Existentially Closed Locally Finite $p$-Groups

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1. INTRODUCTION

Let $\mathcal{X}$ be a class of groups. An $\mathcal{X}$-group $G$ is said to be existentially closed (e.c.) in $\mathcal{X}$, if every system of finitely many equations and inequations with coefficients from $G$, which is solvable in some $\mathcal{X}$-supergroup of $G$, already has a solution in $G$. Denote the class of all locally-$\mathcal{X}$-groups by $\mathcal{LX}$. Then Proposition I.1.3 of J. Hirschfeld and W. H. Wheeler [9] ensures that every $\mathcal{LX}$-group of cardinality $N$ is contained in an e.c. $\mathcal{LX}$-group of cardinality $\max\{K_0, K\}$, if $\mathcal{X}$ is closed under forming subgroups. In [14, 15] we began to study e.c. $\mathcal{LX}$-groups, where $\mathcal{X}$ was supposed to satisfy several properties. The purpose of the present paper is to continue these investigations in the special case in which $\mathcal{X}$ is the class $\mathcal{L}\mathbb{F}_p$ of all locally finite $p$-groups ($p$ a prime).

Existentially closed $\mathcal{L}\mathbb{F}_p$-groups were first studied by B. Maier in [16]. His main tool was a generalization of an amalgamation theorem of G. Higman. If $G$ and $H$ are two groups which intersect in a common subgroup $U = G \cap H$, then the union $G \cup H$ (which is in general not a group!) is called the amalgam $G \cup H \mid U$ of $G$, $H$ over $U$. In [8, Theorem], G. Higman gave a necessary and sufficient condition for an amalgam of finite $p$-groups to be contained in a finite $p$-group, and B. Maier extended this result to amalgams $G \cup H \mid U$ of countable $\mathcal{L}\mathbb{F}_p$-groups $G$, $H$ over a finite subgroup $U$ [16, Satz 1]. This enabled him to construct embeddings of countable $\mathcal{L}\mathbb{F}_p$-groups into countable, e.c. $\mathcal{L}\mathbb{F}_p$-groups, and he proved that there exists up to isomorphism exactly one countable, e.c. $\mathcal{L}\mathbb{F}_p$-group $E_p$ [16, Satz 2]. He also posed the question whether the amalgamation theorem holds for $\mathcal{L}\mathbb{F}_p$-groups of arbitrary cardinality over a finite common subgroup too. By our result [15, Theorem 2.1], this question can be answered in the affirmative (Theorem 3.1). It then follows that any two e.c. $\mathcal{L}\mathbb{F}_p$-groups are partially isomorphic (Theorem 3.2), and that the e.c. $\mathcal{L}\mathbb{F}_p$-groups are exactly the direct limits of $E_p$'s (Theorem 3.3). Moreover we can
axiomatize the class of all e.c. $L\mathfrak{f}_p$-groups by an $L_{\omega_1 \omega}$-sentence (Theorem 3.4); this was suggested by B. Maier [16, pp. 125–126] in the case of an affirmative answer to his question. Furthermore he explained in [16, Satz 8] that the $L_{\omega_1 \omega}$-axiomatizability implies the existence of $2^\aleph$ pairwise nonisomorphic e.c. $L\mathfrak{f}_p$-groups of any cardinality $\aleph > \aleph_0$ (Theorem 3.6). We end Section 3 with some consequences for the structure of e.c. groups in the class $L(\mathfrak{f}_\pi \cap \mathfrak{N})$ of all locally finite-nilpotent $\pi$-groups resp. the class $L\mathfrak{N}_n$ of all locally nilpotent groups, whose torsion-subgroup is a $\pi$-group ($\pi$ a set of primes). The e.c. $L(\mathfrak{f}_\pi \cap \mathfrak{N})$-groups are also $L_{\omega_1 \omega}$-axiomatizeable (Theorem 3.5), and they are exactly the periodic, e.c. $L\mathfrak{f}_p$-groups (Theorem 3.7).

In Section 4 we use the fact that e.c. $L\mathfrak{f}_p$-groups are direct limits of $E_p$'s, in order to carry over all results of [14, Sect. 4] concerning the normal subgroup structure of $E_p$ to e.c. $L\mathfrak{f}_p$-groups of arbitrary cardinality. More precisely we show that every e.c. $L\mathfrak{f}_p$-group $G$ has exactly one chief series (cf. Section 2 for a definition), and that the order type of the chief series is a dense, linear order without endpoints. Moreover the normal subgroups $M$ in the chief factors $M/N$ of $G$ are precisely the normal closures $\langle g^G \rangle$, $g \in G \setminus 1$, and all other nontrivial normal subgroups of $G$ are e.c. $L\mathfrak{f}_p$-groups. The latter holds too for factor groups of $G$ modulo normal subgroups, which do not occur in any chief factor. In contrast to this factor groups by the normal subgroups occurring in the chief factors $M/N$ of $G$ as the group $N$ are not e.c. $L\mathfrak{f}_p$-groups, but they are e.c. in the class of all groups, which are a central extension of the cyclic group of order $p$ by an $L\mathfrak{f}_p$-group. Furthermore $G$ contains no proper subnormal subgroups; i.e., $G$ is a $T$-group. A discussion of these results may be found in Section 4. Section 5 contains the proof that the Schur multiplier of every e.c. $L\mathfrak{f}_p$-group is trivial.

In the next section we give a rather detailed description of the automorphism group of $E_p$ by using Cantor's method of "back and forth" together with embedding techniques developed in [14, Construction 4.1; 15, Theorem 3.1]. Since every automorphism of $E_p$ must map the unique chief series $\Sigma = \{(M_q, N_q)|q \in \mathbb{Q}\}$ of $E_p$ onto itself, there exist two special normal subgroups in Aut($E_p$). First, the group Inv($\Sigma$) of all automorphisms, which leave every normal subgroup of $E_p$ invariant; and, second, the group Stab($\Sigma$) of all automorphisms which centralize every chief factor of $E_p$. We show that Stab($\Sigma$) is the group of all locally inner automorphisms of $E_p$ (Theorem 6.1). Specific information about the structure of Stab($\Sigma$) is therefore provided by [14, Theorem 5.1]. Next we prove that the factors Aut($E_p$)/Inv($\Sigma$) and Inv($\Sigma$)/Stab($\Sigma$) are as large as possible; i.e., for every order-preserving permutation $\pi$ of $\mathbb{Q}$ there exists $\alpha \in \text{Aut}(E_p)$ such that $(M_q \alpha, N_q \alpha) = (M_{\pi q}, N_{\pi q})$ for all $q \in \mathbb{Q}$, and for every tuple $(\gamma_q)_{q \in \mathbb{Q}}$ of automorphisms $\gamma_q \in \text{Aut}(M_q/N_q)$ there exists $\beta \in \text{Inv}(\Sigma)$
such that \((mβ)N_q=(mN_q)γ_q\) for all \(q \in \mathbb{Q}\) (Theorem 6.2). In particular \(\text{Aut}(E_p)/\text{Stab}(Σ)\) is isomorphic to the unrestricted wreath product \(C_{p-1} \text{Wr}_Q A(\mathbb{Q})\), where \(C_{p-1}\) denotes the cyclic group of order \(p-1\) and \(A(\mathbb{Q})\) is the group of all order-preserving permutations of \(\mathbb{Q}\).

By Theorem 4.1 there exist two different kinds of embeddings of \(E_p\) onto a proper normal subgroup of itself. In Section 7 we apply these embeddings successively in order to construct \(2^{\aleph_1}\) e.c. \(\mathcal{L}G_p\)-groups of cardinality \(\aleph_1\), which are pairwise non-isomorphic, because the order-types of their chief series are pairwise distinct (Theorem 7.1). In these groups all proper normal subgroups are countable; dually we can also construct an uncountable e.c. \(\mathcal{L}G_p\)-group, all of whose non-trivial normal subgroups have countable index (Theorem 7.2). Finally, Section 8 contains the observation that e.c. \(\mathcal{L}G_p\)-groups are definitely distinct from the groups which may be constructed by using P. Hall's generalization of the restricted (regular) wreath product (cf. [6]).

**Notation.** As far as basic definitions in group theory are concerned the reader is referred to B. Huppert [10] and D.J.S. Robinson [18]. For definitions in logic and set theory the reader may consult J. Barwise [11], C. C. Chang and H. J. Keisler [4], and K. Kunen [12]. We use the following symbols:

- \(\mathbb{N}, \mathbb{Q}, \mathbb{R}\) sets of natural numbers, rationals, and real numbers;
- \(A(\mathbb{Q})\) group of all order-preserving permutations of \(\mathbb{Q}\);
- \(C_m\) cyclic group of order \(m\);
- \(E_p\) B. Maier's unique countable, e.c. \(\mathcal{L}G_p\)-group;
- \(\leq, \unlhd\) subgroup, normal subgroup;
- \(U \leq G\) \(U\) is isomorphic to a subgroup of \(G\);
- \(x^y = y^{-1}xy\);
- \([x, y] = x^{-1}y^{-1}xy\);
- \(|S|\) cardinality of the set \(S\);
- \(\langle S \rangle\) subgroup generated by the set \(S\);
- \(\langle S^T \rangle = \langle s^t | s \in S, t \in T \rangle\);
- \(G'\) commutator subgroup of \(G\) (= \(\langle [x, y] | x, y \in G \rangle\));
- \(C_G(U), Z(G)\) centralizer of \(U\) in \(G\), centre of \(G\);
- \(\text{Inn}(G)\) group of inner automorphisms of \(G\);
- \(\text{Aut}(G)\) automorphism group of \(G\);
- \(\theta | U\) restriction of the map \(\theta\) to \(U\);
- \(\text{Ker} \ θ\) kernel of the homomorphism \(θ\);
- \(G \cup H | U\) amalgam of \(G\) and \(H\) over \(U = G \cap H\) (see above).
2. Chief Series

In [14] we gave the definition of a chief series. For the convenience of the reader this definition is repeated now, since chief series are of fundamental importance for the description of the structure of e.c. $L_{p^n}$-groups.

Let $I$ be a totally ordered set. A family $\Sigma = \{(M_i, N_i) | i \in I\}$ of pairs of (normal) subgroups $M_i, N_i$ of a group $G$ is a (normal) series of order-type $I$ in $G$, if

(a) $G \setminus 1 = \bigcup_{i \in I} M_i \setminus N_i$,
(b) $N_i \subseteq M_i$ for all $i \in I$, and
(c) $M_k \leq N_i$ for all $i, k \in I$ with $k < i$.

The $M_i/N_i$ are the factors of $\Sigma$. The normal series $\Sigma$ is called a chief series, if each $M_i/N_i$ is a minimal normal subgroup of $G/N_i$; in this case the $M_i/N_i$ are called chief factors. An application of Zorn's lemma yields that every group has a chief series.

If $\bar{I}$ denotes the Dedekind-completion of $I$, we define the Dedekind-completion of the normal series $\Sigma$ to be $\bar{\Sigma} = \{(K_i, L_i) | i \in \bar{I}\}$, where

(a) $K_i = \begin{cases} \bigcap_{j \in I, j < i} N_j & \text{if } i \text{ is not maximal in } \bar{I}, \\ G & \text{otherwise}; \end{cases}$
(b) $L_i = \begin{cases} \bigcup_{j \in I, j < i} M_j & \text{if } i \text{ is not minimal in } \bar{I}, \\ \{1\} & \text{otherwise}. \end{cases}$

Note that the $K_i$ and $L_i$ are normal subgroups of $G$ satisfying

(a) $(K_i, L_i) = (M_i, N_i)$ for all $i \in I$,
(b) $K_i = L_i$ for all $i \in I \setminus I$, and
(c) $K_k \leq L_i$ for all $i, k \in I$ with $k < i$

(cf. [14, Lemma 4.6]).

In the case $G \in L_{\mathfrak{B}_p}$ all chief factors of $G$ are cyclic of order $p$ (cf. O. H. Kegel and B. A. F. Wehrfritz [11, Corollary 1.8]). Hence, if $U \leq G \in L_{\mathfrak{B}_p}$, and if $\Sigma = \{(M_i, N_i) | i \in I\}$ is a chief series in $G$, then $\Sigma \cap U = \{(M_i \cap U, N_i \cap U) | i \in I, M_i \cap U \neq N_i \cap U\}$ is a chief series in $U$, which we call the chief series induced by $\Sigma$ on $U$. For finite $U$, B. Maier introduced in [16, p. 114] the notion of a $G$-chief series as follows: A chief series $\Sigma_U$ in $U$ is called a $G$-chief series, if every finite group $X$ with
U ≤ X ≤ G has a chief series Σ_X such that Σ_X ∩ U = Σ_U. We can show that these two notions are equivalent.

**Lemma 2.1.** For any finite U ≤ G ∈ L G_p algebra, a chief series in U is a G-chief series, if and only if it is induced by a chief series of G on U.

**Proof.** If Σ_G and Σ_U are chief series in G and U (resp.) with Σ_G ∩ U = Σ_U, then any group X with U ≤ X ≤ G has the chief series Σ_X = Σ_G ∩ X satisfying Σ_X ∩ U = Σ_G ∩ U = Σ_U. Conversely let Σ_U = {(K_i, L_i)}_{i ∈ I} be a G-chief series in U. Regard the normal series \{ (KG_i, LG_i) \}_{i ∈ I} in G. Since G ∈ L G_p algebra, there exists for every i ∈ I and every g ∈ < K_i^G > ∩ U a finite group X with U ≤ X ≤ G and g ∈ < K_i^X > ∩ U. Because Σ_U is a G-chief series, we have V_i ≤ X such that V_i ∩ U = K_i, and we obtain g ∈ < K_i^G > ∩ U ≤ V_i ∩ U = K_i. This shows that < K_i^G > ∩ U = K_i for all i ∈ I; similarly < L_i^G > ∩ U = L_i. By Zorn's lemma there exists a chief series Σ_G = \{ (M_j, N_j) \}_{j ∈ J} in G such that we can find for every j ∈ J an i ∈ I with < L_j^G > ≤ N_j < M_j < < K_i^G >. But then Σ_G ∩ U = Σ_U.

Because of Lemma 2.1 we will also denote the chief series induced by some chief series of G on an infinite subgroup U of G as G-chief series in U.

Finally the reader should note that by [15, Theorem 2.3] every e.c. L G_p algebra G has a unique chief series; hence every subgroup of G has a unique G-chief series. Moreover, if G = E_p, then [15, Corollary 3.3] yields for every countable L G_p algebra H an embedding θ: H → G = E_p, which maps any prescribed chief series of H onto the G-chief series of Hθ. This has the following consequence.

**Lemma 2.2.** If Σ₁,...,Σ_{γ(U)} are the distinct chief series of the finite p-group U, then there exists for every k ∈ \{ 1,...,γ(U) \} a finite p-supergroup U_k^* of U such that every chief series of U_k^* induces Σ_k on U.

**Proof.** Use [15, Corollary 3.3] to find an embedding θ_k: U → E_p, which maps Σ_k onto the E_p-chief series of Uθ_k. By B. Maier [16, Hilfssatz 1] there exists a finite group X_k with Uθ_k ≤ X_k ≤ E_p such that every chief series of X_k induces the E_p-chief series on Uθ_k. Identify U with Uθ_k via θ_k and choose U_k^* = X_k.

3. **Axiomatizeability in L G_p**

As stated in the Introduction we can generalize the amalgamation theorems of G. Higman [8, Theorem] and B. Maier [16, Satz 1] for finite p-groups and countable L G_p algebras (resp.) as follows.
THEOREM 3.1. An amalgam $G \cup H \cup U$ of two $\mathcal{L}_{\mathcal{E}}^p$-groups $G$, $H$ over a finite group $U$ is contained in an $\mathcal{L}_{\mathcal{E}}^p$-group, if and only if there exist chief series $\Sigma_G$ in $G$ and $\Sigma_H$ in $H$ such that $\Sigma_G \cap U = \Sigma_H \cap U$.

Proof. If $W$ is an $\mathcal{L}_{\mathcal{E}}^p$-supergroup of $G \cup H \cup U$, then any chief series $\Sigma_W$ in $W$ induces the chief series $\Sigma_G \cap G$ on $G$ and $\Sigma_H \cap H$ on $H$, and we obtain $\Sigma_G \cap U = \Sigma_W \cap U = \Sigma_H \cap U$. For the converse, note that chief factors of $\mathcal{L}_{\mathcal{E}}^p$-groups are always central and cyclic of order $p$; hence [15, Theorem 2.1] can be applied.

The above theorem fails if $U$ is infinite: For in [14, Corollary 5.2] we gave an example of an amalgam $G \cup H \cup U$ with $G \cong H \cong U \cong F_p$, which is not contained in any periodic group, although the unique chief series in $U$ is both the $G$-chief series and the $H$-chief series in $U$.

We can now combine Theorem 3.1 with techniques of B. Maier [16] to obtain

THEOREM 3.2. For any two e.c. $\mathcal{L}_{\mathcal{E}}^p$-groups $G$ and $H$ the set $\Phi = \{ \varphi \mid \varphi \text{ is an isomorphism between finite subgroups } G_{\varphi} \text{ of } G \text{ and } H_{\varphi} \text{ of } H, \text{ which maps the } G\text{-chief series in } G_{\varphi} \text{ onto the } H\text{-chief series in } H_{\varphi} \}$ provides a partial isomorphism $\Phi : G \cong_p H$.

Proof. Obviously $\Phi$ contains the trivial map $1 \rightarrow 1$; hence $\Phi \neq \emptyset$.

In order to show the forth property fix $\varphi \in \Phi$ and $g \in G$. By B. Maier [16, Hilfssatz 1] there exists a finite group $X$ with $\langle G_{\varphi}, g \rangle \leq X \leq G$ such that every chief series in $X$ induces the $G$-chief series on $\langle G_{\varphi}, g \rangle$. In particular $\varphi$ maps the $X$-chief series in $G_{\varphi}$ onto the $H$-chief series in $H_{\varphi}$. Therefore Theorem 3.1 ensures that the amalgam $X \cup H \cup G_{\varphi} \cong H_{\varphi}$, where $G_{\varphi}$ and $H_{\varphi}$ are identified via $\varphi$, is contained in an $\mathcal{L}_{\mathcal{E}}^p$-supergroup of the e.c. $\mathcal{L}_{\mathcal{E}}^p$-group $H$. But the group table of $X$ may be expressed as a system of finitely many equations and inequations with coefficients from $G_{\varphi} \cong H_{\varphi}$. Hence we obtain an embedding $\varphi^* : X \rightarrow H$ with $\varphi^*|G_{\varphi} = \varphi$. By choice of $X$ the restriction $\varphi^*|\langle G_{\varphi}, g \rangle$ maps the $G$-chief series of $\langle G_{\varphi}, g \rangle$ onto the $H$-chief series of $\langle G_{\varphi}, g \rangle \varphi^*$. Therefore $\varphi^*|\langle G_{\varphi}, g \rangle \in \Phi$.

The same argument applied to the set $\Phi^{-1} = \{ \varphi^{-1} \mid \varphi \in \Phi \}$ shows that $\Phi$ has the back property.

In particular, Theorem 3.2 yields that any two e.c. $\mathcal{L}_{\mathcal{E}}^p$-groups are $\omega$-elementarily equivalent (cf. J. Barwise [1, Theorem 3]). We can use Theorem 3.2 in order to “cover” e.c. $\mathcal{L}_{\mathcal{E}}^p$-groups by copies of $E_p$.

THEOREM 3.3. For any group $G$ the following are equivalent:

1. $G$ is an e.c. $\mathcal{L}_{\mathcal{E}}^p$-group.
(2) Every finite subset of $G$ is contained in a subgroup of $G$ isomorphic to $E_p$.

(3) Every countable subset of $G$ is contained in a subgroup of $G$ isomorphic to $E_p$.

(4) Every subset $S$ of $G$ is contained in a subgroup of $G$ which is an e.c. $L\mathcal{F}_p$-group of cardinality $\max \{ N, |S| \}$.

Proof. (4) $\Rightarrow$ (3) $\Rightarrow$ (2): trivial.

(2) $\Rightarrow$ (1): Clearly (2) implies $G \in L\mathcal{F}_p$. If $\mathcal{S}$ is any system of finitely many equations and inequations with coefficients from $G$, which has a solution in some $L\mathcal{F}_p$-supergroup of $G$, then the (finite) set of coefficients is contained in a subgroup $V \cong E_p$ of $G$. But then there exists already a solution for $\mathcal{S}$ in $V \leq G$.

(1) $\Rightarrow$ (2): Since $G$ is locally finite, it suffices to prove (2) for any finite subgroup $U$ of $G$. By [15, Corollary 3.3] there exists an embedding $\varphi: U \to E_p$, which maps the $G$-chief series in $U$ onto the $E_p$-chief series in $U\varphi$. Choose $H = E_p$ and $\Phi$ as in Theorem 3.2. Then $\varphi \in \Phi$ and $\Phi: G \cong E_p$. Since $E_p$ is countable, the back property of $\Phi$ can be applied successively in order to extend $\varphi^{-1}: U\varphi \to U$ to an embedding $\theta: E_p \to G$.

(1) and (2) $\Rightarrow$ (4): Starting with $W_1 = \langle S \rangle$ we will construct inductively an ascending chain $\{ W_n \}_{n \in \mathbb{N}}$ of subgroups of cardinality $\max \{ N, |S| \}$ of $G$ such that every finite subset of $W_n$ is contained in a subgroup of $W_{n+1}$ isomorphic to $E_p$. By (2) $\Rightarrow$ (1) it then follows that $W = \bigcup_{n \in \mathbb{N}} W_n$ is an e.c. $L\mathcal{F}_p$-group of cardinality $\max \{ N, |S| \}$.

When performing the step $n \to n + 1$ we choose $\Sigma_n$ to be the local system of all finite subgroups of $W_n$. By (2) there exists for every $U \in \Sigma_n$ a subgroup $V_{U, n} \cong E_p$ of $G$ with $U \leq V_{U, n} \leq G$. Let $W_{n+1} = \langle V_{U, n} \mid U \in \Sigma_n \rangle$.

Theorem 3.3 shows that the e.c. $L\mathcal{F}_p$-groups are exactly those groups which have a local system of copies of $E_p$, i.e., which are direct limits of $E_p$'s. This characterization will enable us, in Section 4, to carry over all properties of $E_p$ which we proved in [14, Sect. 4] to e.c. $L\mathcal{F}_p$-groups of arbitrary cardinality.

As B. Maier observed in [16, pp. 125–126], Theorem 3.2 provides the key for the following axiomatization of all e.c. $L\mathcal{F}_p$-groups by an $L_{\omega_1 \omega}$-sentence.

Theorem 3.4. Let the set $\mathcal{M}_p$ of finite $p$-groups contain exactly one representative of every isomorphism type, and fix for every $U \in \mathcal{M}_p$ $p$-supergroups $U_k$, $1 \leq k \leq \gamma(U)$, of $U$ with the properties given by Lemma 2.2. Then the class of all e.c. $L\mathcal{F}_p$-groups is axiomatizable by the $L_{\omega_1 \omega}$-sentence, which is the conjunction of the following axioms:
(a) group axioms;
(b) \( \wedge_{n \in \mathbb{N}} \forall x_1 \cdots x_n \forall U \in \mathcal{M}_p \exists U: x_1, \ldots, x_n \in U; \)
(c) \( \wedge_{n \in \mathbb{N}} \forall x_1 \cdots x_n \forall U \in \mathcal{M}_p \land \exists U \subseteq U^*_k \rightarrow \land_{U \subseteq U^*_k} \forall v \in \mathcal{M}_p \exists V: U \leq V) \].

Remark. (1) If \( U = \{u_1, \ldots, u_m\} \), then the abbreviation "\( \exists U: x_1, \ldots, x_n \in U " \) means: Write down "\( \exists u_1 \cdots u_m " \) followed by the group table of \( U \) as a conjunction of equations and inequations in the variables \( u_1, \ldots, u_m \), and add "\( \wedge_{i=1}^n \wedge_{j=1}^m x_i = u_j. " \) Clearly (a) and (b) axiomatize the class \( L_{\mathcal{F}_p} \).

(2) The abbreviation "\( U = \{x_1, \ldots, x_n\} " \) means: Write down the group table of \( U \) in the variables \( x_1, \ldots, x_n \).

If \( U^*_k \setminus U = \{x_{n+1}, \ldots, x_m\} \), then "\( \exists U^*_k: U \leq U^*_k " \) indicates that "\( \exists x_{n+1}, \ldots, x_m " \) has to be followed by the group table of \( U^*_k \) in the variables \( x_1, \ldots, x_m \).

The expression "\( \exists V: U \leq V " \) abbreviates a corresponding kind of formula.

(3) A group \( G \) satisfies the axiom (c), if and only if

(i) every embedding \( \theta: U \rightarrow G \) of any \( U \in \mathcal{M}_p \) can be extended to an embedding \( \theta_k: U^*_k \rightarrow G \) for some \( k \in \{1, \ldots, \gamma(U)\} \), and

(ii) every pair of embeddings \( \varphi: U^*_k \rightarrow G \) and \( \psi: U^*_k \rightarrow V \), where \( V \) is a finite \( p \)-group, implies the existence of an embedding \( \eta: V \rightarrow G \) such that \( \varphi | U = (\psi \circ \eta) | U. \)

Proof of Theorem 3.4. It suffices to show that an \( L_{\mathcal{F}_p} \)-group \( G \) is e.c. in \( L_{\mathcal{F}_p} \), if and only if it satisfies the conditions (i) and (ii) of Remark (3).

First, let \( G \) be an e.c. \( L_{\mathcal{F}_p} \)-group. Fix \( U \in \mathcal{M}_p \) and an embedding \( \theta: U \rightarrow G \). Then there exists \( k \in \{1, \ldots, \gamma(U)\} \) such that every chief series of \( U^*_k \) induces the chief series on \( U \), which is mapped via \( \theta \) onto the \( G \)-chief series in \( U \theta \). By Theorem 3.1 the amalgam \( U^*_k \cup G | U \cong U \theta \), where \( U \) and \( U \theta \) are identified via \( \theta \), is contained in an \( L_{\mathcal{F}_p} \)-supergroup of the e.c. \( L_{\mathcal{F}_p} \)-group \( G \). Hence \( \theta \) can be extended to an embedding \( \theta_k: U^*_k \rightarrow G \). Moreover, if \( \varphi: U^*_k \rightarrow G \) and \( \psi: U^*_k \rightarrow V \in \mathcal{M}_p \) are embeddings, then the choice of \( U^*_k \) ensures that the \( V \)-chief series in \( U \psi \) is mapped via \( \psi^{-1} \circ \varphi \) onto the \( G \)-chief series in \( U \varphi \). Hence Theorem 3.1 yields that the amalgam \( V \cup G | U \psi \cong U \varphi \), where \( U \psi \) and \( U \varphi \) are identified via \( \psi^{-1} \circ \varphi \) is contained in an \( L_{\mathcal{F}_p} \)-supergroup of the e.c. \( L_{\mathcal{F}_p} \)-group \( G \). Therefore we obtain an embedding \( \eta: V \rightarrow G \) with \( (\psi \circ \eta) | U = \varphi | U. \)

Conversely suppose that the \( L_{\mathcal{F}_p} \)-group \( G \) satisfies (i) and (ii). Because of Theorem 3.3 it suffices to show that any finite subgroup of \( G \) is contained in a subgroup of \( G \) isomorphic to \( E_p \). Fix \( U \in \mathcal{M}_p \) and an embedding \( \theta: U \rightarrow G \). By (i) there exists an extension \( \theta_k: U^*_k \rightarrow G \) of \( \theta \) for some \( k \in \{1, \ldots, \gamma(U)\} \). Furthermore J. Hirschfeld and W. H. Wheeler [9,
Proposition 1.1.3] give an embedding \( \rho : U_k \to E_p \). Choose an ascending chain \( \{ H_n \}_{n \in \mathbb{N}} \) of finite subgroups of \( E_p \) with \( H_1 = U_k^* \rho \) and \( E_p = \bigcup_{n \in \mathbb{N}} H_n \). For every \( n \in \mathbb{N} \) we can find \( L_n \in \mathcal{M}_p \) and an isomorphism \( \rho_n : L_n \to H_n \). Moreover there exists \( k_n \in \{ 1, \ldots, \gamma(L_n) \} \) such that the \( (L_n)^k_\mu \)-chief series in \( L_n \) is mapped via \( \rho_n \) onto the \( E_p \)-chief series in \( L_n \). Hence the amalgam \( (L_n)^k_\mu \cup E_p \) \( L_n \cong L_n \rho_n \) is embeddable into an \( L\mathcal{F}_p \)-supergroup of \( E_p \) by Theorem 3.1, and we obtain an extension \( \varphi_n : (L_n)^k_\mu \to E_p \) of \( \rho_n \). Let \( H_n^* = (L_n)^k_\mu \varphi_n \).

We will now construct inductively embeddings \( \eta_n : H_n^* \to G \) with \( \emptyset \leq H_1 \eta_1 \) and \( \eta_{n+1} \mid H_n = \eta_n \mid H_n \). Then \( \eta \mid H_n = \eta_n \mid H_n \) defines an embedding \( \eta : E_p \to G \) with \( \emptyset \leq E_p \eta \). For \( n = 1 \) we obtain from (ii) that the pair of embeddings \( \theta_k : U_k \to G \) and \( \rho : U_k \to H_1 \eta_1 \) yields an embedding \( \eta_1 : H_1 \to G \) with \( \emptyset = \theta_k \mid U = (\rho \circ \eta_1) \mid U \), which means that \( \emptyset = \theta \circ \rho \eta_1 \leq H_1 \eta_1 \). During the induction step \( n \to n + 1 \) we obtain from (ii) that the pair of embeddings \( \varphi_n \cdot \eta_n : (L_n)^k_\mu \to G \) and \( \varphi_n : (L_n)^k_\mu \to (H_n^* \cup G_\mu) \) \( \eta_{n+1} \mid H_{n+1} = \eta_{n+1} \mid H_{n+1} \) yields an embedding \( \eta_{n+1} : (H_n^* \cup G_\mu) \to G \) with \( (\varphi_n \circ \eta_n) \mid L_n = (\varphi_n \circ \eta_{n+1}) \mid L_n \), which means that \( \eta_n \mid H_n = \eta_{n+1} \mid H_n \).

By B. Maier [16, Satz 5] the e.c. \( L(\mathcal{F}_n, \mathfrak{R}) \)-groups are exactly the direct products \( \otimes_{p \in \pi} G_p \), where \( G_p \) is an e.c. \( L\mathcal{F}_p \)-group. Since axiom (c) of Theorem 3.4 is just a statement about continuations of embeddings of finite \( p \)-groups, the following theorem follows easily from the above.

**THEOREM 3.5.** Let the set \( \mathcal{M}_\pi \) of finite, nilpotent \( \pi \)-groups contain exactly one representative of every isomorphism type, and denote the axiom (c) of Theorem 3.4 by \( \Psi_p \). Then the class of all e.c. \( L(\mathcal{F}_n, \mathfrak{R}) \)-groups is axiomatizable by the \( L_{\omega_1 \omega} \)-sentence which is the conjunction of the following axioms:

(a) \( \text{group axioms} \),

(b) \( \bigwedge_{n \in \mathbb{N}} \forall x_1 \cdots x_n \bigvee_{U \in \mathfrak{R}_n} \exists U : x_1, \ldots, x_n \in U \),

(c) \( \bigwedge_{p \in \pi} \Psi_p \).

As B. Maier has pointed out in [16, Satz 8], the \( L_{\omega_1 \omega} \)-axiomatizability of the e.c. \( L\mathcal{F}_p \)-groups allows the use of model theoretic methods, which yield

**THEOREM 3.6.** There exist exactly \( 2^\kappa \) pairwise non-isomorphic, e.c. \( L\mathcal{F}_p \)-groups of each cardinality \( \kappa > \kappa_0 \). If \( \kappa \) is regular, then there exist even \( 2^\kappa \) pairwise non-embeddable e.c. \( L\mathcal{F}_p \)-groups of cardinality \( \kappa \).

In [16, Satz 6], B. Maier has shown that the unique countable, e.c. \( L(\mathcal{F}_n, \mathfrak{R}) \)-group \( E_\pi = \otimes_{p \in \pi} E_p \) is e.c. in \( L\mathcal{R}_\pi \) too. In the proof he needed the countability of \( E_\pi \) only because he had to amalgamate every \( p \)-com-
ponent $E_p$ of $E_\pi$ with some finite $p$-group. We can therefore follow his proof and use Theorem 3.1 in order to establish

**Theorem 3.7.** The e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{A})$-groups are exactly the periodic, e.c. $L\mathfrak{N}_\pi$-groups.

Since the e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{A})$-groups are the direct products of e.c. $L\mathfrak{F}_p$-groups, $p \in \pi$, it is a direct consequence that Theorem 3.6 holds also for e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{A})$-groups and e.c. $L\mathfrak{N}_\pi$-groups.

4. Normal Subgroup Structure

We will now collect all the information about the normal subgroup structure of e.c. $L\mathfrak{F}_p$-groups from [15, Sect. 21 and transpose the remaining properties, which we proved for $E_p$ in [14, Sect. 43, to e.c. $L\mathfrak{F}_p$-groups of arbitrary cardinality.

**Theorem 4.1.** The following holds in every e.c. $L\mathfrak{F}_p$-group $G$.

(a) If $M/N$ is a chief factor in $G$, then $M/N$ is central and cyclic of order $p$, and $N = M'$ holds as well as $M = \langle g^G \rangle$ for all $g \in M \setminus N$. In particular the group $G$ has a unique chief series $\Sigma$, the order-type of $\Sigma$ is a dense, linear order without endpoints, and the normal subgroups of $G$ form a chain.

(b) If $M/N$ is a chief factor in $G$, then there exists for every coset $gN$, $g \in M \setminus N$, and for every $p$-power $p^n \neq 1$ an element of order $p^n$ in $gN$, and any two elements of order $p$ in $gN$ are conjugate in $G$.

(c) Every non-trivial normal subgroup $K$ of $G$ with $K \neq \langle g^G \rangle$ for all $g \in G$ is an e.c. $L\mathfrak{F}_p$-group, and every automorphism of $K$ induced by conjugation with an element from $G$ is locally inner.

(d) For every normal subgroup $K$ of $G$ which does not occur in any chief factor of $G$, the factor group $G/K$ is an e.c. $L\mathfrak{F}_p$-group.

(e) There exist no proper subnormal subgroups in $G$; i.e., every subnormal subgroup of $G$ is normal in $G$.

**Proof.** Part (a) follows from O. H. Kegel and B. A. F. Wehrfritz [11, Corollary 1.B.8] and from [15, Theorems 2.3 and 2.6].

(b) By [15, Theorem 2.5] the coset $gN$ contains an element of order $p^n$ for every $\mu \in N$. Now let $x, y \in gN$ have order $p$. Because of Theorem 3.3 the finite group $U = \langle g, x, y \rangle$ is contained in a subgroup $V \cong E_p$ of $G$. Now $g \in V$ ensures that $M \cap V/N \cap V$ is a chief factor in $V$ with $x, y \in g(N \cap V)$. Therefore [14, Theorem 4.10] yields that $x$ and $y$ are conjugate in $V \leq G$. 
(c) Let $H$ be an $L\mathfrak{S}_p$-supergroup of $K$, in which a system $\mathcal{S}$ of finitely many equations and inequations with coefficients $u_1, \ldots, u_r \in K$ has a solution. Choose $U = \langle u_1, \ldots, u_r \rangle$ and $L = \langle U^G \rangle$. Because of (a) there exists $k \in \{1, \ldots, r\}$ with $L = \langle u_k^G \rangle$. Hence $L \leq K$ by assumption. Let $g \in L \setminus K$ and denote by $M/N$ the unique chief factor in $G$ with $g \in M \setminus N$. Then (a) yields $L \leq N \leq M \leq K$. By Theorem 3.3 the finite group $\langle U, g \rangle$ is contained in a subgroup $V \cong E_p$ of $G$. Now $g \in V$ ensures that $M \cap V/N \cap V$ is a chief factor in $V$ with $U \leq N \cap V$.

Consider the amalgam $H \cup V \cup U$. By [14, Theorem 4.11(c)] the group $N \cap V$ has a unique chief series. Hence $U \leq N \cap V \leq V$ implies that the chief series in $V$ must induce the $(N \cap V)$-chief series on $U$. Similarly any chief series in $H$ must induce the $(N \cap V)$-chief series on $U$. Therefore Theorem 3.1 ensures the existence of an $L\mathfrak{S}_p$-supergroup of the amalgam $H \cup V \cup U$. But $V$ is an e.c. $L\mathfrak{S}_p$-group. Hence, by [14, Theorem 4.11(c)], the system $\mathcal{S}$ with coefficients from $U \leq N \cap V$ must already have a solution in $M \cap V \leq K$. This shows that $K$ is an e.c. $L\mathfrak{S}_p$-group. As in [14, Theorem 4.8] it follows that every automorphism of $K$ induced by conjugation with an element from $G$ is locally inner.

(d) By Theorem 3.3 it suffices to show that any finite subset $\{u_1, K, \ldots, u_r, K\}$ of $G/K$ is contained in a subgroup of $G/K$ isomorphic to $E_p$. Fix the representatives $u_1, \ldots, u_r$. By Theorem 3.3 the finite group $U = \langle u_1, \ldots, u_r \rangle$ is contained in a subgroup $V \cong E_p$ of $G$. Starting with $V_1 = V$ we will construct inductively an ascending chain $\{V_n\}_{n \in \mathbb{N}}$ of subgroups $V_n \cong E_p$ of $G$.

In the step $n \rightarrow n + 1$ we choose $V_{n+1} = V_n$, if $K \cap V_n$ does not occur in any chief factor of $V_n$. Otherwise there exists a chief factor $X/Y$ in $V$ such that either $X = K \cap V_n$ or $Y = K \cap V_n$. Moreover the uniqueness of the chief series in $V_n$ ensures the existence of a chief factor $M/N$ in $G$ such that $(X, Y) = (M \cap V_n, N \cap V_n)$. In the case $X = K \cap V_n$ choose $g_n \in K \setminus M$, in the case $Y = K \cap V_n$ choose $g_n \in N \setminus K$. Then apply Theorem 3.3 to find a subgroup $V_{n+1} \cong E_p$ of $G$ containing $\langle V_n, g_n \rangle$.

Again by Theorem 3.3 the union $W = \bigcup_{n \in \mathbb{N}} V_n$ must be a countable, e.c. $L\mathfrak{S}_p$-group, and hence $W \cong E_p$. Moreover the choice of the elements $g_n$, $n \in \mathbb{N}$, ensures that $K \cap W$ does not occur in any chief factor of $W$. Therefore [15, Theorem 4.3] yields $E_p \cong W/K \cap W \cong WK/K \geq UK/K$.

(e) Let $S$ be a subnormal subgroup in $G$. Fix $s \in S$ and $g \in G$. By Theorem 3.3 there exists a subgroup $V \cong E_p$ of $G$ with $s, g \in V$. Now $S \cap V$ is a subnormal subgroup in $V$ with $s \in S \cap V$. Since all chief factors of $V$ have order $p$, [14, Theorem 4.11(f)] yields $S \cap V \leq V$, and hence $s^g \in S \cap V \leq S$. This shows that $S \leq G$.

Part (a) of Theorem 4.1 shows that an e.c. $L\mathfrak{S}_p$-group $G$ has three types of non-trivial, proper normal subgroups: First, the normal closures $\langle g^G \rangle$,
g \in G \setminus 1
groups \( M \) in the chief factors \( M/N \) of \( G \); second, the derived groups \( \langle g^G \rangle \)
groups \( N \) in the chief factors \( M/N \) of \( G \); and finally the normal subgroups \( K \), which
do not occur in any chief factor—they give rise to the pairs
\((K, K) \in \Sigma_G \setminus \Sigma_G\), where \( \Sigma_G \) denotes the chief series in \( G \) (cf. Section 2).

**Question.** Does every dense, linear order without endpoints occur as
the order-type of the chief series in some e.c. \( L\bar{\mathcal{F}}_p \)-group?

If the answer to this question were affirmative, then every complete,
dense, linear order without endpoints would yield an e.c. \( L\bar{\mathcal{F}}_p \)-group, which
do not contain any normal subgroup of “type \( K \).” It should also be noted
that an e.c. \( L\bar{\mathcal{F}}_p \)-group is not uniquely determined by the order-type of its
chief series, since S. Thomas has constructed in [19] e.c. \( L\bar{\mathcal{F}}_p \)-groups of
cardinality \( \aleph_1 \), whose chief series have order-type \( \aleph \) (which occurs also for
\( E_p \)).

By part (e) of Theorem 4.1, e.c. \( L\bar{\mathcal{F}}_p \)-groups have no proper subnormal
subgroups. Groups with this property are called \( T \)-groups,
and they have been studied in many papers. A survey may be found in [18, Part I,
p. 174], where D. J. S. Robinson poses the question whether every locally
nilpotent \( T \)-group is soluble. This can now be answered in the negative:
Every e.c. \( L\bar{\mathcal{F}}_p \)-group \( G \) is a locally nilpotent \( T \)-group, but it cannot be
soluble, since it contains a copy of every finite \( p \)-group \( P \) (simply consider
\( G \times P \)).

With respect to part (c) of Theorem 4.1 we may ask

**Question.** Which e.c. \( L\bar{\mathcal{F}}_p \)-groups can be embedded onto a proper normal
subgroup of a given e.c. \( L\bar{\mathcal{F}}_p \)-group?

By an argument of [14] it can be shown that \( C_G(H) = 1 \) for every non-trivial
normal subgroup \( H \) of an e.c. \( L\bar{\mathcal{F}}_p \)-group \( G \). Hence Theorem 4.1
implies that an e.c. \( L\bar{\mathcal{F}}_p \)-group \( H \) occurs as normal subgroup of \( G \), if and
only if the canonical embedding \( H \to \text{Inn}(H) \) can be extended to an
embedding of \( G \) into the group of all locally inner automorphisms of \( H \). In
particular \( |G| \leq 2^{|H|} \). Under the assumption \( \diamond \), S. Thomas has constructed
in [19] e.c. \( L\bar{\mathcal{F}}_p \)-groups of cardinality \( \aleph_1 \) which admit no outer
automorphism. These groups cannot be a proper normal subgroup of any
e.c. \( L\bar{\mathcal{F}}_p \)-group.

Corresponding to parts (c) and (d) of Theorem 4.1 one could expect that
for any chief factor \( M/N \) of an e.c. \( L\bar{\mathcal{F}}_p \)-group \( G \) the factor group \( G/M \)
is e.c. in \( L\bar{\mathcal{F}}_p \) too. However, we have shown in [15, Therem 4.3] that this is
not the case (the proof of [15, Theorem 4.3(b)] can be generalized to
uncountable e.c. \( L\bar{\mathcal{F}}_p \)-groups by using Theorem 3.3). But let us consider the
class \( \mathcal{C} \) of all \( L\bar{\mathcal{F}}_p \)-groups \( H \), whose centre contains the cyclic group \( C_p \)
of order \( p \). Note that inclusion in \( \mathcal{C} \) is defined as follows: For \( H_1, H_2 \in \mathcal{C} \) we
have $H_1 \subseteq H_2$, if and only if $H_1 \leq H_2$ and $C_p$ as a subgroup of $\mathbb{Z}(H_1)$ is equal to $C_p$ as a subgroup of $\mathbb{Z}(H_2)$. Then $\mathbb{C}$ is inductive, and every $\mathbb{C}$-group $H$ is contained in an e.c. $\mathbb{C}$-group of cardinality $\max\{\aleph_0, |H|\}$.

**Theorem 4.2.** Let $M/N$ be a chief factor of an e.c. $L\mathbb{F}_p$-group $G$. If we identify $M/N$ with $C_p$, then $G/N$ is an e.c. $\mathbb{C}$-group.

**Proof.** Let $H$ be a $\mathbb{C}$-subgroup of $G/N$, in which a system $\mathcal{S}$ of finitely many equations and inequations with coefficients $g_1, \ldots, g_r \in G/N$ has a solution $h_1, \ldots, h_r \in H$. We may assume that $H$ is e.c. in $\mathbb{C}$. Fix the representatives $g_1, \ldots, g_r$ as well as some $g_0 \in M \setminus N$. Then $\langle g_0 \rangle = C_p \leq \mathbb{Z}(H)$. Let $U = \langle g_0, \ldots, g_r \rangle$. By Theorem 3.3 the $U$ group is contained in a subgroup $V \cong E_p$ of $G$. Now $g_0 \in V$ implies that $M \cap V/N \cap V$ is a chief factor in $\mathbb{C}$. Hence [16, Theorem 4.2] yields that $V$ splits over $N \cap V$ with some complement $\tilde{C}$. Define $c_0, \ldots, c_r \in \tilde{C}$ via $c_1 N = g_1 N$, let $C = \langle c_0, \ldots, c_r \rangle$ and denote by $\mathcal{S}$ the system which we get from $\mathcal{S}$ via replacing $g_i N$ by $c_i$ for $1 \leq i \leq r$. By Theorem 4.1(a) we have $g_0 \in \langle c^V \rangle \leq \langle c^G \rangle$ for every $c \in C \setminus 1$, and hence $N \cap C = 1$.

Regard the amalgam $G \cup H | C \cong UN/N$, where $c \in C$ and $uN \in UN/N$ are identified, if and only if $cN = uN$. Since $G/N \leq H$, every chief series in $H$ must induce the $G/N$-chief series on $UN/N$, which is the $G$-chief series on $C$. Hence Theorem 3.1 ensures the existence of an $L\mathbb{F}_p$-group $W$, which contains the amalgam. Moreover we can follow the proof of [15, Theorem 2.3] to show that $g_0 N \in \langle h^W \rangle \leq \langle h^W \rangle$ for all $h \in \langle UN/N, h_1, \ldots, h_r \rangle \setminus 1$. But $G$ is an e.c. $L\mathbb{F}_p$-group, and therefore we can find already in $G$ elements $x_1, \ldots, x_r$ solving $\mathcal{S}$ such that $c_0 \in \langle x^G \rangle$ for all $x \in \langle C, x_1, \ldots, x_r \rangle \setminus 1$. In particular $\langle C, x_1, \ldots, x_r \rangle \cap N = 1$ by Theorem 4.1(a), and $x_1 N, \ldots, x_r N$ must be a solution for $\mathcal{S}$ in $G/N$. \[\]

5. **Schur Multipliers**

In Section 4 we stated that for any chief factor $M/N$ of an e.c. $L\mathbb{F}_p$-group $G$ the factor group $G/M$ is not e.c. in $L\mathbb{F}_p$. Now $M/N \leq \mathbb{Z}(G/N)$, and $G/N$ does not split over $M/N$, since $G$ has a unique chief series. Therefore the above result may be seen as a consequence of

**Theorem 5.** The Schur multiplier of every e.c. $L\mathbb{F}_p$-group $G$ is trivial, and $G$ is its own covering group.

**Proof.** Theorem 3.1 of [14] ensures that $G$ is perfect. Hence $G$ is its own covering group, if the Schur multiplier of $G$ is trivial. By Theorem 3.3 the group $G$ is a direct limit of $E_p$'s. Therefore Proposition I.5.10 of F. R. Beyl and J. Tappe [3] yields that it suffices to prove that the Schur
multiplier of $E_p$ is trivial. Let \( \{ U_n \}_{n \in \mathbb{N}} \) be an ascending chain of finite subgroups of $E_p$ with $U_1 = 1$ and $E_p = \bigcup_{n \in \mathbb{N}} U_n$. Because $E_p$ is perfect, we also have $E_p = \bigcup_{n \in \mathbb{N}} U'_n$.

Now assume that $H$ is any perfect group with $H/Z = E_p$ for some $Z \leq Z(H)$. We will construct inductively embeddings $\sigma_n : U_n \to H$ such that $(u \sigma_n)Z = u$ for all $u \in U_n$ and $\sigma_n | U'_n = \sigma_n | U'_n$. The definition of $\sigma_1$ is trivial. In the induction step $n \to n + 1$, B. Maier [16, Hilfssatz 1] gives the existence of a finite group $U_{n+1}^*$ with $U_{n+1}^* \leq U_{n+1}^* \leq E_p$ and such that every chief series in $U_{n+1}^*$ induces the $E_p$-chief series on $U_{n+1}$. Now let $V$ be a vector space over the field $GF(p)$ with $p$ elements, whose basis elements $v_q$ are indexed by the rationals $q \in \mathbb{Q}$. Regard McLain’s group $M(\mathbb{Q}, GF(p))$, i.e., the group of all linear transformations of $V$ onto itself generated by the elements $1 + q_2 v_q$, $a \in GF(p)$, $q_1 < q_2$ in $\mathbb{Q}$, where $v_{q_1}v_{q_2}$ sends $v_{q_1}$ onto $v_{q_2}$, and annihilates the rest of the basis. $M(\mathbb{Q}, GF(p))$ is the union of an ascending chain of groups of finite upper unitriangular matrices over $GF(p)$. Hence $M(\mathbb{Q}, GF(p))$ is a countable $L\mathbb{F}_p$-group, which contains a copy of every finite $p$-group. In particular we may identify $U_{n+1}^*$ with some subgroup of $M(\mathbb{Q}, GF(p))$. By choice of $U_{n+1}^*$ every chief series in $M(\mathbb{Q}, GF(p))$ induces the $E_p$-chief series on $U_{n+1}$. Therefore B. Maier [16, Satz 2] ensures that $U_{n+1}$ is contained in a subgroup $M_{n+1} \cong M(\mathbb{Q}, GF(p))$ of $E_p$. From A. J. Berrick and R. G. Downey [2, Proposition 1.4] we obtain an embedding $\tau_{n+1} : M_{n+1} \to H$ such that $(m \tau_{n+1})Z = m$ for all $m \in M_{n+1}$. Choose $\sigma_{n+1} = \tau_{n+1} | U_{n+1}$. Then there exists for every $u \in U_n$ an element $z_u \in Z \leq Z(H)$ such that $u \sigma_{n+1} = u \sigma_{n} \cdot z_u$. Therefore $\sigma_{n+1} | U'_n = \sigma_n | U'_n$.

Finally define $\sigma : E_p \to H$ via $\sigma | U'_n = \sigma_n$. Clearly the image $E_p \sigma$ is a complement to $Z$ in $H$. Therefore $E_p \sigma = H' = H$ and $Z = 1$. This shows that $E_p$ is its own covering group, and hence the Schur multiplier of $E_p$ is trivial.

Theorem 5 corresponds nicely to the result of R. E. Phillips [17, Theorem A] that the Schur multiplier of every countable, e. c. locally finite group (and therefore the Schur multiplier of every e. c. locally finite group) is trivial.

6. THE AUTOMORPHISM GROUP OF $E_p$

Since $E_p$ has a unique chief series $\Sigma = \{(M_q, N_q) | q \in \mathbb{Q}\}$, every automorphism of $E_p$ must permute the pairs $(M_q, N_q) \in \Sigma$. In particular $\text{Aut}(E_p)$ has the normal subgroups

$$\text{Inv}(\Sigma) = \{ \alpha | \alpha \in \text{Aut}(E_p) \text{ with } (M_q \alpha, N_q \alpha) = (M_q, N_q) \text{ for all } q \in \mathbb{Q} \}$$
174

\[ \text{and} \]

\[ \text{Stab}(\Sigma) = \{ \alpha | \alpha \in \text{Inv}(\Sigma) \text{ with } (m\alpha) \ N_q = mN_q \text{ for all } m \in M_q \text{ and all } q \in \mathbb{Q} \}. \]

Moreover the chief factors \( M_q/N_q \) are central, and every locally inner automorphism of \( E_p \) must be contained in \( \text{Stab}(\Sigma) \). Conversely [14, Theorem 5.3] yields that every \( \alpha \in \text{Stab}(\Sigma) \) of finite order is locally inner. We can extend this result to

**Theorem 6.1.** \( \text{Stab}(\Sigma) \) is the group of all locally inner automorphisms of \( E_p \).

**Proof.** Fix \( \alpha \in \text{Stab}(\Sigma) \). It suffices to show that there exists for every finite subgroup \( U \) of \( E_p \) an element \( g \in E_p \) with \( u^\alpha = u \alpha \) for all \( u \in U \). This will be proved by induction on \( |U| \). The case \( |U| = 1 \) is trivial. Now assume \( |U| > 1 \). Let \( (M, N) \in \Sigma \) be the unique pair with \( U = M \cap U \) and \( U \supseteq U \cap N \). Let \( V = U \cap N \), and choose \( w \in U \setminus V \). By induction there exists \( h \in E_p \) such that \( v^h = v \alpha \) for every \( v \in V \). If \( h \) denotes the inner automorphism induced by \( h \) on \( E_p \), then \( \beta = \alpha h^{-1} \in \text{Stab}(\Sigma) \).

Suppose we can find \( g \in E_p \) such that \( w^g = w \beta \) and \( [V, g] = 1 \). It then follows that \( w^{gh} = w \beta h = w \alpha \) as well as \( v^{gh} = v^h = v \alpha \) for all \( v \in V \). Since \( U = \langle V, w \rangle \) this will complete the induction step. Hence it is enough to find \( g \in E_p \) with \( w^g = w \beta \) and \( [V, g] = 1 \). In the following we suppose the reader to be familiar with the technical part of [14, Sect. 4]. Let \( \{ G_n \}_{n \in \mathbb{N}} \) be an ascending chain of finite subgroups of \( E_p \) with \( G_1 = V, G_2 = \langle U, w \beta \rangle \) and \( E_p = \bigcup_{n \in \mathbb{N}} G_n \). Denote the canonical epimorphism \( E_p \to E_p/N \) by \( \theta \). By [14, Construction 4.1] the standard embedding \( \sigma : G_1 \to G_1 \ Wr E_p/N \) with respect to the countermap \( \theta^{-1} \) (for \( \theta(G_1) \)) can be extended to an embedding \( \sigma : E_p \to W = \bigcup_{n \in \mathbb{N}} \left[ (G_n \lambda G_n) \ Wr E_p/N \right] \), where \( W \in \mathcal{L} \bar{S}_p \). (See [14, Sect. 4] for notation and definitions.) Because of \( \beta \in \text{Stab}(\Sigma) \) and \( w \subset M \setminus N \) we have \( w \theta = w \beta \theta \), and in the notation of [14, Construction 4.1] we obtain \( w \sigma = (w\theta, s_w^2) \) and \( w \beta \sigma = (w\theta, s_w^2) \). Now let \( T \) be a right transversal of \( \langle w \theta \rangle \) in \( E_p/N \). By [14, Lemma 4.2] the element \( z_1 = (1, s_1) \in W \), where \( s_1((w\theta)^r t) = s_w^2(t) \) for all \( t \in T, 0 \leq r \leq p - 1 \), conjugates \( w \sigma \) onto \((w\theta, s_w^2) \in W \), where

\[ \bar{s}_w(h) = \begin{cases} (1, (w\theta)^r t \cdot t \theta^{-1}) = (1, w^p) & \text{for } h = (w\theta)^{p-1} t, t \in T, \\ 1 & \text{else.} \end{cases} \]

Similarly \( z_2 = (1, s_2) \in W \), where \( s_2((w\theta)^r t) = s_w^2 r(t) \) for all \( t \in T, 0 \leq r \leq p - 1 \), conjugates \( w \beta \sigma \) onto \((w\theta, s_w^2) \in W \), where

\[ \bar{s}_{w\beta}(h) = \begin{cases} (1, (w\beta)^p) = (1, w^p \beta) & \text{for } h = (w\theta)^{p-1} t, t \in T, \\ 1 & \text{else.} \end{cases} \]
But \( w^p \in V \); hence we have \( \tilde{s}_w = \tilde{s}_{w\beta} \) and

(i) \( (w\sigma)^{\tilde{s}_{w\beta}^{-1}} = w\beta\sigma \).

Let \( \tilde{g} = z_1z_2^{-1} \). We want to show \( [V\sigma, \tilde{g}] = 1 \). Straightforward calculations yield that \( \tilde{g} = (1, s) \), where \( s((w\theta)^{-1}t) = (1, (w \cdot (w\beta)^{-1})t) \) for all \( t \in T, 0 \leq r \leq p - 1 \). Moreover, \( \sigma \) maps \( V = G_1 \) onto the diagonal of \( (V \lambda_1)_{Wr E_p/N \leq W} \). Therefore \( [V\sigma, \tilde{g}] = 1 \) holds, if and only if

\[ [V, w \cdot (w\beta)^{-1}] = 1. \]

But the latter is true, since the fact that \( \beta|V = id \) forces \( w \) to act on \( V \subseteq U \) in the same way as \( w\beta \) acts on \( V = V\beta \subseteq U\beta \). Hence

(ii) \( [V\sigma, \tilde{g}] = 1 \).

The conditions (i) and (ii) may be expressed by a system of finitely many equations and inequations with coefficients from \( V\sigma \cup \{w\sigma, w\beta\sigma\} \), which have the solution \( \tilde{g} \) in the \( L_{E_p}\)-supergroup \( W \) of \( E_p \). But then there must already exist \( g \in E_p \) with \( w^g = w\beta \) and \( [V, g] = 1 \).

By P. Hall and B. Hartley [7, Lemma 4] the group \( Stab(\Sigma) \) has a series with cyclic factors of order \( p \). Moreover [14, Sect. 5] contains some information about the structure of the group \( Stab(\Sigma) \) of all locally inner automorphisms of \( E_p \). It should also be noted that \( Stab(\Sigma) \) contains the group \( Inn(E_p) \) of all inner automorphisms of \( E_p \), which is isomorphic to \( E_p \), since \( E_p \) has trivial centre.

Now let \( A(\Omega) \) be the group of all order-preserving permutations of \( \Omega \). Define the wreath product \( C_{p - 1} \_ WR_\Omega A(\Omega) \) to be the set \( \{(\pi, \gamma) | \pi \in A(\Omega), \gamma : A(\Omega) \to C_{p - 1} \} \) with multiplication \( (\pi_1, \gamma_1) \cdot (\pi_2, \gamma_2) = (\pi_1 \pi_2, \gamma_1^{\pi_2} \gamma_2) \), where \( (\gamma_1^{\pi_2} \gamma_2)(q) = \gamma_1(q\pi_2^{-1}) \cdot \gamma_2(q) \) for all \( q \in \Omega \). In [14, Sect. 5] we have already indicated that \( Aut(E_p)/Inv(\Sigma) \) is isomorphic to a subgroup of \( A(\Omega) \), while \( Inv(\Sigma)/Stab(\Sigma) \) is isomorphic to a subgroup of \( \prod_{q \in \Omega} Aut(M_q/N_q) \). Moreover \( M_q/N_q \cong C_p \) implies \( Aut(M_q/N_q) \cong C_{p - 1} \). It is now our aim to prove \( Aut(E_p)/Stab(\Sigma) \cong C_{p - 1} \_ WR_\Omega A(\Omega) \).

By Theorem 4.1(b) we can fix for every \( q \in \Omega \) an element \( x_q \in M_q \_ N_q \) of order \( p \). Define

\[ \Phi : Aut(E_p) \to C_{p - 1} \_ WR_\Omega A(\Omega) \]

via

\[ \Phi : \alpha \to (\pi, \gamma), \]

where \( \pi \in A(\Omega) \) is given by

\[ (M_{qa}, N_{qa}) = (M_q x, N_q x) \]

for all \( q \in \Omega \),

and where \( \gamma(q) \in Aut(C_p) \cong C_{p - 1} \) is given by

\[ (c)(\gamma(q)) = c^{\gamma(q)} \]

for all \( c \in C_p \),
if and only if

$$\alpha = (x_q N_q)^{r_q} \quad (1 \leq r_q \leq p - 1).$$

Straightforward calculations yield that $\Phi$ is a homomorphism with $\text{Ker } \Phi = \text{Stab}(\Sigma)$. All that remains is to establish

**Theorem 6.2.** The above homomorphism $\Phi$ is surjective; in particular, $\text{Aut}(E_p)/\text{Stab}(\Sigma) \cong C_{p-1} \text{Wr}_Q A(Q)$.

**Proof.** Fix $(\pi, \gamma) \in C_{p-1} \text{Wr}_Q A(Q)$, choose $y_q \in (x_q N_q)^{r_q} \cap \langle x_q \rangle$ for every $q \in Q$, and denote by $\Psi$ the set of all isomorphisms $\psi : U \to V$ between two finite subgroups $U$ and $V$ of $E_p$, which satisfy $(M_q \cap U) \psi = M_{q\pi} \cap V$ and $(N_q \cap U) \psi = N_{q\pi} \cap V$ as well as $(x_q N_q \cap U) \psi = y_q N_{q\pi} \cap V$ for all $q \in Q$. It suffices to show that $\Psi$ provides a partial isomorphism of $E_p$ onto itself, since Cantor's method of "back and forth" will then ensure the existence of $\alpha \in \text{Aut}(E_p)$ with $\alpha \Phi = (\pi, \gamma)$.

Obviously $\Psi$ contains the trivial map $1 \to 1$; hence $\Psi \neq \emptyset$.

In order to show the *forth* property fix $\psi : U \to V$ from $\Psi$ and $g \in E_p$. Let $\bar{U} = \langle U, g \rangle$, and denote by $\{q_1, \ldots, q_r\}$ the set of all $q \in Q$, which satisfy $M_q \cap \bar{U} \neq N_q \cap \bar{U}$. Since the chief factors of $E_p$ are cyclic of order $p$, we have $x_q N_q \cap U \neq \emptyset$, if and only if $M_q \cap U \neq N_q \cap U$. In this case we choose $h_j \in x_q N_q \cap U$ and $g_j = h_j \psi$; then $\psi \in \Psi$ implies $g_j \in y_q N_{q\pi}$. Otherwise choose $h_j \in x_q N_q \cap \bar{U}$ and apply [15, Lemma 2.21] to find an element $g_j \in M_{q\pi} \setminus N_{q\pi}$, which centralizes $V$ and is conjugate in $E_p$ to $y_{q\pi}$. Because the chief factors of $E_p$ are central, we have $g_j \in y_q N_{q\pi}$. Now we can follow the proof of [15, Theorem 3.11] to find an embedding $\tilde{\psi} : \bar{U} \to E_p$ with $\tilde{\psi} | U = \psi$ and $h_j \tilde{\psi} \in g_j N_{q\pi}$ for $1 \leq j \leq r$. The latter implies $(M_q \cap U) \tilde{\psi} = M_{q\pi} \cap \bar{U} \tilde{\psi}$ and $(N_q \cap U) \tilde{\psi} = N_{q\pi} \cap \bar{U} \tilde{\psi}$ for all $q \in Q$. It remains to show that $(x_q N_q \cap \bar{U}) \tilde{\psi} = y_q N_{q\pi} \cap \bar{U} \tilde{\psi}$ for all $q \in Q$.

In the case $q \notin \{q_1, \ldots, q_r\}$ we have $M_q \cap \bar{U} = N_q \cap \bar{U}$ and $M_{q\pi} \cap \bar{U} \tilde{\psi} = N_{q\pi} \cap \bar{U} \tilde{\psi}$; therefore $x_q N_q \cap \bar{U} = \emptyset = y_q N_{q\pi} \cap \bar{U} \tilde{\psi}$, and $(x_q N_q \cap \bar{U}) \tilde{\psi} = y_q N_{q\pi} \cap \bar{U} \tilde{\psi}$ holds trivially. In the case $q = q_j$ we have $x_q N_q = h_j N_q$ and $y_{q\pi} N_{q\pi} = (h_j \tilde{\psi}) N_{q\pi}$. Hence for every $w \in x_q N_q \cap \bar{U} = h_j N_q \cap \bar{U}$ there exists $n \in N_q$ such that $w = h_j n$, and we obtain $w \tilde{\psi} = h_j \tilde{\psi} \cdot m \tilde{\psi} \in (h_j \tilde{\psi}) N_{q\pi} \cap \bar{U} \tilde{\psi} = y_q N_{q\pi} \cap \bar{U} \tilde{\psi}$; this shows $(x_q N_q \cap \bar{U}) \tilde{\psi} \subseteq y_q N_{q\pi} \cap \bar{U} \tilde{\psi}$. A similar argument yields the converse inclusion. Therefore $\psi \in \Psi$.

In order to show the back property for $\Psi$ we can follow the above proof with $\Psi^{-1} = \{\psi^{-1} | \psi \in \Psi\}$ in the role of $\Psi$.

It should be noted that the proof of Theorem 6.2 actually shows that every isomorphism $\psi : U \to V$ between two finite subgroups $U$ and $V$ of $E_p$ which satisfies $(M_q \cap U) \psi = M_{q\pi} \cap V$ and $(N_q \cap U) \psi = N_{q\pi} \cap V$, as well as $(x_q N_q \cap U) \psi = y_q N_{q\pi} \cap V$ for all $q \in Q$, can be extended to an aut-
morphism $\alpha$ of $E_p$ with $(M_q \alpha, N_q \alpha) = (M_{q^n}, N_{q^n})$ and $(x_q N_q) \alpha = y_{q^n} N_{q^n}$. This generalizes a result of B. Maier (cf. [16, Satz 21].

Clearly $\text{Inv}(\Sigma)/\text{Stab}(\Sigma)$ is isomorphic to the base group of $C_{p-1} \wr \mathbb{A}(\mathbb{Q})$, which is a cartesian product of countably many cyclic groups of order $p - 1$, while $\text{Aut}(E_p)/\text{Inv}(\Sigma)$ is isomorphic to $\mathbb{A}(\mathbb{Q})$. Results on the structure of $\mathbb{A}(\mathbb{Q})$ (normal subgroups, conjugacy classes of elements, etc.) may be found in A. M. W. Glass [5, Chap. 2].

7. Groups of Cardinality $\aleph_1$

It is a consequence of Theorem 4.1 that there exist two different types of embeddings of $E_p$ onto a proper normal subgroup of itself:

(a) For every chief factor $M/N$ in $E_p$ we have $N \cong E_p$.

(b) For every non-trivial, proper normal subgroup $K$ of $E_p$ which does not occur in any chief factor of $E_p$, we have $K \cong E_p$.

We call an embedding of the first type rational embedding since $\mathbb{Q}$ is the order-type of the chief series $\Sigma$ of $E_p$, while embeddings of the second type are said to be irrational embeddings since the corresponding normal subgroups of $E_p$ occur in the pairs of $\Sigma \setminus \Sigma$, and since $\Sigma$ has $\mathbb{R}$ as index set.

By constructing partial isomorphisms of $\mathbb{Q}$ onto itself it can be shown that there exists for any two irrational (rational) numbers $r_1$ and $r_2$ an order-preserving permutation $\pi$ of $\mathbb{Q}$ with

$$\pi(q) < r_2 \iff q < r_1 \quad \text{for all } q \in \mathbb{Q} \quad (\text{resp. } \pi(r_1) = r_2).$$

Therefore Theorem 6.2 yields that rational and irrational embeddings of $E_p$ into itself do not depend on the choice of the involved chief factor $M/N$ resp. normal subgroup $K$.

The aim of this section is the construction of certain e.c. $L\mathfrak{P}_\mu$-groups of cardinality $\aleph_1$ by successive use of rational or irrational embeddings of $E_p$ into itself. Therefore we need some combinatorial set theory.

A set $C$ of ordinals $\langle \omega_1$ is called a club, if

(i) for every $\mu < \omega_1$ there exists $\tau \in C$ with $\mu < \tau$, and

(ii) for every limit ordinal $\lambda < \omega_1$ the following holds: If $\{\mu \mid \mu \in C$ and $\mu < \lambda\}$ contains no maximal element, then $\lambda \in C$.

A set $S$ of ordinals $\langle \omega_1$ is called stationary, if $S \cap C \neq \emptyset$ for every club $C$.

**Lemma 7.1.** There exists a set $\mathcal{A}$ of $2^{\aleph_1}$ different maps $\delta : \{\mu \mid \mu < \omega_1\} \to \{0, 1\}$ such that $\{\mu \mid \mu < \omega_1$ and $\gamma(\mu) \neq \delta(\mu)\}$ is stationary for all $\gamma, \delta \in \mathcal{A}$ with $\gamma \neq \delta$. 


Proof. \( \{ \mu \mid \mu < \omega_1 \} \) is the disjoint union of \( \omega_1 \) stationary sets \( S_\tau, \tau < \omega_1 \), by a well-known theorem of Ulam (cf. K.Kunen [12, Theorem II.6.11 and remark after Corollary II.6.12]). For every \( X \subseteq \{ \tau \mid \tau < \omega_1 \} \) define

\[
\delta_X(\mu) = \begin{cases} 1 & \text{if } \mu \in S, \text{ for some } \tau \in X, \\ 0 & \text{else}. \end{cases}
\]

Then choose \( A = \{ \delta_X \mid X \subseteq \{ \tau \mid \tau < \omega_1 \} \} \).

**Lemma 7.2.** Let \( \{ I_\mu \}_{\mu < \omega_1} \) and \( \{ J_\mu \}_{\mu < \omega_1} \) be ascending chains of sets such that \( I_\lambda = \bigcup_{\mu < \lambda} I_\mu \) and \( J_\lambda = \bigcup_{\mu < \lambda} J_\mu \) for every limit ordinal \( \lambda < \omega_1 \). Then for any bijection \( \theta: \bigcup_{\mu < \omega_1} I_\mu \to \bigcup_{\mu < \omega_1} J_\mu \) the set \( C_\theta = \{ \mu \mid \mu < \omega_1 \text{ and } I_\mu \theta = J_\mu \} \) is a club.

**Proof.** Condition (ii) of the definition of a club holds because of \( I_\lambda = \bigcup_{\mu < \lambda} I_\mu \) and \( J_\lambda = \bigcup_{\mu < \lambda} J_\mu \) for limit ordinals \( \lambda < \omega_1 \). In order to verify (i) fix \( \mu \in C_\theta \). The regularity of \( \omega_1 \) ensures that there exists for every \( \tau < \omega_1 \) an ordinal \( \sigma < \omega_1 \) with \( \tau \leq \sigma \) and \( I_\tau \theta \subseteq J_\sigma \) (resp. \( J_\tau \theta \subseteq I_\sigma \)). In particular we can find an ascending sequence \( \{ \tau_n \}_{n \in \mathbb{N}} \) of ordinal numbers \( \tau_n < \omega_1 \) with \( \tau_1 = \mu + 1 \) and \( J_{\tau_1} \subseteq I_{\tau_2} \subseteq J_{\tau_3} \subseteq I_{\tau_4} \subseteq \cdots \). Let \( \tau \) be the smallest ordinal with \( \tau_n \leq \tau \) for all \( n \in \mathbb{N} \). Then \( I_\tau \theta = J_\tau \), and the regularity of \( \omega_1 \) ensures that \( \tau < \omega_1 \).

We now define for every \( \delta: \{ \mu \mid \mu < \omega_1 \} \to \{ 0, 1 \} \) the group \( G_\delta \) to be the direct limit of groups \( G_\mu^\delta \cong E_\mu, \mu < \omega_1 \), subject to the following embeddings:

(i) For every \( \mu < \omega_1 \) let \( G_\mu^\delta \to G_{\mu+1}^\delta \) be

- a rational embedding, if \( \delta(\mu) = 0 \),
- an irrational embedding, if \( \delta(\mu) = 1 \).

(ii) For every limit ordinal \( \lambda < \omega_1 \) let \( G_\lambda^\delta \) be the direct limit of the \( G_\mu^\delta, \mu < \lambda \), and embed \( G_\mu^\delta \to G_\lambda^\delta \) in the obvious way. (Theorem 3.3 ensures that \( G_\mu^\delta \cong E_\mu \).)

For convenience we identify every \( G_\mu^\delta, \mu < \omega_1 \), with the corresponding subgroup of \( G^\delta \). Then \( G^\delta \) is the union of the ascending chain \( \{ G_\mu^\delta \}_{\mu < \omega_1} \), where \( G_\lambda = \bigcup_{\mu < \lambda} G_\mu^\delta \) for every limit ordinal \( \lambda < \omega_1 \), and where \( G_\mu^\delta \subseteq G_{\mu+1}^\delta \) such that

- there exists a minimal normal subgroup in \( G^\delta/G_\mu^\delta \), if \( \delta(\mu) = 0 \);
- there exists no minimal normal subgroup in \( G^\delta/G_\mu^\delta \), if \( \delta(\mu) = 1 \).

By Theorem 3.3 every \( G^\delta \) is an e.c. \( L\aleph_1 \)-group, and since the chain \( \{ G_\mu^\delta \}_{\mu < \omega_1} \) is strictly ascending, \( G^\delta \) has cardinality \( \aleph_1 \).
THEOREM 7.3. Let \( \mathcal{A} \) be the set of maps \( \delta: \{ \mu | \mu < \omega_1 \} \to \{0, 1\} \) given by Lemma 7.1. Then the groups \( G^\delta, \delta \in \mathcal{A}, \) are \( 2^{\aleph_1} \) pairwise non-isomorphic e.c. \( L\Delta_p \)-groups of cardinality \( \aleph_1 \) with the following properties:

(a) The order-types of the unique chief series of the \( G^\delta, \delta \in \mathcal{A}, \) are pairwise distinct.

(b) Every proper normal subgroup of \( G^\delta \) is countable.

(c) If \( K \not\subsetneq G^\delta \) does not occur in any chief factor of \( G^\delta, \) then \( G/K \cong G^\epsilon \) for some \( \epsilon: \{ \mu | \mu < \omega_1 \} \to \{0, 1\}. \)

Proof. Clearly (a) implies that the \( G^\delta, \delta \in \mathcal{A}, \) are pairwise non-isomorphic. Hence it suffices to prove (a), (b), and (c). By transfinite induction using Theorem 4.1(e) it can be shown that every normal subgroup of any of the \( G^\mu, \mu < \omega_1, \) is already normal in \( G^\delta. \)

(a) Let \( \Sigma^\delta \) be the unique chief series in \( G^\delta. \) The above implies that \( M/N \) is a chief factor in \( G^\delta \) if and only if \( M/N \) is a chief factor in some \( G^\mu, \mu < \omega_1. \) Hence \( \Sigma^\delta = \{(M, N)|(M, N) \text{ occurs in the chief series of some } G^\mu, \mu < \omega_1\}, \) and the order-type \( I^\delta \) of \( \Sigma^\delta \) is the union of the ascending series \( \{I^\mu\}_{\mu < \omega_1} \) of the order-types \( I^\mu \) of the chief series in \( G^\mu, \mu < \omega_1. \) Moreover \( I^\delta = \bigcup\mu \leq \lambda I^\mu \) for every limit ordinal \( \lambda < \omega_1, \) and \( I^\delta \setminus I^\mu \) contains

\[
\begin{align*}
\text{a minimal element,} & \quad \text{if } \delta(\mu) = 0; \\
\text{no minimal element,} & \quad \text{if } \delta(\mu) = 1.
\end{align*}
\]

By choice of \( \mathcal{A} \) the set \( \{ \mu | \mu < \omega_1 \text{ and } \gamma(\mu) \neq \delta(\mu) \} \) is stationary for any \( \gamma, \delta \in \mathcal{A} \) with \( \gamma \neq \delta. \) Assume that there exists an order-preserving bijection \( \theta: I^\mu \to I^\delta. \) Because of Lemma 7.2 the set \( \{ \mu | \mu < \omega_1 \text{ and } I^\mu \theta = I^\mu \} \) is a club. Hence there exists \( \mu < \omega_1 \) such that \( \gamma(\mu) \neq \delta(\mu) \) and \( I^\mu \theta = I^\mu \). But this is a contradiction, since \( \gamma(\mu) \neq \delta(\mu) \) yields that one of \( I^\mu \setminus I^\mu \) and \( I^\delta \setminus I^\delta \) contains a minimal element while the other does not.

(b) Let \( K \not\subsetneq G. \) Choose \( x \in G^\delta \setminus K. \) Then \( x \in G^\mu \) for some \( \mu, \delta < \omega_1. \) Now we obtain from Theorem 4.1.(a) that \( K \leq \langle x^G \rangle \cong G^\mu \cong E_\mu. \) Hence \( K \) must be countable.

(c) As in (b) we have \( K \not\subsetneq G^\mu \) for some \( \mu < \omega_1. \) Since every chief factor in \( G^\mu \) is a chief factor in \( G^\delta, \) the normal subgroup \( K \) cannot occur in any chief factor of \( G^\delta. \) Hence Theorem 4.1.d) yields \( G^\delta/K \cong E_\mu. \) This holds for every \( \mu < \omega_1 \) with \( K \not\subsetneq G^\delta. \) Choose \( \mu_0 < \omega_1 \) minimal with respect to \( K \not\subsetneq G^\delta. \) Then \( G^\delta/K \) is the union of the ascending chain \( \{G^\mu/K\}_{\mu_0 \leq \mu < \omega_1} \) of subgroups \( G^\mu/K \cong E_\mu, \) where \( G^\delta/K = \bigcup_{\mu_0 \leq \mu < \omega_1} G^\mu/K \) for every limit ordinal \( \lambda \) with \( \mu_0 < \lambda < \omega_1, \) and where \( \text{id}: G^\mu/K \to G^{\mu+1}/K \) is a rational resp. irrational embedding, if and only if \( \delta(\mu) = 0 \) resp. \( \delta(\mu) = 1. \)

Since \( \omega_1 \) is the smallest uncountable ordinal, there exists an order-preserving bijection \( \alpha: \{ \mu | \mu < \omega_1 \} \to \{ \mu | \mu_0 \leq \mu < \omega_1 \} \) (cf. K. Kunen [12,
Theorem 1.6.3), and hence $G^\delta/K \cong G^\varepsilon$, where $\varepsilon$ denotes the composition of $\alpha$ and $\delta$.

Let $\aleph_{s+1}$ be any successor cardinal. If we could find an e.c. $L\aleph_p$-group $G$ of cardinality $\aleph_s$, which is embeddable onto two normal subgroups of itself, one of which occurs in a chief factor of $G$ while the other does not, then the above method would yield $2^{\aleph_{s+1}}$ e.c. $L\aleph_p$-groups of cardinality $\aleph_{s+1}$, which are pairwise non-isomorphic, because the order-types of their chief series are pairwise distinct.

Finally, fix any $K \not\subseteq E_p$, which is non-trivial and does not occur in any chief factor of $E_p$. By [15, Theorem 4.1] the group $E_p$ splits over $K$. Fix any complement $H$ for $K$ in $E_p$. From Theorem 4.1(d) we know that $H \cong E_p$. Define the group $G$ to be the direct limit of groups $G_\mu \cong E_p$, $\mu < \omega_1$, subject to the following embeddings:

(i) For $\mu < \omega_1$, if $E_p = KH$ gives rise to $G_{\mu+1} = K_{\mu+1}H_{\mu+1}$, then let $G_\mu \to G_{\mu+1}$ be an embedding of $G_\mu$ onto $H_{\mu+1}$.
(ii) For limit ordinals $\lambda < \omega_1$ let $G_\lambda = \bigcup_{\mu < \lambda} G_\mu$.

Again we identify each $G_\mu$ with the corresponding subgroup in $G$. As a counterpart to Theorem 7.3(b) we can now show

**Theorem 7.4.** The above construction yields an e.c. $L\aleph_p$-group $G$ of cardinality $\aleph_1$, all of whose non-trivial proper normal subgroups have countable index.

**Proof.** Let $1 \not\subseteq N \subseteq G$, and choose $\tau < \omega_1$ with $N \cap G_\tau \not= 1$. Define an ascending chain $\{L_\mu\}_{\tau < \mu < \omega_1}$ of subgroups of $G$ via

(i) $L_\tau = 1$,
(ii) $L_{\mu+1} = K_{\mu+1}L_\mu$ for $\tau < \mu < \omega_1$, and
(iii) $L_\lambda = \bigcup_{\tau < \mu < \lambda} L_\mu$ for limit ordinals $\lambda$ with $\tau < \lambda < \omega_1$.

By transfinite induction it can be shown that $L_\mu \cong G_\mu$ and that $G_\mu$ splits over $L_\mu$ with complement $G_\tau$ for $\tau < \mu < \omega_1$. Hence $G$ splits over $L = \bigcup_{\tau < \mu < \omega_1} L_\mu$ with complement $G_\tau$.

Now choose $x \in (N \cap G_\tau) \setminus 1$. Then $x \not\subseteq L$. Therefore Theorem 4.1(a) yields $L \leq \langle x^G \rangle \leq N$, and it suffices to show that $L$ has countable index in $G$. But $G/L \cong G_\tau \cong E_p$.

8. **P. Hall's Generalized Wreath Products**

The restricted wreath product $W = A \wr_\Omega B$ of a group $A$ and a permutation group $B$ on $\Omega$ is the set $\{(b, f) \mid b \in B; f : \Omega \to A$ with finite
support} with multiplication \((b_1,f_1) \cdot (b_2,f_2) = (b_1b_2,f_1^{b_2}f_2)\), where 
\((f_1^{b_2}f_2)(\omega) = f_1(\omega b_2^{-1}) \cdot f_2(\omega)\) for all \(\omega \in \Omega\). It is a split extension of its base group \(\{(1,f) \in W \mid f : \Omega \to A\}\) by its top group \(\{(b,1) \mid b \in B\}\). Fix some element \(1 \in \Omega\). We identify in the obvious way \(B\) with the top group and \(A\) with the \(1\)-component \(\{(1,f) \mid a \in A\}\), where

\[ f_a(\omega) = \begin{cases} a & \text{if } \omega = 1, \\ 1 & \text{else.} \end{cases} \]

The restricted wreath product may also be defined as a permutation group on the set \(\{(a, \omega) \mid a \in A, \omega \in \Omega\}\). In the case when \(B\) acts right regularly on \(\Omega = B\), this leads to the following generalization due to P. Hall (cf. [6]).

Let \(\Lambda\) be a totally ordered set, and let \(\{H_\lambda \mid \lambda \in \Lambda\}\) be a family of groups. Denote by \(V\) the set of all tuples \((v_\lambda)_{\lambda \in \Lambda}\) with \(v_\lambda \in H_\lambda\), where only finitely many \(v_\lambda\) are non-trivial. We define for every \(h \in H_\lambda\) a permutation \(p_h : (v_\lambda)_{\lambda \in \Lambda} \to (v'_\lambda)_{\lambda \in \Lambda}\) on \(V\) via

\[ v'_\mu = v_\mu \quad \text{if } \mu \neq \lambda, \]

\[ v'_\lambda = \begin{cases} v_\lambda h & \text{if } v_\mu = 1 \text{ for all } \mu > \lambda, \\ v_\lambda & \text{else.} \end{cases} \]

The (regular) generalized restricted wreath product \(W = \text{Wr}_{\Lambda \in \Lambda} H_\lambda\) is now the group of permutations of \(V\) generated by the groups \(J_\lambda = \{p_h \mid h \in H_\lambda\} \cong H_\lambda\), \(\lambda \in \Lambda\).

P. Hall has shown in [6] that the structure of the commutator subgroup \(W'\) of \(W\) can be very homogeneous. For example, if \(H_\lambda \cong C_p\) for all \(\lambda \in \Lambda\), and if \(\Lambda\) has the property that for any \(\lambda, \kappa \in \Lambda\) with \(\lambda \neq \kappa\) there exists an order-preserving permutation \(\alpha\) of \(\Lambda\) with \(\kappa < \lambda\alpha\), then \(W'\) is characteristically simple, has trivial Baer-radical, and contains a copy of every finite \(p\)-group. \(E_p\) has all these properties too (cf. B. Maier [16, Satz 6] and [14, Theorem 3.1]). Therefore it seems to be worthwhile to point out that e.c. \(L\tilde{\mathfrak{F}}_p\)-groups are definitely different from commutator subgroups of generalized wreath products.

**Theorem 8.** If the totally ordered set \(\Lambda\) contains at least two elements, then the commutator subgroup \(W'\) of \(W = \text{Wr}_{\Lambda \in \Lambda} H_\lambda\) is not an e.c. \(L\tilde{\mathfrak{F}}_p\)-group for every family \(\{H_\lambda \mid \lambda \in \Lambda\}\) of non-trivial groups.

**Proof.** Assume that \(W'\) is e.c. in \(L\tilde{\mathfrak{F}}_p\). Choose \(\lambda < \kappa\) from \(\Lambda\). By P. Hall [6, Sect. 2.2] we have \(W = W^-_\lambda \text{ wr}_\Lambda (H_\lambda \text{ wr}_\lambda W^+_\mu)\), where \(W^-_\lambda = \text{Wr}_{\mu < \lambda} H_\mu\) and \(W^+_\lambda = \text{Wr}_{\mu > \lambda} H_\mu\), and where \(\Lambda = \otimes_{\mu > \lambda} H_\mu\) and \(I = H_\lambda \times \Lambda\). Denote by \(D_\lambda\) the base group of the top group \(H_\lambda \text{ wr}_\lambda W^+_\lambda\) of \(W\).
Assume, that \( H \notin \mathbb{L}_p \). Then \( \langle u_1, \ldots, u_r \rangle \) is not a finite \( p \)-group for some \( u_1, \ldots, u_r \in H \). Choose \( x \in W_\lambda \setminus \{1\} \). Then the elements \( [x, u_i] = (1, f_i) \in H \wr A W^+_{\lambda i}, 1 \leq i \leq r \), satisfy

\[
 f_i(\delta) = \begin{cases} 
 u_i^{-1} & \text{if } \delta = 1x, \\
 u_i & \text{if } \delta = 1, \\
 1 & \text{else},
\end{cases}
\]

and therefore they generate a subgroup of \( W' \), which is not a finite \( p \)-group. This contradiction shows that \( H \in \mathbb{L}_p \).

Let \( M/N \) be a chief factor in \( H \). Then \( M/N \cong C_p \) implies that \( K = \{ (1, f) \in D \mid f(A) \subseteq M \} \) and \( L = \{ (1, f) \in D \mid f(A) \subseteq N \} \) are normal subgroups in \( D \) with elementary abelian \( p \)-factor group \( K/L \). Assume, that \( x_1, x_2 \in W^+_{\lambda} \setminus \{1\} \) are distinct. Choose a right transversal \( T \) of \( N \) in \( M \), and define for every \( t \in T \) elements \( (1, f_1), (1, g_1) \in W' \cap D \) by \( (1, f_1) = [x_1, t] \) and \( (1, g_1) = [x_2, t] \). Clearly

\[
 f_1(\delta) = \begin{cases} 
 t^{-1} & \text{if } \delta = 1x_1, \\
 t & \text{if } \delta = 1, \\
 1 & \text{else},
\end{cases} \quad \quad g_1(\delta) = \begin{cases} 
 t^{-1} & \text{if } \delta = 1x_2, \\
 t & \text{if } \delta = 1, \\
 1 & \text{else}.
\end{cases}
\]

If \( D \) denotes the base group of \( W = W_{\lambda} \wr T (H \wr A W^+_{\lambda}) \), then the \( (1, f_1), (1, g_1) \), where \( t \in T \), are elements in \( D \wr K \cap W' \) which lie in \( 2p - 1 \) pairwise different cosets modulo \( D \wr L \). Therefore \( D \wr K \cap W'/D \wr L \cap W' \) must be an elementary-abelian factor of order \( \geq p^2 \) in \( W' \). This contradiction to Theorem 4.1(a) shows that \( |W_{\lambda}^+| = 2 \) and \( H_{\kappa} = W_{\lambda}^+ \cong C_2 \). Moreover the above argumentation applied to any \( \mu \in \Lambda \) with \( \mu < \lambda \) would yield the contradiction \( W^+_{\mu} \cong C_2 \); hence \( \Lambda = \{\lambda, \kappa\} \) and \( W = H_{\lambda} \wr C_2 \).

Now the e.c. \( L\mathbb{L}_p \)-group \( W' \) contains a copy of every finite \( p \)-group, and therefore \( H_{\lambda} \) cannot be abelian. Fix any chief factor \( M/N \) in \( H_{\lambda} \) with \( M \subseteq H \), and define \( K, L \subseteq D_\lambda \) as above. Then \( K/L \) is an elementary-abelian factor of order \( p^2 \) in \( W' \). But that contradicts Theorem 4.1(a).

Since every e.c. \( L\mathbb{L}_p \)-group is perfect (cf. [14, Theorem 2.1]), it follows from Theorem 8 that e.c. \( L\mathbb{L}_p \)-groups are also different from generalized wreath products.

R. E. Phillips and J. Roseblade (private communication) have shown recently that \( W = \text{Wr}_{\lambda \in \Lambda} H_{\lambda} \) contains a Prüfer-\( p \)-group, if and only if one of the groups \( H_{\lambda}, \lambda \in \Lambda \), contains a Prüfer-\( p \)-group. In particular a generalized wreath product of finite groups can never have a subgroup which is e.c. in \( L\mathbb{L}_p \).
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