality and integrability in a simultaneous manner. The main advantage of this approach lies in the application of different techniques based on the factorization problem used frequently in the theory of integrable systems. The key finding of this paper is, therefore, the characterization of a semi-infinite moment matrix whose Gauss–Borel factorization leads directly to multiple orthogonality. This makes sense when factorization can be performed, which is the case for perfect combinations $(\mu, \vec{w}_1, \vec{w}_2)$, which allows us to consider some sets of multiple orthogonal polynomials (called ladders) very much in the same manner as in the (non-multiple) orthogonal polynomial setting. The Gauss–Borel factorization of this moment matrix leads, when one takes into account the Hankel type symmetry of the moment matrix, to results like: 1. Recursion relations, 2. ABC theorems and 3. Christoffel–Darboux formulas. The first two are new results while the third is not new, as it was derived from the Riemann–Hilbert problem in [14]. However, our derivation of the Christoffel–Darboux formula is based exclusively on the Gauss–Borel factorization, and its uniqueness and existence for the multiple orthogonality problem are the only requirements. Thus, it is sufficient to have a perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$, and there are examples of this type which do not have a well defined Riemann–Hilbert problem in the spirit of [14].

When we seek for the appropriate integrable hierarchy linked with multiple orthogonality we are lead to the multicomponent 2D Toda lattice hierarchy which extends the construction of the multicomponent KP hierarchy considered by M.J. Bergvelt and A.P.E. ten Kroode in [9]; not to the multicomponent 2D Toda lattice hierarchy as described in [37] or [28]. In the spirit of this last mentioned articles, and complementing the continuous flows of the integrable hierarchy, we also introduce discrete flows, that could be viewed as Darboux transformations, and which correspond to Miwa transformations implying the addition of a zero/pole to the set of weights. Moreover, the Hankel type symmetry is related to an invariance under a number of flows, and to string equations. Bilinear equations can be derived from the Gauss–Borel factorization problem and moreover the τ-function representation is available leading to a bridge to the results of [6] in which no semi-infinite matrix or Gauss–Borel factorization was used.

This paper is divided into three sections, Section 1 is this introduction which contains Section 1.1 in where we review the application of the *LU* factorization of the moment matrix to the theory of orthogonal polynomials in the real line. Next, Section 2 is devoted to the presentation of the moment matrix and the discussion of the Gauss–Borel factorization. In this form we obtain perfect systems in terms of Nikishin systems, determinantal expressions for the multiple orthogonal polynomials, their duals and second type functions, bi-orthogonality for the associated linear forms, recursion relations, ABC type theorems and the Christoffel–Darboux formula. Flows and the integrable hierarchy are studied in Section 3 in which an integrable hierarchy *a la* Bergvelt–ten Kroode is linked with the multiple orthogonality problem. We not only derive from the Gauss–Borel factorization the Lax and Zakharov–Shabat equations, but also we introduce discrete integrable flows, described by Miwa shifts, or Darboux transformations, and also construct an appropriate bilinear equation. Finally, we find the τ functions corresponding to the multiple orthogonality and link them to those of [6]. At the end of the paper, we have added two appendices: the first one collects the more technical proofs of the results in this paper. In Appendix B we consider discrete flows for the case of a measure μ with unbounded support $\operatorname{supp}\mu$.

1.1. The Gauss–Borel factorization of the moment matrix and orthogonal polynomials

Here we discuss how the *LU* factorization of the standard moment matrix $g = (\int x^{i+j}\, \mathrm{d}\mu)$ of a constant sign finite Borel measure μ leads to traditional results in the theory of orthogonal polynomials, namely recursion relation and Christoffel–Darboux formula. In spite that these results are well established we repeat them here because in their derivation is encoded the set of arguments we will use in the multiple orthogonality setting. In the forthcoming exposition it will become clear the *LU* factorization approach is just a compact way of using the orthogonality relations.

The moment matrix can be written as the following Grammian matrix

$$g=\int \chi(x)\chi(x)^\top \, \mathrm{d}\mu(x)$$

in terms of the monomial string $\chi(x):=(1,x,x^2,\dots)^\top$.

The Borel–Gauss factorization of g is

$$g=S^{-1}\bar{S}, \qquad S=\begin{pmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \cdots \\ S_{2,0} & S_{2,1} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \bar{S}^{-1}=\begin{pmatrix} \bar{S}'_{0,0} & \bar{S}'_{0,1} & \bar{S}'_{0,2} & \cdots \\ 0 & \bar{S}'_{1,1} & \bar{S}'_{1,2} & \cdots \\ 0 & 0 & \bar{S}'_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The reader should notice that

- It makes sense whenever the truncated moment matrix $g^{[l]}=(g_{i,j})_{0\leqslant i,j<l}$ is an invertible matrix for any $l=1,2,\dots$. If the factorization exists it is unique.
- Although the truncated matrices $g^{[l]}$ are invertible it can be shown that g itself is not invertible.
- The matrix product of S^{-1} with $\bar{S}$ involves only finite sums, but if we reverse the order of the factors we get series (with an infinite number of summands).

Given the factors S and $\bar{S}$ we consider the following polynomial strings, the semi-infinite vectors,

$$P:=S\chi=(P_0,P_1,\dots)^\top, \qquad \bar{P}:=\left(\bar{S}^{-1}\right)^\top\chi=(\bar{P}_0,\bar{P}_1,\dots)^\top.$$

The families of polynomials $\{P_l\}_{l=0}^\infty$ and $\{\bar{P}_k\}_{k=0}^\infty$ are biorthogonal:

$$\begin{aligned}\int P(x)\bar{P}(x)^\top \, \mathrm{d}\mu(x) &= \int S\chi(x)\chi(x)^\top\bar{S}^{-1}\,\mathrm{d}\mu(x)=S\int \chi(x)\chi(x)^\top\,\mathrm{d}\mu(x)\bar{S}^{-1} \\ &= \mathbb{I} \quad \Rightarrow \quad \int P_l(x)\bar{P}_k(x)\,\mathrm{d}\mu(x)=\delta_{l,k}.\end{aligned}$$

In this simple proof relies the basic connection between orthogonality and the *LU* factorization, which we consider as the very same thing dressed in different manners. From the above orthogonality we conclude that

$$\begin{aligned}&\int P_l(x)x^j\,\mathrm{d}\mu(x)=0, \quad j=0,\dots,l-1, \\ &\int \bar{P}_l(x)x^j\,\mathrm{d}\mu(x)=0, \quad j=0,\dots,l-1,\end{aligned} \tag{4}$$

and we also have that $P_l(x)$ and $\bar{P}_l$ are l-th degree polynomials where P_l is monic and $\bar{P}_l$ satisfies $\int x^l \bar{P}_l(x)\,\mathrm{d}\mu(x)=1$, i.e. we have type II and type I normalizations. Given that the moment matrix is symmetric, $g=g^\top$ and the uniqueness of the LU factorization we deduce that $\bar{S}=H(S^{-1})^\top$, with $H=\operatorname{diag}(h_0,h_1,\dots)$; i.e., $g=S^{-1}H(S^{-1})^\top$ and the factorization is a Cholesky factorization (but this does not extend to the multiple orthogonal case). Therefore $\bar{P}_l=h_l^{-1}P_l$ so that $\int P_l(x)P_k(x)\,\mathrm{d}\mu(x)=h_l\delta_{l,k}$, and $\{P_l\}_{l=0}^\infty$ is a family of monic orthogonal polynomials with respect to the measure μ.

Considering the orthogonality relations as a linear system for the coefficients of the polynomials one concludes that polynomials and their duals can be expressed as

$$
\begin{aligned}
P_l &= \chi^{(l)} - \begin{pmatrix} g_{l,0} & g_{l,1} & \cdots & g_{l,l-1}\end{pmatrix}\left(g^{[l]}\right)^{-1}\chi^{[l]}\\
&= \bar{S}_{l,l}\begin{pmatrix}0 & 0 & \cdots & 0 & 1\end{pmatrix}\left(g^{[l+1]}\right)^{-1}\chi^{[l+1]}\\
&= \frac{1}{\det g^{[l]}}\det\left(\begin{array}{cccc|c}
g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & 1\\
g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & x\\
\vdots & \vdots & & \vdots & \vdots\\
g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & x^{l-1}\\ \hline
g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & x^l
\end{array}\right), \quad l\geqslant 1,
\end{aligned}
$$

and similar expressions for the dual polynomials. We are now ready to get the recursion relations for orthogonal polynomials:

- First, we notice that the moment matrix g is a Hankel matrix, $g_{i+1,j}=g_{i,j+1}$, which in terms of the shift matrix $\Lambda:=\begin{pmatrix}0&1&0&0&\cdots\\0&0&1&0&\cdots\\0&0&0&1&\ddots\\ \vdots&\vdots&\vdots&\ddots&\ddots\end{pmatrix}$ can be written as $\Lambda g=g\Lambda^\top$.[1]
- Second, we observe the eigen-value property $\Lambda\chi(x)=x\chi(x)$.
- Third, we introduce the LU factorization to get $\Lambda S^{-1}\bar{S}=S^{-1}\bar{S}\Lambda^\top \Rightarrow S\Lambda S^{-1}=\bar{S}\Lambda^\top\bar{S}^{-1}=:J$. From this last relation we deduce that the matrix $J=\begin{pmatrix}a_0&1&0&0&\cdots\\b_1&a_1&1&0&\cdots\\0&b_2&a_2&1&\ddots\\ \vdots&\vdots&\ddots&\ddots&\ddots\end{pmatrix}$ is a tridiagonal matrix, i.e. a Jacobi matrix.
- Finally, we notice that the polynomial strings are eigenvectors of the Jacobi matrix: $JP(x)=S\Lambda S^{-1}S\chi(x)=S\Lambda\chi(x)=Sx\chi(x)=xP(x)$; i.e., the recursion relations $xP_k(x)=P_{k+1}(x)+a_kP_k(x)+b_kP_{k-1}(x)$, $k>0$, hold.

We now consider the Aitken–Berg–Collar (ABC) theorem (here we follow the nomenclature used [34]) for orthogonal polynomials. First we introduce the Christoffel–Darboux kernel and therefore we consider

[1] From this symmetry property it follows, by contradiction, that the moment matrix is not invertible; i.e. the assumption of the existence of g^{-1} leads to $g^{-1}\Lambda=\Lambda^\top g^{-1}$, and therefore the first row and column of g^{-1} are identically zero, so that g^{-1} is not invertible.

$$\mathcal{H}^{[l]} = \mathbb{R}\{0, \dots, x^{l-1}\}, \qquad \mathcal{H} = \left\{ \sum_{0 \leqslant l \ll \infty} c_l x^l,\ c_l \in \mathbb{R} \right\},$$

$$\left(\mathcal{H}^{[l]}\right)^{\perp} = \left\{ \sum_{l \leqslant k \ll \infty} c_l P^l(x), c_l \in \mathbb{R} \right\}$$

and the resolution of the identity $\mathcal{H} = \mathcal{H}^{[l]} \oplus (\mathcal{H}^{[l]})^{\perp}$, with the corresponding orthogonal projector $\pi^{(l)}$ such that $\ker \pi^{(l)} = (\mathcal{H}^{[l]})^{\perp}$ and $\operatorname{Ran} \pi^{(l)} = \mathcal{H}^{[l]}$. Then, the Christoffel–Darboux is defined as

$$K^{[l]}(x, y) := \sum_{k=0}^{l-1} P_k(y) \bar{P}_k(x) = \sum_{k=0}^{l-1} h_k^{-1} P_k(y) P_k(x),$$

which, according to the bi-orthogonality property, gives the following integral representation of the projection operator

$$\left(\pi^{(l)} f\right)(y) = \int K^{[l]}(x, y) f(x)\, \mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}.$$

Any semi-infinite vector v can be written in block form as follows

$$v = \left(\frac{v^{[l]}}{v^{[\geqslant l]}} \right)$$

$v^{[l]}$ is the finite vector formed with the first l coefficients of v and $v^{[\geqslant l]}$ the semi-infinite vector formed with the remaining coefficients. This decomposition induces the following block structure for any semi-infinite matrix.

$$g = \left(\begin{array}{c|c} g^{[l]} & g^{[l, \geqslant l]} \\ \hline g^{[\geqslant l, l]} & g^{[\geqslant l]} \end{array} \right).$$

Given a factorizable moment matrix g we have

$$g^{[l]} = \left(S^{[l]}\right)^{-1} \bar{S}^{[l]}, \qquad \left(S^{-1}\right)^{[l]} = \left(S^{[l]}\right)^{-1}, \qquad \left(\bar{S}^{-1}\right)^{[\geqslant l]} = \left(\bar{S}^{[\geqslant l]}\right)^{-1}.$$

The Christoffel–Darboux kernel is related to the moment matrix in the following way

$$K^{[l]}(x, y) = \left(\chi^{[l]}(x)\right)^{\top} \left(g^{[l]}\right)^{-1} \chi^{[l]}(y)$$

which is a consequence of the following identities

$$\begin{aligned} K^{[l]}(x, y) &= \left(\Pi^{[l]} \bar{P}(x)\right)^{\top} \left(\Pi^{[l]} P(y)\right) \\ &= \chi^{\top}(x) \bar{S}^{-1} \Pi^{[l]} S \chi(y) \\ &= \chi^{\top}(x) \left(\Pi^{[l]} \bar{S}^{-1} \Pi^{[l]}\right) \left(\Pi^{[l]} S \Pi^{[l]}\right) \chi(y) \\ &= \left(\chi^{[l]}(x)\right)^{\top} \left(g^{[l]}\right)^{-1} \chi^{[l]}(y). \end{aligned}$$

The relations

$$\big(g^{[l]}\big)^{-1}\Lambda^{[l]}-\big(\Lambda^{[l]}\big)^{\top}\big(g^{[l]}\big)^{-1}=\big(g^{[l]}\big)^{-1}\big(g^{[l,\geqslant l]}\big(\Lambda^{[l,\geqslant l]}\big)^{\top}-\Lambda^{[l,\geqslant l]}g^{[\geqslant l,l]}\big)\big(g^{[l]}\big)^{-1}$$

follow from the block equation

$$\Lambda^{[l]}g^{[l]}+\Lambda^{[l,\geqslant l]}g^{[\geqslant l,l]}=g^{[l]}\big(\Lambda^{[l]}\big)^{\top}+g^{[l,\geqslant l]}\big(\Lambda^{[l,\geqslant l]}\big)^{\top}.$$

We also have

$$\Lambda^{[l]}\chi^{[l]}(x)=x\chi^{[l]}(x)-\Lambda^{[l,\geqslant l]}\chi^{[\geqslant l]}(x),\qquad \Lambda^{[l,\geqslant l]}=e_{l-1}e_0^{\top},$$

where $\{e_i\}_{i=0}^{\infty}$ is the canonical linear basis of $\mathcal{H}$. With all these at hand we deduce

$$\begin{aligned}&\big(\chi^{[l]}(x)\big)^{\top}\big(\big(g^{[l]}\big)^{-1}\Lambda^{[l]}-\big(\Lambda^{[l]}\big)^{\top}\big(g^{[l]}\big)^{-1}\big)\chi^{[l]}(y)\\&\quad=\big(\chi^{[l]}(x)\big)^{\top}\big(g^{[l]}\big)^{-1}\big(g^{[l,\geqslant l]}\big(\Lambda^{[l,\geqslant l]}\big)^{\top}-\Lambda^{[l,\geqslant l]}g^{[\geqslant l,l]}\big)\big(g^{[l]}\big)^{-1}\chi^{[l]}(y)\end{aligned}$$

so that,

$$\begin{aligned}(x-y)K^{[l]}(x,y)=&\big(\big(\chi^{[\geqslant l]}(x)\big)^{\top}-\big(\chi^{[l]}(x)\big)^{\top}\big(g^{[l]}\big)^{-1}g^{[l,\geqslant l]}\big)e_0e_{l-1}^{\top}\big(g^{[l]}\big)^{-1}\chi^{[l]}(y)\\&-\big(\chi^{[l]}(x)\big)^{\top}\big(g^{[l]}\big)^{-1}e_{l-1}e_0^{\top}\big(\chi^{[\geqslant l]}(y)-g^{[\geqslant l,l]}\big(g^{[l]}\big)^{-1}\chi^{[l]}(y)\big).\end{aligned}$$

That using the determinantal expressions for the polynomials presented before leads to the Christoffel–Darboux formula

$$(x-y)K^{[l]}(x,y)=h_{l-1}^{-1}\big(P_l(x)P_{l-1}(y)-P_{l-1}(x)P_l(y)\big).$$

2. Multiple orthogonal polynomials and Gauss–Borel factorization

2.1. The moment matrix

In this section we define the moment matrix in terms of a measure $\mu\in\mathcal{M}(\Delta)$ and two systems of weights $\vec{w}_1$ and $\vec{w}_2$ on $\Delta\subset\mathbb{R}$, as well as corresponding compositions (the order matters) $\vec{n}_1=(n_{1,1},\dots,n_{1,p_1})\in\mathbb{N}^{p_1}$ and $\vec{n}_2=(n_{2,1},\dots,n_{2,p_2})\in\mathbb{N}^{p_2}$ [36]. We will consider multi-indices of positive integers $\vec{n}=(n_1,\dots,n_p)$, where $p\in\mathbb{N}$ and $n_a\in\mathbb{N}_+$, $a=1,\dots,p$ and define $|\vec{n}|:=n_1+\dots+n_p$. Following [9,36] we observe that any $i\in\mathbb{Z}_+:=\{0,1,2,\dots\}$ determines unique non-negative integers $q(i),a(i),r(i)$, such that the composition

$$i=q(i)|\vec{n}|+n_1+\dots+n_{a(i)-1}+r(i),\quad 0\leqslant r(i)<n_{a(i)},\tag{5}$$

holds. Hence, given i there is a unique $k(i)$ with

$$k(i)=q(i)n_{a(i)}+r(i),\quad 0\leqslant r(i)<n_{a(i)}.\tag{6}$$

Let us introduce the function integer part function $[\cdot] : \mathbb{R}_+ \to \mathbb{Z}_+$, $[x] = \max\{y \in \mathbb{Z}_+, y \leqslant x\}$. Combining (5) and (6) we can obtain a formula which expresses explicitly the dependence between the quantities i, k and a

$$i = \left[\frac{k}{n_a}\right]\big(|\vec{n}| - n_a\big) + n_1 + \cdots + n_{a-1} + k. \tag{7}$$

Let $\mathbb{R}^\infty$ denote the vector space of all sequences with elements in $\mathbb{R}$. An element $\lambda \in \mathbb{R}^\infty$ may be interpreted as a column semi-infinity vector as follows

$$\lambda = \big(\lambda^{(0)}, \lambda^{(1)}, \dots\big)^\top, \quad \lambda^{(j)} \in \mathbb{R}, \quad j = 0, 1, \dots.$$

We consider the set $\{e_j\}_{j\geqslant 0} \subset \mathbb{R}^\infty$ with

$$e_j = (\overbrace{0, 0, \dots, 0}^{j}, 1, 0, 0, \dots)^\top.$$

Here $(\cdot)^\top$ denotes the transposition function on vectors and matrices. Analogously, we denote by $(\mathbb{R}^p)^\infty$ the set of all sequences of vectors with p components and observe that each sequence which belongs to $(\mathbb{R}^p)^\infty$ can also be understood as semi-infinity column vector: given the vector sequence $(\vec{v}_0, \vec{v}_1, \dots)$ with $\vec{v}_j = (v_{j,1}, \dots, v_{j,p})^\top$ we have the corresponding sequence in $\mathbb{R}^\infty$ given by $(v_{0,1}, \dots, v_{0,p}, v_{1,1}, \dots, v_{1,p}, \dots)$; i.e., $\mathbb{R}^\infty \cong (\mathbb{R}^p)^\infty$. Therefore, we consider also the set $\{e_a(k)\}_{\substack{a=1,\dots,p\\ k=0,1,\dots}} \subset (\mathbb{R}^p)^\infty$ where for each pair $(a,k) \in \{1,\dots,p\} \times \mathbb{Z}_+$ $e_a(k) = e_{i(k,a)}$ and the function $i(a,k) \in \mathbb{Z}_+$ satisfies the equality (7).

Now, we are ready to introduce the monomial strings

$$\chi_a := \sum_{k=0}^{\infty} e_a(k) z^k, \qquad \chi_a^{(l)} = \begin{cases} z^{k(l)}, & a = a(l), \\ 0, & a \neq a(l), \end{cases} \qquad \chi_a^* := z^{-1}\chi_a\big(z^{-1}\big). \tag{8}$$

These vectors may be understood as sequences of monomials according to the composition $\vec{n}$ introduced previously. We also define the following weighted monomial string

$$\xi := \sum_{a=1}^{p} \chi_a w_a, \qquad \xi^{(l)} = w_{a(l)} z^{k(l)}, \tag{9}$$

which is a sequence of weighted monomials for each given composition $\vec{n}$. Sometimes, when we what to stress the dependence in the composition we write $\chi_{\vec{n},a}$, $\chi_{\vec{n}}$ and $\xi_{\vec{n}}$. Given the weighted monomials $\xi_{\vec{n}_1}$ and $\xi_{\vec{n}_2}$, associated to the compositions $\vec{n}_1$ and $\vec{n}_2$, we introduce the moment matrix in the following manner

Definition 1. The moment matrix is given by

$$g_{\vec{n}_1,\vec{n}_2} := \int \xi_{\vec{n}_1}(x) \xi_{\vec{n}_2}(x)^\top \, \mathrm{d}\mu(x). \tag{10}$$

In terms of the canonical basis $\{E_{i,j}\}$ of the linear space of semi-infinite matrices and for each pair $(i,j)\in\mathbb{Z}_+^2$ we consider the binary permutations or transpositions $\pi_{i,j}=E_{i,j}+E_{j,i}$. Observe that $\pi_{i,j}^2=\mathbb{I}$ and therefore $\pi_{i,j}^{-1}=\pi_{i,j}$. Given two transpositions $\pi_{i,j}$ and $\pi_{k,l}$ the permutation endomorphism corresponding to its product is well defined $\pi_{i,j}\pi_{k,l}=\pi_{k,l}\pi_{i,j}$. Taking a sequence of pairs $I=\{(i_s,j_s)\}_{s\in\mathbb{Z}_+}$, $i_s,j_s\in\mathbb{Z}_+$, we introduce the permutation endomorphism π_I as the infinite product $\pi_I=\prod_{s\in\mathbb{Z}_+}\pi_{i_s,j_s}$, with $\pi_I\pi_I^\top=\pi_I^\top\pi_I=\mathbb{I}$. Given two compositions, $\vec n'$, $\vec n$, there exists a permutation $\pi_{\vec n',\vec n}$ such that $\xi_{\vec n'}=\pi_{\vec n',\vec n}\xi_{\vec n}$ through a permutation semi-infinite matrix as just described.

The change in the compositions is modeled as follows

Proposition 1. *Given two set of weights $\vec w_\ell=(w_{\ell,1},\dots,w_{\ell,p_\ell})$ and compositions $\vec n_\ell$ and $\vec n'_\ell$, $\ell=1,2$, there exist permutation matrices $\pi_{\vec n'_\ell,\vec n_\ell}$ such that*

$$g_{\vec n'_1,\vec n'_2}=\pi_{\vec n'_1,\vec n_1}g_{\vec n_1,\vec n_2}\pi_{\vec n'_2,\vec n_2}^\top. \tag{11}$$

Proof. For any set of weights $\vec w=w_1,\dots,w_p$ and two compositions $\vec n$ and $\vec n'$ we have that the corresponding vectors of weighted monomials are connected,

$$\xi_{\vec n'}=\pi_{\vec n',\vec n}\xi_{\vec n}$$

trough a permutation semi-infinite matrix; i.e., $\pi_{\vec n',\vec n}^\top=\pi_{\vec n',\vec n}^{-1}$. Therefore, the announced result follows. □

For the sake of notation simplicity and when the context is clear enough we will drop the subindex indicating the two compositions and just write g for the moment matrix. Let us discuss in more detail the block Hankel structure of the moment matrix. For each pair $(i,j)\in\mathbb{Z}_+^2$ there exists a unique combination of three others pairs $(q_1,q_2)\in\mathbb{Z}_+^2$, $(a_1,a_2)\in\{1,\dots,p_1\}\times\{1,\dots,p_2\}$ and $(r_1,r_2)\in\{0,\dots,n_{1,a_1}-1\}\times\{0,\dots,n_{2,a_2}-1\}$, such that

$$i=q_1|\vec n_1|+n_{1,1}+\dots+n_{1,a_1-1}+r_1 \quad\text{and}\quad j=q_2|\vec n_2|+n_{2,1}+\dots+n_{2,a_2-1}+r_2.$$

Hence taking $k_\ell=q_\ell n_{\ell,a_\ell}+r_\ell$, $\ell=1,2$, the coefficients $g_{i,j}\in\mathbb{R}$ of the moment matrix $g=(g_{i,j})$ have the following explicit form

$$g_{i,j}=\int x^{k_1+k_2}w_{1,a_1}(x)w_{2,a_2}(x)\,\mathrm{d}\mu(x). \tag{12}$$

Observe that pairs (k_1,a_1) and (k_2,a_2) are univocally determined by i and j respectively.

Before we continue with the study of this moment matrix it is necessary to introduce some auxiliary objects associated with the vector space $\mathbb{R}^\infty$. First, we have the unity matrix $\mathbb{I}=\sum_{k=0}^\infty e_ke_k^\top$ and the shift matrix $\Lambda:=\sum_{k=0}^\infty e_ke_{k+1}^\top$. We also define the projections $\Pi^{[l]}:=\sum_{k=0}^{l-1}e_ke_k^\top$, and with the help of the set $\{e_a(k)\}_{\substack{a=1,\dots,p\\k=0,1,\dots}}$ we construct the projections $\Pi_a:=\sum_{k=0}^\infty e_a(k)e_a(k)^\top$ with $\sum_{a=1}^p\Pi_a=\mathbb{I}$, and

$$\begin{aligned} P_1 &:= \operatorname{diag}(\mathbb{I}_{n_1}, 0_{n_2}, \dots, 0_{n_p}), \quad P_2 := \operatorname{diag}(0_{n_1}, \mathbb{I}_{n_2}, \dots, 0_{n_p}), \quad \dots \\ P_p &:= \operatorname{diag}(0_{n_1}, 0_{n_2}, \dots, \mathbb{I}_{n_p}), \end{aligned} \tag{13}$$

where $\mathbb{I}_{n_s}$ is the $n_s \times n_s$ identity matrix. Finally we introduce the notation

$$x^{\vec{n}} := x^{n_1} P_1 + \cdots + x^{n_p} P_p = \operatorname{diag}\big(x^{n_1}\mathbb{I}_{n_1}, \dots, x^{n_p}\mathbb{I}_{n_p}\big) : \mathbb{R} \to \mathbb{R}^{|\vec{n}| \times |\vec{n}|}. \tag{14}$$

For a better insight of the moment matrix let us introduce the following $n_{1,a} \times n_{2,b}$ matrices

$$m_{a,b}(x) = w_{1,a}(x) w_{2,b}(x) \begin{pmatrix} 1 & x & \cdots & x^{n_{2,b}-1} \\ x & x^2 & \cdots & x^{n_{2,b}} \\ \vdots & \vdots & & \vdots \\ x^{n_{1,a}-1} & x^{n_{1,a}} & \cdots & x^{n_{1,a}+n_{2,b}-2} \end{pmatrix}, \qquad \begin{aligned} a &= 1, \dots, p_1, \\ b &= 1, \dots, p_2, \end{aligned} \tag{15}$$

in terms of which we build up the following $|\vec{n}_1| \times |\vec{n}_2|$-matrix

$$m := \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,p_2} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,p_2} \\ \vdots & \vdots & & \vdots \\ m_{p_1,1} & m_{p_1,2} & \cdots & m_{p_1,p_2} \end{pmatrix} : \mathbb{R} \to \mathbb{R}^{|\vec{n}_1| \times |\vec{n}_2|}. \tag{16}$$

Then, the moment matrix g has the following block structure

$$g := (G_{i,j})_{i,j \geqslant 0} \in \mathbb{R}^{\infty \times \infty}, \qquad G_{i,j} := \int x^{i\vec{n}_1} m(x) x^{j\vec{n}_2} \, \mathrm{d}\mu(x) \in \mathbb{R}^{|\vec{n}_1| \times |\vec{n}_2|}. \tag{17}$$

Fix now a number $l \in \mathbb{N}$ and consider the pair $(l, l+1)$. There exists a unique combination of pairs $(q_1, q_2) \in \mathbb{Z}_+^2$, $(a_1, a_2) \in \{1, \dots, p_1\} \times \{1, \dots, p_2\}$ and $(r_1, r_2) \in \{0, \dots, n_{1,a_1} - 1\} \times \{0, \dots, n_{2,a_2} - 1\}$, such that

$$l = q_1 |\vec{n}_1| + n_{1,1} + \cdots + n_{1,a_1 - 1} + r_1 \quad \text{and} \quad l + 1 = q_2 |\vec{n}_2| + n_{2,1} + \cdots + n_{2,a_2 - 1} + r_2.$$

Given the compositions $\vec{n}_1$ and $\vec{n}_2$ we introduce the degree multi-indices $\vec{\nu}_1 \in \mathbb{Z}_+^{p_1}$ and $\vec{\nu}_2 \in \mathbb{Z}_+^{p_2}$ [9] where for each $\ell = 1, 2$, we have

$$\begin{aligned} \vec{\nu}_\ell &= (\nu_{\ell,1}, \dots, \nu_{\ell,a_\ell - 1}, \nu_{\ell,a_\ell}, \nu_{\ell,a_\ell + 1}, \dots, \nu_{\ell,p_\ell}) \\ &= \big((q_\ell + 1) n_{\ell,1}, \dots, (q_\ell + 1) n_{\ell,a_\ell - 1}, q_\ell n_{\ell,a_\ell} + r_\ell, q_\ell n_{\ell,a_\ell + 1}, \dots, q_\ell n_{\ell,p_\ell}\big), \end{aligned} \tag{18}$$

which satisfy

$$k_\ell(i+1) = \nu_{\ell,a_\ell(i)}(i), \qquad \big|\vec{\nu}_\ell(i)\big| = i + 1, \qquad \vec{\nu}_\ell\big(i + l|\vec{n}_\ell|\big) = \vec{\nu}_\ell(i) + l\vec{n}_\ell, \tag{19}$$

and consider the $l \times (l+1)$ block matrix Γ_l from g

$$\Gamma_l = \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l} \end{pmatrix}. \tag{20}$$

Let us study the homogeneous system $\mathit{\Gamma}_l \boldsymbol{x}_{l+1} = \mathbf{0}_l$, where $\boldsymbol{x}_{l+1} \in \mathbb{R}^{l+1}$ and $\mathbf{0}_{l+1}$ is the null vector in $\mathbb{R}^{l+1}$. Taking into account $\mathit{\Gamma}_l$'s structure (12), we see that such equation is exactly the expression of the orthogonality relations (3). We can see now that for each $l \in \mathbb{N}$ the existence of a system of mixed multiple orthogonal polynomials $(A_1^{(l)}, \dots, A_{p_1}^{(l)})$ is ensured; that is because $\mathit{\Gamma}_l$ in (20) is an $l \times (l+1)$ matrix, and the homogeneous matrix equation $\mathit{\Gamma}_l \boldsymbol{x}_{l+1} = \mathbf{0}_l$, which is satisfied by the coefficients of the polynomials corresponds to a system of l homogeneous linear equations for $l+1$ unknown coefficients. Thus, the system always has a non-trivial solution. Obviously, $(A_1^{(l)}, \dots, A_{p_1}^{(l)})$ is not univocally determined by the matrix equation $\mathit{\Gamma}_l \boldsymbol{x}_{l+1} = \mathbf{0}_l$ or equivalently by the orthogonality relations (3), because its solution space has at least dimension 1. Hence, the appropriate question to consider is the uniqueness question without counting constant factors, or equivalently if the solution space has exactly dimension 1. In terms of $\mathit{\Gamma}_l$ the question becomes: Does $\mathit{\Gamma}_l$ have rank l? In order to have a positive answer it is sufficient to ensure that the $l \times l$ square matrix

$$g^{[l]} := \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} \\ \vdots & \vdots & & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} \end{pmatrix}, \quad l \geqslant 1, \tag{21}$$

is invertible, where $g^{[l]}$ results from $\mathit{\Gamma}_l$ after removing its last column. It is easy to prove that such condition is equivalent to require that all possible solutions of (3) satisfy $\deg A_{p_1} = \nu_{1,p_1} - 1$. Obviously this requirement is ensured when the polynomials $(A_1^{(l)}, \dots, A_{p_1}^{(l)})$ fulfill $\deg A_j = \nu_{1,j} - 1$, $j = 1, \dots, p_1$.

2.2. *Perfect combinations and Nikishin systems*

We introduce the concept of a perfect combination.

Definition 2. A combination $(\mu, \vec{w}_1, \vec{w}_2)$ of a measure $\mu \in \mathcal{M}(\Delta)$ and two systems of weights $\vec{w}_1$ and $\vec{w}_2$ on $\Delta \subset \mathbb{R}$ is said to be perfect if for each pair of multi-indices $(\vec{\nu}_1, \vec{\nu}_2)$, with $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$ the orthogonality relations (3) imply that $\deg A_a = \nu_{1,a} - 1$, $a = 1, \dots, p_1$.

For a perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$ and any given $l \in \mathbb{Z}_+$ the solution space of the equation $\mathit{\Gamma}_l \boldsymbol{x}_{l+1} = \mathbf{0}_l$ is one-dimensional. Then, we can determine a unique system of mixed type orthogonal polynomials $(A_1, \dots, A_{p_1})$ satisfying (3) requiring for $a_1 \in \{1, \dots p_1\}$ that A_{a_1} monic. Following [14] we say that we have a type II normalization and denote the corresponding system of polynomials by $A_a^{(\mathrm{II},a_1)}$, $j = 1, \dots, p_1$. Alternatively, we can proceed as follows, since the system of weights is perfect from (3) we deduce that

$$\int x^{\nu_{2,b_2}} \sum_{a=1}^{p_1} A_a(x) w_{1,a}(x) w_{2,b_2}(x) \,\mathrm{d}\mu(x) \neq 0.$$

Then, we can determine a unique system of mixed type of multi-orthogonal polynomials $(A_1^{(\mathrm{I},b_2)}, \dots, A_{p_2}^{(\mathrm{I},b_2)})$ imposing that

$$\int x^{\nu_{2,b_2}} \sum_{a=1}^{p_1} A_a^{(\mathrm{I},b_2)}(x) w_{1,a}(x) w_{2,b}(x) \,\mathrm{d}\mu(x) = 1,$$

which is a type I normalization. We will use the notation $A^{(\mathrm{II},a_1)}_{[\vec{\nu}_1;\vec{\nu}_2],a}$ and $A^{(\mathrm{I},b_2)}_{[\vec{\nu}_1;\vec{\nu}_2],a}$ to denote these multiple orthogonal polynomials with type II and I normalizations, respectively.

A known illustration of perfect combinations $(\mu, \vec{w}_1, \vec{w}_2)$ can be constructed with an arbitrary positive finite Borel measure μ and systems of weights formed with exponentials:

$$\left(\mathrm{e}^{\gamma_1 x}, \dots, \mathrm{e}^{\gamma_p x}\right), \quad \gamma_i \neq \gamma_j, \quad i \neq j,\ i, j = 1, \dots, p, \tag{22}$$

or by binomial functions

$$\left((1-z)^{\alpha_1}, \dots, (1-z)^{\alpha_p}\right), \quad \alpha_i - \alpha_j \notin \mathbb{Z}, \quad i \neq j,\ i, j = 1, \dots, p. \tag{23}$$

or combining both classes, see [30]. Recently a wide class of systems of weights where proven to be perfect [19]; these systems of functions, now called Nikishin systems, were introduced by E.M. Nikishin [30] and initially named MT-systems (after Markov and Tchebycheff).

Given a closed interval Δ let $\mathring{\Delta}$ be the interior set of Δ. Let us take two intervals Δ_α and Δ_β whose interior sets are disjoint, i.e. $\mathring{\Delta}_\alpha \cap \mathring{\Delta}_\beta = \emptyset$. Set two measures $\mu_\alpha \in \mathcal{M}(\Delta_\alpha)$ and $\mu_\beta \in \mathcal{M}(\Delta_\beta)$ such that the measure $\langle \mu_\alpha, \mu_\beta \rangle$ with the following differential form

$$\mathrm{d}\langle \mu_\alpha, \mu_\beta \rangle(x) = \int \frac{\mathrm{d}\mu_\beta(t)}{x-t}\,\mathrm{d}\mu_\alpha(x) = \hat{\mu}_\beta(x)\,\mathrm{d}\mu_\alpha(x),$$

is a finite measure, that implies that $\langle \mu_\alpha, \mu_\beta \rangle \in \mathcal{M}(\Delta_\alpha)$. The function $\hat{\mu}_\beta$ denotes the Cauchy transform corresponding to μ_β. Let us consider then a system of p intervals $\Delta_1, \dots, \Delta_p$ such that $\mathring{\Delta}_j \cap \mathring{\Delta}_{j+1} = \emptyset$, $j \in \{1, \dots, p-1\}$. Take p measures $\mu_j \in \mathcal{M}(\Delta_j)$, which for each $j = 1, \dots, p-1$, the measure $\langle \mu_j, \sigma_{j+1} \rangle$ belongs to $\mathcal{M}(\Delta_j)$. So the system of measures $(\xi_0, \dots, \xi_p)$ where

$$\zeta_1 = \mu_1, \quad \zeta_2 = \langle \mu_1, \mu_2 \rangle, \quad \zeta_3 = \big\langle \mu_1, \langle \mu_2, \mu_3 \rangle \big\rangle = \langle \mu_1, \mu_2, \mu_3 \rangle, \quad \dots, \quad \zeta_p = \langle \mu_1, \dots, \mu_p \rangle,$$

is the Nikishin system of measures generated by the system $(\sigma_1, \dots, \sigma_m)$. So we denote $(\zeta_1, \dots, \zeta_p) = \mathcal{N}(\sigma_1, \dots, \sigma_p)$.

Actually, in [19] the authors shown perfectness for combinations of Nikishin systems where intervals $\Delta_1, \dots, \Delta_p$ are bounded and for each $j \in \{1, \dots, p-1\}$ the intervals Δ_j and Δ_{j+1} are disjoint. The same authors have communicated to us that they were able to prove a generalization of this result to unbounded intervals such that $\Delta_j \cap \Delta_{j+1} \neq \emptyset$. Consequently, in what follows we assume such generalization.

As we have seen, general Nikishin systems have an intricate structure; therefore, in order to make easy the reader we focus on a "simple" class of Nikishin systems which we call M-Nikishin systems. Set the interval $\Delta_1 = [0, 1]$ and let $\mathcal{M}_0(\Delta_1) \subset \mathcal{M}(\Delta_1)$ denote the set of measures in $\mathcal{M}(\Delta_1)$ such that if $\sigma \in \mathcal{M}_0(\Delta_1)$ then, the function

$$\tilde{\sigma}(x) := \int_{\Delta_1} \frac{\mathrm{d}\sigma(t)}{1-tx} \quad \text{satisfies} \quad \lim_{\substack{x\to 1\\ x \in \mathring{\Delta}_1}} \left| \int_{\Delta_1} \frac{\mathrm{d}\sigma(t)}{1-tx} \right| = \lim_{\substack{x\to 1\\ x \in \mathring{\Delta}_1}} \int_{\Delta_1} \frac{|\mathrm{d}\sigma(t)|}{1-tx} < +\infty,$$
$$\text{where } \mathring{\Delta}_1 = (0, 1). \tag{24}$$

The constraint (24) guarantees that the function $\tilde{\sigma}$ is a weight in compact intervals in $(-\infty, 1]$. As $(1-tx)$ does not vanish for $(t,x) \in \Delta_1 \times (\mathbb{C} \setminus [1,+\infty))$ we deduce that $1/(1-tx)$ is a continuous function in x for $t \in \Delta_1$. Therefore, we conclude that $\tilde{\sigma}$ is a holomorphic function on $\mathbb{C} \setminus (1,+\infty)$, having a continuation as continuous function in 1. Taking into account that $\tilde{\sigma}$ does not vanish in $\mathbb{C} \setminus (1,+\infty)$ and that it takes real values on $\mathbb{R} \setminus (1,+\infty) = (-\infty, 1]$, we deduce that it is a continuous weight on $(-\infty, 1]$. Observe that

$$\tilde{\sigma}(x) = \int\limits_{\Delta_1} \frac{d\sigma(t)}{1-tx} = \int\limits_{[1,+\infty)} \frac{\zeta\, d\sigma(1/\zeta)}{x-\zeta} = \int\limits_{[1,+\infty)} \frac{\mathrm{d}\mu(\zeta)}{x-\zeta}, \tag{25}$$

is the Cauchy transform of another measure $\mu \in \mathcal{M}([1,+\infty))$, such that $|\hat{\mu}(1)| = |\tilde{\sigma}(1)| < +\infty$.

Given two measures $\sigma_\alpha \in \mathcal{M}_0(\Delta_1), \sigma_\beta \in \mathcal{M}_0(\Delta_1)$ we define a third one as follows (using the differential notation)

$$\mathrm{d}[\sigma_\alpha, \sigma_\beta](x) = \tilde{\sigma}_\beta(x)\,\mathrm{d}\sigma_\alpha(x), \qquad \tilde{\sigma}_\beta(x) = \int\limits_{\Delta_1} \frac{\mathrm{d}\sigma_\beta(\zeta)}{1-x\zeta}.$$

As $\tilde{\sigma}_\beta$ is a continuous weight on Δ_1 we conclude that $[\sigma_\alpha, \sigma_\beta] \in \mathcal{M}_0(\Delta_1)$. If we take a system of measures $(\sigma_1, \dots, \sigma_p)$ such that $\sigma_j \in \mathcal{M}_0(\Delta_1)$, $j = 1, \dots, p$, we say that $(s_1, \dots, s_p) = \mathcal{MN}(\sigma_1, \dots, \sigma_p)$, where

$$\begin{aligned} &s_1 = \sigma_1, \quad s_2 = [\sigma_1, \sigma_2], \quad s_3 = \big[\sigma_1, [\sigma_2, \sigma_3]\big] = [\sigma_1, \sigma_2, \sigma_3], \quad \dots, \\ &\quad s_p = [\sigma_1, \sigma_2, \dots, \sigma_p] \end{aligned} \tag{26}$$

is the M-Nikishin system of measures generated by $(\sigma_1, \dots, \sigma_p)$, with corresponding M-Nikishin system of functions given by $\vec{w} = (w_1, \dots, w_p) = (\tilde{s}_1, \dots, \tilde{s}_p) = \mathcal{M}\hat{\mathcal{N}}(\sigma_1, \dots, \sigma_p)$.

Notice that $s_i \in \mathcal{M}_0(\Delta_1)$ which implies that for each arbitrary compact subinterval of $(-\infty, 1]$ the system of functions $\vec{w}$ conforms a system of continuous weights. M-Nikishin systems are included in the class of Nikishin systems. Taking into account the identity (25) we see that the M-Nikishin system defined in (26) can be written as a classical Nikishin system. Let us take a system $(\mu_1, \dots, \mu_p)$ where

$$\begin{aligned} &\mathrm{d}\mu_1(x) = x^2\sigma_1(1/x), \quad \mu_2 = -\sigma_2, \quad \mathrm{d}\mu_3(x) = x^2\,\mathrm{d}\sigma_3(1/x), \quad \dots, \\ &\quad \mathrm{d}\mu_{2[p/2]-1}(x) = x^2\,\mathrm{d}\sigma_{2[p/2]-1}, \quad \mu_{2[p/2]} = -\sigma_{2[p/2]}, \end{aligned}$$

and if p is odd $\mathrm{d}\mu_p(x) = x^2\,\mathrm{d}\sigma_p(x)$. Notice then

$$s_1 = \zeta_1 = \sigma_1, \quad s_2 = \zeta_2 = \langle \mu_1, \mu_2 \rangle, \quad \dots, \quad s_p = \zeta_p = \langle \mu_1, \mu_2, \dots, \mu_p \rangle.$$

Hence $(s_1, \dots, s_p) = \mathcal{MN}(\sigma_1, \dots, \sigma_p) = \mathcal{N}(\mu_1, \dots, \mu_p) = (\zeta_1, \dots, \zeta_p)$. Fixing two M-Nikishin systems of functions $\vec{w}_\ell(x) = (\tilde{s}_{\ell,1}(x), \dots \tilde{s}_{\ell,p}(x))$ whose elements are weights on $\Delta_0 = [-1, 1]$, and a measure $\mu \in \mathcal{M}(\Delta_0)$ we have at our disposal the perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$. We can also obtain a perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$ choosing $\vec{w}_1$ and $\vec{w}_2$ between two different of the classes mentioned in (22) and (23) (not necessarily the same).

Proposition 2. *The Taylor series at $\zeta = 0$ corresponding to the functions $\tilde{s}_j(\zeta)$ and $f_j(\zeta) := \log \tilde{s}_j(\zeta)$ converge uniformly to $\tilde{s}_j$ and f_j respectively on Δ_1, i.e.*

$$\tilde{s}_j(x) = \sum_{i=0}^{\infty} \lambda_{i,j} x^i = e^{\sum_{i=0}^{\infty} t_{i,j} x^i}, \quad x \in \Delta_0, \ j = 1, \ldots, p. \tag{27}$$

where $\lambda_{i,j}$ and $t_{i,j}$ are constants.

Proof. For each $j \in \{1, \ldots, p\}$, $\tilde{s}_j$ is a holomorphic function on the open unitary disc centered on the origin. That implies that

$$\tilde{s}_j(x) = \sum_{i=0}^{\infty} \lambda_{i,j} x^i = e^{\sum_{i=0}^{\infty} t_{i,j} x^i}, \quad x \in \bigl\{|\zeta| < 1\bigr\}, \ j = 1, \ldots, p.$$

Notice that

$$\left| \sum_{i=0}^{\infty} \lambda_{i,j} \right| = \lim_{\substack{x \to 1 \\ x \in [0,1)}} \left| \sum_{i=0}^{\infty} \lambda_{i,j} x^i \right| = \lim_{\substack{x \to 1 \\ x \in \mathring{\Delta}_1}} \left| \int \frac{\mathrm{d}s_j(t)}{1 - xt} \right| < +\infty.$$

So the first equality in (27) is proved. The second one comes immediately from the fact that the functions $\tilde{s}_j$ do not vanish on Δ_0. That implies that $\sum_{i=0}^{\infty} t_{i,j} x^i$ are also bounded and therefore continuous. Hence we can proceed analogously as in the first equality. □

2.2.1. The inverse problem

Given the series

$$w_j(x) = \sum_{i=0}^{\infty} \lambda_{i,j} x^i = e^{\sum_{i=0}^{\infty} t_{i,j} x^i}, \quad x \in \Delta_0, \ j = 1, \ldots, p, \tag{28}$$

we consider the problem of finding conditions over $\{\lambda_{i,j}\}$ such that the set of series $\{w_j\}_{j=1}^{p}$ form a M-Nikishin system of functions. The reader should notice that $\lambda_{i,j} = S_i(t_{i,0}, t_{i,1}, \ldots, t_{i,j})$ where S_i is the i-th elementary Schur polynomial. Elementary Schur polynomials $S_j(t_1, \ldots, t_j)$ are defined by the following generating relation $\exp(\sum_{j=1}^{\infty} t_j z^j) = \sum_{j=0}^{\infty} S_j(t_1, t_2, \ldots, t_j) z^j$, and therefore $S_j = \sum_{p=1}^{j} \sum_{j_1 + \cdots + j_p = j} t_{j_1} \cdots t_{j_p}$. Given a partition $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_+^r$ we have the Schur function $s_{\vec{n}}(t) = \det(S_{n_i - i + j}(t))_{1 \leqslant i,j \leqslant r}$. For more on the relation of these Schur functions and those in [25], see [31].

In order to state sufficient conditions in this direction we need some preliminary definitions and results.

Definition 3. Given a sequence $C = \{c_i\}_{i=0}^{\infty} \subset \mathbb{R}$ its Hausdorff moment problem consists in finding a measure $\sigma \in \mathcal{M}(\Delta)$ such that

$$c_i = \int \zeta^i \, \mathrm{d}\sigma(\zeta), \quad i \in \mathbb{Z}_+.$$

Moreover, if we further impose the constraint $\sigma \in \mathcal{M}_0(\Delta)$ we say that we have a restricted Hausdorff moment problem.

Here we have made a variation in the classical definition of a Hausdorff problem, where the solutions are positive measures. In our Hausdorff problem we look for measures in a wider class where they do not change their sign. Obviously, since $\mathcal{M}_0(\Delta) \subset \mathcal{M}(\Delta)$ each solution of a restricted Hausdorff problem is also a solution of a Hausdorff problem. In pages 8 and 9 in [33] J.A. Shohat and J.D. Tamarkin study Hausdorff problems and give a sufficient and necessary condition over the sequences to have solution. Using this result we deduce the following lemma.

Lemma 1. *The Hausdorff moment problem for a sequence $C = \{c_i\}_{i=0}^{\infty} \subset \mathbb{R}$ has a solution if and only if*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i c_{i+k} \geqslant 0 \quad \forall (n,k) \in \mathbb{Z}_+^2 \quad \textit{or} \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^i c_{i+k} \leqslant 0 \quad \forall (n,k) \in \mathbb{Z}_+^2. \tag{29}$$

When (29) *holds a necessary and sufficient condition that ensures solution for the restricted Hausdorff moment problem of C is*

$$\left| \sum_{i=0}^{\infty} c_i \right| < +\infty. \tag{30}$$

Proof. Theorem 1.5 in [33] states that the first set of inequalities in (29) is a necessary and sufficient condition to have a positive measure σ solving the classical Hausdorff problem. Following their proof it is not hard to conclude that adding the second set of inequalities leads to a solution in $\mathcal{M}(\Delta)$. Let us take a measure $\sigma \in \mathcal{M}(\Delta)$ and observe that $\int \frac{\mathrm{d}\sigma(t)}{1-xt}$ is a holomorphic function on $\bar{\mathbb{C}} \setminus [1, +\infty)$, then if C is its moment sequence we deduce

$$\int_{\Delta} \frac{\mathrm{d}\sigma(t)}{1 - xt} = \sum_{i=0}^{\infty} c_i x^i, \quad x \in \{|\zeta| < 1\}.$$

Thus, since all the c_i's have the same sign, by Lebesgue's dominated convergence theorem we have

$$\lim_{\substack{x \to 1 \\ x \in [0,1)}} \left| \int_{\Delta} \frac{\mathrm{d}\sigma(t)}{1 - xt} \right| = \lim_{\substack{x \to 1 \\ x \in [0,1)}} \left| \sum_{i=0}^{\infty} x^i c_i \right| = \left| \sum_{i=0}^{\infty} c_i \right|.$$

Thus $\sigma \in \mathcal{M}_0(\Delta)$ if and only if (30) takes place. $\square$

Given the series

$$w_j(x) = \sum_{i=0}^{\infty} \lambda_{i,j,1} x^i, \quad x \in \Delta_1, \ j = 1, \ldots, p, \tag{31}$$

we introduce a set of semi-infinite matrices Θ_k and semi-infinite vectors $\theta_{j,k}$, $j = k, \ldots, p$, $k = 1, \ldots, p$ in the following recursive way. First, we define

$$\Theta_1 := \begin{pmatrix} \lambda_{0,1,1} & \lambda_{1,1,1} & \lambda_{2,1,1} & \cdots \\ \lambda_{1,1,1} & \lambda_{2,1,1} & \lambda_{3,1,1} & \cdots \\ \lambda_{2,1,1} & \lambda_{3,1,1} & \lambda_{4,1,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \theta_{j,1} := \begin{pmatrix} \lambda_{0,j,1} \\ \lambda_{1,j,1} \\ \lambda_{2,j,1} \\ \vdots \end{pmatrix}, \quad j = 1, \dots, p.$$

Then, we seek solutions $\theta_{j,2} := \begin{pmatrix} \lambda_{0,j,2} \\ \lambda_{1,j,2} \\ \lambda_{2,j,2} \\ \vdots \end{pmatrix}$ of $\Theta_1\theta_{j,2} = \theta_{j,1}$, for $j = 2, \dots, p$, and if these solutions exist we define $\Theta_2 := \begin{pmatrix} \lambda_{0,2,2} & \lambda_{1,2,2} & \lambda_{2,2,2} & \cdots \\ \lambda_{1,2,2} & \lambda_{2,2,2} & \lambda_{3,2,2} & \cdots \\ \lambda_{2,2,2} & \lambda_{3,2,2} & \lambda_{4,2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. Then, we look for $\theta_{j,3} = \begin{pmatrix} \lambda_{0,j,3} \\ \lambda_{1,j,3} \\ \lambda_{2,j,3} \\ \vdots \end{pmatrix}$ which solves $\Theta_2\theta_{j,3} = \theta_{j,2}$, for $j = 3, \dots, p$, and when such solutions exist we introduce $\Theta_3 = \begin{pmatrix} \lambda_{0,3,3} & \lambda_{1,3,3} & \lambda_{2,3,3} & \cdots \\ \lambda_{1,3,3} & \lambda_{2,3,3} & \lambda_{3,3,3} & \cdots \\ \lambda_{2,3,3} & \lambda_{3,3,3} & \lambda_{4,3,3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$. In this way we get for $k \in \{1, \dots, p\}$ the matrices Θ_k and vectors $\theta_{j,k}$, $j = k, \dots, p$, linked by $\Theta_k\theta_{j,k+1} = \theta_{j,k}$ with expressions

$$\Theta_{k+1} = \begin{pmatrix} \lambda_{0,k+1,k+1} & \lambda_{1,k+1,k+1} & \lambda_{2,k+1,k+1} & \cdots \\ \lambda_{1,k+1,k+1} & \lambda_{2,k+1,k+1} & \lambda_{3,k+1,k+1} & \cdots \\ \lambda_{2,k+1,k+1} & \lambda_{3,k+1,k+1} & \lambda_{4,k+1,k+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \theta_{j,k+1} = \begin{pmatrix} \lambda_{0,j,k+1} \\ \lambda_{1,j,k+1} \\ \lambda_{2,j,k+1} \\ \vdots \end{pmatrix}.$$

Here we understand $\Theta_k\theta_{j,k+1} = \theta_{j,k}$ as

$$\sum_{i=0}^{\infty} \lambda_{l+i,k,k}\lambda_{i,j,k+1} = \lambda_{l,j,k}, \quad l \in \mathbb{Z}_+.$$

We now consider the sequences

$$C_{k,k} := \{\lambda_{i,k,k}\}_{i=0}^{\infty}, \qquad C_{j,k} := \{\lambda_{i,j,k}\}_{i=0}^{\infty}, \quad j = k, \dots, p, \ k = 1, \dots, p. \tag{32}$$

Later, we will prove that none of the semi-infinite Hankel matrices Θ_k, $k = 1, \dots, p$, are invertible. Hence such infinite linear systems are either undetermined or incompatible. In this last case we say that the systems of sequences $(C_{k,k}, \dots, C_{p,k})$, $k = 1, \dots, p$, do not exist.

First we need the following preliminary

Lemma 2. *The series*

$$w(x) = \sum_{i=0}^{\infty} \lambda_i x^i, \quad x \in \Delta_0,$$

converges uniformly on Δ_0 to a function $\tilde{\sigma}(x) = \int \mathrm{d}\sigma(t)/(1 - tx)$ corresponding to a measure $\sigma \in \mathcal{M}_0(\Delta_1)$ on Δ_0 if and only if the restricted Hausdorff moment problem corresponding to the sequence $\{\lambda_i \colon i \in \mathbb{Z}_+\}$ has a solution.

Proof. Let us assume that the restricted Hausdorff moment problem of a sequence $\{\lambda_i \colon i \in \mathbb{Z}_+\}$ has a solution. That means that there exists a measure $\sigma \in \mathcal{M}_0(\Delta)$ such that

$$\lambda_i = \int\limits_{\Delta_1} t^i \,\mathrm{d}\sigma(t), \quad i \in \mathbb{Z}_+, \qquad \lim_{\substack{x\to 1\\ x\in[0,1)}} \left| \int\limits_{\Delta_1} \frac{\mathrm{d}\sigma(t)}{1-tx} \right| = \left| \sum_{i=0}^{\infty} \lambda_i \right| < +\infty.$$

Since $|\lambda_i x^i| \leqslant |\lambda_i|$, $|x| \leqslant 1$ and $\sum_{i=0}^{\infty} |\lambda_i| < +\infty$, by Weirestrass' Theorem $\sum_{i=0}^{\infty} \lambda_i x^i$ converges uniformly on Δ_0. This proves the *if* implication in Lemma 2. On the other hand

$$\lim_{\substack{x\to 1\\ x\in[0,1)}} \left| \int\limits_{\Delta} \frac{\mathrm{d}\sigma(t)}{1-tx} \right| = \lim_{\substack{x\to 1\\ x\in[0,1)}} \left| \sum_{i=0}^{\infty} \lambda_i x^i \right| = \left| \sum_{i=0}^{\infty} \lambda_i \right|$$

because $|\sum_{i=0}^{\infty} \lambda_i x^i|$ must be continuous on Δ_0. λ_i coincides with the i-th moment corresponding to the measure σ which completes the proof. $\square$

Theorem 1. *The system of weights $\{w_{1,j}\}_{j=1}^{p}$, as in* (31)*, converges uniformly in Δ_0 to an M-Nikishin system of functions $\{\hat{s}_j\}_{j=1}^{p}$ if and only if for each $k = 1, \ldots, p-1$, there exists a system of sequences $(C_{k,k}, \ldots, C_{p,k})$ as in* (32)*, such that their restricted Hausdorff moment problems have solutions.*

Proof. The proof of this theorem goes as follows. From Lemma 2 we have that for each $j = 1, \ldots, p$,

$$w_{j,1}(x) = \sum_{i=0}^{\infty} \lambda_{i,j,1} x^i, \quad x \in \Delta_0,$$

converges in Δ_0 to a function $\tilde{s}_j(x) = \int \mathrm{d}s_j(t)/(1-tx)$ if and only if the restricted Hausdorff moment problem corresponding to $\{\lambda_{i,j,1}\colon i \in \mathbb{Z}_+\}$ has a solution. We assume that $w_{j,1}$ converges uniformly on Δ_0 to the function $\tilde{s}_j$ corresponding to the $s_j \in \mathcal{M}(\Delta_1)$. In order to prove the necessity in theorem's statement we suppose that $(s_1, \ldots, s_p) = \mathcal{MN}(\sigma_1, \ldots, \sigma_p)$ is an M-Nikishin system of measures as it was defined in Section 2.2. Fixed $k \in \{1, \ldots, p\}$ we define another M-Nikishin system $(s_{k,k}, \ldots, s_{k,p}) = \mathcal{MN}(\sigma_k, \ldots, \sigma_p)$. Let us observe that $(s_{1,1}, \ldots, s_{1,p}) = (s_1, \ldots, s_p)$.

By construction for each $k \in \{1, \ldots, p\}$, we have that $\mathrm{d}s_{k,j} = \tilde{s}_{k+1,j}\,\mathrm{d}s_{k,k}$, $j = k, \ldots, p$. When $j = k$ we understand $\tilde{s}_{k+1,k} \equiv 1$. Fixed $j \in \{k+1, \ldots, p\}$, $\tilde{s}_{k+1,j}$ is a holomorphic function on $\bar{\mathbb{C}} \setminus (-\infty, 1)$; hence, its Taylor's series

$$w_{j,k+1}(t) = \sum_{i=0}^{\infty} \lambda_{i,j,k+1} t^i, \quad t \in \Delta_1 \subset \Delta_0$$

converges uniformly to $\tilde{s}_{k+1,j}$ on Δ_1. Then, for each $x \in \Delta_1$

$$\begin{aligned}\tilde{s}_{k,j}(x) &= \sum_{l=0}^{\infty} \lambda_{l,j,k} x^l = \int \tilde{s}_{k+1,j}(t) \frac{\mathrm{d}s_{k,k}(t)}{1-tx} = \int \sum_{i=0}^{\infty} \lambda_{i,j,k+1} t^i \frac{\mathrm{d}s_{k,k}(\zeta)}{1-tx} \\ &= \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \lambda_{i,j,k+1} \int t^{i+l}\,\mathrm{d}s_{k,k}(t) = \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \lambda_{i,j,k+1} \lambda_{l+i,k,k} x^l,\end{aligned}$$

which proves one implication of the equivalence. The other implication comes immediately from Lemma 2. □

We remark from the statements of Lemma 1 that the conditions in Theorem 1 are equivalent to the inequalities in (29). Hence, by continuity criteria, such conditions are stable under perturbations of the coefficients $\lambda_{i,1,1}$, $i \in \mathbb{Z}_+$. We will come to this later in Section 3, when we consider deformations of the weights leading to the multicomponent 2D Toda flows in the precise form discussed in this section.

2.3. The Gauss–Borel factorization and multiple orthogonal polynomials

Given a perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$ we consider [2]

Definition 4. The Gauss–Borel factorization (also known as *LU* factorization) of a semi-infinite moment matrix g, determined by $(\mu, \vec{w}_1, \vec{w}_2)$, is the problem of finding the solution of

$$g = S^{-1}\bar{S}, \qquad S = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \cdots \\ S_{2,0} & S_{2,1} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\bar{S}^{-1} = \begin{pmatrix} \bar{S}'_{0,0} & \bar{S}'_{0,1} & \bar{S}'_{0,2} & \cdots \\ 0 & \bar{S}'_{1,1} & \bar{S}'_{1,2} & \cdots \\ 0 & 0 & \bar{S}'_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad S_{i,j}, \bar{S}'_{i,j} \in \mathbb{R}. \tag{33}$$

In terms of these matrices we construct the polynomials

$$A_a^{(l)} := \sum\nolimits'_i S_{l,i} x^{k_1(i)}, \tag{34}$$

where the sum $\sum'$ is taken for a fixed $a = 1, \ldots, p_1$ over those i such that $a = a_1(i)$ and $i \leqslant l$. We also construct the dual polynomials

$$\bar{A}_b^{(l)} := \sum\nolimits'_j x^{k_2(j)} \bar{S}'_{j,l}, \tag{35}$$

where the sum $\sum'$ is taken for a given b over those j such that $b = a_2(j)$ and $j \leqslant l$.

This factorization makes sense whenever all the principal minors of g do not vanish, i.e., if $\det g^{[l]} \neq 0$, $l = 1, 2, \ldots$, and in our case it is true because $(\mu, \vec{w}_1, \vec{w}_2)$ is a perfect combination. It can be shown that the following sets

$$G_- := \left\{ S = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \cdots \\ S_{2,0} & S_{2,1} & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ S_{i,j} \in \mathbb{R} \right\},$$

$$G_+ := \left\{ \bar S = \begin{pmatrix} \bar S_{0,0} & \bar S_{0,1} & \bar S_{0,2} & \cdots \\ 0 & \bar S_{1,1} & \bar S_{1,2} & \cdots \\ 0 & 0 & \bar S_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \bar S_{i,j} \in \mathbb{R}, \bar S_{i,i} \neq 0 \right\}$$

are groups. Indeed, the multiplication of two arbitrary semi-infinite matrices is, in general, not well defined as it involves, for each coefficient of the product, a series; however if the two matrices lie on G_-, the mentioned series collapses into a finite sum, and the same holds for G_+. Moreover, the inverse of a matrix in $S \in G_-$ can be found to be in G_- in a recursive way: first we express $S = \mathbb{I} + \sum_{i>0} S_i(\Lambda^\top)^i$ with $S_i = \operatorname{diag}(S_i(0), S_i(1), \dots)$ a diagonal matrix, then we assume $S^{-1} = \mathbb{I} + \sum_{i>0} \tilde S_i(\Lambda^\top)^i$ to have the same form, and finally we find that the diagonal matrix unknown coefficients $\tilde S_i$ are expressed in terms of $S_0, \dots, S_i$ in a unique way; the same holds in G_+. Given, two elements $S \in G_-$ and $\bar S \in G_+$ the coefficients of the product $S\bar S$ are finite sums. However, this is not the case for $\bar S S$, where the coefficients are series. Therefore, given an LU factorizable element $g = S^{-1}\bar S$ we can not ensure that g has an inverse, observe that in spite of the existence of S and $\bar S^{-1}$, the existence of $\bar S^{-1} S = g^{-1}$ is not ensured as this product involves the evaluation of series instead of finite sums.

With the use of the coefficients of the matrices S and $\bar S$ we construct multiple orthogonal polynomials of mixed type with normalizations of types I and II.

Proposition 3. *We have the following identifications*

$$A_a^{(l)} = A^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)],a}, \qquad \bar A_b^{(l)} = A^{(\mathrm{I},a_1(l))}_{[\vec\nu_2(l);\vec\nu_1(l-1)],b},$$

in terms of multiple orthogonal polynomials of mixed type with two normalizations I *and* II*, respectively.*

Proof. From the LU factorization we deduce

$$\sum_{i=0}^{l} S_{l,i} g_{i,j} = 0, \quad j = 0, 1, \dots, l-1, \qquad S_{ii} := 1. \tag{36}$$

With the aid of (18) and (34) we express (36) as follows

$$\int \left(\sum_{a=1}^{p_1} A_a^{(l)}(x) w_{1,a}(x) \right) w_{2,b}(x) x^k \, \mathrm{d}\mu(x) = 0, \qquad \deg A_a^{(l)} \leqslant \nu_{1,a}(l) - 1,$$
$$0 \leqslant k \leqslant \nu_{2,b}(l-1) - 1. \tag{37}$$

We recognize these equations as those defining a set of multiple orthogonal polynomials of mixed type as discussed in [14]. This fact leads to $A_{[\vec\nu_1;\vec\nu_2],a} := A_a^{(l)}$ where $\vec\nu_1 = \vec\nu_1(l)$ and $\vec\nu_2 = \vec\nu_2(l-1)$. Observe that for a given l each polynomial $A_{[\vec\nu_1;\vec\nu_2],a}$ has at much $\nu_{1,a}(l)$ coefficients, and therefore we have $|\vec\nu_1(l)| = l+1$ unknowns, while we have $|\vec\nu_2(l-1)| = l$ equations. Moreover, from the normalization condition $S_{ii} = 1$ we get that the polynomial $A_{[\vec\nu_1;\vec\nu_2],a_1(l)}$ is monic with $\deg A_{[\vec\nu_1;\vec\nu_2],a_1(l)} = \nu_{1,a_1(l)}(l) - 1 = k_1(l+1) - 1$, so that we are dealing with a type II normalization and therefore we can write $A_a^{(l)} = A^{(\mathrm{II},a_1(l))}_{[\vec\nu_1;\vec\nu_2],a}$.

Dual equations to (36) are

$$\sum_{j=0}^{l} g_{i,j}\bar{S}'_{j,l} = 0, \quad i = 0, 1, \dots, l-1, \tag{38}$$

$$\sum_{j=0}^{l} g_{l,j}\bar{S}'_{j,l} = 1. \tag{39}$$

Now, using again (18) and (35), (38) becomes

$$\int \left(\sum_{b=1}^{p_2} \bar{A}_b^{(l)}(x) w_{2,b}(x)\right) w_{1,a}(x) x^k \,\mathrm{d}\mu(x) = 0, \qquad \deg \bar{A}_b^{(l)} \leqslant \nu_{2,b}(l) - 1,$$
$$0 \leqslant k \leqslant \nu_{1,a}(l-1) - 1, \tag{40}$$

while (39) reads

$$\int \left(\sum_{b=1}^{p_2} \bar{A}_b^{(l)}(x) w_{2,b}(x)\right) w_{1,a_1(l)}(x) x^{k_1(l)} \,\mathrm{d}\mu(x) = 1, \tag{41}$$

where using (19) we obtain

$$k_1(l) = \nu_{1,a_1(l)}(l-1). \tag{42}$$

As above we are dealing with multiple orthogonal polynomials and therefore $\bar{A}_b^{(l)} = \bar{A}_{[\vec{\nu}_2;\vec{\nu}_1],b}$, with $\vec{\nu}_1 = \vec{\nu}_1(l-1)$ and $\vec{\nu}_2 = \vec{\nu}_2(l)$, which now happens to have a normalization of type I and consequently we write $\bar{A}_b^{(l)} = \bar{A}^{(\mathrm{I},a_1(l))}_{[\vec{\nu}_2;\vec{\nu}_1],b}$. □

Given a definite sign finite Borel measure the corresponding set of monic orthogonal polynomials $\{p_l\}_{l=0}^{\infty}$, $\deg p_l = l$, can be viewed as a ladder of polynomials, in which to get up to a given degree one needs to ascend l steps in the ladder. For multiple orthogonality the situation is different as we have, instead of a chain, a multi-dimensional lattice of degrees. Let us consider a perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$ and the corresponding set of multiple orthogonal polynomials $\{A_{[\vec{\nu}_1;\vec{\nu}_2],a}\}_{a=1}^{p_1}$, with degree vectors such that $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$. There always exist compositions $\vec{n}_1, \vec{n}_2$ and an integer l with $|\vec{\nu}_1| = l+1$ and $|\vec{\nu}_2| = l$ such that the polynomials $\{A_a^{(l)}\}_{a=1}^{p_1}$ coincides with $\{A_{[\vec{\nu}_1;\vec{\nu}_2],a}\}_{a=1}^{p_1}$. Therefore, the set of sets of multiple orthogonal polynomials $\{\{A_a^{(k)}\}_{a=1}^{p_1},\ k = 0, \dots, l\}$, can be understood as a ladder leading to the desired set of multiple orthogonal polynomials $\{A_{[\vec{\nu}_1;\vec{\nu}_2],a}\}_{a=1}^{p_1}$ after ascending l steps in the ladder, very much in same style as in standard orthogonality (non-multiple) setting. The ladder can be identified with the compositions $(\vec{n}_1, \vec{n}_2)$. However, by no means there is always a unique ladder to achieve this, in general there are several compositions that do the job. A particular ladder, which we refer to as the simplest $[\vec{\nu}_1; \vec{\nu}_2]$ ladder, is given by the choice $\vec{n}_1 = \vec{\nu}_1$ and $\vec{n}_2 = \vec{\nu}_2 + \vec{e}_{2,p_2}$. Many of the expressions that will be derived later on in this paper for multiple orthogonal polynomials and second kind functions only depend on the integers $(\vec{\nu}_1, \vec{\nu}_2)$ and not on the particular ladder chosen, and therefore compositions, one uses to reach to it.

2.4. Linear forms and multiple bi-orthogonality

We introduce linear forms associated with multiple orthogonal polynomials as follows

Definition 5. Strings of linear forms and dual linear forms associated with multiple orthogonal polynomials and their duals are defined by

$$Q := \begin{pmatrix} Q^{(0)} \\ Q^{(1)} \\ \vdots \end{pmatrix} = S\xi_1, \qquad \bar{Q} := \begin{pmatrix} \bar{Q}^{(0)} \\ \bar{Q}^{(1)} \\ \vdots \end{pmatrix} = \left(\bar{S}^{-1}\right)^\top \xi_2. \tag{43}$$

It can be immediately checked that

Proposition 4. *The linear forms and their duals, introduced in Definition 5, are given by*

$$Q^{(l)}(x) := \sum_{a=1}^{p_1} A_a^{(l)}(x) w_{1,a}(x), \qquad \bar{Q}^{(l)}(x) := \sum_{b=1}^{p_2} \bar{A}_b^{(l)}(x) w_{2,b}(x). \tag{44}$$

Sometimes we use the alternative notation $Q^{(l)} = Q_{[\vec{\nu}_1;\vec{\nu}_2]}$ and $\bar{Q}^{(l)} = \bar{Q}_{[\vec{\nu}_2;\vec{\nu}_1]}$. It is also trivial to check the following

Proposition 5. *The orthogonality relations*

$$\begin{aligned} &\int Q^{(l)}(x) w_{2,b}(x) x^k \, \mathrm{d}\mu(x) = 0, \quad 0 \leqslant k \leqslant \nu_{2,b}(l-1) - 1,\ b = 1, \ldots, p_2, \\ &\int \bar{Q}^{(l)}(x) w_{1,a}(x) x^k \, \mathrm{d}\mu(x) = 0, \quad 0 \leqslant k \leqslant \nu_{1,a}(l-1) - 1,\ a = 1, \ldots, p_1, \end{aligned} \tag{45}$$

are fulfilled.

Moreover, we have that these linear forms are bi-orthogonal.

Proposition 6. *The following multiple bi-orthogonality relations among linear forms and their duals*

$$\int Q^{(l)}(x) \bar{Q}^{(k)}(x) \, \mathrm{d}\mu(x) = \delta_{l,k}, \quad l, k \geqslant 0, \tag{46}$$

hold.

Proof. Observe that

$$\begin{aligned} \int Q(x) \bar{Q}(x)^\top \, \mathrm{d}\mu(x) &= \int S\xi_1(x) \xi_2(x)^\top \bar{S}^{-1} \, \mathrm{d}\mu(x) \quad \text{from (43)} \\ &= S\left(\int \xi_1(x)\xi_2(x)^\top \, \mathrm{d}\mu(x)\right)\bar{S}^{-1} \end{aligned}$$

$$= Sg\bar{S}^{-1} \quad \text{from (10)}$$
$$= \mathbb{I} \quad \text{from (33).} \qquad \square$$

Definition 6. Denote by $\xi_i^{[l]}$, $i = 1, 2$ the truncated vector formed with the first l components of ξ_i.

We are ready to give different expressions for these linear forms and their duals

Proposition 7. *The linear forms can be expressed in terms of the moment matrix in the following different ways*

$$\begin{aligned}
Q^{(l)} &= \xi_1^{(l)} - \begin{pmatrix} g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} \end{pmatrix} \bigl(g^{[l]}\bigr)^{-1} \xi_1^{[l]} \\
&= \bar{S}_{l,l} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \bigl(g^{[l+1]}\bigr)^{-1} \xi_1^{[l+1]} \\
&= \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c}
g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \xi_1^{(0)} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \xi_1^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \xi_1^{(l-1)} \\
\hline
g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \xi_1^{(l)}
\end{array}\right), \quad l \geqslant 1,
\end{aligned} \tag{47}$$

and the dual linear forms as

$$\begin{aligned}
\bar{Q}^{(l)} &= (\bar{S}_{l,l})^{-1} \left(\xi_2^{(l)} - \bigl(\xi_2^{[l]}\bigr)^{\top} \bigl(g^{[l]}\bigr)^{-1} \begin{pmatrix} g_{0,l} \\ g_{1,l} \\ \vdots \\ g_{l-1,l} \end{pmatrix} \right) \\
&= \bigl(\xi_2^{[l+1]}\bigr)^{\top} \bigl(g^{[l+1]}\bigr)^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\det g^{[l+1]}} \det \left(\begin{array}{cccc|c}
g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l} \\
\vdots & \vdots & & \vdots & \vdots \\
g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l} \\
\hline
\xi_2^{(0)} & \xi_2^{(1)} & \cdots & \xi_2^{(l-1)} & \xi_2^{(l)}
\end{array}\right), \quad l \geqslant 0.
\end{aligned} \tag{48}$$

Proof. See Appendix A. $\square$

As a consequence we get different expressions for the multiple orthogonal polynomials and their duals

Corollary 1. *The multiple orthogonal polynomials and their duals have the following alternative expressions*

$$\begin{aligned}
A_a^{(l)} &= \chi_{1,a}^{(l)} - \begin{pmatrix} g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} \end{pmatrix} \left(g^{[l]}\right)^{-1} \chi_{1,a}^{[l]} \\
&= \bar{S}_{l,l} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \left(g^{[l+1]}\right)^{-1} \chi_{1,a}^{[l+1]} \\
&= \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \chi_{1,a}^{(0)} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \chi_{1,a}^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \chi_{1,a}^{(l-1)} \\ \hline g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \chi_{1,a}^{(l)} \end{array}\right), \quad l \geqslant 1,
\end{aligned} \tag{49}$$

and

$$\bar{A}_b^{(l)} = (\bar{S}_{l,l})^{-1} \left(\chi_{2,b}^{(l)} - \left(\chi_{2,b}^{[l]}\right)^\top \left(g^{[l]}\right)^{-1} \begin{pmatrix} g_{0,l} \\ g_{1,l} \\ \vdots \\ g_{l-1,l} \end{pmatrix} \right) \tag{50}$$

$$\begin{aligned}
&= \left(\chi_{2,b}^{[l+1]}\right)^\top \left(g^{[l+1]}\right)^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\det g^{[l+1]}} \det \left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l} \\ \hline \chi_{2,b}^{(0)} & \chi_{2,b}^{(1)} & \cdots & \chi_{2,b}^{(l-1)} & \chi_{2,b}^{(l)} \end{array}\right), \quad l \geqslant 0.
\end{aligned} \tag{51}$$

Observe that (208), Appendix A, implies

$$\bar{S}_{l,l} = \frac{\det g^{[l+1]}}{\det g^{[l]}}. \tag{52}$$

2.5. *Functions of the second kind*

The Cauchy transforms of the linear forms (44) play a crucial role in the Riemann–Hilbert problem associated with the multiple orthogonal polynomials of mixed type [14]. Following the approach of Adler and van Moerbeke we will show that these Cauchy transforms are also related to the *LU* factorization considered in this paper.

Observe that the construction of multiple orthogonal polynomials performed so far is synthesized in the following strings of multiple orthogonal polynomials and their duals

$$\mathscr{A}_a := \begin{pmatrix} A_a^{(0)} \\ A_a^{(1)} \\ \vdots \end{pmatrix} = S\chi_{1,a}, \qquad \bar{\mathscr{A}}_b := \begin{pmatrix} \bar{A}_b^{(0)} \\ \bar{A}_b^{(1)} \\ \vdots \end{pmatrix} = \left(\bar{S}^{-1}\right)^\top \chi_{2,b},$$
$$a = 1, \dots, p_1, \; b = 1, \dots, p_2. \tag{53}$$

In order to complete these formulae and in terms of χ^* as in (8) we consider

Definition 7. Let us introduce the following formal semi-infinite vectors

$$\mathscr{C}_b = \begin{pmatrix} C_b^{(0)} \\ C_b^{(1)} \\ \vdots \end{pmatrix} = \bar{S}\chi^*_{2,b}(z), \qquad \bar{\mathscr{C}}_a = \begin{pmatrix} \bar{C}_a^{(0)} \\ \bar{C}_a^{(1)} \\ \vdots \end{pmatrix} = \left(S^{-1}\right)^\top \chi^*_{1,a}(z),$$
$$b = 1, \dots, p_2, \; a = 1, \dots, p_1, \tag{54}$$

that we call strings of second kind functions.

These objects are actually Cauchy transforms of the linear forms $Q^{(l)}$, $l \in \mathbb{Z}_+$, whenever the series converge and outside the support of the measures involved. Notice that fixed $z \in \mathbb{C}$ the entries in each string $\bar{\mathscr{C}}_a$ and $\mathscr{C}_b$ are series not necessarily convergent. In the non-convergent case we obviously understand the definition only formally. For each $l \in \mathbb{Z}_+$ we denote by $\bar{D}_a^{(l)}$ and $D_b^{(l)}$ on $\mathbb{C}$ the domains where the series $\bar{C}_a^{(l)}$ and $C_b^{(l)}$ are uniform convergent, respectively, and we understand them as their corresponding limits. From properties of Taylor's series, we know that uniform convergence of these series holds only on $\bar{D}_a^{(l)}$ and $D_b^{(l)}$ when they are the biggest open disks around $z = \infty$ which do not contain the respectively supports, $\operatorname{supp}(w_{2,a}\,\mathrm{d}\mu)$ and $\operatorname{supp}(w_{2,b}\mathrm{d}\mu)$. Outside the sets $\bar{D}_a^{(l)}$ and $D_b^{(l)}$ the series diverges at every point. Hence to have non-empty sets in $\bar{D}_a^{(l)}$ and $D_b^{(l)}$ the corresponding supports $\operatorname{supp}(w_{2,a}\,\mathrm{d}\mu)$ and $\operatorname{supp}(w_{2,b}\,\mathrm{d}\mu)$ must be bounded.

Proposition 8. *For each $l \in \mathbb{Z}_+$ the second kind functions can be expressed as follows*

$$C_b^{(l)}(z) = \int \frac{Q^{(l)}(x) w_{2,b}(x)}{z - x}\,\mathrm{d}\mu(x), \quad z \in D_b^{(l)} \setminus \operatorname{supp}\big(w_{1,b}\,\mathrm{d}\mu(x)\big),$$
$$\bar{C}_a^{(l)}(z) = \int \frac{\bar{Q}^{(l)}(x) w_{1,a}(x)}{z - x}\,\mathrm{d}\mu(x), \quad z \in \bar{D}_a^{(l)} \setminus \operatorname{supp}\big(w_{2,a}\,\mathrm{d}\mu(x)\big). \tag{55}$$

Proof. The Gauss–Borel factorization leads to

$$\begin{aligned} C_b^{(l)}(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{l} S_{lk} g_{kn} \big(\Pi_{2,b}\chi_2^*(z)\big)_n \\ &= \sum_{n=0}^{\infty} \int \sum_{k=0}^{l} S_{lk} x^{k_1(k)} w_{1,a_1(k)}(x) w_{2,b}(x) \frac{x^n}{z^{n+1}}\,\mathrm{d}\mu(x) \quad \text{use (12)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int x^n Q^{(l)}(x) w_{2,b}(x)\,\mathrm{d}\mu(x) \quad \text{use (44).} \end{aligned}$$

When $D_b^{(l)} \setminus \operatorname{supp}(w_{2,b}\,\mathrm{d}\mu) = \emptyset$ the proof is trivial. Given a non-empty compact set $\mathcal{K} \subset D_b^{(l)} \setminus \operatorname{supp}(w_{2,b}\,\mathrm{d}\mu) \neq \emptyset$ and recalling the closed character of $\operatorname{supp}(w_{2,b}\,\mathrm{d}\mu)$, we have that the distance between them $d_b^{(l)}(K) := \operatorname{distance}(\mathcal{K}, \operatorname{supp}(w_{2,b}\,\mathrm{d}\mu)) > 0$ is positive and that $\sup\{|z|\colon z \in \mathcal{K}\} =: M_{\mathcal{K}} < +\infty$. Taking into account that the series

$$C_b^{(l)}(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int x^n Q^{(l)}(x) w_{2,b}(x)\,\mathrm{d}\mu(x)$$

converges uniformly on $\mathcal{K}$ we can ensure

$$\lim_{n\to\infty} \sup_{|z|\in\mathcal{K}} \left\{ \left| \frac{1}{z^{n+1}} \int x^n Q^{(l)}(x) w_{2,b}(x)\,\mathrm{d}\mu(x) \right| \right\} = 0. \tag{56}$$

Hence, we have the bound

$$\begin{aligned}
&\left| \sum_{i=0}^{n} \frac{1}{z^{i+1}} \int x^i Q^{(l)}(x) w_{2,b}(x)\,\mathrm{d}\mu(x) - \int Q^{(l)}(x) w_{2,b}(x) \frac{1}{z-x}\,\mathrm{d}\mu(x) \right| \\
&\quad = \left| z \frac{1}{z^{n+1}} \int x^n Q^{(l)}(x) w_{2,b}(x) \frac{\mathrm{d}\mu(x)}{z-x} \right| \\
&\quad \leqslant \frac{M_{\mathcal{K}}}{d_b^{(l)}(\mathcal{K})} \sup_{|z|\in\mathcal{K}} \left\{ \left| \frac{1}{z^{n+1}} \int x^n Q^{(l)}(x) w_{2,b}(x)\,\mathrm{d}\mu(x) \right| \right\}, \quad \forall z \in \mathcal{K}.
\end{aligned} \tag{57}$$

Taking into account (56) we deduce from (57) the first equality for any compact set $\mathcal{K}$. Therefore, we get the first claim of the proposition; the second equality can be proved analogously. □

Given $l \geqslant 1$ and $a = 1, \dots, p$ the $+$ $(-)$ associated integer is the smallest (largest) integer l_{+a} (l_{-a}) such that $l_{+a} \geqslant l$ $(l_{-a} \leqslant l)$ and $a(l_{+a}) = a$ $(a(l_{-a}) = a)$. It can be shown that

$$\begin{aligned}
l_{-a} &:= \begin{cases} q(l)|\vec{n}| + \sum_{i=1}^{a} n_i - 1, & a < a(l), \\ l, & a = a(l), \\ q(l)|\vec{n}| - \sum_{i=a+1}^{p} n_i - 1, & a > a(l-1), \end{cases} \\
l_{+a} &:= \begin{cases} (q(l)+1)|\vec{n}| + \sum_{i=1}^{a-1} n_i, & a < a(l), \\ l, & a = a(l), \\ (q(l)+1)|\vec{n}| - \sum_{i=a}^{p} n_i, & a > a(l). \end{cases}
\end{aligned} \tag{58}$$

To give a determinantal expression for these second kind formal series we need

Definition 8. We introduce

$$\Gamma^{(l)}_{k,a} := \sum_{k'=l_{+a}}^{\infty} g_{k',k} z^{-k_1(k')-1} \delta_{a_1(k'),a}, \qquad \bar{\Gamma}^{(l)}_{k,b} := \sum_{k'=\bar{l}_{+b}}^{\infty} g_{k,k'} z^{-k_2(k')-1} \delta_{a_2(k'),b}. \tag{59}$$

Here l_{+a} is the $+$ associated integer within the $\vec{n}_1$ composition, while $\bar{l}_{+b}$ is the $+$ associated integer for the $\vec{n}_2$ composition.

With these definitions we can state

Proposition 9. *The following determinantal expressions for the functions of the second kind hold*

$$C^{(l)}_b = \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \bar{\Gamma}^{(l)}_{0,b} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \bar{\Gamma}^{(l)}_{1,b} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \bar{\Gamma}^{(l)}_{l-1,b} \\ \hline g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \bar{\Gamma}^{(l)}_{l,b}, \end{array}\right), \quad l \geqslant 1, \tag{60}$$

$$\bar{C}^{(l)}_a = \frac{1}{\det g^{[l+1]}} \det \left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l} \\ \hline \Gamma^{(l)}_{0,a} & \Gamma^{(l)}_{1,a} & \cdots & \Gamma^{(l)}_{l-1,a} & \Gamma^{(l)}_{l,a} \end{array}\right), \quad l \geqslant 1. \tag{61}$$

Proof. See Appendix A. □

Following [19] we consider the Markov–Stieltjes functions and polynomials of the second type.

Definition 9. The Markov–Stieltjes functions are defined by

$$\hat{\mu}_{a,b}(z) := \int \frac{w_{1,a}(x) w_{2,b}(x)}{z-x} \, \mathrm{d}\mu(x), \tag{62}$$

in terms of which we define

$$\begin{aligned} H^{(l)}_b(z) &:= \sum_{a=1}^{p_1} A^{(l)}_a(z) \hat{\mu}_{a,b}(z) - C^{(l)}_b(z), \\ \bar{H}^{(l)}_a(z) &:= \sum_{b=1}^{p_2} \hat{\mu}_{a,b}(z) \bar{A}^{(l)}_b(z) - \bar{C}^{(l)}_a(z). \end{aligned} \tag{63}$$

Proposition 10. *The functions $H^{(l)}_b$ and $\bar{H}^{(l)}_a$ are polynomials in z.*

Proof. The reader should notice that the functions $H_b^{(l)}$ and $\bar{H}_a^{(l)}$ are

$$H_b^{(l)}(z) = \int \sum_{a=1}^{p_1} w_{1,a}(x) \frac{A_a^{(l)}(z) - A_a^{(l)}(x)}{z-x} w_{2,b}(x)\, \mathrm{d}\mu(x),$$

$$\bar{H}_a^{(l)}(z) = \int \sum_{b=1}^{p_2} w_{1,a}(x) \frac{\bar{A}_b^{(l)}(z) - \bar{A}_b^{(l)}(x)}{z-x} w_{2,b}(x)\, \mathrm{d}\mu(x),$$

and as $z = x$ is a zero of the polynomials $A_a^{(l)}(z) - A_a^{(l)}(x)$ and $\bar{A}_b^{(l)}(z) - \bar{A}_b^{(l)}(x)$ from the above formulae we conclude that they are indeed polynomials in z. □

2.6. Recursion relations

The moment matrix has a Hankel type symmetry that implies the recursion relations and the Christoffel–Darboux formula. We consider the shift operators defined by

$$\Lambda_a := \sum_{k=0}^{\infty} e_a(k) e_a(k+1)^\top, \tag{64}$$

notice that

- Λ_a leaves invariant the subspaces $\Pi_{a'}\mathbb{R}^\infty$, for $a' = 1, \ldots, p$, and $\Pi_{a'}\Lambda_a = \Lambda_a \Pi_{a'}$.
- The set of semi-infinite matrices $\{\Lambda_a^j\}_{\substack{a=1,\ldots,p\\ j=1,2,\ldots}}$ is commutative.
- We have the eigenvalue property

$$\Lambda_a \chi_{a'} = \delta_{a,a'} z \chi_a. \tag{65}$$

Definition 10. We define the following multiple shift matrices

$$\Upsilon_1 := \sum_{a=1}^{p_1} \Lambda_{1,a}, \qquad \Upsilon_2 := \sum_{b=1}^{p_2} \Lambda_{2,b}, \tag{66}$$

and we also introduce the integers

$$N_{1,a} := |\vec{n}_1| - n_{1,a} + 1 = \sum_{\substack{a'=1,\ldots,p_1\\ a'\neq a}} n_{1,a'} + 1, \quad a = 1, \ldots, p_1, \; N_1 := \max_{a=1,\ldots,p_1} N_{1,a},$$

$$N_{2,b} := |\vec{n}_2| - n_{2,b} + 1 = \sum_{\substack{b'=1,\ldots,p_2\\ b'\neq b}} n_{2,b'} + 1, \quad b = 1, \ldots, p_2, \; N_2 := \max_{b=1,\ldots,p_2} N_{2,b}.$$

A careful but straightforward computation leads to

Proposition 11. *We have the following structure for Υ_1 and Υ_2*

$$\Upsilon_1 = D_{1,0}\Lambda + D_{1,1}\Lambda^{N_{1,1}} + \cdots + D_{1,p_1}\Lambda^{N_{1,p_1}},$$
$$\Upsilon_2 = D_{2,0}\Lambda + D_{2,1}\Lambda^{N_{2,1}} + \cdots + D_{2,p_2}\Lambda^{N_{2,p_2}},$$

where $D_{1,a}$, $a = 1, \dots, p_1$, and $D_{2,b}$, $b = 1, \dots, p_2$, are the following semi-infinite diagonal matrices:

$$D_{1,a} = \operatorname{diag}\bigl(D_{1,a}(0), D_{1,a}(1), \dots\bigr),$$
$$D_{1,a}(n) := \begin{cases} 1, & n = k|\vec{n}_1| + \sum_{a'=1}^{a} n_{1,a'} - 1,\ k \in \mathbb{Z}_+, \\ 0, & n \neq k|\vec{n}_1| + \sum_{a'=1}^{a} n_{1,a'} - 1,\ k \in \mathbb{Z}_+, \end{cases} \qquad D_{1,0} = \mathbb{I} - \sum_{a=1}^{p_1} D_{1,a},$$
$$D_{2,b} = \operatorname{diag}\bigl(D_{2,b}(0), D_{2,b}(1), \dots\bigr),$$
$$D_{2,b}(n) := \begin{cases} 1, & n = k|\vec{n}_2| + \sum_{b'=1}^{b} n_{2,b'} - 1,\ k \in \mathbb{Z}_+, \\ 0, & n \neq k|\vec{n}_2| + \sum_{b'=1}^{b} n_{2,b'} - 1,\ k \in \mathbb{Z}_+, \end{cases} \qquad D_{2,0} = \mathbb{I} - \sum_{b=1}^{p_1} D_{2,b}.$$

In terms of these shift matrices we can describe the particular Hankel symmetries for the moment matrix.

Proposition 12. *The moment matrix g satisfies the Hankel type symmetry*

$$\Upsilon_1 g = g\Upsilon_2^\top. \tag{67}$$

Proof. With the use of (19) and (12) we get

$$\Lambda_{1,a} g \Pi_{2,b} = \Pi_{1,a} g \Lambda_{2,b}^\top, \tag{68}$$

and summing up in $a = 1, \dots, p_1$ and $b = 1, \dots, p_2$ we get the desired result. $\square$

Observe that from (67) we deduce that in spite of being all the truncated moment matrices $g^{[l]}$, $l = 1, 2, \dots$ invertible, the moment matrix $g = \lim_{l\to\infty} g^{[l]}$ is not invertible. Suppose that the inverse $g^{-1} = (\tilde{g}_{i,j})_{1,j=0,1,\dots}$ of g exists so that (67) implies $g^{-1}\Upsilon_1 = \Upsilon_2^\top g^{-1}$, and therefore $\tilde{g}_{i,0} = \tilde{g}_{0,j} = 0$ for all i, $j = 0, 1, \dots$, which is contradictory with the invertibility of g.

Proposition 13. *From the symmetry of the moment matrix one derives*

$$S\Upsilon_1 S^{-1} = \bar{S}\Upsilon_2^\top \bar{S}^{-1}. \tag{69}$$

Proof. If we introduce (33) into (67) we get

$$\Upsilon_1 S^{-1}\bar{S} = S^{-1}\bar{S}\Upsilon_2^\top \quad \Rightarrow \quad S\Upsilon_1 S^{-1} = \bar{S}\Upsilon_2^\top \bar{S}^{-1}. \qquad \square$$

Definition 11. We define the matrices

$$J := J_+ + J_-, \qquad J_+ := \left(S\Upsilon_1 S^{-1}\right)_+, \qquad J_- := \left(\bar{S}\Upsilon_2^\top \bar{S}^{-1}\right)_-,$$

where the sub-indices $+$ and $-$ denote the upper triangular and strictly lower triangular projections.

Thus, J_+ is an upper triangular matrix and J_- a strictly lower triangular matrix. Moreover, from the string Eq. (69) we have the alternative expressions

$$J = S\Upsilon_1 S^{-1} = \bar{S}\Upsilon_2^\top \bar{S}^{-1}.$$

We now analyze the structure of $J_+ := (SD_{1,0}\Lambda S^{-1})_+ + (SD_{1,1}\Lambda^{N_{1,1}}S^{-1})_+ + \cdots + (SD_{1,p_1}\Lambda^{N_{1,p_1}}S^{-1})_+$. It is clear that we need to evaluate expressions of the form $SE_{i,j}S^{-1}$ with $i = \kappa_1(k,a) - 1$ and $j = \kappa_1(k+1,a-1)$ being $\kappa_1(k,a) := k|\vec{n}_1| + \sum_{a'=1}^{a} n_{1,a'}$. Given the form of S, see (33), we have

$$\left(SE_{i,j}S^{-1}\right)_+ = E_{i,j} + \sum_{l,l \in \mathbb{L}_{i,j}} s_{l,l'} E_{l,l'},$$

$$\mathbb{L}_{i,j} := \left\{(l,l') \in \mathbb{Z}_+^2 \;\middle|\; l < i,\ l' < j,\ l' \geqslant l\right\},$$

for some numbers $s_{l,l'} \in \mathbb{R}$ depending on the coefficients of S and on i, j; this matrix has zeroes everywhere but on a region of it that can be represented as a right triangle with hypotenuse lying on the main diagonal, this hypotenuse has its opposite vertex precisely on the (i,j) position. Therefore

$$J_+ = \left(SD_{1,0}\Lambda S^{-1}\right)_+ + \sum_{a=1}^{p_1}\sum_{k=0}^{\infty}\left(E_{\kappa_1(k,a)-1,\kappa_1(k+1,a-1)} + \sum_{l,l' \in \mathbb{L}_{1,k,a}} s_{l,l'} E_{l,l'}\right),$$

$$\mathbb{L}_{1,k,a} := \mathbb{L}_{\kappa_1(k,a)-1,\kappa_1(k+1,a-1)}.$$

We see that J_+ can be schematically represented as a staircase, the $\vec{n}_1$-staircase, descending over the main diagonal with steps—which are built with right triangles with hypotenuse lying on the main diagonal and opposite vertex (and therefore corner of the step) located at the $(\kappa_1(k,a) - 1, \kappa_1(k+1,a-1))$ position of the matrix—having width and height given by the integers in the composition $\vec{n}_1$. For example, the j-th step has width $n_{1,\frac{j}{p_1}-[\frac{j}{p_1}]}$ and height $n_{1,\frac{j+1}{p_1}-[\frac{j+1}{p_1}]}$. A similar description holds for $J_-^\top$ but replacing the composition $\vec{n}_1$ by $\vec{n}_2$. Therefore, the matrix J is a generalized Jacobi matrix and, in contrast with the non-multiple case, now is multi-diagonal (having in general more than three diagonals) and has a diagonal band of length $N_1 + N_2 + 1$. Moreover, this band has a number of zeroes on it, according to the $\vec{n}_1$-stair on the upper part and to the $\vec{n}_2$-stair in the lower part, we refer to this as a double $(\vec{n}_1, \vec{n}_2)$-staircase shape. To illustrate this *snake* shape let us write for the case $\vec{n}_1 = (4,3,2)$ and $\vec{n}_2 = (3,2)$ the corresponding truncated, $l = 27$, Jacobi type matrix

$$J^{[27]} = \left(\begin{array}{ccccccccccccccccccccccccccc}
\ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast
\end{array}\right), \tag{70}$$

where $\ast$ denotes a non-necessarily null real number. We can write

$$J = J_{N_1} \Lambda^{N_1} + \cdots + J_1 \Lambda + J_0 + J_{-1} \Lambda^\top + \cdots + J_{-N_2} \left(\Lambda^\top\right)^{N_2}, \tag{71}$$

where $J_i = \operatorname{diag}(J_i(0), J_i(1), \ldots)$. For convenience we extend the notation with $J_r(s) = 0$ whenever $r + s < 0$ or $s < 0$. We introduce

Definition 12. The semi-infinite vectors c_b and $\bar{c}_a$ are given by

$$\begin{aligned}
c_b &:= \bar{S} e_{2,b}(0), \quad b = 1, \ldots, p_2, \\
\bar{c}_a &:= \left(S^{-1}\right)^\top e_{1,a}(0), \quad a = 1, \ldots, p_1.
\end{aligned} \tag{72}$$

It is not difficult to show that

$$\begin{aligned}
c_b &= \sum_{l=0}^{n_{2,1}+\cdots+n_{2,b}-1} \bar{S}_{l, n_{2,1}+\cdots+n_{2,b}-1} e_l, \\
\bar{c}_a &= \sum_{l=0}^{n_{1,1}+\cdots+n_{1,a}-1} \left(S^{-1}\right)_{n_{1,1}+\cdots+n_{1,a}-1, l} e_l.
\end{aligned} \tag{73}$$

The semi-infinite matrices J and $J^\top$ have the following important property

Proposition 14. *The following equations are fulfilled*

$$\begin{aligned}
J \mathscr{A}_a(z) &= z \mathscr{A}_a(z), & J^\top \bar{\mathscr{A}_b}(z) &= z \bar{\mathscr{A}_b}(z), \\
J \mathscr{C}_b(z) &= z \mathscr{C}_b(z) - c_b, & J^\top \bar{\mathscr{C}_a}(z) &= z \mathscr{C}_a(z) - \bar{c}_a.
\end{aligned} \tag{74}$$

Proof. From (53) and (54)

$$\begin{aligned} J\mathscr{A}_a(z) &= S\Upsilon_1 S^{-1}S\chi_{1,a}(z) = zS\chi_{1,a} = z\mathscr{A}_a(z), \\ J\mathscr{C}_b(z) &= \bar{S}\Upsilon_2^\top \bar{S}^{-1}\bar{S}\chi^*_{2,b}(z) = \bar{S}\big(z\chi^*_{2,b}(z) - e_b(0)\big) = z\mathscr{C}_b(z) - c_b, \end{aligned}$$

where we have taken into account that $\Upsilon_2^\top \chi^*_{2,b}(z) = z\chi^*_{2,b}(z) - e_b(0)$. For $J^\top$ we proceed similarly:

$$\begin{aligned} J^\top \bar{\mathscr{A}}_b(z) &= \big(\bar{S}^{-1}\big)^\top \Upsilon_2 \bar{S}^\top \big(\bar{S}^{-1}\big)^\top \chi_{2,b}(z) = z\big(\bar{S}^{-1}\big)^\top \chi_{2,a} = z\bar{\mathscr{A}}_b(z), \\ J^\top \bar{\mathscr{C}}_a(z) &= \big(S^{-1}\big)^\top \Upsilon_1^\top S^\top \big(S^{-1}\big)^\top \chi^*_{1,a}(z) = \big(S^{-1}\big)^\top \big(z\chi^*_{1,a}(z) - e_a(0)\big) = z\bar{\mathscr{C}}_a(z) - \bar{c}_a. \qquad \square \end{aligned}$$

Theorem 2. *The multiple orthogonal polynomials and their associated second kind functions fulfill the following recursion relations*

$$\begin{aligned} zA_a^{(l)}(z) &= J_{-N_2}(l)A_a^{(l-N_2)}(z) + \cdots + J_{N_1}(l)A_a^{(l+N_1)}(z), \\ zC_b^{(l)}(z) - c_b^{(l)} &= J_{-N_2}(l)C_b^{(l-N_2)}(z) + \cdots + J_{N_1}(l)C_b^{(l+N_1)}(z), \end{aligned} \tag{75}$$

while the dual relations are

$$\begin{aligned} z\bar{A}_b^{(l)}(z) &= J_{-N_2}(l+N_2)\bar{A}_b^{(l+N_2)}(z) + \cdots + J_{N_1}(l-N_1)\bar{A}_b^{(l-N_1)}(z), \\ z\bar{C}_a^{(l)}(z) - \bar{c}_a &= J_{-N_2}(l+N_2)\bar{C}_a^{(l+N_2)}(z) + \cdots + J_{N_1}(l-N_1)\bar{C}_a^{(l-N_1)}(z). \end{aligned} \tag{76}$$

We see that given integers $(\vec{\nu}_1, \vec{\nu}_2)$ there are several recursion relations associated with $A_{[\vec{\nu}_1;\vec{\nu}_2],a}$. In fact they are as many as different ladders exists leading to this set of degrees. For the simplest ladder, i.e. $\vec{n}_1 = \vec{\nu}_1$ and $\vec{n}_2 = \vec{\nu}_2 + \vec{e}_{2,p_2}$, we get the longest recursion, in the sense that we have more polynomials contributing in the recursion relation, as smaller are the integers in the compositions shorter is the recursion. Observe also that the multiple orthogonal polynomials involved in each case are different.

Attending to (70) we get that the recursion relations corresponding to $l = 8$ an $l = 14$ are of the form

$$\begin{aligned} zA_a^{(8)}(z) &= *A_a^{(4)}(z) + \cdots + *A_a^{(15)}(z) + A_a^{(16)}(z), \quad a = 1,2,3, \\ zA_a^{(14)}(z) &= *A_a^{(12)}(z) + \cdots + *A_a^{(18)}(z), \quad a = 1,2,3. \end{aligned}$$

We see that the first recursion has 13 terms while the second one only 7 terms.

In order to identify these polynomials with mops of the form $A_{[\vec{\nu}_1;\vec{\nu}_2,a]}$ we use the following table of degrees for the compositions $\vec{n}_1 = (4,3,2)$ and $\vec{n}_2 = (3,2)$ is

1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\vec{\nu}_1(l)$	(4,0,0)	(4,1,0)	(4,2,0)	(4,3,0)	(4,3,1)	(4,3,2)	(5,3,2)	(6,3,2)	(7,3,2)	(8,3,2)	(8,4,2)	(8,5,2)	(8,6,2)	(8,6,3)	(8,6,4)
$\vec{\nu}_2(l-1)$	(2,0)	(3,0)	(3,1)	(3,2)	(4,2)	(5,2)	(6,2)	(6,3)	(6,4)	(7,4)	(8,4)	(9,4)	(9,5)	(9,6)	(10,6)

2.7. Christoffel–Darboux type formulae

From (53) and (54) we can infer the value of the following series constructed in terms of multiple orthogonal polynomials and corresponding functions of the second kind.

Proposition 15. *The following relations hold*

$$\sum_{l=0}^{\infty}\bar{C}_a^{(l)}(z)A_{a'}^{(l)}(z')=\frac{\delta_{a,a'}}{z-z'},\quad |z'|<|z|,$$

$$\sum_{l=0}^{\infty}C_b^{(l)}(z)\bar{A}_{b'}^{(l)}(z')=\frac{\delta_{b,b'}}{z-z'},\quad |z'|<|z|,$$

$$\sum_{l=0}^{\infty}\bar{C}_a^{(l)}(z)C_b^{(l)}(z')=-\frac{\hat{\mu}_{a,b}(z)-\hat{\mu}_{a,b}(z')}{z-z'},\quad |z|,|z'|>R_{a,b},\tag{77}$$

where $R_{a,b}$ is the radius of any origin centered disk containing $\operatorname{supp}(w_{1,a}w_{a,b}\,\mathrm{d}\mu)$.

Proof. See Appendix A. □

2.7.1. Projection operators and the Christoffel–Darboux kernel

To introduce the Christoffel–Darboux kernel we need

Definition 13. We will use the following spans

$$\mathcal{H}_1^{[l]}=\mathbb{R}\{\xi_1^{(0)},\dots,\xi_1^{(l-1)}\},\qquad \mathcal{H}_2^{[l]}=\mathbb{R}\{\xi_2^{(0)},\dots,\xi_2^{(l-1)}\},\tag{78}$$

and their limits

$$\mathcal{H}_1=\left\{\sum_{0\leqslant l\ll\infty}c_l\xi_1^{(l)},\ c_l\in\mathbb{R}\right\},\qquad \mathcal{H}_2=\left\{\sum_{0\leqslant l\ll\infty}c_l\xi_2^{(l)},\ c_l\in\mathbb{R}\right\}.\tag{79}$$

The corresponding splittings

$$\mathcal{H}_1=\mathcal{H}_1^{[l]}\oplus\left(\mathcal{H}_1^{[l]}\right)^{\perp},\qquad \mathcal{H}_2=\mathcal{H}_2^{[l]}\oplus\left(\mathcal{H}_2^{[l]}\right)^{\perp},\tag{80}$$

induce the associated orthogonal projections

$$\pi_1^{(l)}:\mathcal{H}_1\to\mathcal{H}_1^{[l]},\qquad \pi_2^{(l)}:\mathcal{H}_2\to\mathcal{H}_2^{[l]}.\tag{81}$$

In the previous definition $l\ll\infty$ means that in the series there are only a finite number of nonzero contributions. It is easy to realize that

Proposition 16. *We have the following characterization of the previous linear subspaces*

$$\mathcal{H}_1^{[l]} = \mathbb{R}\{Q^{(0)}, \dots, Q^{(l-1)}\}, \qquad \mathcal{H}_2^{[l]} = \mathbb{R}\{\bar{Q}^{(0)}, \dots, \bar{Q}^{(l-1)}\},$$
$$\big(\mathcal{H}_1^{[l]}\big)^{\perp} = \Big\{ \sum_{l \leqslant j \ll \infty} c_j Q^{(j)},\ c_j \in \mathbb{R} \Big\}, \qquad \big(\mathcal{H}_2^{[l]}\big)^{\perp} = \Big\{ \sum_{l \leqslant j \ll \infty} c_j \bar{Q}^{(j)}, c_j \in \mathbb{R} \Big\}, \tag{82}$$

and

$$\mathcal{H}_1 = \Big\{ \sum_{0 \leqslant l \ll \infty} c_l Q^{(l)},\ c_l \in \mathbb{R} \Big\}, \qquad \mathcal{H}_2 = \Big\{ \sum_{0 \leqslant l \ll \infty} c_l \bar{Q}^{(l)},\ c_l \in \mathbb{R} \Big\}. \tag{83}$$

Definition 14. The Christoffel–Darboux kernel is

$$K^{[l]}(x, y) := \sum_{k=0}^{l-1} Q^{(k)}(y) \bar{Q}^{(k)}(x). \tag{84}$$

This is the kernel of the integral representation of the projections introduced in Definition 13.

Proposition 17. *The integral representation*

$$\big(\pi_1^{(l)} f\big)(y) = \int K^{[l]}(x, y) f(x)\, \mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}_1,$$
$$\big(\pi_2^{(l)} f\big)(y) = \int K^{[l]}(y, x) f(x)\, \mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}_2, \tag{85}$$

holds.

Proof. It follows from the bi-orthogonality condition (46). □

This Christoffel–Darboux kernel has the reproducing property

Proposition 18. *The kernel* $K^{[l]}(x, y)$ *fulfills*

$$K^{[l]}(x, y) = \int K^{[l]}(x, v) K^{[l]}(v, y)\, \mathrm{d}\mu(v). \tag{86}$$

Proof. From

$$f(y) = \int K^{[l]}(x, y) f(x)\, \mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}_1^{[l]},$$
$$f(y) = \int K^{[l]}(y, x) f(x)\, \mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}_2^{[l]},$$

and $K^{[l]}(x, y) \in \mathcal{H}_1^{[l]}$ as a function of y and $K^{[l]}(x, y) \in \mathcal{H}_2^{[l]}$ as a function of x we conclude the reproducing property. □

2.7.2. The ABC type theorem

We also have an ABC (Aitken–Berg–Collar) type theorem—here we follow [34]—for the Christoffel–Darboux kernel

Definition 15. The partial Christoffel–Darboux kernels are defined by

$$K^{[l]}_{b,a}(x,y) := \sum_{k=0}^{l-1} \bar{A}^{(k)}_b(x) A^{(k)}_a(y). \tag{87}$$

Observe that

$$K^{[l]}(x,y) = \sum_{\substack{a=1,\dots,p_1\\ b=1,\dots,p_2}} K^{[l]}_{b,a}(x,y) w_{1,a}(y) w_{2,b}(x). \tag{88}$$

We introduce the notation

Definition 16. Any semi-infinite vector v can be written in block form as follows

$$v = \left(\frac{v^{[l]}}{v^{[\geqslant l]}}\right), \tag{89}$$

where $v^{[l]}$ is the finite vector formed with the first l coefficients of v and $v^{[\geqslant l]}$ the semi-infinite vector formed with the remaining coefficients. This decomposition induces the following block structure for any semi-infinite matrix.

$$g = \left(\begin{array}{c|c} g^{[l]} & g^{[l,\geqslant l]} \\ \hline g^{[\geqslant l,l]} & g^{[\geqslant l]} \end{array}\right). \tag{90}$$

From (33) we get

Proposition 19. *Given a moment matrix g satisfying* (33) *we have*

$$g^{[l]} = \left(S^{[l]}\right)^{-1} \bar{S}^{[l]}, \tag{91}$$

and $(S^{-1})^{[l]} = (S^{[l]})^{-1}$, $(\bar{S}^{-1})^{[\geqslant l]} = (\bar{S}^{[\geqslant l]})^{-1}$.

Proof. Use the block structure of g, S and $\bar{S}$. □

Then, we are able to conclude the following result

Theorem 3. *The Christoffel–Darboux kernel is related to the moment matrix in the following way*

$$K^{[l]}_{b,a}(x,y) = \left(\chi^{[l]}_{2,b}(x)\right)^{\top} \left(g^{[l]}\right)^{-1} \chi^{[l]}_{1,a}(y). \tag{92}$$

Proof. The ABC theorem is a consequence of the following chain of identities

$$\begin{aligned}
K^{[l]}_{b,a}(x,y) &= \big(\Pi^{[l]}\bar{\mathscr{A}}_b(x)\big)^\top\big(\Pi^{[l]}\mathscr{A}_a(y)\big) \quad \text{the sum is over the first } l \text{ components}\\
&= \chi_{2,b}^\top(x)\bar{S}^{-1}\Pi^{[l]}S\chi_{1,a}(y) \quad \text{see (43)}\\
&= \chi_{2,b}^\top(x)\big(\Pi^{[l]}\bar{S}^{-1}\Pi^{[l]}\big)\big(\Pi^{[l]}S\Pi^{[l]}\big)\chi_{1,a}(y) \quad \text{lower and upper form of } S \text{ and } \bar{S}\\
&= \big(\chi_{2,b}^{[l]}(x)\big)^\top\big(\bar{S}^{[l]}\big)^{-1}S^{[l]}\chi_{1,a}^{[l]}(y)\\
&= \big(\chi_{2,b}^{[l]}(x)\big)^\top\big(g^{[l]}\big)^{-1}\chi_{1,a}^{[l]}(y) \quad LU \text{ factorization (33).} \qquad \square
\end{aligned}$$

We immediately deduce the

Corollary 2. *For the Christoffel–Darboux kernel we have*

$$K^{[l]}(x,y) = \big(\xi_2^{[l]}(x)\big)^\top\big(g^{[l]}\big)^{-1}\xi_1^{[l]}(y). \tag{93}$$

2.7.3. Christoffel–Darboux formula

In this subsection we derive a Christoffel–Darboux type formula from the symmetry property (67) of the moment matrix g. We need some preliminary lemmas

Lemma 3. *The relations*

$$\big(g^{[l]}\big)^{-1}\Upsilon_1^{[l]} - \big(\Upsilon_2^{[l]}\big)^\top\big(g^{[l]}\big)^{-1} = \big(g^{[l]}\big)^{-1}\big(g^{[l,\geqslant l]}\big(\Upsilon_2^{[l,\geqslant l]}\big)^\top - \Upsilon_1^{[l,\geqslant l]}g^{[\geqslant l,l]}\big)\big(g^{[l]}\big)^{-1}, \tag{94}$$

hold true.

Proof. The first block of (67) is

$$\Upsilon_1^{[l]}g^{[l]} + \Upsilon_1^{[l,\geqslant l]}g^{[\geqslant l,l]} = g^{[l]}\big(\Upsilon_2^{[l]}\big)^\top + g^{[l,\geqslant l]}\big(\Upsilon_2^{[l,\geqslant l]}\big)^\top,$$

from where the result follows immediately. $\square$

Lemma 4. *We have*

$$\Upsilon_\ell^{[l]}\xi_\ell^{[l]}(x) = x\xi_\ell^{[l]}(x) - \Upsilon_\ell^{[l,\geqslant l]}\xi_\ell^{[\geqslant l]}(x), \quad \ell = 1,2. \tag{95}$$

Proof. It follows from the block decomposition of Definition 16 and the eigen-value property of Υ_ℓ. $\square$

After a careful computation from Definition 10 we get

Lemma 5. *If we assume that* $l \geqslant \max(|\vec{n}_1|, |\vec{n}_2|)$ *we can write*

$$\Upsilon_1^{[l,\geqslant l]} = \sum_{a=1}^{p_1} e_{(l-1)_{-a}}e_{l+a-l}^\top, \qquad \Upsilon_2^{[l,\geqslant l]} = \sum_{b=1}^{p_2} e_{\overline{(l-1)}_{-b}}e_{\bar{l}+b-l}^\top. \tag{96}$$

Here $l_{\pm a}$ is the $\pm$ associated integer within the $\vec{n}_1$ composition, while $\bar{l}_{\pm b}$ is the $\pm$ associated integer for the $\vec{n}_2$ composition.

Finally, to derive a Christoffel–Darboux type formula we need the following objects

Definition 17. Associated polynomials are given by

$$\begin{aligned}
A^{(l)}_{+a,a'}(y) &:= \chi^{(l_{+a})}_{1,a'}(y) - \begin{pmatrix} g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1} \end{pmatrix} \left(g^{[l]}\right)^{-1} \chi^{[l]}_{1,a'}(y),\\
\bar{A}^{(l)}_{-a,b'}(x) &:= \left(\chi^{[l+1]}_{2,b'}(x)\right)^\top \left(g^{[l+1]}\right)^{-1} e_{l_{-a}},\\
A^{(l)}_{-b,a'}(y) &:= e^\top_{\bar{l}_{-b}} \left(g^{[l+1]}\right)^{-1} \chi^{[l+1]}_{1,a'}(y),\\
\bar{A}^{(l)}_{+b,b'}(x) &:= \left(\chi^{(\bar{l}_{+b})}_{2,b'}(x) - \left(\chi^{[l]}_{2,b'}(x)\right)^\top \left(g^{[l]}\right)^{-1} \begin{pmatrix} g_{0,\bar{l}_{+b}} \\ g_{1,\bar{l}_{+b}} \\ \vdots \\ g_{l-1,\bar{l}_{+b}} \end{pmatrix} \right),
\end{aligned} \tag{97}$$

with the corresponding linear forms given by

$$\begin{aligned}
Q^{(l)}_{+a} &:= \sum_{a'=1}^{p_1} A^{(l)}_{+a,a'} w_{1,a'}, & \bar{Q}^{(l)}_{-a} &:= \sum_{b'=1}^{p_2} \bar{A}^{(l)}_{-a,b'} w_{2,b'}, & a &= 1,\dots,p_1,\\
Q^{(l)}_{-b} &:= \sum_{a'=1}^{p_1} A^{(l)}_{-b,a'} w_{1,a'}, & \bar{Q}^{(l)}_{+b} &:= \sum_{b'=1}^{p_2} \bar{A}^{(l)}_{+b,b'} w_{2,b'}, & b &= 1,\dots,p_2.
\end{aligned} \tag{98}$$

Then, we can show that

Theorem 4. *Whenever $l \geqslant \max(|\vec{n}_1|, |\vec{n}_2|)$ the following Christoffel–Darboux type formulae*

$$\begin{aligned}
(x-y)K^{[l]}_{a',b'}(x,y) &= \sum_{b=1}^{p_2} \bar{A}^{(l)}_{+b,b'}(x) A^{(l-1)}_{-b,a'}(y) - \sum_{a=1}^{p_1} \bar{A}^{(l-1)}_{-a,b'}(x) A^{(l)}_{+a,a'}(y),\\
(x-y)K^{[l]}(x,y) &= \sum_{b=1}^{p_2} \bar{Q}^{(l)}_{+b}(x) Q^{(l-1)}_{-b}(y) - \sum_{a=1}^{p_1} \bar{Q}^{(l-1)}_{-a}(x) Q^{(l)}_{+a}(y),
\end{aligned}$$

hold.

Proof. From Lemma 3 we deduce

$$\begin{aligned}
&\left(\chi^{[l]}_{2,b'}(x)\right)^\top \left(\left(g^{[l]}\right)^{-1} \Upsilon^{[l]}_1 - \left(\Upsilon^{[l]}_2\right)^\top \left(g^{[l]}\right)^{-1}\right) \chi^{[l]}_{1,a'}(y)\\
&\quad = \left(\chi^{[l]}_{2,b'}(x)\right)^\top \left(g^{[l]}\right)^{-1} \left(g^{[l,\geqslant l]} \left(\Upsilon^{[l,\geqslant l]}_2\right)^\top - \Upsilon^{[l,\geqslant l]}_1 g^{[\geqslant l,l]}\right) \left(g^{[l]}\right)^{-1} \chi^{[l]}_{1,a'}(y),
\end{aligned}$$

so that, recalling Theorem 3, we get

$$
\begin{aligned}
(y-x)K^{[l]}_{b',a'}(x,y) &= \big(\chi^{[l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}\big(g^{[l,\geqslant l]}\big(\Upsilon_2^{[l,\geqslant l]}\big)^\top-\Upsilon_1^{[l,\geqslant l]}g^{[\geqslant l,l]}\big)\big(g^{[l]}\big)^{-1}\chi^{[l]}_{1,a'}(y)\\
&\quad+\big(\chi^{[l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}\Upsilon_1^{[l,\geqslant l]}\chi^{[\geqslant l]}_{1,a'}(y)\\
&\quad-\big(\Upsilon_2^{[l,\geqslant l]}\chi^{[\geqslant l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}\big(\chi^{[l]}_{1,a'}(y)\big),
\end{aligned}
\tag{99}
$$

or

$$
\begin{aligned}
(x-y)K^{(l-1)}_{b',a'}(x,y) &= \big(\big(\chi^{[\geqslant l]}_{2,b'}(x)\big)^\top-\big(\chi^{[l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}g^{[l,\geqslant l]}\big)\big(\Upsilon_2^{[l,\geqslant l]}\big)^\top\big(g^{[l]}\big)^{-1}\chi^{[l]}_{1,a'}(y)\\
&\quad-\big(\chi^{[l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}\Upsilon_1^{[l,\geqslant l]}\big(\chi^{[\geqslant l]}_{1,a'}(y)-g^{[\geqslant l,l]}\big(g^{[l]}\big)^{-1}\chi^{[l]}_{1,a'}(y)\big).
\end{aligned}
$$

Finally, from Lemma 5 we conclude

$$
\begin{aligned}
\Upsilon_1^{[l,\geqslant l]}\chi^{[\geqslant l]}_{1,a'}(y) &= \sum_{a=1}^{p_1} e_{(l-1)_{-a}}\chi^{(l_{+a})}_{1,a'}(y),\\
\Upsilon_1^{[l,\geqslant l]}g^{[\geqslant l,l]} &= \sum_{a=1}^{p_1} e_{(l-1)_{-a}}\begin{pmatrix} g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1}\end{pmatrix},\\
\big(\chi^{[\geqslant l]}_{2,b'}(x)\big)^\top\big(\Upsilon_2^{[l,\geqslant l]}\big)^\top &= \sum_{b=1}^{p_2}\chi^{(\bar l_{+b})}_{2,b'}(x)e^\top_{\overline{(l-1)}_{-b}},\\
g^{[l,\geqslant l]}\big(\Upsilon_2^{[l,\geqslant l]}\big)^\top &= \sum_{b=1}^{p_2}\begin{pmatrix} g_{0,\bar l_{+b}}\\ g_{1,\bar l_{+b}}\\ \vdots\\ g_{l-1,\bar l_{+b}}\end{pmatrix}e^\top_{\overline{(l-1)}_{-b}},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&(x-y)K^{[l]}_{b',a'}(x,y)\\
&\quad=\sum_{b=1}^{p_2}\left(\chi^{(\bar l_{+b})}_{2,b'}(x)-\big(\chi^{[l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}\begin{pmatrix} g_{0,\bar l_{+b}}\\ g_{1,\bar l_{+b}}\\ \vdots\\ g_{l-1,\bar l_{+b}}\end{pmatrix}\right)e^\top_{\overline{(l-1)}_{-b}}\big(g^{[l]}\big)^{-1}\chi^{[l]}_{1,a'}(y)\\
&\qquad-\sum_{a=1}^{p_1}\big(\chi^{[l]}_{2,b'}(x)\big)^\top\big(g^{[l]}\big)^{-1}e_{(l-1)_{-a}}\big(\chi^{(l_{+a})}_{1,a'}(y)\\
&\qquad-\begin{pmatrix} g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1}\end{pmatrix}\big(g^{[l]}\big)^{-1}\chi^{[l]}_{1,a'}(y)\big).
\end{aligned}
\tag{100}
$$

Recalling Definition 17 we get the announced result. □

The associated linear forms are identified with linear forms of multiple orthogonal polynomials as follows

Proposition 20. *We have the formulae*

$$
\begin{aligned}
Q^{(l)}_{+a} &= Q^{(\mathrm{II},a)}_{[\vec{\nu}_1(l-1)+\vec{e}_{1,a};\vec{\nu}_2(l-1)]}, & Q^{(l)}_{-b} &= Q^{(\mathrm{I},b)}_{[\vec{\nu}_1(l);\vec{\nu}_2(l)-\vec{e}_{2,b}]},\\
\bar{Q}^{(l)}_{+b} &= \bar{Q}^{(\mathrm{II},b)}_{[\vec{\nu}_2(l-1)+\vec{e}_{2,b};\vec{\nu}_1(l-1)]}, & \bar{Q}^{(l)}_{-a} &= Q^{(\mathrm{I},a)}_{[\vec{\nu}_2(l);\vec{\nu}_1(l)-\vec{e}_{1,a}]}.
\end{aligned}
\tag{101}
$$

Proof. See Appendix A. $\square$

Proposition 20 allows us to give the following form of the Christoffel–Darboux formula stated in Theorem 4

Proposition 21. *For $l \geqslant \max(|\vec{n}_1|, |\vec{n}_2|)$ the following*

$$
\begin{aligned}
(x-y)K^{[l]}(x,y) = {}& \sum_{b=1}^{p_2} \bar{Q}^{(\mathrm{II},b)}_{[\vec{\nu}_2(l-1)+\vec{e}_{2,b};\vec{\nu}_1(l-1)]}(x) Q^{(\mathrm{I},b)}_{[\vec{\nu}_1(l-1);\vec{\nu}_2(l-1)-\vec{e}_{2,b}]}(y)\\
&- \sum_{a=1}^{p_1} \bar{Q}^{(\mathrm{I},a)}_{[\vec{\nu}_2(l-1);\vec{\nu}_1(l-1)-\vec{e}_{1,a}]}(x) Q^{(\mathrm{II},a)}_{[\vec{\nu}_1(l-1)+\vec{e}_{1,a};\vec{\nu}_2(l-1)]}(y).
\end{aligned}
\tag{102}
$$

$$
\begin{aligned}
(x-y)K^{[l]}_{b',a'}(x,y) = {}& \sum_{b=1}^{p_2} \bar{A}^{(\mathrm{II},b)}_{[\vec{\nu}_2(l-1)+\vec{e}_{2,b};\vec{\nu}_1(l-1)],b'}(x) A^{(\mathrm{I},b)}_{[\vec{\nu}_1(l-1);\vec{\nu}_2(l-1)-\vec{e}_{2,b}],a'}(y)\\
&- \sum_{a=1}^{p_1} \bar{A}^{(\mathrm{I},a)}_{[\vec{\nu}_2(l-1);\vec{\nu}_1(l-1)-\vec{e}_{1,a}],b'}(x) A^{(\mathrm{II},a)}_{[\vec{\nu}_1(l-1)+\vec{e}_{1,a};\vec{\nu}_2(l-1)],a'}(y),
\end{aligned}
\tag{103}
$$

holds.

Relation (102) is precisely the Christoffel–Darboux formula derived in [14], the difference here is that derivation is based on the Gauss–Borel factorization problem for the moment matrix; i.e. only on algebraic arguments, and not in the Riemann–Hilbert problem found in [14], and hence the conditions on the weights are not so restrictive. However, the reader should notice that the Christoffel–Darboux kernel does not depend on the ladder determined by the composition vectors $\vec{n}_1, \vec{n}_2$, but only on the degree vectors $\vec{\nu}_1(l-1)$ and $\vec{\nu}_2(l-1)$. This was noticed in [13] for type I multiple orthogonality.

Proposition 22. *The associated polynomials introduced in Definition* 17 *have the following determinantal expressions*

$$
A^{(l)}_{+a,a'} = \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c}
g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \chi^{(0)}_{1,a'}\\
g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \chi^{(1)}_{1,a'}\\
\vdots & \vdots & & \vdots & \vdots\\
g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \chi^{(l-1)}_{1,a'}\\
\hline
g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1} & \chi^{(l_{+a})}_{1,a'}
\end{array}\right),
\tag{104}
$$

$$\bar{A}^{(l)}_{-a,b'}(x)=\frac{(-1)^{l+l_{-a}}}{\det g^{[l+1]}}\det\left(\begin{array}{ccc|c} g_{0,0} & \cdots & g_{0,l-1} & g_{0,l}\\ \vdots & & \vdots & \vdots\\ g_{l_{-a}-1,0} & \cdots & g_{l_{-a}-1,l-1} & g_{l_{-a}-1,l}\\ \hdashline g_{l_{-a}+1,0} & \cdots & g_{l_{-a}+1,l-1} & g_{l_{-a}+1,l}\\ \vdots & & \vdots & \vdots\\ g_{l-1,0} & \cdots & g_{l-1,l-1} & g_{l-1,l}\\ \hline \chi^{(0)}_{2,b'} & \cdots & \chi^{(l-1)}_{2,b'} & \chi^{(l)}_{2,b'} \end{array}\right), \tag{105}$$

$$A^{(l)}_{-b,a'}=\frac{(-1)^{l+\bar{l}_{-b}}}{\det g^{[l+1]}}\det\left(\begin{array}{ccc:ccc|c} g_{0,0} & \cdots & g_{0,\bar{l}_{-b}-1} & g_{0,\bar{l}_{-b}+1} & \cdots & g_{0,l-1} & \chi^{(0)}_{1,a'}\\ \vdots & & \vdots & \vdots & & \vdots & \vdots\\ g_{l-1,0} & \cdots & g_{l-1,\bar{l}_{-b}-1} & g_{l-1,\bar{l}_{-b}+1} & \cdots & g_{l-1,l-1} & \chi^{(l-1)}_{1,a'}\\ \hline g_{l,0} & \cdots & g_{l,\bar{l}_{-b}-1} & g_{l,\bar{l}_{-b}+1} & \cdots & g_{l,l-1} & \chi^{(l)}_{1,a'} \end{array}\right), \tag{106}$$

$$\bar{A}^{(l)}_{+b,b'}=\frac{1}{\det g^{[l]}}\det\left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,\bar{l}_{+b}}\\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,\bar{l}_{+b}}\\ \vdots & \vdots & & \vdots & \vdots\\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,\bar{l}_{+b}}\\ \hline \chi^{(0)}_{2,b'} & \chi^{(1)}_{2,b'} & \cdots & \chi^{(l-1)}_{2,b'} & \chi^{(\bar{l}_{+b})}_{2,b'} \end{array}\right). \tag{107}$$

3. Connection with the multi-component 2D Toda lattice hierarchy

In this section we introduce deformations of the Gauss–Borel factorization problem that give the connection with the theory of the integrable hierarchies of 2D Toda lattice type, in the multi-component flavor case. First, we introduce the continuous flows and then the discrete ones. Let us stress that both flows could be considered simultaneously but we consider them separately for the sake of simplicity and clearness in the exposition.

3.1. Continuous deformations of the moment matrix

Definition 18. The deformed moment matrix is given by

$$g_{\vec{n}_1,\vec{n}_2}(t):=W_{0,\vec{n}_1}(t)g\bar{W}_{0,\vec{n}_2}(t)^{-1}, \tag{108}$$

where we use the following semi-infinite matrices

$$W_{0,\vec{n}_1}(t):=\sum_{a=1}^{p_1}\exp\left(\sum_{j=1}^{\infty}t_{j,a}\Lambda^j_{1,a}\right)\in G_+,\qquad \bar{W}_{0,\vec{n}_2}(t):=\sum_{b=1}^{p_2}\exp\left(\sum_{j=1}^{\infty}\bar{t}_{j,b}\big(\Lambda^{\top}_{2,b}\big)^j\right)\in G_-$$

depending on $t=(t_{j,a},\bar{t}_{j,b})_{j,a,b}$ with $t_{j,a},\bar{t}_{j,b}\in\mathbb{R}$, $j=1,2,\dots$, $a=1,\dots,p_1$ and $b=1,\dots,p_2$.

As in the previous section and when the context is clear enough we will drop the subscripts associated with the compositions $\vec{n}_1$ and $\vec{n}_2$. The reader should notice that the following semi-infinite matrices are well defined

$$\big(W_{0,\vec{n}_1}(t)^{-1}\big)^\top=\sum_{a=1}^{p_1}\exp\left(-\sum_{j=1}^{\infty}t_{j,a}\big(\Lambda_{1,a}^\top\big)^j\right)\in G_-,$$

$$\big(\bar{W}_{0,\vec{n}_2}(t)^{-1}\big)^\top=\sum_{b=1}^{p_2}\exp\left(-\sum_{j=1}^{\infty}\bar{t}_{j,b}\Lambda_{2,b}^j\right)\in G_+.$$

This deformation preserves the structure that characterizes g as a moment matrix, in fact we have

Theorem 5. *The matrix $g(t)$ is a moment matrix with new "deformed weights" given by*

$$w_{1,a}(x,t)=\mathscr{E}_a(x,t)w_{1,a}(x),\qquad \mathscr{E}_a:=\exp\left(\sum_{j=1}^{\infty}t_{j,a}x^j\right),$$

$$w_{2,b}(x,t)=\bar{\mathscr{E}}_b(x,t)^{-1}w_{2,b}(x),\qquad \bar{\mathscr{E}}_b:=\exp\left(\sum_{j=1}^{\infty}\bar{t}_{j,b}x^j\right). \tag{109}$$

Proof. Observe that

$$W_0(t)=\sum_{j\geqslant 0}\sum_{a=1}^{p_1}\sigma_j^{(a)}(t)\Lambda_{1,a}^j,\qquad \bar{W}_0(t)^{-1}=\sum_{j\geqslant 0}\sum_{b=1}^{p_2}\big(\Lambda_{2,b}^\top\big)^j\bar{\sigma}_j^{(b)}(t),$$

where $\sigma_j^{(a)}$ is the j-th elementary Schur polynomial in the variables t_{ja} and $\bar{\sigma}_b^{(j)}$ is also an elementary Schur polynomial but now in the variables $-\bar{t}_{j,b}$. To prove (109) we first discuss the action of $\Lambda_{1,a}$ and $\Lambda_{2,b}^\top$ on g explicitly. Recalling (12) it is straightforward to see that

$$\big(\Lambda_{1,a}g\Lambda_{2,b}^\top\big)_{i,j}=\int x^{k_1(i)+1}w_{1,a_1(i)}(x)w_{2,a_2(j)}(x)x^{k_2(j)+1}\delta_{a_1(i),a}\delta_{a_2(j),b}\,\mathrm{d}\mu(x),$$

and consequently the following expression holds

$$\begin{aligned}\big(W_0g\bar{W}_0^{-1}\big)_{i,j}=\sum_{a=1}^{p_1}\sum_{b=1}^{p_2}\int x^{k_1(i)}\left(\sum_{l\geqslant 0}\sigma_l^{(a)}x^l\right)w_{1,a_1(i)}(x)w_{2,a_2(j)}(x)\\ \times\left(\sum_{m\geqslant 0}\bar{\sigma}_m^{(b)}x^m\right)x^{k_2(j)}\delta_{a_1(i),a}\delta_{a_2(j),b}\,\mathrm{d}\mu(x),\end{aligned}$$

that leads directly to (109). □

That the sign definition of the weights is preserved under deformations is ensured by the fact that all times t are real. Let us comment that these deformations could also be considered as evolutions, and from hereon we indistinctly talk about deformation/evolution. If the initial measures have bounded support then there is no problem with the exponential behavior at ∞ of the $\mathscr{E}$ factors; however, for unbounded situations a discussion is needed for each case.

The Gauss–Borel factorization problem

$$g_{\vec{n}_1,\vec{n}_2}(t) = S(t)^{-1}\bar{S}(t), \qquad S(t) \in G_-, \qquad \bar{S}(t) \in G_+, \tag{110}$$

with $S(t)$ lower triangular and $\bar{S}(t)$ upper triangular, will give the connection with integrable systems of Toda type.

Let us assume that the weights in $(\vec{w}_1, \vec{w}_2)$ are of the form (28) and that conform an M-Nikishin system, then Theorem 1 indicates that for small values of the times the new weights are also in the M-Nikishin class, ensuring that $(\vec{w}_1(t), \vec{w}_2(t), \mu)$ is a perfect system and therefore the Gauss–Borel factorization makes sense.

3.2. Lax equations and the integrable hierarchy

Let us introduce the Lax machinery associated with the Gauss–Borel factorization that will lead to a multi-component 2D Toda lattice hierarchy as described in [28]:

Definition 19. Associated with the deformed Gauss–Borel factorization we consider

1. Wave semi-infinite matrices

$$W(t) := S(t)W_0(t), \qquad \bar{W}(t) := \bar{S}(t)\bar{W}_0(t). \tag{111}$$

2. Wave

$$\Psi_a(z,t) := W(t)\chi_{1,a}(z), \qquad \bar{\Psi}_b(z,t) := \bar{W}(t)\chi^*_{2,b}(z), \tag{112}$$

   and adjoint wave semi-infinite vector functions[2]

$$\Psi^*_a(z,t) := \big(W(t)^{-1}\big)^\top \chi^*_{1,a}(z), \qquad \bar{\Psi}^*_b(z,t) := \big(\bar{W}(t)^{-1}\big)^\top \chi_{2,b}(z). \tag{113}$$

3. Lax semi-infinite matrices

$$L_a(t) := S(t)\Lambda_{1,a}S(t)^{-1}, \qquad \bar{L}_b(t) := \bar{S}(t)\Lambda^\top_{2,b}\bar{S}(t)^{-1}. \tag{114}$$

[2] In this point the reader should notice that there are two differences between this definition of wave functions (also known as Baker–Akheizer functions) and the one common in the literature, see for example [4]. Our modifications are motivated by two facts, i) we prefer $\bar{\Psi}^*_b$ to be a polynomial in z and not in z^{-1}, up to plane–wave factors, ii) we choose to have a direct connection between wave functions and Cauchy transforms of polynomials, with no z^{-1} factors multiplying the Cauchy transforms when identified with wave functions. If we denote by small ψ the wave functions corresponding to the scheme of for example [4] then we should have the following correspondence $\Psi^{(l)}_a(z) \leftrightarrow \psi^{(l)}_a(z)$, $z(\Psi^*_a)^{(l)}(z) \leftrightarrow (\psi^*_a)^{(l)}(z)$, $z^{-1}\bar{\Psi}^{(l)}_b(z^{-1}) \leftrightarrow \bar{\psi}^{(l)}_b(z)$ and $(\bar{\Psi}^*_b)^{(l)}(z^{-1}) \leftrightarrow (\bar{\psi}^*_b)^{(l)}(z)$.

4. Zakharov–Shabat semi-infinite matrices

$$B_{j,a} := \left(L_a^j\right)_+, \qquad \bar{B}_{j,b} := \left(\bar{L}_b^j\right)_-, \tag{115}$$

where the subindex $+$ indicates the projection in the upper triangular matrices while the subindex $-$ the projection in the strictly lower triangular matrices.

Observe that

$$L_a \Psi_{a'} = \delta_{a,a'} z \Psi_{a'}, \qquad \bar{L}_b^\top \bar{\Psi}_{b'}^* = \delta_{b,b'} z^{-1} \bar{\Psi}_{b'}^*. \tag{116}$$

We also mention that the matrices S and $\bar{S}$ correspond to the Sato operators (also known as gauge operators) of the integrable hierarchy we are dealing with. Some times [9] the operators L_a are referred as resolvents and the Lax name is reserved only for a convenient linear combination of the resolvents.

The reader should notice that as $S(t) \in G_-$ and $W_0(t) \in G_+$ the product $W(t) = S(t)W_0(t)$ is well defined as its coefficients are finite sums instead of series, for $(\bar{W}(t)^{-1})^\top = (\bar{S}(t)^{-1})^\top(\bar{W}_0(t)^{-1})^\top$ we can apply the previous argument and therefore the product is well defined. However, $(W(t)^{-1})^\top = (S(t)^{-1})^\top(W_0(t)^{-1})^\top$ is a product of elements which involves series instead of finite sum and its existence is not in principle ensured. The situation is reproduced with $\bar{W}(t) = \bar{S}(t)\bar{W}_0(t)$, and the existence of the product is not guaranteed. However, we notice that the simultaneous consideration of the factorization problems (110) and (33) leads $S(t)^{-1}\bar{S}(t) = W_0(t)S^{-1}\bar{S}\bar{W}_0(t)^{-1}$ that shows two products involving series, namely $W_0(t)S^{-1}$ and $\bar{S}\bar{W}_0(t)^{-1}$, but they are well defined if we assume the existence of both LU factorizations. From hereon we give for granted the existence of $\bar{W}$ and W^{-1}, and as we will see they indeed involve series, which in the convergent situation lead to Cauchy transforms.

Proposition 23. *For the wave functions we have*

$$\Psi_a^{(k)}(z,t) = A_a^{(k)}(z,t)\mathscr{E}_a(z,t), \qquad \left(\bar{\Psi}_b^*\right)^{(k)}(z,t) = \bar{A}_b^{(k)}(z,t)\bar{\mathscr{E}}_b(z,t)^{-1}, \tag{117}$$

where $A_a^{(k)}(x,t)$, $\bar{A}_b^{(k)}(x,t)$ *are the multiple orthogonal polynomials and dual polynomials* (*in the* x *variable*) *corresponding to* (109). *The evolved linear forms, associated with weights* (109), *are*

$$Q^{(k)}(x,t) := \sum_{a=1}^{p_1} A_a^{(k)}(x,t) w_{1,a}(x,t) = \sum_{a=1}^{p_1} \Psi_a^{(k)}(x,t) w_{1,a}(x), \tag{118}$$

$$\bar{Q}^{(k)}(x,t) := \sum_{b=1}^{p_2} \left(\bar{A}_b^*\right)^{(k)}(x,t) w_{2,b}(x,t) = \sum_{b=1}^{p_2} \left(\bar{\Psi}_b^*\right)^{(k)}(x,t) w_{2,b}(x), \tag{119}$$

which are bi-orthogonal polynomials of mixed type for each t

$$\int Q^{(l)}(t,x)\bar{Q}^{(k)}(t,x)\,\mathrm{d}\mu(x) = \delta_{l,k}, \quad l,k \geqslant 0, \tag{120}$$

and

$$\bar{\Psi}_b^{(k)}(z,t) = \int \frac{Q^{(k)}(x,t)}{z-x} w_{2,b}(x)\,\mathrm{d}\mu(x),$$
$$\left(\Psi_a^*\right)^{(k)}(z,t) = \int \frac{\bar{Q}^{(k)}(x,t)}{z-x} w_{1,a}(x)\,\mathrm{d}\mu(x). \tag{121}$$

Proof. From the definitions (112) and (113), and the factorization problem $Wg = \bar{W}$ we conclude

$$\bar{\Psi}_b = \bar{W}\chi_{2,b}^* = S(W_0 g)\chi_{2,b}^*, \qquad \Psi_a^* = \left(W^{-1}\right)^\top \chi_{1,a}^* = \left(\bar{S}^{-1}\right)^\top \left(g\bar{W}_0^{-1}\right)^\top \chi_{1,a}^*. \tag{122}$$

We get, in terms of the linear forms, the following identities

$$\bar{\Psi}_b^{(k)}(z,t) = \int \frac{Q^{(k)}(x,t)}{z-x} w_{2,b}(x)\,\mathrm{d}\mu(x), \qquad \left(\Psi_a^*\right)^{(k)}(z,t) = \int \frac{\bar{Q}^{(k)}(x,t)}{z-x} w_{1,a}(x)\,\mathrm{d}\mu(x),$$

where the Cauchy transforms are understood as before.[3] $\square$

We must stress in this point that these functions are not the evolved second kind functions of the linear forms

$$\bar{C}_b^{(k)}(z,t) := \int \frac{Q^{(k)}(x,t)}{z-x} w_{2,b}(x,t)\,\mathrm{d}\mu(x),$$
$$(C_a)^{(k)}(z,t) := \int \frac{\bar{Q}^{(k)}(x,t)}{z-x} w_{1,a}(x,t)\,\mathrm{d}\mu(x). \tag{123}$$

Theorem 6. *For $j, j' = 1, 2, \ldots,$ $a, a' = 1, \ldots, p_1$ and $b, b' = 1, \ldots, p_2$ the following differential relations hold*

[3] The reader should notice that there is a difference in this semi-infinite context, appropriate for the construction of multiple orthogonal polynomials, and the bi-infinite case which is the one considered in [37]. In the present context we do not have expressions, as we do have in the bi-infinite situation, of the form

$$\bar{\Psi}_b^{(k)}(z,t) = \left(P_0 + P_1 z^{-1} + \cdots\right)\exp\left(\sum_{j>0} \bar{t}_{j,b} z^j\right),$$
$$\left(\Psi_a^*\right)^{(k)}(z,t) = \left(Q_0 + Q_1 z^{-1} + \cdots\right)\exp\left(-\sum_{j>0} t_{j,a} z^j\right).$$

The reason for this issue is rooted into non-invertibility of Λ_a. Indeed, for the semi-infinite case, we have

$$\left(\Lambda_a^\top\right)^j \chi_a^* = \left[z^j \chi_a^*\right]_- \quad\Longrightarrow\quad \exp\left(\sum_{j=1}^\infty c_j \left(\Lambda_a^\top\right)^j\right)\chi_a^* = \left[\exp\left(\sum_{j=1}^\infty c_j z^j\right)\chi_a^*\right]_-$$

where the subindex $-$ stands for the negative powers in z in the Laurent expansion; while in the bi-infinite case we drop the $-$ subindex in the previous formulae.

1. *Auxiliary linear systems for the wave matrices*

$$\frac{\partial W}{\partial t_{j,a}} = B_{j,a}W, \qquad \frac{\partial W}{\partial \bar{t}_{j,b}} = \bar{B}_{j,b}W,$$
$$\frac{\partial \bar{W}}{\partial t_{j,a}} = B_{j,a}\bar{W}, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_{j,b}} = \bar{B}_{j,b}\bar{W}. \tag{124}$$

2. *Linear systems for the wave and adjoint wave semi-infinite matrices*

$$\frac{\partial \Psi_{a'}}{\partial t_{j,a}} = B_{j,a}\Psi_{a'}, \qquad \frac{\partial \Psi_{a'}}{\partial \bar{t}_{j,b}} = \bar{B}_{j,b}\Psi_{a'},$$
$$\frac{\partial \bar{\Psi}_{b'}}{\partial t_{j,a}} = B_{j,a}\bar{\Psi}_{b'}, \qquad \frac{\partial \bar{\Psi}_{b'}}{\partial \bar{t}_{j,b}} = \bar{B}_{j,b}\bar{\Psi}_{b'}, \tag{125}$$
$$\frac{\partial \Psi^*_{a'}}{\partial t_{j,a}} = -B^{\top}_{j,a}\Psi^*_{a'}, \qquad \frac{\partial \Psi^*_{a'}}{\partial \bar{t}_{j,b}} = -\bar{B}^{\top}_{j,b}\Psi^*_{a'},$$
$$\frac{\partial \bar{\Psi}^*_{b'}}{\partial t_{j,a}} = -B^{\top}_{j,a}\bar{\Psi}^*_{b'}, \qquad \frac{\partial \bar{\Psi}^*_{b'}}{\partial \bar{t}_{j,b}} = -\bar{B}^{\top}_{j,b}\bar{\Psi}^*_{b'}. \tag{126}$$

3. *Linear systems for multiple orthogonal polynomials and their duals*

$$\frac{\partial \mathscr{A}_{a'}}{\partial t_{j,a}} = \big(B_{j,a} - \delta_{a,a'}x^j\big)\mathscr{A}_{a'}, \qquad \frac{\partial \mathscr{A}_{a'}}{\partial \bar{t}_{j,b}} = (\bar{B}_{j,b})\mathscr{A}_{a'},$$
$$\frac{\partial \bar{\mathscr{A}}_{b'}}{\partial t_{j,a}} = -B^{\top}_{j,a}\bar{\mathscr{A}}_{b'}, \qquad \frac{\partial \bar{\mathscr{A}}_{b'}}{\partial \bar{t}_{j,b}} = \big(-\bar{B}^{\top}_{j,b} + \delta_{b,b'}x^j\big)\bar{\mathscr{A}}_{b'}. \tag{127}$$

4. *Lax equations*

$$\frac{\partial L_{a'}}{\partial t_{j,a}} = [B_{j,a}, L_{a'}], \qquad \frac{\partial L_{a'}}{\partial \bar{t}_{j,b}} = [\bar{B}_{j,b}, L_{a'}],$$
$$\frac{\partial \bar{L}_{b'}}{\partial t_{j,a}} = [B_{j,a}, \bar{L}_{b'}], \qquad \frac{\partial \bar{L}_{b'}}{\partial \bar{t}_{j,b}} = [\bar{B}_{j,b}, \bar{L}_{b'}]. \tag{128}$$

5. *Zakharov–Shabat equations*

$$\frac{\partial B_{j,a}}{\partial t_{j',a'}} - \frac{\partial B_{j',a'}}{\partial t_{j,a}} + [B_{j,a}, B_{j',a'}] = 0, \tag{129}$$

$$\frac{\partial \bar{B}_{j,b}}{\partial \bar{t}_{j',b'}} - \frac{\partial \bar{B}_{j',b'}}{\partial \bar{t}_{j,b}} + [\bar{B}_{j,b}, \bar{B}_{j',b'}] = 0, \tag{130}$$

$$\frac{\partial B_{j,a}}{\partial \bar{t}_{j',b'}} - \frac{\partial \bar{B}_{j',b'}}{\partial t_{j,a}} + [B_{j,a}, \bar{B}_{j',b'}] = 0. \tag{131}$$

Proof. To prove (124) we proceed as follows. In the first place we compute

$$\frac{\partial W_0}{\partial t_{j,a}} = \Lambda_{1,a}^j W_0, \qquad \frac{\partial \bar{W}_0}{\partial \bar{t}_{j,b}} = \left(\Lambda_{2,b}^\top\right)^j \bar{W}_0,$$

and in the second place we observe that

$$\frac{\partial W}{\partial t_{j,a}} = \left(\frac{\partial S}{\partial t_{j,a}} S^{-1} + L_a^j\right) W, \qquad \frac{\partial \bar{W}}{\partial t_{j,a}} = \left(\frac{\partial \bar{S}}{\partial t_{j,a}} S^{-1}\right) \bar{W}, \tag{132}$$

$$\frac{\partial W}{\partial \bar{t}_{j,b}} = \left(\frac{\partial S}{\partial \bar{t}_{j,b}} S^{-1}\right) W, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_{j,b}} = \left(\frac{\partial \bar{S}}{\partial \bar{t}_{j,b}} \bar{S}^{-1} + \bar{L}_b^j\right) \bar{W}. \tag{133}$$

Now, using the factorization problem we get

$$\begin{aligned} \frac{\partial S}{\partial t_{j,a}} S^{-1} + L_a^j &= \frac{\partial \bar{S}}{\partial t_{j,a}} \bar{S}^{-1}, \\ \frac{\partial \bar{S}}{\partial \bar{t}_{j,b}} \bar{S}^{-1} + \bar{L}_b^j &= \frac{\partial S}{\partial \bar{t}_{j,b}} S^{-1}, \end{aligned}$$

which, taking the $+$ part (upper triangular) and the $-$ part (strictly lower triangular) imply

$$\begin{aligned} \frac{\partial S}{\partial t_{j,a}} S^{-1} &= -\left(L_a^j\right)_-, & \frac{\partial \bar{S}}{\partial t_{j,a}} \bar{S}^{-1} &= \left(L_a^j\right)_+, \\ \frac{\partial \bar{S}}{\partial \bar{t}_{j,b}} \bar{S}^{-1} &= -\left(\bar{L}_b^j\right)_+, & \frac{\partial S}{\partial \bar{t}_{j,b}} S^{-1} &= \left(\bar{L}_b^j\right)_-, \end{aligned} \tag{134}$$

so using (134) into (132) and (133) with the definitions (115) we obtain (124). The linear system (125) is obtained by inserting (112) into (124).

To obtain the Lax equations (128) we take derivatives of (114)

$$\begin{aligned} \frac{\partial L_{a'}}{\partial t_{j,a}} &= \left[\frac{\partial S}{\partial t_{j,a}} S^{-1}, L_{a'}\right] = [B_{j,a}, L_{a'}], & \frac{\partial \bar{L}_{b'}}{\partial t_{j,a}} &= \left[\frac{\partial \bar{S}}{\partial t_{j,a}} \bar{S}^{-1}, \bar{L}_{b'}\right] = [B_{j,a}, \bar{L}_{b'}], \\ \frac{\partial L_{a'}}{\partial \bar{t}_{j,b}} &= \left[\frac{\partial S}{\partial \bar{t}_{j,b}} S^{-1}, L_{a'}\right] = [\bar{B}_{j,b}, L_{a'}], & \frac{\partial \bar{L}_{b'}}{\partial \bar{t}_{j,b}} &= \left[\frac{\partial \bar{S}}{\partial \bar{t}_{j,b}} \bar{S}^{-1}, \bar{L}_{b'}\right] = [\bar{B}_{j,b}, \bar{L}_{b'}]. \end{aligned}$$

Finally, (129) are obtained as compatibility conditions for (124). □

All these equations provide us with different descriptions of a multi-component integrable hierarchy of the 2D Toda lattice hierarchy type that rules the flows of the multiple orthogonal polynomials with respect to deformed weights. This integrable hierarchy is the Toda type extension of the multi-component KP hierarchy considered in [9].

3.3. Darboux–Miwa discrete flows

We complete the previously considered continuous flows with discrete flows, which we introduce through an iterated application of Darboux transformations [5].

Definition 20. Given sequences of complex numbers

$$\begin{aligned}\lambda_a &:= \{\lambda_a(n)\}_{n\in\mathbb{Z}} \subset \mathbb{C}, \quad a = 1,\dots,p_1,\\ \bar\lambda_b &:= \{\bar\lambda_b(n)\}_{n\in\mathbb{Z}} \subset \mathbb{C}, \quad b = 1,\dots,p_2,\end{aligned} \tag{135}$$

(where $\bar\lambda$ is not intended to denote the complex conjugate of λ) and two vectors, $(s_1,\dots,s_{p_1}) \in \mathbb{Z}^{p_1}$ and $(\bar s_1,\dots,\bar s_{p_2}) \in \mathbb{Z}^{p_2}$, we construct the following semi-infinite matrices

$$\begin{aligned} D_0 &:= \sum_{a=1}^{p_1} D_{0,a}, & D_{0,a} &:= \begin{cases} \prod_{n=1}^{s_a}(\Lambda_{1,a} - \lambda_a(n)\Pi_{1,a}), & s_a > 0,\\ \Pi_{1,a}, & s_a = 0,\\ \prod_{n=1}^{|s_a|}(\Lambda_{1,a} - \lambda_a(-n)\Pi_{1,a})^{-1}, & s_a < 0,\end{cases}\\ \bar D_0^{-1} &:= \sum_{b=1}^{p_2} \bigl(\bar D_0^{-1}\bigr)_b, & \bigl(\bar D_0^{-1}\bigr)_b &:= \begin{cases} \prod_{n=1}^{\bar s_b}(\Lambda_{2,b}^\top - \bar\lambda_b(n)\Pi_{2,b}), & \bar s_b > 0,\\ \Pi_{2,b}, & \bar s_b = 0,\\ \prod_{n=1}^{|\bar s_b|}(\Lambda_{2,b}^\top - \bar\lambda_b(-n)\Pi_{2,b})^{-1}, & \bar s_b < 0,\end{cases}\end{aligned} \tag{136}$$

where $s := \{s_a, \bar s_b\}_{\substack{a=1,\dots,p_1\\ b=1,\dots,p_2}}$ denotes the set of discrete times, in terms of which we define the deformed moment matrix

$$g(s) = D_0(s) g \bar D_0(s)^{-1}. \tag{137}$$

Proposition 24. *The moment matrix $g(s)$ has the same form as the moment matrix g but with new weights*

$$\begin{aligned} w_{1,a}(s,x) &= \mathscr{D}_a(x,s_a) w_{1,a}(x), & \mathscr{D}_a &:= \begin{cases} \prod_{n=1}^{s_a}(x - \lambda_a(n)), & s_a > 0,\\ 1, & s_a = 0,\\ \prod_{n=1}^{|s_a|}(x - \lambda_a(-n))^{-1}, & s_a < 0,\end{cases}\\ w_{2,b}(s,x) &= \bar{\mathscr{D}}_b(x,\bar s_b)^{-1} w_{2,b}(x), & \bar{\mathscr{D}}_b^{-1} &:= \begin{cases} \prod_{n=1}^{\bar s_b}(x - \bar\lambda_b(n)), & \bar s_b > 0,\\ 1, & \bar s_b = 0,\\ \prod_{n=1}^{|\bar s_b|}(x - \bar\lambda_b(-n))^{-1}, & \bar s_b < 0.\end{cases}\end{aligned} \tag{138}$$

Thus, the proposed discrete evolution introduces new zeroes and poles in the weights at the points defined by sequences of λ's. For example, in the a-th direction, the s_a flow in the positive direction, $s_a \to s_a + 1$, introduces a new zero at the point $\lambda_a(s_a + 1)$, while if we move in the negative direction, $s_a \to s_a - 1$, it introduces a simple pole at $\lambda_a(s_a - 1)$. Let us stress that for the time being we have not ensured the reality and positiveness/negativeness of the evolved weights, this will be considered later on.

3.3.1. Miwa transformations

Here we show that the discrete flows just introduced can be reproduced with the aid of Miwa shifts in the continuous variables.

Definition 21. We consider two types of Miwa transformations:

1. We introduce the following time shifts

$$t \to t \mp \left[z^{-1}\right]_a := \left\{ t_{j,a'} \mp \delta_{a',a} \frac{1}{j z^j},\ \bar{t}_{j,b'} \right\}_{\substack{j=1,2,\dots,\\ a'=1,\dots,p_1,\\ b'=1,\dots,p_2.}} \tag{139}$$

2. Dual time shifts are

$$t \to t \pm \overline{\left[z^{-1}\right]}_b := \left\{ t_{j,a'},\ \bar{t}_{j,b'} \pm \delta_{b',b} \frac{1}{j z^j} \right\}_{\substack{j=1,2,\dots,\\ a'=1,\dots,p_1,\\ b'=1,\dots,p_2.}} \tag{140}$$

Proposition 25. *The Miwa transformations produce the following effect on the weights*

$$\begin{aligned} w_{1,a'}\left(x, t \mp \left[z^{-1}\right]_a, s\right) &= \left(1 - \frac{x}{z}\right)^{\pm\delta_{a,a'}} w_{1,a'}(x,t,s), \\ w_{2,b'}\left(x, t \mp \left[z^{-1}\right]_a, s\right) &= w_{2,b'}(x,t,s), \end{aligned} \tag{141}$$

$$\begin{aligned} w_{1,a'}\left(x, t \pm \overline{\left[z^{-1}\right]}_b, s\right) &= w_{1,a'}(x,t,s), \\ w_{2,b'}\left(x, t \pm \overline{\left[z^{-1}\right]}_b, s\right) &= \left(1 - \frac{x}{z}\right)^{\pm\delta_{b,b'}} w_{2,b'}(x,t,s). \end{aligned} \tag{142}$$

Proof. When we consider what happens to the evolutionary factors under these shifts we find

$$\begin{aligned} \exp\left(\sum_j t_{j,a'} x^j\right) &\to \exp\left(\sum_j \left(t_{j,a'} \mp \delta_{a',a} \frac{x^j}{j z^j}\right)\right) \\ &= \left(1 - \frac{x}{z}\right)^{\mp\delta_{a',a}} \exp\left(\sum_j t_{j,a'} x^j\right), \end{aligned} \tag{143}$$

and therefore the weights transform according to

$$w_{1,a'}(x,t,s) \to \left(1 - \frac{x}{z}\right)^{\pm\delta_{a,a'}} w_{1,a'}(x,t,s), \tag{144}$$

which is like the Darboux transformations considered previously. For the dual Miwa shifts we consider what happens to the evolutionary factors under these shifts

$$\exp\Bigl(-\sum_{j,b'}\bar t_{j,b'}x^j\Bigr)\to\exp\Bigl(-\sum_{j,b'}\Bigl(\bar t_{j,b'}\pm\delta_{b',b}\frac{x^j}{jz^j}\Bigr)\Bigr)$$
$$=\Bigl(1-\frac{x}{z}\Bigr)^{\pm 1}\exp\Bigl(-\sum_{j,b'}\bar t_{j,b'}x^j\Bigr), \tag{145}$$

and the transformation for the weights is

$$w_{2,b'}(x,t,s)\to\Bigl(1-\frac{x}{z}\Bigr)^{\pm\delta_{b,b'}}w_{2,a'}(x,t,s). \qquad \square \tag{146}$$

Thus, a comparison of (138), (141) and (142) leads to

Proposition 26. *Miwa transformations and discrete flows can be identified as follows*

$$c_a w_{1,a}(x,t,s_a)=\begin{cases} w_{1,a}(x,t-\sum_{n=1}^{s_a}[\lambda_a(n)^{-1}]_a), & s_a>0,\\ w_{1,a}(x,t), & s_a=0,\\ w_{1,a}(x,t+\sum_{n=1}^{|s_a|}[\lambda_a(-n)^{-1}]_a), & s_a<0,\end{cases}$$
$$c_a:=\begin{cases}\prod_{n=1}^{s_a}(-\lambda_a(n))^{-1}, & s_a>0,\\ 1, & s_a=0,\\ \prod_{n=1}^{|s_a|}(-\lambda_a(-n)), & s_a<0,\end{cases}$$
$$\bar c_b w_{2,b}(x,t,\bar s_b)=\begin{cases} w_{2,b}(x,t+\sum_{n=1}^{\bar s_b}\overline{[\bar\lambda_b(n)^{-1}]_b},x), & \bar s_b>0,\\ w_{2,b}(x,t), & \bar s_b=0,\\ w_{2,b}(x,t-\sum_{n=1}^{|\bar s_b|}\overline{[\bar\lambda_b(-n)^{-1}]_b}), & \bar s_b<0,\end{cases}$$
$$\bar c_b:=\begin{cases}\prod_{n=1}^{\bar s_b}(-\bar\lambda_b(n))^{-1}, & \bar s_b>0,\\ 1, & \bar s_b=0,\\ \prod_{n=1}^{|\bar s_b|}(-\bar\lambda_b(-n)), & \bar s_b<0.\end{cases} \tag{147}$$

As a conclusion, the discrete flows and Miwa shifts in the continuous flows are the very same thing, and therefore we could work with continuous flows and Miwa transformations or with continuous/discrete flows. This discussion justifies the Miwa part in the name we gave to these discrete flows.

3.3.2. Bounded from below measures

Of course, in order to preserve the link with multiple orthogonal polynomials, these discrete flows must preserve the reality, regularity and sign constance of the weights, which generically is not the case. When the support of the weights is bounded from below, i.e. there are finite real numbers K_a and K_b, such that $\operatorname{supp}(w_{1,a}\,\mathrm{d}\mu)\subset[K_a,\infty)$ and $\operatorname{supp}(w_{2,b}\,\mathrm{d}\mu)\subset[\bar K_b,\infty)$, a possible solution is to place all the new zeroes and poles in the real line but outside the corresponding support, $\lambda_a(n)<\inf(\operatorname{supp}(w_{1,a}\,\mathrm{d}\mu))$ and $\bar\lambda_b(n)<\inf(\operatorname{supp}(w_{2,b}\,\mathrm{d}\mu))$. A different approach, which will be considered in Appendix B, is to arrange the zeroes in complex conjugate pairs.

To analyze the consequence of the discrete flows on the integrable hierarchy we introduce two sets of shifts operators:

Definition 22.

1. Let us consider the sets of shift operators $\{T_a\}_{a=1}^{p_1}$ and $\{\bar{T}_b\}_{b=1}^{p_2}$, where T_a stands for the shift $s_a \mapsto s_a + 1$ and $\bar{T}_b$ stands for $s_b \mapsto \bar{s}_b + 1$. The rest of the variables $\{s_{a'}, \bar{s}_{b'}\}$ will remain constant.
2. We introduce

$$\begin{aligned} q_a &:= \mathbb{I} - \Pi_{1,a}\big(\mathbb{I} + \lambda_a(s_a+1)\big) + \Lambda_{1,a}, \\ \bar{q}_b &:= \mathbb{I} - \Pi_{2,b}\big(\mathbb{I} + \bar{\lambda}_b(\bar{s}_b+1)\big) + \Lambda_{2,b}^{\top}. \end{aligned} \tag{148}$$

3. We also define the operators

$$\begin{aligned} \delta_a &:= S q_a S^{-1} = \mathbb{I} - C_a\big(\mathbb{I} + \lambda_a(s_a+1)\big) + L_a, \\ \bar{\delta}_b &:= \bar{S}\bar{q}_b\bar{S}^{-1} = \mathbb{I} - \bar{C}_b\big(\mathbb{I} + \bar{\lambda}_b(\bar{s}_b+1)\big) + \bar{L}_b, \\ C_a &:= S\Pi_{1,a}S^{-1}, \qquad \bar{C}_b := \bar{S}\Pi_{2,b}\bar{S}^{-1}. \end{aligned} \tag{149}$$

   Here the matrices δ_a and $\bar{\delta}_b$ are called lattice resolvents.
4. Finally the semi-infinite wave matrices

$$W := SD_0, \qquad \bar{W} := \bar{S}\bar{D}_0. \tag{150}$$

Observe that

$$\begin{aligned} (T_a D_0)D_0^{-1} &= q_a, \qquad \bar{D}_0^{-1}(T_a\bar{D}_0) = \mathbb{I}, \\ \bar{D}_0\big(\bar{T}_b\bar{D}_0^{-1}\big) &= \bar{q}_b, \qquad (\bar{T}_b D_0)D_0^{-1} = \mathbb{I}. \end{aligned} \tag{151}$$

When we assume that the semi-infinite matrices δ_a and $\bar{\delta}_b$ are *LU* factorizable as in (33), i.e. all their principal minors do not vanish, we can write

$$\delta_a = \delta_{a,-}^{-1}\delta_{a,+}, \qquad \bar{\delta}_b = \bar{\delta}_{b,-}^{-1}\bar{\delta}_{b,+}, \tag{152}$$

where $\delta_{a,-}$ and $\bar{\delta}_{b,-}$ are lower matrices as is S in (33), and $\delta_{a,+}$ and $\bar{\delta}_{b,+}$ are upper matrices as $\bar{S}$ in (33). We now show that when the deformed moment matrix $g(s)$ is factorizable, and therefore the multiple orthogonality makes sense, the following holds

Proposition 27. *If the deformed moment matrix $g(s)$ is factorizable for all values of s then so is δ_a and $\bar{\delta}_b$ with*

$$\begin{aligned} \delta_{a,+} &= (T_a\bar{S})\bar{S}^{-1}, \qquad \delta_{a,-} = (T_a S)S^{-1}, \\ \bar{\delta}_{b,+} &= (\bar{T}_b\bar{S})\bar{S}^{-1}, \qquad \bar{\delta}_{b,+} = (\bar{T}_b S)S^{-1}. \end{aligned} \tag{153}$$

Proof. When we apply the discrete shifts to the Gauss–Borel factorization problem $g(s) = S^{-1}(s)\bar{S}(s)$ we get

$$T_a(S^{-1})T_a(\bar{S}) = T_a g(s) = (T_a D_0)D_0^{-1}g(s) = q_a g(s) \quad \Rightarrow \quad \big((T_a S)S^{-1}\big)^{-1}(T_a\bar{S})\bar{S}^{-1} = \delta_a,$$
$$\bar{T}_b(S^{-1})\bar{T}_b(\bar{S}) = \bar{T}_b g(s) = g(s)\bar{D}_0(T_b\bar{D}_0^{-1}) = g(s)\bar{q}_b \quad \Rightarrow \quad \big((\bar{T}_b S)S^{-1}\big)^{-1}(\bar{T}_b\bar{S})\bar{S}^{-1} = \bar{\delta}_b,$$

and the desired result follows. □

Therefore, we can consider the following

Definition 23. The semi-infinite matrices ω_a and $\bar{\omega}_b$ are given by

$$\omega_a := \delta_{a,-}\delta_a = \delta_{a,+}, \qquad \bar{\omega}_b := \bar{\delta}_{b,-} = \bar{\delta}_{b,+}\bar{\delta}_b^{-1}, \tag{154}$$

and show that

Proposition 28.

1. *The following auxiliary linear systems*

$$\begin{aligned} T_a W &= \omega_a W, & T_a\bar{W} &= \omega_a\bar{W},\\ \bar{T}_b W &= \bar{\omega}_b W, & \bar{T}_b\bar{W} &= \bar{\omega}_b\bar{W}, \end{aligned} \tag{155}$$

   are satisfied.

2. *The Lax matrices fulfill the following relations*

$$\begin{aligned} T_a L_{a'} &= \omega_a L_{a'}\omega_a^{-1}, & T_a\bar{L}_b &= \omega_a\bar{L}_b\omega_a^{-1},\\ \bar{T}_b L_a &= \bar{\omega}_b L_a\bar{\omega}_b^{-1}, & \bar{T}_b\bar{L}_{b'} &= \bar{\omega}_b\bar{L}_{b'}\bar{\omega}_b^{-1}. \end{aligned} \tag{156}$$

3. *The following discrete Zakharov–Shabat compatibility conditions hold*

$$\begin{gathered} (T_a\omega_{a'})\omega_a = (T_{a'}\omega_a)\omega_{a'}, \qquad (T_a\bar{\omega}_b)\omega_a = (\bar{T}_b\omega_a)\bar{\omega}_b,\\ (\bar{T}_b\bar{\omega}_{b'})\bar{\omega}_b = (\bar{T}_{b'}\bar{\omega}_b)\bar{\omega}_{b'}. \end{gathered} \tag{157}$$

4. *When the discrete and continuous flows are considered simultaneously, the following equations*

$$\begin{aligned} T_{a'}B_{j,a} &= (\partial_{j,a}\omega_{a'})\omega_{a'}^{-1} + \omega_{a'}B_{j,a}\omega_{a'}^{-1}, & \bar{T}_b B_{j,a} &= (\partial_{j,a}\bar{\omega}_b)\bar{\omega}_b^{-1} + \bar{\omega}_b B_{j,a}\bar{\omega}_b^{-1},\\ T_a\bar{B}_{j,b} &= (\bar{\partial}_{j,b}\omega_a)\omega_a^{-1} + \omega_a\bar{B}_{j,b}\omega_a^{-1}, & \bar{T}_{b'}\bar{B}_{j,b} &= (\bar{\partial}_{j,b}\bar{\omega}_{b'})\bar{\omega}_{b'}^{-1} + \bar{\omega}_{b'}\bar{B}_{j,b}\bar{\omega}_{b'}^{-1}, \end{aligned} \tag{158}$$

   are obtained.

Proof. We compute

$$\begin{aligned}T_a W &= (T_a S)(T_a D_0) = (T_a S)S^{-1}Sq_a S^{-1}SD_0 = \delta_{a,-}\delta_a W = \delta_{a,+}W,\\ T_a\bar W &= (T_a\bar S)\bar D_0 = (T_a\bar S)\bar S^{-1}\bar S\bar D_0 = \delta_{a,+}\bar W,\\ \bar T_b W &= (\bar T_b S)D_0 = (\bar T_b S)S^{-1}SD_0 = \bar\delta_{b,-}W,\\ \bar T_b\bar W &= (\bar T_b\bar S)(T_b\bar D_0) = (T_b\bar S)\bar S^{-1}\bar S\bar q_b^{-1}\bar S^{-1}\bar S\bar D_0 = \bar\delta_{b,+}\bar\delta_b^{-1}\bar W = \bar\delta_{b,-}\bar W,\end{aligned}$$

from where we deduce (155), which in turn imply (156) and (157). □

The simultaneous consideration of continuous and discrete flows leads to the replacement $W_0 \to W_0 D_0$ and $\bar W_0 \to \bar W_0\bar D_0$, and the corresponding modification of the weight's flows is achieved by the multiplication of the continuous and discrete evolutionary factors, in this context we also have (158). These discrete flows could be understood as a sequence of Darboux transformations of LU and UL types in the terminology of [5], which motivates the Darboux part in name we give to these discrete flows. In fact, we have that the lattice resolvents satisfy

$$\begin{aligned}\delta_a &= \delta_{a,-}^{-1}\delta_{a,+} \quad\Rightarrow\quad T_a\delta_a = \omega_a\delta_a\omega_a^{-1} = \delta_{a,+}\delta_{a,-}^{-1}\delta_{a,+}\delta_{a,+}^{-1} = \delta_{a,+}\delta_{a,-}^{-1},\\ \bar\delta_b &= \bar\delta_{b,-}^{-1}\bar\delta_{b,+} \quad\Rightarrow\quad \bar T_b\bar\delta_b = \bar\omega_b\bar\delta_b\bar\omega_b^{-1} = \bar\delta_{b,-}\bar\delta_{b,-}^{-1}\bar\delta_{b,+}\bar\delta_{b,-}^{-1} = \bar\delta_{b,+}\bar\delta_{b,-}^{-1},\end{aligned}$$

which amounts to the typical permutation of the LU factorization to the UL factorization. When there is only one component we have $\delta = L + \lambda$ and $\bar\delta = \bar\lambda + \bar L$ and the shift corresponds to the classical LU or UL Darboux transformations.

If $A_a^{(k)}(x,s)$, $\bar A_b^{(k)}(x,s)$ are the multiple orthogonal polynomials and dual polynomials in the x variable corresponding to the discrete evolution of the weights (138) respectively we have the discrete version of Proposition 23

Proposition 29. *The wave and adjoint wave functions* (150) *are*

$$\Psi_a^{(k)}(z,s) = A_a^{(k)}(z,s)\mathscr{D}_a(z,s_a), \qquad \big(\bar\Psi_b^*\big)^{(k)}(z,s) = \bar A_b^{(k)}(z,s)\bar{\mathscr{D}}_b(z,\bar s_b)^{-1}, \tag{159}$$

and the linear forms

$$Q^{(k)}(x,s) = \sum_{a=1}^{p_1} A_a^{(k)}(x,s)w_{1,a}(x,s), \qquad \bar Q^{(k)}(x,s) = \sum_{b=1}^{p_2}(\bar A_b)^{(k)}(x,s)w_{2,b}(x,s), \tag{160}$$

associated with the weights $w_{1,a}(x,s)$, $w_{2,b}(x,s)$, can be expressed as

$$Q^{(k)}(x,s) := \sum_{a=1}^{p_1}\Psi_a^{(k)}(x,s)w_{1,a}(x), \qquad \bar Q^{(k)}(x,s) := \sum_{b=1}^{p_2}\big(\bar\Psi_b^*\big)^{(k)}(x,s)w_{2,b}(x), \tag{161}$$

in terms of which we have the equations

$$\bar{\Psi}_b^{(k)}(z,s) = \int \frac{Q^{(k)}(x,s)}{z-x} w_{2,b}(x)\,\mathrm{d}\mu(x),$$
$$\left(\Psi_a^*\right)^{(k)}(z,s) = \int \frac{\bar{Q}^{(k)}(x,s)}{z-x} w_{1,a}(x)\,\mathrm{d}\mu(x). \tag{162}$$

Here the Cauchy transforms must be interpreted in exactly the same terms as in Proposition 8. Observe that (162) do not correspond to the functions of the second kind

$$\bar{C}_b^{(k)}(z,s) := \int \frac{Q^{(k)}(x,s)}{z-x} w_{2,b}(x,s)\,\mathrm{d}\mu(x),$$
$$C_a^{(k)}(z,s) := \int \frac{\bar{Q}^{(k)}(x,s)}{z-x} w_{1,a}(x,s)\,\mathrm{d}\mu(x). \tag{163}$$

Notice also that from (149)–(152), relations that hold true for any g and not only for the moment matrix, we get

Lemma 6. *We have that*

$$\omega_a = \omega_{a,0}\Lambda^{|\vec{n}_1|-n_{1,a}+1} + \omega_{a,1}\Lambda^{|\vec{n}_1|-n_{1,a}} + \cdots + \omega_{a,|\vec{n}_1|-n_{1,a}+1},$$
$$\bar{\omega}_b = \bar{\omega}_{b,0}\left(\Lambda^\top\right)^{|\vec{n}_2|-n_{2,b}+1} + \bar{\omega}_{b,1}\left(\Lambda^\top\right)^{|\vec{n}_2|-n_{2,b}} + \cdots + \bar{\omega}_{b,|\vec{n}_2|-n_{2,b}+1},$$
$$\omega_a^\top = \rho_{a,0}\left(\Lambda^\top\right)^{|\vec{n}_1|-n_{1,a}+1} + \rho_{a,1}\left(\Lambda^\top\right)^{|\vec{n}_1|-n_{1,a}} + \cdots + \rho_{a,|\vec{n}_1|-n_{1,a}+1},$$
$$\bar{\omega}_b^\top = \bar{\rho}_{b,0}\Lambda^{|\vec{n}_2|-n_{2,b}+1} + \bar{\rho}_{b,1}\Lambda^{|\vec{n}_2|-n_{2,b}} + \cdots + \rho_{b,|\vec{n}_2|-n_{2,b}+1}, \tag{164}$$

for some diagonal semi-infinite matrices

$$\omega_{a,j} = \operatorname{diag}\left(\omega_{a,j}(0), \omega_{a,j}(1), \ldots\right),$$
$$\bar{\omega}_{b,j} = \operatorname{diag}\left(\bar{\omega}_{b,j}(0), \bar{\omega}_{b,j}(1), \ldots\right),$$
$$\rho_{a,j} = \operatorname{diag}\left(\rho_{a,j}(0), \rho_{a,j}(1), \ldots\right),$$
$$\bar{\rho}_{b,j} = \operatorname{diag}\left(\bar{\rho}_{b,j}(0), \bar{\rho}_{b,j}(1), \ldots\right), \tag{165}$$

with

$$\rho_{a,j}(k) := \omega_{a,j}(k - |\vec{n}_1| + n_{1,a} - 1 + j),$$
$$\bar{\rho}_{b,j}(k) := \bar{\omega}_{b,j}(k + |\vec{n}_2| - n_{2,b} + 1 - j), \tag{166}$$

that with

Definition 24. We define

$$\gamma_{a,a'}(s,x) := \left(1 - \delta_{a,a'}\left(1 + \lambda_a(s_a+1) - x\right)\right),$$
$$\gamma_{b,b'}(s,x) := \left(1 - \delta_{b,b'}\left(1 + \bar{\lambda}_b(\bar{s}_b+1) - x\right)\right), \tag{167}$$

leads to

Proposition 30. *The following equations*

$$\begin{aligned}\big(T_{a'}A_a^{(k)}\big)\gamma_{a,a'} &= \omega_{a',0}(k)A_a^{(k+|\vec n_1|-n_{1,a'}+1)}+\cdots+\omega_{a',|\vec n_1|-n_{1,a'}+1}(k)A_a^{(k)},\\ \bar T_{b'}A_a^{(k)} &= \bar\omega_{b',0}(k)A_a^{(k-|\vec n_2|+n_{2,b'}-1)}+\cdots+\bar\omega_{b',|\vec n_2|-n_{2,b'}+1}(k)A_a^{(k)},\end{aligned}\tag{168}$$

$$\begin{aligned}\rho_{a',0}(k)\big(T_{a'}\bar A_b^{(k-|\vec n_1|+n_{1,a'}-1)}\big)+\cdots+\rho_{a',|\vec n_1|-n_{1,a'}+1}(k)\big(T_{a'}\bar A_b^{(k)}\big) &= \bar A_b^{(k)},\\ \big(\bar\rho_{b',0}(k)\big(\bar T_{b'}\bar A_b^{(k+|\vec n_2|-n_{2,b'}+1)}\big)+\cdots+\bar\rho_{b',|\vec n_2|-n_{2,b'}+1}(k)\big(\bar T_{b'}\bar A_b^{(k)}\big)\big)\bar\gamma_{b,b'} &= \bar A_b^{(k)},\end{aligned}\tag{169}$$

are fulfilled.

Proof. For (168) recall the discrete auxiliary systems for W, while for (169) just consider that

$$\omega_a^\top T_a\big(\big(\bar W^{-1}\big)^\top\big)=\big(\bar W^{-1}\big)^\top,\qquad \bar\omega_b^\top \bar T_b\big(\big(\bar W^{-1}\big)^\top\big)=\big(\bar W^{-1}\big)^\top.\qquad\square$$

Notice that relations (168) and (169) are among multiple orthogonal polynomials in the same ladder but with different weights, they link the polynomials for the weights $w_{1,a}$, $w_{2,b}$ with those with $T_{a'}w_{1,a}$, $T_{a'}w_{2,b}$ or $\bar T_{b'}w_{1,a}$, $\bar T_{b'}w_{2,b}$.

3.4. Symmetries, recursion relations and string equations

We now return to the discussion of the symmetry of the moment matrix that we started in Section 2.6 but with evolved weights and the use of Lax matrices. The first observation is the following

Proposition 31. *The j-th power of the evolved Jacobi type matrix introduced in Section* 2.6 *is related with Lax matrices through what we call a string equation*:

$$J^j=\sum_{a=1}^{p_1}L_a^j=\sum_{b=1}^{p_2}\bar L_b^j,\quad j=1,2,\ldots,\tag{170}$$

and the multiple orthogonal polynomials are eigen-vectors:

$$J^j\mathscr A_{a'}=x^j\mathscr A_{a'},\qquad \big(J^j\big)^\top\bar{\mathscr A}_{b'}=x^j\bar{\mathscr A}_{b'},\tag{171}$$

for $a'=1,\ldots,p_1$ and $b'=1,\ldots,p_2$.

Proof. Using (68) it can be proven by induction on j that for any $j\geqslant 1$ the following equation holds

$$\Lambda_{1,a}^j g\Pi_{2,b}=\Pi_{1,a}g\big(\Lambda_{2,b}^\top\big)^j,\tag{172}$$

so that

$$L_a^j\bar C_b=C_a\bar L_b^j.\tag{173}$$

Summing over a, b we deduce (170). Moreover (171) is obtained as follows

$$J^j \mathscr{A}_{a'} = S \sum_{a=1}^{p_1} \Lambda_{1,a}^j S^{-1} S \chi_{1,a'} = x^j \mathscr{A}_{a'}, \tag{174}$$

$$\left(J^j\right)^\top \bar{\mathscr{A}}_{a'} = \left(\bar{S}^{-1}\right)^\top \sum_{b=1}^{p_2} \Lambda_{2,a}^j \bar{S}^\top \left(\bar{S}^{-1}\right)^\top \chi_{2,b'} = x^j \bar{\mathscr{A}}_{b'}. \qquad \square \tag{175}$$

We are ready to show that the symmetry (68) induces a corresponding invariance on Lax matrices and multiple orthogonal polynomials

Proposition 32. *The following relations hold for $j = 1, 2, \ldots$*

$$\left(\sum_{a=1}^{p_1} \frac{\partial}{\partial t_{j,a}} + \sum_{b=1}^{p_2} \frac{\partial}{\partial \bar{t}_{j,b}}\right) L_{a'} = 0, \qquad \left(\sum_{a=1}^{p_1} \frac{\partial}{\partial t_{j,a}} + \sum_{b=1}^{p_2} \frac{\partial}{\partial \bar{t}_{j,b}}\right) \bar{L}_{b'} = 0, \tag{176}$$

$$\left(\sum_{a=1}^{p_1} \frac{\partial}{\partial t_{j,a}} + \sum_{b=1}^{p_2} \frac{\partial}{\partial \bar{t}_{j,b}}\right) \mathscr{A}_{a'} = 0, \qquad \left(\sum_{a=1}^{p_1} \frac{\partial}{\partial t_{j,a}} + \sum_{b=1}^{p_2} \frac{\partial}{\partial \bar{t}_{j,b}}\right) \bar{\mathscr{A}}_{b'} = 0. \tag{177}$$

Proof. See Appendix A. $\square$

3.5. Bilinear equations and τ-functions

The proof of the bilinear identity needs three lemmas. For the first one, let $W_{\vec{n}_1,\vec{n}_2}, \bar{W}_{\vec{n}_1,\vec{n}_2}$ be the wave matrices associated with the moment matrix $g_{\vec{n}_1,\vec{n}_2}$; so that, $W_{\vec{n}_1,\vec{n}_2} g_{\vec{n}_1,\vec{n}_2} = \bar{W}_{\vec{n}_1,\vec{n}_2}$. Then, we have

Lemma 7. *The wave matrices associated with different compositions and times satisfy*

$$W_{\vec{n}_1,\vec{n}_2}(t,s) \pi_{\vec{n}_1',\vec{n}_1}^\top W_{\vec{n}_1',\vec{n}_2'}\left(t',s'\right)^{-1} = \bar{W}_{\vec{n}_1,\vec{n}_2}(t,s) \pi_{\vec{n}_2',\vec{n}_2}^\top \bar{W}_{\vec{n}_1',\vec{n}_2'}\left(t',s'\right)^{-1}. \tag{178}$$

Proof. We consider simultaneously the following equations

$$W_{\vec{n}_1,\vec{n}_2}(t,s) g = \bar{W}_{\vec{n}_1,\vec{n}_2}(t,s),$$
$$W_{\vec{n}_1',\vec{n}_2'}\left(t',s'\right) \pi_{\vec{n}_1',\vec{n}_1} g \pi_{\vec{n}_2',\vec{n}_2}^\top = \bar{W}_{\vec{n}_1',\vec{n}_2'}\left(t',s'\right),$$

where $g = g_{\vec{n}_1,\vec{n}_2}$, and we get

$$W_{\vec{n}_1,\vec{n}_2}(t,s)^{-1} \bar{W}_{\vec{n}_1,\vec{n}_2}(t,s) = \pi_{\vec{n}_1',\vec{n}_1}^\top W_{\vec{n}_1',\vec{n}_2'}\left(t',s'\right)^{-1} \bar{W}_{\vec{n}_1',\vec{n}_2'}\left(t',s'\right) \pi_{\vec{n}_2',\vec{n}_2} = g,$$

and the result becomes evident. $\square$

For the second one, let $(\cdot)_{-1}$ denote the coefficient in z^{-1} in the Laurent expansion around $z = \infty$ (place where the Cauchy transforms make sense).

Lemma 8. *For the vectors χ_a the following formulae hold*

$$\left(\sum_{a=1}^{p}\chi_a\big(\chi_a^*\big)^\top\right)_{-1}=\left(\sum_{a=1}^{p}\chi_a^*\chi_a^\top\right)_{-1}=\mathbb{I},$$

and therefore

Lemma 9. *For any couple of semi-infinite matrices U and V we have*

$$UV=\left(\sum_{a=1}^{p_1}(U\chi_{1,a})\big(V^\top\chi_{1,a}^*\big)^\top\right)_{-1} \tag{179}$$

$$=\left(\sum_{b=1}^{p_2}\big(U\chi_{2,b}^*\big)\big(V^\top\chi_{2,b}\big)^\top\right)_{-1}. \tag{180}$$

Proof. It follows easily from Lemma 8:

$$\left(\sum_{a=1}^{p_1}(U\chi_{1,a})\big(V^\top\chi_{1,a}^*\big)^\top\right)_{-1}=U\left(\sum_{a=1}^{p_1}\chi_{1,a}\big(\chi_{1,a}^*\big)^\top\right)_{-1}V=UV,$$

$$\left(\sum_{b=1}^{p_2}\big(U\chi_{2,b}^*\big)\big(V^\top\chi_{2,b}\big)^\top\right)_{-1}=U\left(\sum_{b=1}^{p_2}\chi_{2,b}^*\chi_{2,b}^\top\right)_{-1}V=UV. \qquad \square$$

We have the following

Theorem 7.

1. *The wave functions and their companions satisfy*

$$\sum_{a=1}^{p_1}\oint_\infty \Psi_{\vec n_1,\vec n_2,a}^{(k)}(z,t,s)\big(\Psi_{\vec n_1',\vec n_2',a}^*\big)^{(l)}\big(z,t',s'\big)\,\mathrm{d}z$$

$$=\sum_{b=1}^{p_2}\oint_\infty \bar\Psi_{\vec n_1,\vec n_2,b}^{(k)}(z,t,s)\big(\bar\Psi_{\vec n_1',\vec n_2',b}^*\big)^{(l)}\big(z,t',s'\big)\,\mathrm{d}z.$$

2. *Multiple orthogonal polynomials, their duals and the corresponding second kind functions are linked by*

$$\sum_{a=1}^{p_1}\oint_\infty A_{\vec n_1,\vec n_2,a}^{(k)}(z,t,s)\bar C_{\vec n_1',\vec n_2',a}^{(l)}\big(z,t',s'\big)E_a(z)\,\mathrm{d}z$$

$$=\sum_{b=1}^{p_2}\oint_\infty C_{\vec n_1,\vec n_2,b}^{(k)}(z,t,s)\bar A_{\vec n_1',\vec n_2',b}^{(l)}\big(z,t',s'\big)\bar E_b(z)\,\mathrm{d}z, \tag{181}$$

where

$$E_a := (\mathscr{E}_a\mathscr{D}_a)(z,t,s)\big((\mathscr{E}_a\mathscr{D}_a)\big(z,t',s'\big)\big)^{-1}, \qquad \bar{E}_b := (\bar{\mathscr{E}}_b\bar{\mathscr{D}}_b)(z,t,s)\big((\bar{\mathscr{E}}_b\bar{\mathscr{D}}_b)\big(z,t',s'\big)\big)^{-1}.$$

Proof. 1. If we set in (179) $U = W_{\vec{n}_1,\vec{n}_2}(t,s)$ and $V = \pi^\top_{\vec{n}'_1,\vec{n}_1} W_{\vec{n}'_1,\vec{n}'_2}(t',s')^{-1}$ and in (180) we put $U = \bar{W}_{\vec{n}_1,\vec{n}_2}(t,s)$ and $V = \pi^\top_{\vec{n}'_2,\vec{n}_2}\bar{W}_{\vec{n}'_1,\vec{n}'_2}(t',s')^{-1}$ attending to (178), recalling that $\Psi_{\vec{n}_1,\vec{n}_2,a} = W_{\vec{n}_1,\vec{n}_2}\chi_{\vec{n}_1,a}$, $\bar{\Psi}_{\vec{n}_1,\vec{n}_2,b} = \bar{W}_{\vec{n}_1,\vec{n}_2}\chi^*_{\vec{n}_2,b}$ and observing that $\Psi^*_{\vec{n}'_1,\vec{n}'_2,a} = (W^{-1}_{\vec{n}'_1,\vec{n}'_2})^\top \pi_{\vec{n}'_1,\vec{n}_1}\chi^*_{\vec{n}_1,a}$ and $\bar{\Psi}^*_{\vec{n}'_1,\vec{n}'_2,b} = (\bar{W}^{-1}_{\vec{n}'_1,\vec{n}'_2})^\top\pi_{\vec{n}'_2,\vec{n}_2}\chi_{\vec{n}_2,b}$ we get the stated bilinear equation for the wave functions.[4]

2. We can write

$$\begin{aligned} &W_{\vec{n}_1,\vec{n}_2}(t,s)\pi^\top_{\vec{n}'_1,\vec{n}_1} W_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)^{-1}\\ &\quad= \big(S_{\vec{n}_1,\vec{n}_2}(t,s)W_{0,\vec{n}_1}(t,s)\pi^\top_{\vec{n}'_1,\vec{n}_1}\big(W_{0,\vec{n}'_1}\big(t',s'\big)\big)^{-1}\pi_{\vec{n}'_1,\vec{n}_1}\big)\pi^\top_{\vec{n}'_1,\vec{n}_1}S_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)^{-1}, \end{aligned}$$

which strongly suggests to consider in (179)

$$U = S_{\vec{n}_1,\vec{n}_2}(t,s)W_{0,\vec{n}_1}(t,s)\pi^\top_{\vec{n}'_1,\vec{n}_1}\big(W_{0,\vec{n}'_1}\big(t',s'\big)\big)^{-1}\pi_{\vec{n}'_1,\vec{n}_1}, \qquad V = \pi^\top_{\vec{n}'_1,\vec{n}_1}S_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)^{-1}.$$

Analogously

$$\begin{aligned} &\bar{W}_{\vec{n}_1,\vec{n}_2}(t,s)\pi^\top_{\vec{n}'_1,\vec{n}_1} \bar{W}_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)^{-1}\\ &\quad= \big(\bar{S}_{\vec{n}_1,\vec{n}_2}(t,s)\bar{W}_{0,\vec{n}_1}(t,s)\pi^\top_{\vec{n}'_1,\vec{n}_1}\big(\bar{W}_{0,\vec{n}'_1}\big(t',s'\big)\big)^{-1}\pi_{\vec{n}'_1,\vec{n}_1}\big)\pi^\top_{\vec{n}'_1,\vec{n}_1}\bar{S}_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)^{-1}, \end{aligned}$$

suggest to set in (180)

$$U = \bar{S}_{\vec{n}_1,\vec{n}_2}(t,s)\bar{W}_{0,\vec{n}_2}(t,s)\pi^\top_{\vec{n}'_2,\vec{n}_2}\big(\bar{W}_{0,\vec{n}'_2}\big(t',s'\big)\big)^{-1}\pi_{\vec{n}'_2,\vec{n}_2}, \qquad V = \pi^\top_{\vec{n}'_2,\vec{n}_2}\bar{S}_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)^{-1}.$$

The application of (179), (180) and (178) gives the alternative bilinear relations (181) where we have used the evolved Cauchy transforms (123) and introduce the evolutionary factors

$$\begin{aligned} E_a &:= (\mathscr{E}_a\mathscr{D}_a)(z,t,s)\big((\mathscr{E}_a\mathscr{D}_a)\big(z,t',s'\big)\big)^{-1},\\ \bar{E}_b &:= (\bar{\mathscr{E}}_b\bar{\mathscr{D}}_b)(z,t,s)\big((\bar{\mathscr{E}}_b\bar{\mathscr{D}}_b)\big(z,t',s'\big)\big)^{-1}. \qquad \Box \end{aligned}$$

The factors involved in this definition were introduced in (109) and (138), so that we assume the discrete flows within the bounded from below support scenario, while if we consider the two-step discrete flows the replacement of the $\mathscr{D}$-factors by the $\mathscr{D}'$-factors (220) is required.

[4] The reader familiarized with Toda bilinear equations should notice that in the right hand term we are working at $z=\infty$ instead of, as customary, at $z=0$; the reason is that for the definition of $\bar{\mathscr{A}}_b$ we have used χ_2 instead of χ_2^*, in order to get polynomials in z, while normally one gets polynomials in z^{-1}. See footnote 2. Moreover, due to the redefinition of the wave functions there is no $\frac{\mathrm{d}z}{2\pi\mathrm{i}z}$ factor.

It can be shown that for certain weights, for which we can use the Fubini and Cauchy theorems, and when one only considers a finite number of continuous flows that the r.h.s and l.h.s. in this bilinear relations are proportional to $\int_{\mathbb{R}} Q^{(k)}_{\vec{n}_1,\vec{n}_2}(x,t)\bar{Q}^{(l)}_{\vec{n}'_1,\vec{n}'_2}(x,t')\,\mathrm{d}\mu(x)$. This is a direct consequence of

Proposition 33. *We have the following identity*

$$\int Q_{\vec{n}_1,\vec{n}_2}(x,t,s)\bar{Q}^{\top}_{\vec{n}'_1,\vec{n}'_2}\big(x,t',s'\big)\,\mathrm{d}\mu(x) = W_{\vec{n}_1,\vec{n}_2}(t,s)\pi^{\top}_{\vec{n}'_1,\vec{n}_1}\big(W_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)\big)^{-1} \tag{182}$$

$$= \bar{W}_{\vec{n}_1,\vec{n}_2}(t,s)\pi^{\top}_{\vec{n}'_2,\vec{n}_2}\big(\bar{W}_{\vec{n}'_1,\vec{n}'_2}\big(t',s'\big)\big)^{-1}. \tag{183}$$

Proof. See Appendix A. □

Now, we will perform a full characterization of the τ-functions associated with the multiple orthogonal polynomials defined in this paper.

Definition 25. Let us define the following matrices

$$g^{[l+1]}_{+a} := \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l} \\ \hdashline g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1} & g_{l_{+a},l} \end{pmatrix},$$

$$\bar{g}^{[l+1]}_{+b} := \left(\begin{array}{cccc:c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,\bar{l}_{+b}} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,\bar{l}_{+b}} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,\bar{l}_{+b}} \\ g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & g_{l,\bar{l}_{+b}} \end{array}\right). \tag{184}$$

The matrix $g^{[l+1]}_{+a}$ is obtained from $g^{[l+1]}$ replacing the last row (operation denoted by a dashed line) by

$$(g_{l_{+a},0}, g_{l_{+a},1}, \ldots, g_{l_{+a},l-1}, g_{l_{+a},l}),$$

and $\bar{g}^{[l+1]}_{+b}$ is obtained from $g^{[l+1]}$ replacing the last column by $(g_{0,\bar{l}_{+b}}, g_{1,\bar{l}_{+b}}, \ldots, g_{l-1,\bar{l}_{+b}}, g_{l,\bar{l}_{+b}})^{\top}$. It is clear that if $a_1(l)=a$ then $g^{[l+1]}_{+a}=g^{[l+1]}$ and if $a_2(l)=b$ then $\bar{g}^{[l+1]}_{+b}=g^{[l+1]}$.

The minors of the these matrices (184) will be denoted as $M^{[l+1]}_{i,j}=\bar{M}^{[l+1]}_{i,j}$ for $g^{[l+1]}$, $M^{[l+1]}_{+a,i,j}$ for $g^{[l+1]}_{+a}$ and $\bar{M}^{[l+1]}_{+b,i,j}$ for $\bar{g}^{[l+1]}_{+b}$. Now we introduce the following determinants that are cofactors of the previously defined matrices

Definition 26. The τ-functions are defined as follows

$$\tau^{(l)}_{+a,-a'} := (-1)^{l+l_{-a'}} M^{[l+1]}_{+a,l_{-a'},l}, \qquad \tau^{(l)}_{-b,-a} := (-1)^{\bar{l}_{-b}+l_{-a}} M^{[l+1]}_{l_{-a},\bar{l}_{-b}}, \tag{185}$$

$$\bar{\tau}^{(l)}_{+b,-b'} := (-1)^{l+\bar{l}_{-b'}} \bar{M}^{[l+1]}_{+b,l,\bar{l}_{-b'}}, \qquad \bar{\tau}^{(l)}_{-a,-b} := (-1)^{l_{-a}+\bar{l}_{-b}} \bar{M}^{[l+1]}_{l_{-a},\bar{l}_{-b}}. \tag{186}$$

Moreover,

1. If $a_1(l) = a$ then we denote $\tau^{(l)}_{-a'} := \tau^{(l)}_{+a,-a'}$ and $\bar{\tau}^{(l)}_{-b} := \bar{\tau}^{(l)}_{-a,-b}$.
2. If $a_2(l) = b$ then we denote $\tau^{(l)}_{-a} := \tau^{(l)}_{-b,-a}$ and $\bar{\tau}^{(l)}_{-b'} := \bar{\tau}^{(l)}_{+b,-b'}$.
3. We also introduce $\tau^{(l)} = \bar{\tau}^{(l)} := \det g^{[l]}$ and

$$\tau^{(l+1)}_{+a} := \det g^{[l+1]}_{+a}, \qquad \bar{\tau}^{(l+1)}_{+b} := \det \bar{g}^{[l+1]}_{+b}.$$

If $a_1(l) = a$ then $\tau^{(l+1)}_{+a} = \tau^{(l+1)}$, and if $a_2(l) = b$ then $\bar{\tau}^{(l+1)}_{+b} = \tau^{(l+1)}$.

Given a perfect combination $(\mu, \vec{w}_1, \vec{w}_2)$ and the corresponding set of multiple orthogonal polynomials $\{A_{[\vec{\nu}_1;\vec{\nu}_2],a}\}_{a=1}^{p_1}$, with degree vectors such that $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$, there exist a $(\vec{n}_1, \vec{n}_2)$ ladder and an integer l with $|\vec{\nu}_1| = l + 1$ and $|\vec{\nu}_2| = l$ such that the polynomials $\{A^{(l)}_a\}_{a=1}^{p_1}$ coincide with $\{A_{[\vec{\nu}_1;\vec{\nu}_2],a}\}_{a=1}^{p_1}$. The final result does not depend upon the particular $(\vec{n}_1, \vec{n}_2)$ ladder we choose to get up to the given degrees in the ladder; however, the τ-functions do indeed depend on the ladder chosen through a global sign. A simple sign-fixing rule is to choose the ladder $\vec{n}_1 = \vec{\nu}_1$ and $\vec{n}_2 = \vec{\nu}_2 + \vec{e}_{p_2}$. We define

$$\tau_{[\vec{\nu}_1;\vec{\nu}_2]} := \tau^{(l)}_{\vec{\nu}_1,\vec{\nu}_2}, \qquad l = |\vec{\nu}_1| - 1 = |\vec{\nu}_2|,$$

and we deduce

Proposition 34. *Given degree vectors $(\vec{\nu}_1, \vec{\nu}_2)$ such that $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$, a composition with $\vec{n}_1 = \vec{\nu}_1$ and $\vec{n}_2 = \vec{\nu}_2 + \vec{e}_{p_2}$ and $l = |\vec{\nu}_1| - 1 = |\vec{\nu}_2|$, we have the following identities*

$$\tau^{(l)}_{+a,-a'} = \varepsilon_{1,1}\big(a, a'\big)\tau_{[\vec{\nu}_1-\vec{e}_{1,a'}+\vec{e}_{1,a};\vec{\nu}_2]}, \qquad \bar{\tau}^{(l)}_{+b,-b'} = \varepsilon_{2,2}\big(b, b'\big)\tau_{[\vec{\nu}_1;\vec{\nu}_2-\vec{e}_{2,b'}+\vec{e}_{2,b}]},$$
$$\tau^{(l)}_{-b,-a} = \bar{\tau}^{(l)}_{-a,-b} = \varepsilon_{2,1}(b, a)\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1}-\vec{e}_{1,a};\vec{\nu}_2+\vec{e}_{2,p_2}-\vec{e}_{2,b}]},$$

where

$$\varepsilon_{1,1}\big(a, a'\big) := (-1)^{\sum_{j=1}^{a}\nu_{1,j}+\sum_{j=1}^{a'}\nu_{1,j}+\delta_{a,p_1}-1}, \quad a' < a,$$

$$\varepsilon_{1,1}\big(a, a'\big) := (-1)^{\sum_{j=1}^{a}\nu_{1,j}+\sum_{j=1}^{a'}\nu_{1,j}+\delta_{a',p_1}}, \quad a' > a,$$

$$\varepsilon_{2,2}\big(b, b'\big) := (-1)^{\sum_{j=1}^{b}\nu_{2,j}+\sum_{j=1}^{b'}\nu_{2,j}-1}, \quad b' < b,$$

$$\varepsilon_{2,2}\big(b, b'\big) := (-1)^{\sum_{j=1}^{b}\nu_{2,j}+\sum_{j=1}^{b'}\nu_{2,j}}, \quad b' > b,$$

$$\varepsilon_{2,1}(b, a) := (-1)^{\sum_{j=1}^{b}\nu_{2,j}+\sum_{j=1}^{a}\nu_{1,j}+\delta_{b,p_2}},$$

$$\varepsilon_{1,1}(a, a) := 1 = \varepsilon_{2,2}(b, b).$$

In particular

$$\tau^{(l)}_{-a}=\varepsilon_{1,1}(p_1,a)\tau_{[\vec\nu_1+\vec e_{1,p_1}-\vec e_{1,a};\vec\nu_2]},\qquad \bar\tau^{(l)}_{-b}=\varepsilon_{2,2}(p_2,b)\tau_{[\vec\nu_1;\vec\nu_2+\vec e_{2,p_2}-\vec e_{2,b}]},$$
$$\tau^{(l+1)}_{+a}=\varepsilon_{1,1}(a,p_1)\tau_{[\vec\nu_1+\vec e_{1,a};\vec\nu_2+\vec e_{2,p_2}]},\qquad \bar\tau^{(l+1)}_{+b}=\varepsilon_{2,2}(b,p_2)\tau_{[\vec\nu_1+\vec e_{1,p_1};\vec\nu_2+\vec e_{2,b}]}.$$

We now proceed to give the τ-function representation of multiple orthogonal polynomials, their duals, second kind functions and bilinear equations. The τ-functions allow for compact expressions for the multiple orthogonal polynomials:

Proposition 35. *The mixed multiple orthogonal polynomials $A^{(l)}_a$, $A^{(l)}_{+a',a}$ and $A^{(l)}_{-b,a}$ have the following τ-function representation*

$$A^{(l)}_a(z)=A^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)],a}=z^{\nu_{1,a}(l)-1}\frac{\tau^{(l)}_{-a}(t-[z^{-1}]_a)}{\tau^{(l)}(t)},\quad l\geqslant 1,\tag{187}$$

$$A^{(l)}_{+a',a}(z)=A^{(\mathrm{II},a')}_{[\vec\nu_1(l-1)+\vec e_{1,a'};\vec\nu_2(l-1)],a}=z^{\nu_{1,a}(l-1)+\delta_{a,a'}-1}\frac{\tau^{(l)}_{+a',-a}(t-[z^{-1}]_a)}{\tau^{(l)}(t)},\quad l\geqslant 1,\tag{188}$$

$$A^{(l)}_{-b,a}(z)=A^{(\mathrm{I},b)}_{[\vec\nu_1(l);\vec\nu_2(l)-\vec e_{2,b}],a}=z^{\nu_{1,a}(l)-1}\frac{\tau^{(l)}_{-b,-a}(t-[z^{-1}]_a)}{\tau^{(l+1)}(t)},\quad l\geqslant 1.\tag{189}$$

The dual polynomials $\bar A^{(l)}_b$, $\bar A^{(l)}_{+b',b}$, and $\bar A^{(l)}_{-a,b}$ have the following τ-function representation

$$\bar A^{(l)}_b(z)=\bar A^{(\mathrm{I},a_2(b))}_{[\vec\nu_2(l);\vec\nu_1(l-1)],b}=z^{\nu_{2,b}(l)-1}\frac{\bar\tau^{(l)}_{-b}(t+\overline{[z^{-1}]}_b)}{\tau^{(l+1)}(t)},\quad l\geqslant 1,\tag{190}$$

$$\bar A^{(l)}_{+b',b}(z)=\bar A^{(\mathrm{II},b')}_{[\vec\nu_2(l-1)+\vec e_{2,b'};\vec\nu_1(l-1)],b}=z^{\nu_{2,b}(l-1)+\delta_{b,b'}-1}\frac{\bar\tau^{(l)}_{+b',-b}(t+\overline{[z^{-1}]}_b)}{\tau^{(l)}(t)},\quad l\geqslant 1,\tag{191}$$

$$\bar A^{(l)}_{-a,b}(z)=\bar A^{(\mathrm{I},a)}_{[\vec\nu_2(l);\vec\nu_1(l)-\vec e_{1,a}],b}=z^{\nu_{2,b}(l)-1}\frac{\bar\tau^{(l)}_{-a,-b}(t+\overline{[z^{-1}]}_b)}{\tau^{(l+1)}(t)},\quad l\geqslant 1.\tag{192}$$

Proof. See Appendix A $\square$

Observe that in the simple ladder defined above $(\vec\nu_1,\vec\nu_2+\vec e_{2,p_2})$ with $l=|\vec\nu_2|=|\vec\nu_1|-1$ we have

$$\vec\nu_1(l)=\vec\nu_1,\qquad \vec\nu_2(l-1)=\vec\nu_2,$$
$$\vec\nu_1(l-1)=\vec\nu_1-\vec e_{1,p_1},\qquad \vec\nu_2(l)=\vec\nu_2+\vec e_{2,p_2}.$$

From Proposition 35 we get

$$A^{(\mathrm{II},p_1)}_{[\vec\nu_1;\vec\nu_2],a}(z)=\varepsilon_{1,1}(p_1,a)z^{\nu_{1,a}-1}\frac{\tau_{[\vec\nu_1+\vec e_{1,p_1}-\vec e_{1,a}a;\vec\nu_2]}(t-[z^{-1}]_a)}{\tau_{[\vec\nu_1;\vec\nu_2]}(t)},$$

$$A^{(\mathrm{II},a')}_{[\vec{\nu}_1-\vec{e}_{1,p_1}+\vec{e}_{1,a'};\vec{\nu}_2],a}(z)=\varepsilon_{1,1}\big(a',a\big)z^{\nu_{1,a}-\delta_{a,p_1}+\delta_{a,a'}-1}\frac{\tau_{[\vec{\nu}_1-\vec{e}_{1,a}+\vec{e}_{1,a'};\vec{\nu}_2]}(t-[z^{-1}]_a)}{\tau_{[\vec{\nu}_1;\vec{\nu}_2]}(t)},$$

$$A^{(\mathrm{I},b)}_{[\vec{\nu}_1;\vec{\nu}_2+\vec{e}_{2,p_2}-\vec{e}_{2,b}],a}(z)=\varepsilon_{2,1}(b,a)z^{\nu_{1,a}-1}\frac{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1}-\vec{e}_{1,a};\vec{\nu}_2+\vec{e}_{2,p_2}-\vec{e}_{2,b}]}(t-[z^{-1}]_a)}{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1};\vec{\nu}_2+\vec{e}_{2,p_2}]}(t)}, \tag{193}$$

$$\bar{A}^{(\mathrm{I},p_2)}_{[\vec{\nu}_2+\vec{e}_{2,p_2};\vec{\nu}_1-\vec{e}_{p_1}],b}(z)=\varepsilon_{2,2}(p_2,b)z^{\nu_{2,b}+\delta_{b,p_2}-1}\frac{\tau_{[\vec{\nu}_1;\vec{\nu}_2+\vec{e}_{2,p_2}-\vec{e}_{2,b}]}(t+\overline{[z^{-1}]}_b)}{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1};\vec{\nu}_2+\vec{e}_{2,p_2}]}(t)},$$

$$\bar{A}^{(\mathrm{II},b')}_{[\vec{\nu}_2+\vec{e}_{2,b'};\vec{\nu}_1-\vec{e}_{1,p_1}],b}=\varepsilon_{2,2}\big(b,b'\big)z^{\nu_{2,b}+\delta_{b',b}-1}\frac{\tau_{[\vec{\nu}_1;\vec{\nu}_2-\vec{e}_{2,b}+\vec{e}_{2,b'}]}(t+\overline{[z^{-1}]}_b)}{\tau_{[\vec{\nu}_1;\vec{\nu}_2]}(t)},$$

$$\bar{A}^{(\mathrm{I},a)}_{[\vec{\nu}_2+\vec{e}_{2,p_2};\vec{\nu}_1-\vec{e}_{1,a}],b}=\varepsilon_{2,1}(b,a)z^{\nu_{2,b}+\delta_{b,p_2}-1}\frac{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1}-\vec{e}_{1,a};\vec{\nu}_2+\vec{e}_{2,p_2}-\vec{e}_{2,b}]}(t+\overline{[z^{-1}]}_b)}{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1};\vec{\nu}_2+\vec{e}_{2,p_2}]}(t)}. \tag{194}$$

We now present the τ-representation of the Cauchy transforms of the linear forms.

Proposition 36. *The Cauchy transforms have the following τ-function representation*

$$\bar{C}^{(l)}_a=z^{-\nu_{1,a}(l-1)-1}\frac{\tau^{(l+1)}_{+a}(t+[z^{-1}]_a)}{\tau^{(l+1)}(t)}, \tag{195}$$

$$C^{(l)}_b=z^{-\nu_{2,b}(l-1)-1}\frac{\bar{\tau}^{(l+1)}_{+b}(t-\overline{[z^{-1}]}_b)}{\tau^{(l)}(t)}. \tag{196}$$

Proof. See Appendix A □

We have the representation

$$C^{(\mathrm{I},p_2)}_{[\vec{\nu}_2+\vec{e}_{2,p_2};\vec{\nu}_1-\vec{e}_{1,p_1}],a}(z)=\varepsilon_{1,1}(a,p_1)z^{-\nu_{1,a}-1+\delta_{a,p_1}}\frac{\tau_{[\vec{\nu}_1+\vec{e}_{1,a};\vec{\nu}_2+\vec{e}_{2,p_2}]}(t+[z^{-1}]_a)}{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1};\vec{\nu}_2+\vec{e}_{2,p_2}]}(t)},$$

$$C^{(\mathrm{II},p_1)}_{[\vec{\nu}_1;\vec{\nu}_2],b}(z)=\varepsilon_{2,2}(b,p_2)z^{-\nu_{2,b}-1}\frac{\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1};\vec{\nu}_2+\vec{e}_{2,b}]}(t-\overline{[z^{-1}]}_b)}{\tau_{[\vec{\nu}_1;\vec{\nu}_2]}(t)}.$$

Finally, we consider the τ-function representation of the bilinear equation

Proposition 37. *The τ functions fulfill the following bilinear relation*

$$\sum_{a=1}^{p_1}\oint_{z=\infty}z^{\nu_{1,a}(k)-\nu'_{1,a}(l-1)-2}\tau^{(k)}_{\vec{n}_1,\vec{n}_2,-a}\big(t-\big[z^{-1}\big]_a\big)\tau^{(l+1)}_{\vec{n}'_1,\vec{n}'_2,+a}\big(t'+\big[z^{-1}\big]_a\big)E_a(z)\,\mathrm{d}z$$
$$=\sum_{b=1}^{p_2}\oint_{z=\infty}z^{\nu'_{2,b}(l)-\nu_{2,b}(k-1)-2}\bar{\tau}^{(k+1)}_{\vec{n}_1,\vec{n}_2,+b}\big(t-\overline{\big[z^{-1}\big]}_b\big)\bar{\tau}^{(l)}_{\vec{n}'_1,\vec{n}'_2,-b}\big(t'+\overline{\big[z^{-1}\big]}_b\big)\bar{E}_b(z)\,\mathrm{d}z. \tag{197}$$

Proof. Just consider (181) together with (187), (190), (195) and (196). □

This bilinear relation can also be written as follows

$$
\begin{aligned}
&\sum_{a=1}^{p_1}\varepsilon_{11}(p_1,a)\varepsilon'_{11}(p_1,a)\oint_{z=\infty} z^{\nu_{1,a}-\nu'_{1,a}-\delta_{a,p_1}-2}\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1}-\vec{e}_{1,a};\vec{\nu}_2]}\\
&\quad\times\big(t-\big[z^{-1}\big]_a\big)\tau^{(l+1)}_{\vec{\nu}'_1+\vec{e}_{1,a};\vec{\nu}'_2+\vec{e}_{2,p_2}}\big(t'+\big[z^{-1}\big]_a\big)E_a(z)\,\mathrm{d}z\\
&\quad=\sum_{b=1}^{p_2}\varepsilon_{22}(p_2,b)\varepsilon'_{22}(p_2,b)\oint_{z=\infty} z^{\nu'_{2,b}+\delta_{b,p_2}-\nu_{2,b}-2}\tau_{[\vec{\nu}_1+\vec{e}_{1,p_1};\vec{\nu}_2+\vec{e}_{2,b}]}\\
&\quad\times\big(t-\overline{\big[z^{-1}\big]}_b\big)\tau_{\vec{\nu}'_1;\vec{\nu}'_2+\vec{e}_{2,p_2}-\vec{e}_{2,b}]}\big(t'+\overline{\big[z^{-1}\big]}_b\big)\bar{E}_b(z)\,\mathrm{d}z.
\end{aligned}
\tag{198}
$$

That with the identification $m^*=\vec{\nu}_1+\vec{e}_{1,p_1}$, $n^*=\vec{\nu}_2$, $m=\vec{\nu}'_1$ and $n=\vec{\nu}'_2+\vec{e}_{2,p_2}$, up to signs, is the bilinear relation (41) in [6].

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Appendix A. Proofs

Proof Proposition 7. The orthogonality relations can be recast into two alternative forms

$$
\begin{aligned}
&\begin{pmatrix} S_{l,0} & S_{l,1} & \cdots & S_{l,l-1}\end{pmatrix}\begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1}\\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1}\\ \vdots & \vdots & & \vdots\\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1}\end{pmatrix}\\
&\quad=-\begin{pmatrix} g_{l,0} & g_{l,1} & \cdots & g_{l,l-1}\end{pmatrix},\quad l\geqslant 1,
\end{aligned}
\tag{199}
$$

$$
\begin{aligned}
&\begin{pmatrix} S_{l,0} & S_{l,1} & \cdots & S_{l,l-1} & S_{l,l}\end{pmatrix}\begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l}\\ g_{1,0} & g_{1,1} & \cdots & g_{1,l}\\ \vdots & \vdots & & \vdots\\ g_{l,0} & g_{l,1} & \cdots & g_{l,l}\end{pmatrix}\\
&\quad=\underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 & \bar{S}_{l,l}\end{pmatrix}}_{l+1\text{ components}},\quad l\geqslant 0.
\end{aligned}
\tag{200}
$$

From (43) we get

$$
\begin{aligned}
Q^{(l)} &= \sum_{k=0}^{l} S_{l,k}\xi_1^{(k)} \\
&= \xi_1^{(l)} - \begin{pmatrix} g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} \end{pmatrix} \bigl(g^{[l]}\bigr)^{-1}\xi_1^{[l]} \quad \text{use (199)} && (201)\\
&= \bar{S}_{l,l}\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\bigl(g^{[l+1]}\bigr)^{-1}\xi_1^{[l+1]} \quad \text{use (200).} && (202)
\end{aligned}
$$

Cramer's method solves (199) as follows

$$
S_{l,i} = \frac{1}{\det g^{[l]}} \sum_{j=0}^{l-1} g_{l,j}(-1)^{i+j+1} M_{i,j}^{(l)} = \frac{(-1)^{i+l} M_{i,l}^{(l+1)}}{\det g^{[l]}}, \tag{203}
$$

where $M_{i,j}^{(l)}$ is the (i,j)-minor of the truncated moment matrix $g^{[l]}$ defined in (21). Therefore,

$$
\begin{aligned}
Q^{(l)} &= \frac{1}{\det g^{[l]}} \sum_{i=0}^{l} (-1)^{i+l} M_{i,l}^{(l+1)} \xi_1^{(i)} \\
&= \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c}
g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \xi_1^{(0)} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \xi_1^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \xi_1^{(l-1)} \\
\hline
g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \xi_1^{(l)}
\end{array}\right), \quad l \geqslant 1.
\end{aligned}
$$

The orthogonality relations for the dual system can be written also in two alternative forms

$$
\begin{pmatrix}
g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} \\
\vdots & \vdots & & \vdots \\
g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1}
\end{pmatrix}
\begin{pmatrix} \bar{S}'_{0,l} \\ \bar{S}'_{1,l} \\ \vdots \\ \bar{S}'_{l-1,l} \end{pmatrix}
= -(\bar{S}_{l,l})^{-1}
\begin{pmatrix} g_{0,l} \\ g_{1,l} \\ \vdots \\ g_{l-1,l} \end{pmatrix}, \quad l \geqslant 1, \tag{204}
$$

$$
\begin{pmatrix}
g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\
\vdots & \vdots & & \vdots \\
g_{l,0} & g_{l,1} & \cdots & g_{l,l}
\end{pmatrix}
\begin{pmatrix} \bar{S}'_{0,l} \\ \bar{S}'_{1,l} \\ \vdots \\ \bar{S}'_{l,l} \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad l \geqslant 0. \tag{205}
$$

As before, (43) leads to the following expressions for the dual linear forms

$$
\bar{Q}^{(l)} = \sum_{k=0}^{l} \bar{S}'_{k,l}\xi_2^{(k)}
$$

$$= (\bar{S}_{l,l})^{-1}\left(\xi_2^{(l)} - \left(\xi_2^{[l]}\right)^\top \left(g^{[l]}\right)^{-1}\begin{pmatrix} g_{0,l} \\ g_{1,l} \\ \vdots \\ g_{l-1,l}\end{pmatrix}\right) \quad \text{use (204)} \tag{206}$$

$$= \left(\xi_2^{[l+1]}\right)^\top \left(g^{[l+1]}\right)^{-1}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1\end{pmatrix} \quad \text{use (205).} \tag{207}$$

From (205) we obtain

$$\bar{S}'_{j,l} = \left(g^{[l+1]^{-1}}\right)_{j,l} = \frac{(-1)^{l+j} M_{l,j}^{(l+1)}}{\det(g^{[l+1]})}, \quad j = 0, \dots, l, \tag{208}$$

and consequently

$$\bar{Q}^{(l)} = \sum_{j=0}^{l} \bar{S}'_{j,l}\xi_2^{(j)} = \frac{1}{\det g^{[l+1]}}\sum_{j=0}^{l}(-1)^{l+j} M_{l,j}^{(l+1)}\xi_2^{(j)}$$
$$= \frac{1}{\det g^{[l+1]}} \det\left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l} \\ \hline \xi_2^{(0)} & \xi_2^{(1)} & \cdots & \xi_2^{(l-1)} & \xi_2^{(l)} \end{array}\right), \quad l \geqslant 0.$$

Proof Proposition 9. We have

$$C_b^{(l)} = \frac{1}{\det g^{[l]}}\sum_{k=0}^{l}(-1)^{k+l} M_{k,l}^{(l+1)} \sum_{k_2=\nu_{2,b}(l-1)}^{\infty} z^{-k_2-1}\int x^{k_1(k)} w_{1,a_1(k)}(x) w_{2,b}(x) x^{k_2}\,\mathrm{d}\mu(x),$$

which according to (20) recasts into

$$C_b^{(l)} = \frac{1}{\det g^{[l]}}\sum_{k=0}^{l}(-1)^{k+l} M_{k,l}^{(l+1)} \bar{\Gamma}_{k,b}^{(l)}$$
$$= \frac{1}{\det g^{[l]}} \det\left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \bar{\Gamma}_{0,b}^{(l)} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \bar{\Gamma}_{0,b}^{(l)} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \bar{\Gamma}_{l-1,b}^{(l)} \\ \hline g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \bar{\Gamma}_{l,b}^{(l)}, \end{array}\right), \quad l \geqslant 1. \tag{209}$$

We also obtain

$$\bar{C}_a^{(l)}=\frac{1}{\det g^{[l+1]}}\sum_{k=0}^{l}(-1)^{l+k}M_{l,k}^{(l+1)}\sum_{k_1=\nu_{1,a}(l-1)}^{\infty}z^{-k_1-1}\int x^{k_1}w_{1,a}(x)w_{2,a_2(k)}x^{k_2(k)}\,\mathrm{d}\mu(x),$$

which can be written as (20)

$$\begin{aligned}\bar{C}_a^{(l)}&=\frac{1}{\det g^{[l+1]}}\sum_{k=0}^{l}(-1)^{k+l}M_{l,k}^{(l+1)}\Gamma_k^{(l)}\\&=\frac{1}{\det g^{[l+1]}}\det\left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l}\\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l}\\ \vdots & \vdots & & \vdots & \vdots\\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l}\\ \hline \Gamma_{0,a}^{(l)} & \Gamma_{1,a}^{(l)} & \cdots & \Gamma_{l-1,a}^{(l)} & \Gamma_{l,a}^{(l)}\end{array}\right),\quad l\geqslant 1.\end{aligned}\tag{210}$$

Proof Proposition 15. From (53) and (54) we deduce that

$$\begin{gathered}\big(\bar{\mathscr{C}}_a(z)\big)^\top\mathscr{A}_{a'}\big(z'\big)=\big(\chi_{1,a}^*(z)\big)^\top\chi_{1,a'}\big(z'\big)=\frac{\delta_{a,a'}}{z-z'},\quad \big|z'\big|<|z|,\\ \big(\mathscr{C}_b(z)\big)^\top\bar{\mathscr{A}}_{a'}\big(z'\big)=\big(\chi_{2,b}^*(z)\big)^\top\chi_{2,b'}\big(z'\big)=\frac{\delta_{b,b'}}{z-z'},\quad \big|z'\big|<|z|,\\ \big(\bar{\mathscr{C}}_a(z)\big)^\top\mathscr{C}_b\big(z'\big)=\big(\chi_{1,a}^*(z)\big)^\top g\,\chi_{2,b}^*\big(z'\big).\end{gathered}$$

The two first relations imply the corresponding equations in the Proposition. For the third we observe that from (10) we get

$$\begin{aligned}\big(\chi_{1,a}^*(z)\big)^\top g\,\chi_{2,b'}^*\big(z'\big)&=\int\big(\chi_{1,a}^*(z)\big)^\top\xi_1(x)\big(\xi_2(x)\big)^\top\chi_{2,b'}^*\big(z'\big)\,\mathrm{d}\mu(x)\\&=\int\big(\chi_{1,a}^*(z)\big)^\top\chi_{1,a}(x)\big(\chi_{2,b}(x)\big)^\top\chi_{2,b'}^*\big(z'\big)w_{1,a}(x)w_{2,b}(x)\,\mathrm{d}\mu(x)\\&=\int\frac{1}{(z-x)(z'-x)}w_{1,a}(x)w_{2,b}(x)\,\mathrm{d}\mu(x)\\&=-\frac{1}{z-z'}\int\left(\frac{1}{z-x}-\frac{1}{z'-x}\right)w_{1,a}(x)w_{2,b}(x)\,\mathrm{d}\mu(x).\end{aligned}$$

Proof Proposition 20. Using Definition 17 for the linear forms $Q_{+a}^{(l)}$ and multiplying by $(\xi_2^{[l]}(x))^\top$ we have

$$\begin{aligned}Q_{+a}^{(l)}(x)\big(\xi_2^{[l]}(x)\big)^\top={}&\xi_1^{(l_{+a})}(x)\big(\xi_2^{[l]}(x)\big)^\top-\begin{pmatrix}g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1}\end{pmatrix}\\&\times\big(g^{[l]}\big)^{-1}\xi_1^{[l]}(x)\big(\xi_2^{[l]}(x)\big)^\top,\end{aligned}$$

integrating both sides we get

$$
\begin{aligned}
&\int Q^{(l)}_{+a}(x)\big(\xi^{[l]}_2(x)\big)^\top \mathrm{d}\mu(x)\\
&\quad=\int \xi^{(l_{+a})}_1(x)\big(\xi^{[l]}_2(x)\big)^\top \mathrm{d}\mu(x)\\
&\qquad-\begin{pmatrix} g_{l_{+a},0} & \cdots & g_{l_{+a},l-1}\end{pmatrix}\big(g^{[l]}\big)^{-1}\int \xi^{[l]}_1(x)\big(\xi^{[l]}_2(x)\big)^\top \mathrm{d}\mu(x)\\
&\quad=\int \xi^{(l_{+a})}_1(x)\big(\xi^{[l]}_2(x)\big)^\top \mathrm{d}\mu(x)-\begin{pmatrix} g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1}\end{pmatrix}\big(g^{[l]}\big)^{-1}g^{[l]}\\
&\quad=\begin{pmatrix} g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1}\end{pmatrix}-\begin{pmatrix} g_{l_{+a},0} & g_{l_{+a},1} & \cdots & g_{l_{+a},l-1}\end{pmatrix}\\
&\quad=0,
\end{aligned}
$$

that written componentwise gives the following orthogonality relations

$$\int Q^{(l)}_{+a}(x)w_{2,a_2(k)}(x)x^{k_2(k)}\mathrm{d}\mu(x),\quad k=0,\dots,l-1,$$

or equivalently

$$\int Q^{(l)}_{+a}(x)w_{2,b}(x)x^k\,\mathrm{d}\mu(x)=0,\quad 0\leqslant k\leqslant \nu_{2,b}(l-1)-1,\ b=1,\dots,p_2.$$

Notice that, $A^{(l)}_{+a,a}$ is monic and $\deg A^{(l)}_{+a,a}(x)=k_1(l_{+a})$ but $A^{(l)}_{+a,a'}$ with $a\neq a'$ satisfy $\deg A^{(l)}_{+a,a'}\leqslant k_1((l-1)_{-a'})$. This means that the set of polynomials $A^{(l)}_{+a,a'}(x)$ have degrees determined by $\vec{\nu}_1(l-1)+\vec{e}_{1,a}$ and a normalization with respect to the a-th component of type II; i.e., $Q^{(l)}_{+a}=Q^{(\mathrm{II},a)}_{[\vec{\nu}_1(l-1)+\vec{e}_{1,a};\vec{\nu}_2(l-1)]}$.

In a similar way, the associated linear forms $Q^{(l)}_{-b}(x)$ solve a mixed multiple orthogonal problem that can be obtained as follows. From Definition 17 and multiplying by $(\xi^{[l]}_2(x))^\top$ we get

$$Q^{(l)}_{-b}(x)\big(\xi^{[l+1]}_2(x)\big)^\top=e^\top_{\bar{l}_{-b}}\big(g^{[l+1]}\big)^{-1}\xi^{[l+1]}_1(x)\big(\xi^{[l+1]}_2(x)\big)^\top,$$

integrating both sides

$$\int Q^{(l)}_{-b}(x)\big(\xi^{[l+1]}_2(x)\big)^\top\mathrm{d}\mu(x)=e^\top_{\bar{l}_{-b}}\big(g^{[l+1]}\big)^{-1}\int \xi^{[l+1]}_1(x)\big(\xi^{[l+1]}_2(x)\big)^\top \mathrm{d}\mu(x)=e^\top_{\bar{l}_{-b}},$$

and written componentwise

$$\int Q^{(l)}_{-b}(x)x^{k_2(k)}w_{2,a_2(k)}(x)\,\mathrm{d}\mu(x)=\delta_{k,\bar{l}_{-b}},\quad k=0,\dots,l,$$

that is equivalent to

$$\int_{\mathbb{R}} Q^{(l)}_{-b}(x) w_{2,b}(x) x^k \,\mathrm{d}\mu(x) = \delta_{k,\bar{l}_{-b}}, \quad 0 \leqslant k \leqslant \nu_{2,b}(l) - 1,\ b = 1, \dots, p_2.$$

Hence, the set $A^{(l)}_{-b,a'}$ is a type I normalized to the b-th component solution for a mixed multiple orthogonality problem; the degrees satisfy $\deg A^{(l)}_{-b,a'} \leqslant l_{-a'}$. Moreover, the fact that the last orthogonality condition in the b-th component is missing gives the identification $Q^{(l)}_{-b} = Q^{(\mathrm{I},b)}_{[\vec{\nu}_1(l);\vec{\nu}_2(l)-\vec{e}_{2,b}]}$.

Using Definition 17 and multiplying by $\xi^{[l]}_1(x)$ we have

$$\bar{Q}^{(l)}_{+b}(x)\xi^{[l]}_1(x) = \left(\xi^{[l]}_1(x)\xi^{(\bar{l}_{+b})}_2(x) - \xi^{[l]}_1(x)\big(\xi^{[l]}_2(x)\big)^\top \big(g^{[l]}\big)^{-1} \begin{pmatrix} g_{0,\bar{l}_{+b}} \\ g_{1,\bar{l}_{+b}} \\ \vdots \\ g_{l-1,\bar{l}_{+b}} \end{pmatrix}\right),$$

and integrating both sides

$$\begin{aligned}
&\int_{\mathbb{R}} \bar{Q}^{(l)}_{+b}(x)\xi_1(x)^{[l]} \,\mathrm{d}\mu(x) \\
&\quad = \left(\int \xi^{[l]}_1(x)\xi^{(\bar{l}_{+b})}_2(x)\,\mathrm{d}\mu(x) - \int \xi^{[l]}_1(x)\big(\xi^{[l]}_2(x)\big)^\top \mathrm{d}\mu(x)\big(g^{[l]}\big)^{-1} \begin{pmatrix} g_{0,\bar{l}_{+b}} \\ g_{1,\bar{l}_{+b}} \\ \vdots \\ g_{l-1,\bar{l}_{+b}} \end{pmatrix}\right) \\
&\quad = \left(\int \xi^{[l]}_1(x)\xi^{(\bar{l}_{+b})}_2(x)\,\mathrm{d}\mu(x) - \big(g^{[l]}\big)\big(g^{[l]}\big)^{-1} \begin{pmatrix} g_{0,\bar{l}_{+b}} \\ g_{1,\bar{l}_{+b}} \\ \vdots \\ g_{l-1,\bar{l}_{+b}} \end{pmatrix}\right) \\
&\quad = \begin{pmatrix} g_{0,\bar{l}_{+b}} \\ g_{1,\bar{l}_{+b}} \\ \vdots \\ g_{l-1,\bar{l}_{+b}} \end{pmatrix} - \begin{pmatrix} g_{0,\bar{l}_{+b}} \\ g_{1,\bar{l}_{+b}} \\ \vdots \\ g_{l-1,\bar{l}_{+b}} \end{pmatrix} \\
&\quad = 0,
\end{aligned}$$

that componentwise leads to the following orthogonality relations

$$\int \bar{Q}^{(l)}_{+b}(x) w_{1,a_1(k)}(x) x^{k_1(k)} \,\mathrm{d}\mu(x) = 0, \quad k = 0, \dots, l-1,$$

or alternatively

$$\int \bar{Q}^{(l)}_{+b}(x) w_{1,a}(x) x^k \,\mathrm{d}\mu(x) = 0, \quad 0 \leqslant k \leqslant \nu_{1,a}(l-1) - 1,\ a = 1, \ldots, p_1.$$

Notice that, $\bar{A}^{(l)}_{+b,b}$ is monic and $\deg \bar{A}^{(l)}_{+b,b} = k_2(\bar{l}_{+b})$ but $\bar{A}^{(l)}_{+b,b'}$ with $b \neq b'$ satisfy $\deg A^{(l)}_{+b,b'} \leqslant k_1(\overline{(l-1)}_{-b'})$. This means that the polynomials $\bar{A}^{(l)}_{+b,b'}$ have degrees determined by $\vec{v}_2(l-1) + \vec{e}_{2,b}$ and a normalization with respect to the b-th component of type II; i.e., $\bar{Q}^{(l)}_{+b} = \bar{Q}^{(\mathrm{II},b)}_{[\vec{v}_2(l-1)+\vec{e}_{2,b};\vec{v}_1(l-1)]}$.

Finally, we obtain the orthogonality relations for the linear forms $\bar{Q}^{(l)}_{-a}(x)$. From the definition we get

$$\xi_1^{[l+1]}(x)\bar{Q}^{(l)}_{-a}(x) = \xi_1^{[l+1]}(x)\big(\xi_2^{[l+1]}(x)\big)^\top \big(g^{[l+1]}\big)^{-1} e_{l_{-a}},$$

and integrating both sides

$$\int \xi_1^{[l+1]}(x)\bar{Q}^{(l)}_{-a}(x)\mathrm{d}\mu(x) = \left(\int \xi_1^{[l+1]}(x)\big(\xi_2^{[l+1]}(x)\big)^\top \mathrm{d}\mu(x)\right)\big(g^{[l+1]}\big)^{-1} e_{l_{-a}} = e_{l_{-a}},$$

and componentwise that means

$$\int \bar{Q}^{(l)}_{-a}(x) x^{k_1(k)} w_{1,a_1(k)}(x)\,\mathrm{d}\mu(x) = \delta_{k,l_{-a}}, \quad k = 0, \ldots, l,$$

or equivalently

$$\int \bar{Q}^{(l)}_{-a}(x) x^k w_{1,a}(x)\,\mathrm{d}\mu(x) = \delta_{k,l_{-a}}, \quad 0 \leqslant k \leqslant \nu_{1,a}(l) - 1,\ a = 1, \ldots, p_1,$$

so the set $\bar{A}^{(l)}_{-a,b'}$ is a type I normalized to the a-th component solution for a mixed multiple orthogonality problem. The degrees satisfy $\deg A^{(l)}_{-a,b'} \leqslant l_{-b'}$; and therefore we conclude that $\bar{Q}^{(l)}_{-b} = Q^{(\mathrm{I},b)}_{[\vec{v}_2(l);\vec{v}_1(l)-\vec{e}_{1,a}]}$.

Proof of Proposition 32. Taking $+$ and $-$ parts in (170) we obtain

$$\sum_{a=1}^{p_1} L_a^j = \sum_{a=1}^{p_1}\big(L_a^j\big)_+ + \sum_{a=1}^{p_1}\big(L_a^j\big)_- = \sum_{a=1}^{p_1}\big(L_a^j\big)_+ + \sum_{b=1}^{p_2}\big(\bar{L}_b^j\big)_- = \sum_{a=1}^{p_1} B_{j,a} + \sum_{b=1}^{p_2} \bar{B}_{j,b} = \sum_{b=1}^{p_2} \bar{L}_b^j.$$

Using Lax equations and observing that $L_a L_{a'} = L_{a'} L_a$ and $\bar{L}_b \bar{L}_{b'} = \bar{L}_{b'} \bar{L}_b$ we have the following symmetries for the Lax operators

$$\left(\sum_{a=1}^{p_1}\frac{\partial}{\partial t_{j,a}}+\sum_{b=1}^{p_2}\frac{\partial}{\partial \bar t_{j,b}}\right)L_{a'}=\left[\sum_{a=1}^{p_1}B_{j,a}+\sum_{b=1}^{p_2}\bar B_{j,b},L_{a'}\right]=\sum_{a=1}^{p_1}\left[L_a^j,L_{a'}\right]=0,$$
$$\left(\sum_{a=1}^{p_1}\frac{\partial}{\partial t_{j,a}}+\sum_{b=1}^{p_2}\frac{\partial}{\partial \bar t_{j,b}}\right)\bar L_{b'}=\left[\sum_{a=1}^{p_1}B_{j,a}+\sum_{b=1}^{p_2}\bar B_{j,b},\bar L_{b'}\right]=\sum_{b=1}^{p_2}\left[\bar L_b^j,\bar L_{b'}\right]=0.$$

From (127) we conclude that the multiple orthogonal polynomials and their duals are also invariant

$$\left(\sum_{a=1}^{p_1}\frac{\partial}{\partial t_{j,a}}+\sum_{b=1}^{p_2}\frac{\partial}{\partial \bar t_{j,b}}\right)\mathscr{A}_{a'}=\left(\sum_{a=1}^{p_1}B_{j,a}+\sum_{b=1}^{p_2}\bar B_{j,b}-x^j\right)\mathscr{A}_{a'}=\left(J^j-x^j\right)\mathscr{A}_{a'}=0,$$
$$\left(\sum_{a=1}^{p_1}\frac{\partial}{\partial t_{j,a}}+\sum_{b=1}^{p_2}\frac{\partial}{\partial \bar t_{j,b}}\right)\bar{\mathscr{A}}_{b'}=-\left(\sum_{a=1}^{p_1}B_{j,a}+\sum_{b=1}^{p_2}\bar B_{j,b}-x^j\right)^\top\bar{\mathscr{A}}_{b'}=-\left(J^j-x^j\right)^\top\bar{\mathscr{A}}_{a'}=0.$$

Proof of Proposition 33. We just follow the following chain of identities

$$\begin{aligned}
&W_{\vec n_1,\vec n_2}(t,s)\pi^\top_{\vec n_1',\vec n_1}\left(W_{\vec n_1',\vec n_2'}(t',s')\right)^{-1}\\
&\quad=W_{\vec n_1,\vec n_2}(t,s)\pi^\top_{\vec n_1',\vec n_1}g_{\vec n_1',\vec n_2'}\left(\bar W_{\vec n_1',\vec n_2'}(t',s')\right)^{-1}\\
&\quad=S_{\vec n_1,\vec n_2}(t,s)W_{0,\vec n_1}(t,s)\pi^\top_{\vec n_1',\vec n_1}\left(\int\xi_{\vec n_1'}(x)\xi^\top_{\vec n_2'}(x)\,\mathrm d\mu(x)\right)\left(W_{0,\vec n_2'}(t',s')\right)^{-1}\left(\bar S_{\vec n_1',\vec n_2'}(t',s')\right)^{-1}\\
&\quad=S_{\vec n_1,\vec n_2}(t,s)W_{0,\vec n_1}(t,s)\left(\int\xi_{\vec n_1}(x)\xi^\top_{\vec n_2'}(x)\,\mathrm d\mu(x)\right)\left(W_{0,\vec n_2'}(t',s')\right)^{-1}\left(\bar S_{\vec n_1',\vec n_2'}(t',s')\right)^{-1}\\
&\quad=S_{\vec n_1,\vec n_2}(t,s)\left(\int\xi_{\vec n_1}(x,t,s)\xi^\top_{\vec n_2'}\left(x,t',s'\right)\mathrm d\mu(x)\right)\left(\bar S_{\vec n_1',\vec n_2'}(t',s')\right)^{-1}\\
&\quad=\int\left(S_{\vec n_1,\vec n_2}(t,s)\left(\xi_{\vec n_1}(x,t,s)\right)\left(\bar S^\top_{\vec n_1',\vec n_2'}(t',s')\right)^{-1}\xi_{\vec n_2'}\left(x,t',s'\right)\right)^\top\mathrm d\mu(x)\\
&\quad=\int_{\mathbb R}Q_{\vec n_1,\vec n_2}(x,t,s)\bar Q^\top_{\vec n_1',\vec n_2'}\left(x,t',s'\right)\mathrm d\mu(x),
\end{aligned}$$

where $\xi_{\vec n_1}(x,t,s)$ and $\xi_{\vec n_2'}(x,t',s')$ represent the vectors of weighted monomials but with evolved weights.

Proof of Proposition 35.

To find the τ-function of the multiple orthogonal representation we first need two lemmas

Lemma 10. *Let $R^{(j)}$ be the j-th row of $\tau^{(l)}(t)$ and $R_z^{(j)}$ the j-th row of $\tau^{(l)}(t-[z^{-1}]_a)$, then*

$$R_z^{(j)}=R^{(j)}-\delta_{a_1(j),a}z^{-1}R^{(j')},\tag{211}$$

where $j'=j+1$ if $r_1(j)<n_{1,a}-1$, but $j'=j+(|\vec n_1|-n_{1,a})+1$ if $r_1(j)=n_{1,a}-1$. This is also valid for $\tau^{(l)}_{-a}$, $\tau^{(l)}_{+a,-a'}$ and for $\tau^{(l)}_{-b,-a'}$.

Let now be $C^{(j)}$ the j-th column of $\bar\tau^{(l)}$ and $C_z^{(j)}$ the j-th column of $\bar\tau^{(l)}(t+\overline{[z^{-1}]}_b)$, then

$$C_z^{(j)} = C^{(j)} - \delta_{a_2(j),b} z^{-1} C^{(j')}, \tag{212}$$

where $j' = j+1$ if $r_2(l) < n_{2,b} - 1$ but $j' = j + (|\vec n_2| - n_{2,b}) + 1$ if $r_2(j) = n_{2,b} - 1$. This is also valid for $\bar\tau^{(l)}_{-b}$, $\bar\tau^{(l)}_{+b,-b'}$ and for $\bar\tau^{(l)}_{-a,-b'}$.

Proof. It follows directly from (141) and (142). □

Let us recall the skew multi-linear character of determinants and the consequent formulation in terms of wedge products of covectors. Observe that

Lemma 11. *Given a set of covectors $\{r_1,\dots,r_n\}$ it can be shown that*

$$\bigwedge_{j=1}^{n}(zr_j - r_{j+1}) = \sum_{j=1}^{n+1}(-1)^{n+1-j} z^{j-1} r_1\wedge r_2\wedge\cdots\wedge\hat r_j\wedge\cdots\wedge r_{n+1}, \tag{213}$$

where the notation $\hat r_j$ means that we have erased the covector r_j in the wedge product $r_1\wedge\cdots\wedge r_{n+1}$.

Proof. It can be done directly by induction. □

The proof of Proposition 35 relies on Lemma 10, Lemma 11, Corollary 1 and Proposition 22. First let's focus on (187); it is clear that $z^{\nu_{1,a}(l)-1}\tau^{(l)}_{-a}(t-[z^{-1}]_a)$ expands in z according to (211) for $\tau^{(l)}_{-a}$ and to (213). Now $n = k_1(l_{-a})$ and the covectors r_j should be taken equal to those rows $R^{(j)}$ with $a_1(j) = a$. Observe that there are only $k_1(l_{-a})$ $(=\nu_{1,a}(l)-1)$ rows which are non-trivially transformed. In this form we get the identification of (49) with (187), where the terms corresponding to the wedge with one covector deleted corresponds to the minors $M^{[l+1]}_{j,l}$. Now, looking to (188) and (189) we expand again in z and use the same technique based on (211) for $\tau^{(l)}_{+a,-a'}$ and $\tau^{(l)}_{-b,-a}$ and (213). These allow to link (104) to (188) and (106) to (189).

To prove (190) we proceed similarly. Looking at (212) for $\bar\tau^{(l)}_{-b}$ observe that there are only $k_2(\bar l_{-b})(=\nu_{2,b}(l)-1)$ columns which are non-trivially transformed. Now, recalling (50) and using (213) but with r_j being the columns $C^{(j)}$, such that $a_2(j) = b$, and $n = k_2(\bar l_{-b})$, we get the desired result. Finally for (191) and (192) we expand around z to see the equivalence between (107) and (191) and the equivalence between (105) and (192).

Proof of Proposition 36. We need the following two lemmas:

Lemma 12. *Let $R^{(j)}$ be the j-th row of $g^{[l+1]}_{+a}$ and $R^{(j)}_z$ the j-th row of $g^{[l+1]}_{+a}(t+[z^{-1}]_a)$, we get*

$$R^{(j)}_z = R^{(j)} + \delta_{a_1(j),a}\sum_{j'=1}^{\infty} z^{-k_1(j')} R^{(j+j')}\delta_{a_1(j+j'),a}. \tag{214}$$

Let $C^{(j)}$ *be the* j*-th column of* $\bar{g}_{+b}^{[l+1]}$ *and* $C_z^{(j)}$ *the* j*-th column of* $\bar{g}_{+b}^{[l+1]}(t-\overline{[z^{-1}]}_b)$*, then* (142) *gives*

$$C_z^{(j)}=C^{(j)}+\delta_{a_2(j),b}\sum_{j'=1}^{\infty}z^{-k_2(j')}C^{(j+j')}\delta_{a_2(j+j'),b}. \tag{215}$$

Proof. For the first equality insert the expansion

$$\left(1-\frac{x}{z}\right)^{-1}=\sum_{k=0}^{\infty}\frac{x^k}{z^k}$$

into (142). The other equation is proven similarly. □

Lemma 13. *The following identity*

$$\bigwedge_{j=1}^{n}\left(\sum_{i=0}^{\infty}r_{j+i}z^{-i}\right)=r_1\wedge\cdots\wedge r_{n-1}\wedge\left(\sum_{i=0}^{\infty}r_{n+i}z^{-i}\right) \tag{216}$$

holds.

Proof. Use induction in n. □

Finally Proposition 36 is proven using (210) and (209).

Appendix B. Discrete flows associated with binary Darboux transformations

When the supports of the measures are not bounded from below, (in which case the new "weights" (138) do not have in general a definite sign and therefore should not be considered as such), there is an alternative form of constructing discrete flows which preserve the positiveness/negativeness of the measures. The construction is based in the previous one, but now the shift is the composition of two consecutive shifts associated with the pair $\lambda_a(n)$ and $\lambda_a(n+1)$, being complex numbers conjugate to each other; i.e., we consider

Definition 27. We define a deformed moment matrix

$$g(s)=D_0'(s)g\big(\bar{D}_0'(s)\big)^{-1}, \tag{217}$$

with

$$D_0':=\sum_{a=1}^{p_1}D_{0,a}',$$

$$D_{0,a}':=\begin{cases}\prod_{n=1}^{s_a}(|\lambda_a(n)|^2\varPi_{1,a}-2\operatorname{Re}(\lambda_a(n))\varLambda_{1,a}+\varLambda_{1,a}^2), & s_a>0,\\ \varPi_{1,a}, & s_a=0,\\ \prod_{n=1}^{|s_a|}(|\lambda_a(-n)|^2\varPi_{1,a}-2\operatorname{Re}(\lambda_a(-n))\varLambda_{1,a}+\varLambda_{1,a}^2)^{-1}, & s_a<0,\end{cases} \tag{218}$$

$$\big(\bar{D}_0'\big)^{-1} := \sum_{b=1}^{p_2} \big(\big(\bar{D}_0(s)'\big)^{-1}\big)_b,$$

$$\big(\big(\bar{D}_0(s)'\big)^{-1}\big)_b := \begin{cases} \prod_{n=1}^{\bar{s}_b}(|\bar{\lambda}_b(n)|^2 \Pi_{2,b} - 2\operatorname{Re}(\bar{\lambda}_b(n))\Lambda_{2,b}^\top + (\Lambda_{2,b}^\top)^2), & \bar{s}_b > 0, \\ \Pi_{2,b}, & \bar{s}_b = 0, \\ (\prod_{n=1}^{\bar{s}_b}(|\bar{\lambda}_b(-n)|^2 \Pi_{2,b} - 2\operatorname{Re}(\bar{\lambda}_b(-n))\Lambda_{2,b}^\top) + (\Lambda_{2,b}^\top)^2)^{-1}, & \bar{s}_b < 0. \end{cases} \tag{219}$$

Proposition 38. *The previously defined deformed moment matrix corresponds to a moment matrix with the following positive/negative evolved weights*

$$w_{1,a}(s,x) = \mathscr{D}_a'(x, s_a) w_{1,a}(x),$$

$$\mathscr{D}_a' := \begin{cases} \prod_{n=1}^{s_a} |x - \lambda_a(n)|^2, & s_a > 0, \\ 1, & s_a = 0, \\ \prod_{n=1}^{|s_a|} |x - \lambda_a(-n)|^{-2}, & s_a < 0, \end{cases}$$

$$w_{2,b}(s,x) = \bar{\mathscr{D}}_b'(x, \bar{s}_b)^{-1} w_{2,b}(x),$$

$$\big(\bar{\mathscr{D}}_b'\big)^{-1} := \begin{cases} \prod_{n=1}^{\bar{s}_b} |x - \bar{\lambda}_b(n)|^2, & \bar{s}_b > 0, \\ 1, & \bar{s}_b = 0, \\ \prod_{n=1}^{|\bar{s}_b|} |x - \bar{\lambda}_b(-n)|^{-2}, & \bar{s}_b < 0. \end{cases} \tag{220}$$

Proceeding as in the previous case

Definition 28. We introduce

$$q_a' := \mathbb{I} - \Pi_{1,a}\big(\mathbb{I} - \big|\lambda_a(s_a+1)\big|^2\big) - 2\operatorname{Re}\big(\lambda_a(s_a+1)\big)\Lambda_{1,a} + \Lambda_{1,a}^2,$$

$$\bar{q}_b' := \mathbb{I} - \Pi_{2,b}\big(\mathbb{I} - \big|\bar{\lambda}_b(\bar{s}_b+1)\big|^2\big) - 2\operatorname{Re}\big(\bar{\lambda}_b(\bar{s}_b+1)\big)\big(\Lambda_{2,b}^\top\big) + \big(\Lambda_{2,b}^\top\big)^2, \tag{221}$$

and

$$\delta_a' := \mathbb{I} - C_a\big(\mathbb{I} - \big|\lambda_a(s_a+1)\big|^2\big) - 2\operatorname{Re}\big(\lambda_a(s_a+1)\big)L_a + L_a^2,$$

$$\bar{\delta}_b' := \mathbb{I} - \bar{C}_b\big(\mathbb{I} - \big|\bar{\lambda}_b(\bar{s}_b+1)\big|^2\big) - 2\operatorname{Re}\big(\bar{\lambda}_b(\bar{s}_b+1)\big)\bar{L}_b + \bar{L}_b^2. \tag{222}$$

The wave and adjoint wave functions now have the form

$$\Psi_a^{(k)}(z,s) = A_a^{(k)}(z,s)\mathscr{D}_a'(z,s_a), \qquad \big(\bar{\Psi}_b^*\big)^{(k)}(z,s) = \bar{A}_b^{(k)}(z,s)\bar{\mathscr{D}}_b'(z,\bar{s}_b)^{-1}, \tag{223}$$

and the expressions (161)–(162) still hold.

If we introduce ω_a' and $\bar{\omega}_b'$ as in (154) but replacing δ by δ', Eqs. (155)–(158) hold true by replacement of ω by ω'. Now, the form ω' differs from (164) as now we have

$$\omega_a' = \omega_{a,0}' \Lambda^{2(|\vec{n}_1| - n_{1,a}+1)} + \cdots + \omega_{a,2(|\vec{n}_1| - n_{1,a}+1)}',$$

$$\bar{\omega}_b' = \bar{\omega}_{b,0}'\big(\Lambda^\top\big)^{2(|\vec{n}_2| - n_{2,b}+1)} + \cdots + \bar{\omega}_{b,2(|\vec{n}_2| - n_{2,b}+1)}'. \tag{224}$$

With the definition of

$$\begin{aligned}\gamma'_{a,a'}(s,x) &:= (1-\delta_{a,a'}\big(1-\big|x-\lambda_a(s_a+1)\big|^2\big),\\ \gamma'_{b,b'}(s,x) &:= (1-\delta_{b,b'}\big(1-\big|x-\bar\lambda_b(\bar s_b+1)\big|^2\big),\end{aligned}\tag{225}$$

we have that

Proposition 39. *The present setting* (168) *and* (169) *are replaced by*

$$\begin{cases}\big(T_{a'}A_a^{(k)}\big)\gamma'_{a,a'}=\omega'_{a',0}A_a^{(k+2(|\vec n_1|-n_{1,a}+1))}+\cdots+\omega_{a',2(|\vec n_1|-n_{1,a}+1)}(k)A_a^{(k)},\\ \bar T_{b'}A_a^{(k)}=\bar\omega'_{b,0}(k)A_a^{(k-2(|\vec n_2|-n_{2,b'}+1))}+\cdots+\bar\omega'_{b,2(|\vec n_2|-n_{2,b'}+1)}A_a^{(k)},\end{cases}\tag{226}$$

$$\begin{cases}\rho'_{a',0}\big(T_{a'}\bar A_b^{(k-2(|\vec n_1|-n_{1,a'}+1))}\big)+\cdots+\rho'_{a',2(|\vec n_1|-n_{1,a'}+1)}(k)\big(T_{a'}\bar A_b^{(k)}\big)=\bar A_b^{(k)},\\ \big(\bar\rho'_{b',0}(k)\big(\bar T_{b'}\bar A_b^{(k+2(|\vec n_2|-n_{2,b'}+1)}\big)+\cdots+\bar\rho'_{b',2(|\vec n_2|-n_{2,b'}+1)}\big(\bar T_{b'}\bar A_b^{(k)}\big)\big)\bar\gamma'_{b,b'}=\bar A_b^{(k)}.\end{cases}\tag{227}$$

References

[1] M. Adler, P. van Moerbeke, Group factorization, moment matrices and Toda lattices, Int. Math. Res. Not. IMRN 12 (1997) 556–572.

[2] M. Adler, P. van Moerbeke, Generalized orthogonal polynomials, discrete KP and Riemann–Hilbert problems, Comm. Math. Phys. 207 (1999) 589–620.

[3] M. Adler, P. van Moerbeke, Vertex operator solutions to the discrete KP hierarchy, Comm. Math. Phys. 203 (1999) 185–210.

[4] M. Adler, P. van Moerbeke, The spectrum of coupled random matrices, Ann. of Math. 149 (1999) 921–976.

[5] M. Adler, P. van Moerbeke, Darboux transforms on band matrices, weights and associated polynomials, Int. Math. Res. Not. IMRN 18 (2001) 935–984.

[6] M. Adler, P. van Moerbeke, P. Vanhaecke, Moment matrices and multi-component KP, with applications to random matrix theory, Comm. Math. Phys. 286 (2009) 1–38.

[7] C. Álvarez-Fernández, U. Fidalgo, M. Mañas, The multicomponent 2D Toda hierarchy: Generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems, Inverse Problems 26 (2010) 055009 (17 pp.).

[8] R. Apery, Irrationalite de $\zeta(2)$ et $\zeta(3)$, Astèrisque 61 (1979) 11–13.

[9] M.J. Bergvelt, A.P.E. ten Kroode, Partitions, vertex operators constructions and multi-component KP equations, Pacific J. Math. 171 (1995) 23–88.

[10] F. Beukers, Padé approximation in number theory, in: Lecture Notes in Math., vol. 888, Springer-Verlag, Berlin, 1981, pp. 90–99.

[11] P.M. Bleher, A.B.J. Kuijlaars, Random matrices with external source and multiple orthogonal polynomials, Int. Math. Res. Not. IMRN 2004 (2004) 109–129.

[12] J. Coates, On the algebraic approximation of functions, I, II, III, Indag. Math. 28 (1966) 421–461.

[13] E. Daems, A.B.J. Kuijlaars, A Christoffel–Darboux formula for multiple orthogonal polynomials, J. Approx. Theory 130 (2004) 188–200.

[14] E. Daems, A.B.J. Kuijlaars, Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions, J. Approx. Theory 146 (2007) 91–114.

[15] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Operator approach to the Kadomtsev–Petviashvili equation. Transformation groups for soliton equations. III, J. Phys. Soc. Japan 50 (1981) 3806–3812.

[16] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations. Euclidean Lie algebras and reduction of the KP hierarchy, Publ. Res. Inst. Math. Sci. 18 (1982) 1077–1110.

[17] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, in: M. Jimbo, T. Miwa (Eds.), Nonlinear Integrable Systems–Classical Theory, Quantum Theory, World Scientific, Singapore, 1983, pp. 39–120.

[18] R. Felipe, F. Ongay, Algebraic aspects of the discrete KP hierarchy, Linear Algebra Appl. 338 (2001) 1–17.
[19] U. Fidalgo Prieto, G. López Lagomasino, Nikishin systems are perfect, arXiv:1001.0554, 2010.
[20] A.S. Fokas, A.R. Its, A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Comm. Math. Phys. (1992) 395–430.
[21] Ch. Hermite, Sur la fonction exponentielle, C. R. Acad. Sci. Paris 77 (1873) 18–24, 74–79, 226–233, 285–293; reprinted in his Oeuvres in: Tome III, Gauthier–Villars, Paris, 1912, pp. 150–181.
[22] H. Jager, A simultaneous generalization of the Padé table, I–VI, Indag. Math. 26 (1964) 193–249.
[23] V.G. Kac, J.W. van de Leur, The n-component KP hierarchy and representation theory, J. Math. Phys. 44 (2003) 3245–3293.
[24] A.B.J. Kuijlaars, Multiple orthogonal polynomial ensembles, arXiv:0902.1058v1 [math.PR], 2009.
[25] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, 1995.
[26] K. Mahler, Perfect systems, Compos. Math. 19 (1968) 95–166.
[27] M. Mañas, L. Martínez Alonso, The multicomponent 2D Toda hierarchy: Dispersionless limit, Inverse Problems 25 (2009) 115020 (22 pp.).
[28] M. Mañas, L. Martínez Alonso, C. Álvarez-Fernández, The multicomponent 2D Toda hierarchy: Discrete flows and string equations, Inverse Problems 25 (2009) 065007 (31 pp.).
[29] M. Mulase, Complete integrability of the Kadomtsev–Petviashvili equation, Adv. Math. 54 (1984) 57–66.
[30] E.M. Nikishin, On simultaneous Padé approximants, Mat. Sb. 113 (1980) 499–519 (in Russian), English translation Math. USSR Sb. 41 (1982) 409–425.
[31] A.Yu. Orlov, D.M. Scherbin, Fermionic representation for basic hypergeometric functions related to Schur polynomials, arXiv:nlin/0001001v4 [nlin.SI], 2009.
[32] M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, Res. Inst. Math. Sci. Kokyuroku 439 (1981) 30–46.
[33] J.A. Shohat, J.D. Tamarkin, The Problem of Moments, American Mathematical Society, 1943.
[34] B. Simon, The Christoffel–Darboux kernel, in: Proceedings of Symposia in Pure Mathematics 79: "Perspectives in Partial Differential Equations, Harmonic Analysis and Applications: A Volume in Honor of Vladimir G. Maz'ya's 70th Birthday", 2008, pp. 295–336, arXiv:0806.1528.
[35] V.N. Sorokin, On simultaneous approximation of several linear forms, Vestnik Moskov. Univ. Ser. I Mat. 1 (1983) 44–47.
[36] R.P. Stanley, Enumerative Combinatorics, Cambridge University Press, Cambridge, 1998.
[37] K. Ueno, K. Takasaki, Toda lattice hierarchy, in: Group Representations and Systems of Differential Equations, in: Adv. Stud. Pure Math., vol. 4, Mathematical Society of Japan, 1984, pp. 1–95.
[38] W. Van Assche, J.S. Geromino, A.B.J. Kuijlaars, Riemann–Hilbert problems for multiple orthogonal polynomials, in: Bustoz, et al. (Eds.), Special Functions 2000: Current Perspectives and Future Directions, Kluwer Academic Publishers, Dordrecht, 2001, pp. 23–59.