Stability of impulsive functional differential equations via the Liapunov functional

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\section{Introduction}

Differential equations with impulses are recognized as an adequate mathematical tool for studying evolution processes that are subject to abrupt changes in their states (refer to [1–4]). Many models in biological, physical, economics and engineering applications exhibit impulsive effects [1–10]. An example is a model for the prices of several commodities in a speculative and unscrupulous environment where the customer stocks for speculative reasons and a trader hoards the goods as his/her utility has reached a threshold value. This model contains both impulses and delays. Thus the interest in impulsive differential equations with delay is not just theoretical but also practical.

Qualitative properties of impulsive differential equations have been intensively researched for years. In [11–15], the Liapunov function or the Liapunov functional coupled with a certain Razumikhin technique was suggested for stability of impulsive functional equations with finite delays. These methods were also used to study impulsive functional differential equations with infinite delays [16–18]. It has been shown that these results are very effective for various impulsive functional differential equations with finite delays or infinite delays.

However, when we considered impulsive systems involving both finite and infinite delays, we discovered that it was not entirely convenient to apply these results.

The purpose of the present work is to improve the upper bound of the Liapunov functional $V(t, \phi)$ so that some general stability theorems can be obtained. Our results can be applied to finite delay equations or infinite delay equations or equations involving both finite and infinite delay, in a unified way. Examples are also given to illustrate the advantage of the results.
2. Preliminaries

Consider the system of the impulsively functional differential equation

\[ x'(t) = F(t, x(t)), \quad t \geq t^*, \]

\[ \Delta x(t_k) = l(t_k, x(t_k^+)), \quad k = 1, 2, \ldots \]  

(2.1a)

where \( t^* \geq \alpha \geq -\infty \), \( F \) is a Volterra type functional, its values are in \( R^n \) and are determined by \( t \geq t^* \) and the values of \( x(s) \) for \( [\alpha, t] \) with \( F(t, 0) \equiv 0 \). In the case when \( \alpha = -\infty \), the interval \( [\alpha, t] \) is understood to be replaced by \( (-\infty, t] \). \( x'(t) \) denotes the right-hand derivative of \( x(t) \), \( t^* < t_k < t_{k+1} \) with \( t_k \rightarrow \infty \) as \( k \rightarrow \infty \). \( F: [t^*, \infty) \times R^n \rightarrow R^n \) with \( l(t_k, 0) \equiv 0 \), and

\[ x(t_k^+) = \lim_{t \rightarrow t_k^+} x(t). \]

Let \( J \subset R \) be any interval. Define \( PC(J, R^n) = \{ x: J \rightarrow R^n \mid x \text{ is continuous everywhere except at the points } t = t_k \in J \text{ and } x(t_k^-) \text{ and } x(t_k^+) \text{ exists with } x(t_k^-) = x(t_k^+) \}. \)

For any \( \phi \in PCB(t) \), the norm of \( \phi \) is defined by \( \| \phi \| = \| \phi \|^{[\alpha, t]} = \sup_{0 \leq s \leq t} |\phi(s)| \).

The initial condition for system (2.1) is given by

\[ x(t) = \phi(t), \quad \alpha \leq t \leq \sigma, \]

(2.2)

where \( \sigma \geq t^* \) and \( \phi \in PCB(\sigma) \).

It is shown in [19] that under the following hypotheses (H1)–(H4), the initial value problem (2.1) and (2.2) has a unique solution \( x(t, \sigma, \phi) \) existing in a maximal interval \( [\alpha, \sigma + \beta] \), \( \beta > 0 \). We note that the existence results for impulsive functional differential equations can be established on the basis of the consideration of the piecewise continuous (bounded) initial value functions space \( PC(\sigma) \) (cf. [7,19,20]).

(H1) \( F \) is continuous on \( \{t_{k-1}, t_k\} \times PC(t) \) for \( k = 1, 2, \ldots \), where \( t_0 = t^* \). For all \( \phi \in PC(t) \) and \( k = 1, 2, \ldots \), the limit \( \lim_{(t, \phi) \rightarrow (t_k^-, \phi)} F(t, \phi) = F(t_k^-, \phi) \) exists.

(H2) \( F \) is Lipschitz in \( \phi \) in each compact set in \( PCB(t) \). More precisely, for every \( \gamma \in [\alpha, \sigma + \beta] \) and every compact set \( G \subset PCB(t) \) there exists a constant \( L = L(\gamma, G) \) such that

\[ |F(t, \phi(t)) - F(t, \phi'(t))| \leq L \| \phi - \phi' \|^{[\alpha, t]}, \]

(2.3)

whenever \( t \in [\alpha, \gamma] \) and \( \phi, \phi' \in G \).

(H3) The function \( (t, x) \rightarrow l(t, x) \) is continuous on \( [t^*, \infty) \times R^n \). For any \( \rho > 0 \), there exists a \( \rho_1 > 0 \) \( (0 < \rho_1 < \rho) \) such that \( x \in S(\rho_1) \) implies that \( x + l(t_k, x) \in S(\rho) \) for \( k \in Z^+ \).

(H4) For any \( x(t) \in PC([\alpha, \infty), R^n) \), \( F(t, x(\cdot)) \in PC([t^*, \infty), R^n) \).

**Definition 2.1.** A function \( V(t, x): [\alpha, \infty) \times S(\rho) \rightarrow R^+ \) belongs to class \( \nu_{0} \) if:

(i) \( V \) is continuous on each of the sets \( \{t_{k-1}, t_k\} \times S(\rho) \) and for all \( x \in S(\rho) \) and \( k \in Z^+ \), the limit \( \lim_{(t, y) \rightarrow (t_k^-, x)} V(t, y) = V(t_k^-, x) \) exists.

(ii) \( V \) is locally Lipschitz in \( x \) and \( V(t, 0) \equiv 0 \).

**Definition 2.2.** A functional \( V(t, \phi): [\alpha, \infty) \times PCB(t) \rightarrow R^+ \) belongs to class \( \nu_{0}(-) \) if:

(i) \( V \) is continuous on each of the sets \( \{t_{k-1}, t_k\} \times PCB(t) \) and for all \( \phi \in PCB(t) \) and \( k \in Z^+ \), the limit \( \lim_{(t, \phi) \rightarrow (t_k^-)} V(t, \phi) = V(t_k^-, \phi) \) exists.

(ii) \( V \) is locally Lipschitz in \( \phi \) and \( V(t, 0) \equiv 0 \).

**Definition 2.3.** A functional \( V(t, \phi) \) belongs to class \( \nu_{0}(-) \) if \( V \in \nu_{0}(-) \) and for any \( x \in PC([\alpha, \infty), R^n) \), \( V(t, x(\cdot)) \) is continuous for \( t \geq t^* \).

Let \( V \in \nu_{0} \), for any \( (t, x) \in \{t_{k-1}, t_k\} \times S(\rho) \). The right hand derivative \( V_{(2.1)}(t, x(t)) \) along the solution \( x(t) \) of (2.1) is defined by

\[ V_{(2.1)}(t, x(t)) = \lim_{h \rightarrow 0^+} \frac{V(t + h, x(t + h)) - V(t, x(t))}{h}. \]

(2.4)

Let \( V \in \nu_{0}(-) \). For any \( (t, \phi) \in \{t_{k-1}, t_k\} \times PCB(t) \), the right hand derivative \( V_{(2.1)}(t, \phi(t)) \) along the solution \( x(\xi) = x(\xi; t, \phi) \) of (2.1) is denoted by

\[ V_{(2.1)}(t, \phi(\cdot)) = \lim_{h \rightarrow 0^+} \frac{V(t + h, \phi(\cdot)) - V(t, \phi(\cdot))}{h}. \]

(2.5)

For definitions of uniform stability, uniform asymptotic stability, class \( K \) and class \( PIM \), see [17].
In the proof of our main result, we need the following lemma:

**Lemma 2.1.** Assume that \( x: [0, \infty) \to [0, \rho_1] \) is a piecewise continuous function, \( W_1, W_2 \in K, \ r > 0 \) a constant, and \([t-r, t] \subset [0, \infty)\). If there exists a \( \lambda > 0 \) such that \( \int_{t-r}^t W_1(x(s))\,ds \geq \lambda \), then

\[
\int_{t-r}^t W_2(x(s))\,ds \geq \beta = \frac{\lambda}{2W_1(\rho_1)} W_2 \left( W_1^{-1} \left( \frac{\lambda}{2r} \right) \right).
\]

**Proof.** Let \( l_1 = [s \in [t-r, t] | W_1(x(s)) \geq \lambda / 2r] \), \( l_2 = [t-r, t] - l_1 \). Then,

\[
\lambda \leq \int_{t-r}^t W_1(x(s))\,ds = \int_{l_1} W_1(x(s))\,ds + \int_{l_2} W_1(x(s))\,ds
\]

\[
\leq W_1(\rho_1) \mu(l_1) + \frac{\lambda}{2r} (r - \mu(l_1)),
\]

and so \( \mu(l_1) \geq r \lambda / (2r W_1(\rho_1) - \lambda) \geq \lambda / 2W_1(\rho_1) \). Thus,

\[
\int_{t-r}^t W_2(x(s))\,ds \geq \int_{l_1} W_2(x(s))\,ds \geq \frac{\lambda}{2W_1(\rho_1)} W_2 \left( W_1^{-1} \left( \frac{\lambda}{2r} \right) \right).
\]

\( \square \)

3. Main results

**Theorem 3.1.** Let \( W_i \in K \ (i = 1, \ldots, 7) \), \( \tau > 0 \) be a constant, \( V_1(x, t) \in V_0, V_2(t, \phi) \in V_0(\cdot), \psi: [t^* - \tau, \infty) \to [0, A] \) be continuous with \( A > 0, \ \phi \in C(R^+, R^+), \ \phi \in L^1(0, \infty), \ \Phi(t) \leq L \), and \( q \in C(R^+, R^+) \) such that \( q(s) \) is nonincreasing, \( q(s) > 0, \ s > 0 \). Assume that the following assumptions hold:

(i) \( W_1(\phi(t)) \leq V(t, \phi(t)) \leq W_2(\phi(t)) + W_3 \left( \int_{t-r}^t \psi(s) W_4(\phi(s))\,ds \right) + W_5 \left( \int_{t-r}^t \phi(t) - s \right) W_6(\phi(t))\,ds \), where \( V(t, \phi(t)) = V_1(t, \phi(t)) + V_2(t, \phi(t)) \in V_0(\cdot), \phi \in PCB(t) \).

(ii) For each \( k = 1, 2, \ldots, \) and all \( x \in S(\rho_1), V_1(t_k, x + l(t_k, x)) \leq (1 + b_k) V_1(t, x), \) where \( b_k \geq 0, \sum_{k=1}^\infty b_k < \infty \).

(iii) For any \( \sigma \geq t^*, \phi \in PCB(\sigma),\)

\[
V'(2,1)(t, x(s)) \leq -W_2(\phi(t)), \quad \text{where} \quad W_2(\phi(t)) = W_2(\phi(t)) + W_3 \left( \int_{t-r}^t \psi(s) W_4(\phi(s))\,ds \right) + W_5 \left( \int_{t-r}^t \phi(t) - s \right) W_6(\phi(t))\,ds.
\]

\( \text{where} \quad \psi(s) = \int_s^{\infty} \phi(u)\,du \text{ for max}\{\alpha, t - q(V(t, x(s)))\} \leq s \leq t, \text{where} \quad P(s) \in C(R^+, R^+), P(s) > M s (s > 0), M = \prod_{k=1}^\infty (1 + b_k) \) and \( x(t) = x(t, \sigma, \phi) \) is the solution of (2.1).

Then the zero solution of (2.1) is uniformly asymptotically stable.

**Proof.** Let \( \epsilon > 0 \) be given. Choose a positive number \( \delta < \epsilon \) so that \( W_2(\delta) + W_3(\phi_0 W_4(\delta)) + W_5(\phi_0 W_6(\delta)) \leq W_1(\epsilon)/M, \) where \( J = \int_0^\infty \phi(u)\,du \). Without loss of generality, we may assume \( \sigma \geq t^* > \alpha + \tau \). Let \( x(t) = x(t, \sigma, \phi) \) be the solution of (2.1) and \( \sigma, \sigma + \beta \) be its maximal interval of existence. If \( \beta < +\infty \), then there exists some \( t \in \sigma + \beta \) with \( \|x_j(x)\| > \epsilon \). We prove that \( \|x_j(x)\| \leq \epsilon \) for all \( t \in [\sigma, \sigma + \beta] \), which in turn implies that \( \beta = +\infty \) and the zero solution of (2.1) is uniformly stable.

Suppose that there exists some \( t \in [\sigma, \sigma + \beta] \) with \( \|x(t)\| > \epsilon \). Then let \( \bar{t} = \inf \{t \in [\sigma, \sigma + \beta] | \|x(t)\| > \epsilon \} \). Note that \( \|x(t)\| = \psi(t) \leq \delta < \epsilon \) for \( t \in [\sigma, \sigma] \). We see that \( \bar{t} > \sigma, \|x(t)\| \leq \epsilon \) for \( t \in [\sigma, \bar{t}] \), and \( \|x(\bar{t})\| = \epsilon \) or \( \|x(t)\| > \epsilon \) and \( \bar{t} = t_j \) for some \( j \). In the latter case \( \|x(t)\| \leq \epsilon \) since \( \|x(t)\| \leq \epsilon < \rho_1 \), and by our assumption on the functional \( I \). Thus, in either case \( V(t, x(t)) \) is defined for \( [\alpha, \bar{t}] \). Set \( V_1(t) = V_1(t, x(t)), V_2(t) = V_2(t, x(t)) \) and \( V(t) = V_1(t) + V_2(t) \). Let \( t \in [t_m - 1, t_m] \), where \( t_m = t^* \). Then

\[
W_1(\|x(t)\|) \leq V_1(t) \leq W_2(\delta) + W_3(\phi_0 W_4(\delta)) + W_5(\phi_0 W_6(\delta)) \leq M^{-1} W_1(\epsilon), \ \alpha \leq t \leq \sigma.
\]

Next, we consider two possible cases.

Case 1. \( t_m - 1 \leq \sigma < \bar{t} < t_m \); then \( \|x(t)\| = \epsilon \). We prove that

\[
V(t) \leq M^{-1} W_1(\epsilon), \quad \sigma \leq t \leq \bar{t}.
\]

In fact, if (3.2) does not hold, then there exists some \( \bar{t} \in [\sigma, \bar{t}] \) such that \( V(t) > M^{-1} W_1(\epsilon) \geq V(\sigma) \), which implies that there is a \( \bar{t} \in [\sigma, \bar{t}] \) such that \( V_1(\bar{t}) > 0 \) and \( V(t) \leq V(\bar{t}), \ \alpha \leq s \leq \bar{t} \). Thus, \( P(V(t)) > MV(\bar{t}) \geq V(\sigma), \max\{\alpha, \bar{t} - q(V(\bar{t}))\} \leq s \leq \bar{t} \). By assumption (iii), \( V_1(\bar{t}) \leq -W_2(\phi(\bar{t})) \). This is a contradiction and so (3.2) holds. Thus, for \( s \in [\sigma, \bar{t}] \), \( V(s) \leq M^{-1} W_1(\epsilon) \leq W_1(\epsilon) \), which implies that \( P(V(s)) > MV(\bar{t}) \geq V(\sigma), \max\{\alpha, \bar{t} - q(V(\bar{t}))\} \leq s \leq \bar{t} \). By assumption (iii), \( 0 \leq V_1(\bar{t}) \leq -W_2(\phi(\bar{t})) = -W_2(\phi(\bar{t})) < 0 \), a contradiction.

Case 2. \( t_m + k \leq \bar{t} < t_{m+k+1} \) for some \( k \geq 0 \). Analogously to proving (3.2), we can prove that \( V(t) \leq M^{-1} W_1(\epsilon), \ \alpha \leq t \leq t_m \). By assumption (ii), we have \( V(t_m) = V_1(t_m) + V_2(t_m) \leq 1 + b_m V_1(t_m) + (1 + b_m) V_2(t_m) = (1 + b_m) V(t_m) \leq M^{-1} (1 + b_m) W_1(\epsilon) \).

Similarly one can prove in general that for \( i = 1, 2, \ldots, k \),

\[
V(t) \leq M^{-1} (1 + b_m) \cdots (1 + b_{m+i-1}) W_1(\epsilon), \ t_{m+i-1} \leq t < t_{m+i},
\]

\[
V(t_{m+i}) \leq M^{-1} (1 + b_m) \cdots (1 + b_{m+i}) W_1(\epsilon).
\]
and
\[ V(t) \leq M^{-1} (1 + b_m) \cdots (1 + b_m t) W_1(\varepsilon), \quad t_{m+1} \leq t \leq \bar{t}. \]

Therefore
\[ V(t) \leq W_1(\varepsilon), \quad t \in [\sigma, \bar{t}). \tag{3.3} \]

If \( \bar{t} = t_{m+1}, \) then \( |x(\bar{t})| > \varepsilon. \) By assumption (ii), \( 0 < W_1(\varepsilon) < W_1(|x(\bar{t})|) \leq V(\bar{t}) = V(t_{m+1}) \leq W_1(\varepsilon), \) a contradiction. If \( t_{m+1} < \bar{t} < t_{m+2}, \) then \( |x(\bar{t})| = \varepsilon. \) Applying exactly the same argument as for Case 1, we get a contradiction.

So in either case, we obtain a contradiction, which proves that the zero solution of (2.1) is uniformly stable.

Next we show that it is uniformly asymptotically stable. For \( \rho_1 > 0, \) choose a \( \delta > 0. \) In view of the proof of uniform stability, we know that \( \varphi \in PCB_{\delta}(\sigma) \) implies that \( V(t) \leq W_1(\rho_1), \) \( |x(t)| \leq \rho_1, t \geq \sigma. \)

For any \( \varepsilon > 0 (\varepsilon < \rho_1), \) we will prove that there exists a \( T = T(\varepsilon) > 0 \) such that \( \varphi \in PCB_{\delta}(\sigma) \) implies that \( |x(t)| \leq \varepsilon, t \geq \sigma + T. \)

Since \( \varphi \in L^1[0, \infty), \) it follows that there is a \( r > \text{max}[1, \tau] \) such that
\[ W_6(\rho_1) \int_{-\infty}^{\infty} \varphi(u) du < \frac{1}{3} W_5^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right). \tag{3.4} \]

Without loss of generality, we may assume that \( A \) is so large that
\[ W_4^{-1} \left[ \frac{1}{A} W_5^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right) \right] < W_2^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right), \tag{3.5} \]
and
\[ \frac{1}{A} W_3^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right) < \frac{\lambda}{2W_6(\rho_1)} W_4 \left( W_6^{-1} \left( \frac{\lambda}{4r} \right) \right), \tag{3.6} \]
where \( \lambda = (2/3M) W_5^{-1} ((1/3M) W_1(\varepsilon)). \)

It is clear that for \( t \geq \sigma + r, \)
\[ W_1(|x(t)|) \leq V(t) \leq W_2(|x(t)|) + W_3 \left( A \int_{t-r}^{t} W_4(|x(s)|) ds \right) + W_5 \left[ \frac{1}{3} W_5^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right) + L \int_{t-r}^{t} W_6(|x(s)|) ds \right]. \]

Let \( 0 < d < \inf \{ P(s) - Ms: M^{-1} W_1(\varepsilon) \leq s \leq W_1(\rho_1) \}, \) \( N \) be the first positive integer such that \( W_1(\rho_1) \leq M^{-1}[W_1(\varepsilon) + Nd]. \)

Set \( \tau_i = \sigma + iT^* \), \( i = 0, 1, \ldots, N, \) where \( T^* \) will be given later and is independent of \( \sigma \) and \( \varphi. \) We shall prove that
\[ V(t) \leq W_1(\varepsilon) + (N - i - 1)d, \quad t \geq \tau_i, \quad i = 0, 1, \ldots, N. \tag{3.7} \]

Clearly, (3.7)\(_0\) holds. Suppose that (3.7)\(_i\) holds for some \( 0 \leq i < N. \) We prove that
\[ V(t) \leq W_1(\varepsilon) + (N - i - 1)d, \quad t \geq \tau_{i+1}. \tag{3.7}_{i+1} \]

Set \( h = \text{max}[r, q(M^{-1}W_1(\varepsilon)), 1] \), \( l_i = [\tau_i + h, \tau_{i+1}]. \) We first claim that there exists a \( \bar{t} \in l_i \) such that
\[ V(\bar{t}) \leq M^{-1}[W_1(\varepsilon) + (N - i - 1)d]. \tag{3.8} \]

Suppose that for all \( t \in l_i, V(t) > M^{-1}[W_1(\varepsilon) + (N - i - 1)d], \) then for such a \( t \) we have \( M^{-1}W_1(\varepsilon) < V(t) \leq W_1(\rho_1), \) and so \( P(V(t)) > MV(t) + d > W_1(\varepsilon) + (N - i)d \geq V(s), \) \( t - h \leq s \leq t. \) In view of the definition of \( h \) and noting that \( q \) is nonincreasing, one has for \( t \in l_i, \)
\[ P(V(t)) > V(s), \quad \text{max}[\sigma, t - q(V(t))] \leq s \leq t. \]

By assumption (iii), \( V(2.1)(t) \leq -W_7(|x(t)|), \) \( t \in l_i, \) and so for \( t \in l_i, \)
\[ V(t) \leq V(\tau_i + h) - \int_{\tau_i + h}^{t} W_7(|x(s)|) ds + \sum_{\tau_i + h < t \leq t\varepsilon} [V(t_k) - V(\tau_i)] \]
\[ \leq (1 + M^*) W_1(\rho_1) - \int_{\tau_i + h}^{t} W_7(|x(s)|) ds \]
where \( M^* = \sum_{k=1}^{\infty} b_k. \) On the other hand, for \( t \in l_i, \)
\[ W_2(|x(t)|) + W_3 \left[ A \int_{t-r}^{t} W_4(|x(s)|) ds \right] + W_5 \left[ \frac{1}{3} W_5^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right) + L \int_{t-r}^{t} W_6(|x(s)|) ds \right] \]
\[ \geq V(t) > M^{-1}[W_1(\varepsilon) + (N - i - 1)d] \geq M^{-1}W_1(\varepsilon). \]

We claim that for \( t \in [\tau_i + h + 2r, \tau_{i+1}], \)
\[ \int_{t-2r}^{t} W_4(|x(s)|) ds \geq \frac{1}{A} W_5^{-1} \left( \frac{1}{3M} W_1(\varepsilon) \right). \tag{3.9} \]
In fact, if (3.9) does not hold, then there is a \( \tilde{t} \in [\tau_i + h + 2r, \tau_{i+1}] \) such that
\[
\int_{\tilde{t} - 2r}^{\tilde{t}} W_4(|x(s)|) ds < \frac{1}{A} W_3^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right).
\] (3.10)

Then for \( t \in [\tilde{t} - r, \tilde{t}] \),
\[
\int_{\tilde{t} - r}^{\tilde{t}} W_4(|x(s)|) ds \leq \int_{\tilde{t} - 2r}^{\tilde{t}} W_4(|x(s)|) ds < \frac{1}{A} W_3^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right).
\]

So there is a \( \tilde{t} \in [\tilde{t} - r, \tilde{t}] \) such that \( |x(\tilde{t})| < W_4^{-1} \left( \frac{1}{A} W_3^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right) \right) \). Thus
\[
W_5(|x(\tilde{t})|) < W_2 \left[ W_4^{-1} \left( \frac{1}{A} W_3^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right) \right) \right] < \frac{1}{3M} W_1(\epsilon).
\]

On the other hand, according to Lemma 2.1 and (3.10) one has
\[
\int_{\tilde{t} - 2r}^{\tilde{t}} W_6(|x(s)|) ds < \lambda = \frac{2}{3A} W_5^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right),
\]
and so
\[
W_5 \left[ \frac{1}{3} W_5^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right) + L \int_{\tilde{t} - r}^{\tilde{t}} W_6(|x(s)|) ds \right] \leq W_5 \left[ \frac{1}{3} W_5^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right) + L \int_{\tilde{t} - 2r}^{\tilde{t}} W_6(|x(s)|) ds \right] < \frac{1}{3M} W_1(\epsilon).
\]

Therefore
\[
\frac{1}{M} W_1(\epsilon) < V(\tilde{t}) \leq W_2(|x(\tilde{t})|) + W_3 \left[ A \int_{\tilde{t} - r}^{\tilde{t}} W_4(|x(s)|) ds \right] + W_5 \left[ \frac{1}{3} W_5^{-1} \left( \frac{1}{3M} W_1(\epsilon) \right) + L \int_{\tilde{t} - r}^{\tilde{t}} W_6(|x(s)|) ds \right] \leq \frac{1}{3M} W_1(\epsilon) + \frac{1}{3M} W_1(\epsilon) + \frac{1}{3M} W_1(\epsilon) = \frac{1}{M} W_1(\epsilon),
\]
which is a contradiction and so (3.9) holds. By Lemma 2.1, there exists a \( \mu > 0 \) such that
\[
\int_{\tilde{t} - 2r}^{\tilde{t}} W_7(|x(s)|) ds \geq \mu, \tau_i + h + 2r \leq t \leq \tau_{i+1}.
\] (3.11)

Let \( k^* \) be the positive integer such that \( (1 + M^*) W_1(\rho_1) \geq (k^* - 1) \mu \). Now we may choose \( T^* = h + 2k^*r \), and obtain
\[
V(\tau_i + T^*) \leq (1 + M^*) W_1(\rho_1) - \int_{\tau_{i+1}}^{\tau_{i+1} + 2k^*r} W_7(|x(s)|) ds \leq (1 + M^*) W_1(\rho_1) - k^* \mu < 0.
\]

This is a contradiction. Thus (3.8) holds. Let \( \ell = \min \{ k \in Z^+: t_k > \tilde{T} \} \). Analogously to proving (3.3), we can prove that \( V(t) \leq W_1(\epsilon) + (N - i - 1)d, \ t \geq \tilde{T} \). Thus, (3.7) holds. By the induction, we know that (3.7) hold for all \( i = 0, 1, \ldots, N \). In particular, when \( i = N \) we obtain \( W_1(|x(t)|) \leq V(t) \leq W_1(\epsilon), \ t \geq \sigma + NT^* \). Now, let \( T = NT^*; \) then \( |x(t)| \leq \epsilon, \ t \geq \sigma + T \). The proof is complete. \( \square \)

From the proof of the uniform stability part in Theorem 3.1, one can easily obtain the following uniform stability theorem.

**Corollary 3.1.** Let \( W_i \in K \ (i = 1, \ldots, 7) \), \( r > 0 \) be a constant, \( V_i(t, x) \in V_0, V_2(t, \phi) \in V_0^\phi, \psi: [t^* - \tau, \infty) \to [0, A] \) be continuous with \( A > 0, \psi \in C(R^+, R^+), \phi \in L^1[0, \infty), \phi(t) \leq L \). Assume that the following assumptions hold:

(i) \( W_1(\phi(t)) \leq V(t, \phi(t)) \leq W_2(\phi(t)) + W_3 \left( \int_{t^* - \tau}^{t} \phi(s) W_6(\phi(s)) ds \right) + W_5 \left( \int_{t^* - \tau}^{t} \phi(s) W_6(\phi(s)) ds \right) \), where \( V(t, \phi(t)) = V_1(t, \phi(t)) + V_2(t, \phi(t)) \in V_0^\phi, \phi \in PCB(t) \).

(ii) For each \( k = 1, 2, \ldots \) and all \( x \in S(\rho_1), V_i(t_k, x + l(t_k, x)) \leq (1 + b_k)V_1(t_k, x), \) where \( b_k \geq 0, \sum_{k=1}^{\infty} b_k < \infty \).

(iii) For any \( \sigma^*, \varphi \in PCB(\sigma), \)
\[
V_{1, 2}(t, x) \leq -W_7(|x(t)|),
\] (3.12)
whenever \( P(V(t, x(t))) > V(s, x(s)), \) for \( \sigma \leq s \leq t, \) where \( P(s) \in C(R^+, R^+), \ P(s) > M s \ (s > 0), \ M = \prod_{k=1}^{\infty} (1 + b_k) \) and \( x(t) = x(t, \sigma, \varphi) \) is the solution of (2.1).

Then the zero solution of (2.1) is uniformly stable.
Theorem 3.1. Theorem 3.1 extends Theorem 3.1 of [17] by adding a third term to the upper bound of the Liapunov functional $V(t, \phi)$, and so it can be applied conveniently to those kinds of impulsive functional differential equations which involve both finite and infinite delays (see the example below).

Remark 3.2. The following Theorem 3.2 is a generalization of Theorem 3.1. Since the proof is similar as that in Theorem 3.1, we omit it here for the sake of brevity.

Theorem 3.2. Let $\mathcal{W}_i \in \mathbb{K} (i = 1, \ldots, 7), \tau > 0$ be a constant, $V_1(t, x) \in \mathcal{V}_0$, $V_2(t, \phi) \in \mathcal{V}_0^{\infty}$, $\psi: [\tau - \tau, \infty) \to [0, A]$ be continuous with $A > 0$, and $\Phi(t, s)$ be a continuous function defined for $0 \leq s \leq t < \infty, 0 \leq \Phi(t, s) \leq L, \int_0^t \Phi(t, s) ds \leq L^*$ for $t \geq t_0$, for any $\varepsilon > 0$, and let there exist a $T_0 > 0$ with $\int_0^{T_0} \Phi(t, s) ds < \varepsilon$ for $t \geq t_0 + T_0$, and also $q \in C(R^+, \mathbb{R}^+)$ be nonincreasing with $q(s) > 0(s > 0)$. Assume that the following assumptions hold:

(i) $W_i(\Phi(t, s)) \leq V(t, \phi(t)) - W_i(\Phi(t, s)) + W_3 \int_0^s \Phi(t, s) ds$ + $W_5 \int_0^s \Phi(t, s) ds$, where $V(t, \phi(t)) = V_1(t, \phi(t)) + V_2(t, \phi(t)) \in \mathcal{V}_0^{\infty}$, $\phi \in \mathcal{P}(t)$.

(ii) For each $k = 1, 2, \ldots$ and all $x \in \mathcal{S}(\rho_1), V_1(t_k, x + l(k, x)) \leq (1 + b_k)V_1(t_k, x)$, where $b_k \geq 0, \sum_{k=1}^{\infty} b_k < \infty$.

(iii) For any $x \geq t^*$, $\psi \in \mathcal{P}(t)$, $x + q(V(t, x))$ is the solution of (2.1).

Then the zero solution of (2.1) is uniformly asymptotically stable.

Example Consider the equation

\[ x'(t) = -a(t)x(t) + b(t)x(t - \tau) + \int_{t-\tau}^{t} c(s-t)x(s)ds, \quad t \geq 0, \]

\[ x(t_k) = l_k(x(t_k)), \quad k = 1, 2, \ldots, \]

where $a, b, c: [0, \infty) \to \mathbb{R}$ are continuous, $t > 0, |l_k(x)| \leq (1 + b_k)|x|, x \in \mathcal{S}(\rho_1), b_k \geq 0$ and $\sum_{k=1}^{\infty} b_k < \infty$. Assume that:

(i) $b(t)$ is bounded, $\int_0^\infty |c(u)|du = c^* < \infty, \int_0^\infty |c(u)|du \in L^1[0, \infty)$.

(ii) There exists a $T_0 > 0$ with $-a(t) + c^* + |b(t + \tau)| \leq -L$.

Then the zero solution of (3.14) is uniformly asymptotically stable.

In fact, define $V_1(t) \in \mathcal{V}_0^{\infty}$ as $V_1(t, \phi(0)) = |\phi(0)|$,

\[ V_2(t, \psi) = \int_{t-\tau}^{t} |b(t+\tau)||\psi(s)|ds + \int_{t-\tau}^{t} \int_{s}^{\infty} |c(u - s)|ds |\psi(s)|ds. \]

Then $V_1(t_k, l_k(x)) = |l_k(x)| \leq (1 + b_k)|x| = (1 + b_k)V_1(t_k, x)$. For any solution $x(t) = x(t, \sigma, \nu)$ of (3.14), $V_1(t, x(t)) \leq -L|x(t)|$. By Theorem 3.1, the zero solution of (3.14) is uniformly asymptotically stable.

Remark 3.3. When $c(t) \equiv 0$, the system (3.14) becomes

\[ x'(t) = -a(t)x(t) + b(t)x(t - \tau), \quad t \geq 0, \]

\[ x(t_k) = l_k(x(t_k)), \quad k = 1, 2, \ldots. \]

Hence, the zero solution of (3.15) is uniformly asymptotically stable if $b(t)$ is bounded and $-a(t) + |b(t + \tau)| \leq -L$ with some constant $L > 0$; these are weaker than the assumptions in [12].

Therefore, applying our theorems yields better conclusions than the results in the literature.

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References


