Maximal operators and singular integral operators along submanifolds

Hung Viet Le

Department of Mathematics, Southwestern Oklahoma State University, Weatherford, OK 73096, USA

Received 3 September 2003
Available online 10 June 2004
Submitted by J.A. Ball

Abstract

In this paper we prove, for certain values of \( p \), the \( L^p \) boundedness of the maximal operator

\[
\mathcal{I}_f(\bar{x}) = \sup_{h} \left| p.v. \int_{\mathbb{R}^m} \frac{h(|y|)\Omega(y')}{|y|^m} f(\bar{x} - \Gamma(y)) \, dy \right| \quad (\bar{x} \in \mathbb{R}^n; \ n > m \geq 2),
\]

where the supremum is taken over all measurable radial functions \( h \) with \( \|h\|_{L^1(\mathbb{R}^+, \mathbb{R}^+)} \leq 1 \) and \( 1 \leq s \leq 2 \). Here \( \Omega \in H^1(\mathbb{S}^{m-1}) \), \( \Gamma(y) = (\phi(|y|)y', \Psi(|y|)) \). We also obtain the range of \( p \) for which the maximal operator above is unbounded. Moreover, we show that the singular integral

\[
T_f(\bar{x}) = p.v. \int_{\mathbb{R}^m} \frac{h(|y|)\Omega(y')}{|y|^m} f(\bar{x} - \Gamma(y)) \, dy
\]

and its associated maximal function \( T_f^*(x) \) are bounded in \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

© 2004 Elsevier Inc. All rights reserved.

Introduction

The authors of [4] proved the \( L^p \) boundedness \( (p > ms/(ms - 1), \ 1 \leq s \leq 2) \) for the maximal operator

\[
T_f(x) = \sup_{h} \left| p.v. \int_{\mathbb{R}^m} \frac{h(|y|)\Omega(y')}{|y|^m} f(x - y) \, dy \right| \quad (x \in \mathbb{R}^m; \ m \geq 2),
\]

E-mail address: hung.le@swosu.edu.
where the supremum is taken over all measurable radial functions $h$ with

$$\|h\|_{L^s(\mathbb{R}^+, \frac{dr}{r})} \leq 1 \quad \text{and} \quad 1 \leq s \leq 2.$$ 

Here $\Omega$ is a continuous function on $S^{m-1}$, is homogeneous of degree zero, and has mean value zero over the sphere $S^{m-1}$. Note that the range of $p$ obtained above was the best possible range. That is, the operator above fails to be bounded in $L^p(\mathbb{R}^m)$ when $p \leq \frac{ms}{ms - 1}, 1 \leq s \leq 2$ (see [4]).

Their work has motivated us to study the above maximal operator. We wish to extend the results of [4] in some directions: by considering $\Omega \in H^1(S^{m-1})$ instead of $\Omega \in C(S^{m-1})$, by adding some roughness to the kernel, and by considering the maximal operator along some types of submanifolds. We now introduce some notations and definitions, and summarize our results below.

**Definition.** We say that a function $\gamma$ satisfies hypothesis A if

(a) $\gamma : [0, \infty) \to [0, \infty)$ is strictly increasing and $\gamma(2t) \geq \lambda \gamma(t)$ for some fixed $\lambda > 1$.
(b) $|\gamma^{(l)}(t)| \geq \alpha \gamma(t)/tl$ on $(0, \infty)$ for some fixed $l \geq 1$ and $\alpha > 0$. If $l = 1$, then $\gamma'(t)$ is assumed to be monotone.

We say that $\gamma$ satisfies hypothesis B if

(c) $\gamma : (0, \infty) \to (0, \infty)$ is strictly decreasing, $\gamma(t) \geq \lambda \gamma(2t)$ for some fixed $\lambda > 1$.
(d) $|\gamma^{(l)}(t)| \geq \alpha \gamma(t)/tl$ on $(0, \infty)$ for some fixed $l \geq 1$ and $\alpha > 0$. If $l = 1$, then $\gamma'(t)$ is assumed to be monotone.

Finally, $\gamma$ is said to satisfy hypothesis C if

(e) $\gamma$ is $C^1$ and strictly increasing on its compact support, say $[0, b], b < \infty$.
(f) $\gamma'(t)$ is increasing on its support.

For the rest of this paper, we let $\Omega \in H^1(S^{m-1})$ ($m \geq 2$) be homogeneous of degree zero and have the mean value zero property. Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^n$ ($2 \leq m < n$) be defined by $\Gamma(y) = (\phi(|y|) y', \Psi(|y|))$ ($y' = y/|y|$), where $\phi$ and $\Psi$ are radial functions. Define the maximal operator along submanifold $\mathfrak{I}^\Gamma$ on the class of Schwartz functions $S(\mathbb{R}^n)$ by

$$\mathfrak{I}^\Gamma f(\bar{x}) = \sup \left\| \text{p.v.} \int_{\mathbb{R}^m} \frac{h(|y|) \Omega(y')}{|y|^m} f(\bar{x} - \Gamma(y)) \, dy \right\| (\bar{x} \in \mathbb{R}^n),$$

where the supremum is taken over all measurable radial functions $h$ with $\|h\|_{L^s(\mathbb{R}^+, \frac{dr}{r})} \leq 1, 1 \leq s \leq 2$. We define the maximal operator $\mathfrak{I}$ acting on $S(\mathbb{R}^m)$ by

$$\mathfrak{I} f(x) = \sup \left\| \text{p.v.} \int_{\mathbb{R}^m} \frac{h(|y|) \Omega(y')}{|y|^m} f(x - \phi(y)y') \, dy \right\| (y' = y/|y|)$$

where the supremum is taken over all measurable radial functions $h$ with $\|h\|_{L^s(\mathbb{R}^+, \frac{dr}{r})} \leq 1, 1 \leq s \leq 2$. We define the maximal operator $\mathfrak{I}$ acting on $S(\mathbb{R}^m)$ by
Finally, let $Mg$ be the maximal operator, defined on $S(\mathbb{R}^{n-m+1})$ by:

$$
Mg(x_1, x_2) = \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{2^{k+1}} \int_{2^k}^{2^{k+1}} |g(x_1 - \phi(t), x_2 - \psi(t))| \, dt \right\} 
$$

with $x_1, x_2 \in \mathbb{R}^{n-m}$.

We now state the following theorems.

**Theorem 1.** Suppose $\phi$ satisfies either hypothesis A or hypothesis B. If the maximal operator $Mg$ is bounded in $L^p(\mathbb{R}^{n-m+1})$ for all $p > 1$, then $\mathcal{S}_f$ has a bounded extension in $L^p(\mathbb{R}^n)$ for $s/(s-1) \leq p < \infty$ when $1 < s \leq 2$, and for $p = \infty$ when $s = 1$. $\mathcal{S}_f$ is unbounded in $L^p(\mathbb{R}^n)$ for $0 < p < \infty$ when $s = 1$, for $0 < p \leq ms/(ms-1)$ when $1 < s \leq 2$, and for $1 \leq p \leq \infty$ when $0 < s < 1$.

**Corollary 1.** If $\Gamma(y) = (\phi(|y|) y', \Psi(|y|)) \equiv (|y|^{k_0} y', |y|^{k_1}, \ldots, |y|^{k_{n-m}})$ for some positive real numbers $k_0 < k_1 < \cdots < k_{n-m}$, then $\mathcal{S}_f$ is bounded or unbounded in $L^p(\mathbb{R}^n)$ with the same values of $p$ and $s$ as given in Theorem 1.

**Corollary 2.** Suppose $\Gamma(y) = (\phi(|y|) y', \Psi(|y|)) \equiv (|y|^{k_0} y', y_1(|y|), \ldots, y_{n-m}(|y|))$, where $k$ is a positive integer, and $y_1, \ldots, y_{n-m}$ are polynomials in $|y|$. Then $\mathcal{S}_f$ is bounded or unbounded in $L^p$ with the same values of $p$ and $s$ as given in Theorem 1.

**Corollary 3.** Suppose $\Gamma(y) = (\phi(|y|) y', \Psi(|y|)) \equiv (\phi(|y|) y', y_1(|y|), \ldots, y_{n-m}(|y|))$, where $\phi$ satisfies hypothesis A or B, and $\Psi$ has compact support. Assume that for each...
Also, we will only prove it for the case that \( m \) is bounded in \( L^p(\mathbb{R}^m) \) for \( s/(s-1) \leq p < \infty \) when \( 1 < s \leq 2 \), and for \( p = \infty \) when \( s = 1 \). \( \exists f \) is unbounded for \( 0 < p \leq ms/(ms - 1) \) when \( 1 < s \leq 2 \).

**Theorem 2.** Assume \( \phi \) satisfies either hypothesis A or hypothesis B. Then \( \exists f \) is bounded in \( L^p(\mathbb{R}^m) \) for \( s/(s-1) \leq p < \infty \) when \( 1 < s \leq 2 \), and for \( p = \infty \) when \( s = 1 \). \( \exists f \) is unbounded for \( 0 < p \leq ms/(ms - 1) \) when \( 1 < s \leq 2 \).

**Theorem 3.** Let \( \phi \) and \( \Psi \) be given as in Corollaries 1, 2, or 3. Then the singular integral \( T_f \) is bounded in \( L^p(\mathbb{R}^n) \) (\( n > m \geq 2 \)) for \( 1 < p < \infty \), whenever \( h \in L^1(\mathbb{R}^+, d\sigma) \), \( 1 < s \leq 2 \). Moreover, if \( h \in L^1(\mathbb{R}^+, d\sigma) \cap L^\infty(\mathbb{R}^+) \), then its associated maximal function \( T^*_f \) is bounded in \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

**Theorem 4.** If \( \phi \) satisfies either hypothesis A or hypothesis B, then the singular integral \( T_f \) is bounded in \( L^p(\mathbb{R}^m) \) (\( m \geq 2 \)) for \( 1 < p < \infty \), whenever \( h \in L^1(\mathbb{R}^+, d\sigma) \), \( 1 < s \leq 2 \). Moreover, if \( h \in L^1(\mathbb{R}^+, d\sigma) \cap L^\infty(\mathbb{R}^+) \), then its associated maximal function \( T^*_f \) is bounded in \( L^p(\mathbb{R}^m) \) for \( 1 < p < \infty \).

**Example.** Let \( \phi : [0, \infty) \to [0, \infty) \) be a function of the types \( t^\alpha (q \neq 0) \), \( t^\alpha e^{\beta t} (\alpha \geq 1, \beta \geq 0) \) or \( t^{-\alpha} e^{-\beta t} (\alpha > 0, \beta \geq 0) \). For each \( j = 1, \ldots, n-m \), let \( \gamma_j(t) = t^{\alpha j} (q_j \geq 1, \beta_j \geq 0) \) with compact support. Then by Corollary 3, the operator \( \exists_{\Gamma} f \) is bounded in \( L^p \), where the values of \( p \) are given in Theorem 1. Moreover, the operators \( T_{\Gamma} f \) and \( T^*_{\Gamma} f \) are also bounded in \( L^p \) for \( 1 < p < \infty \). Observe that the coordinates of \( (\phi(t), \gamma_1(t), \ldots, \gamma_{n-m}(t)) \) in this example are not necessarily linearly independent.

For the rest of this paper, we will denote \( C \) as a constant, which is not necessarily the same each time it appears. Note that the proof of Theorem 1 follows some ideas in [4].

**Proof of Theorem 1.** In view of the atomic decomposition of \( \Omega \) (see [5,7]) and the fact that \( \exists \) is sublinear, it suffices to prove the \( L^p \) boundedness of the operator

\[
\exists_a f(x, \tilde{x}) = \sup_h \left\| \text{p.v.} \int_{\mathbb{R}^m} h(|y|)a(y') f(x - \phi(|y|)y', \tilde{x} - \Psi(|y|)) dy \right\|,
\]

\((x \in \mathbb{R}^m, \tilde{x} \in \mathbb{R}^{n-m})\),

with the bound independent of the regular \( \infty \)-atom \( a \) in the atomic decomposition of \( \Omega \). Also, we will only prove it for the case that \( m \geq 3 \) and \( \phi \) satisfies hypothesis A, since the proofs of the remaining cases are essentially the same. By Hölder’s inequality, we have

\[
\exists_a f(x, \tilde{x}) = \sup_h \left\| \int_0^\infty h(r) \int_{S^{m-1}} a(y') f(x - \phi(r) y', \tilde{x} - \Psi(r)) d\sigma(y') \frac{dr}{r} \right\|
\]

\[\leq \left\| \int_{S^{m-1}} a(y') f(x - \phi(r) y', \tilde{x} - \Psi(r)) d\sigma(y') \right\|_{L^p(\mathbb{R}^+, \frac{dr}{r})},\]
where $s'$ is the conjugate of $s$. We first consider the case $s = 2$. Then

$$
\mathcal{A}_u f(x, \tilde{x}) \leq \left( \int_0^\infty \left| \int_{\mathbb{S}^{n-1}} a(y') f(x - \phi(r)y', \tilde{x} - \Psi(r)) d\sigma(y') \right|^2 \frac{dr}{r} \right)^{1/2}
$$

$$
= \left( \sum_k \left| \int_1^2 \int_{\mathbb{S}^{n-1}} a(y') f(x - \phi(2^k r)y', \tilde{x} - \Psi(2^k r)) d\sigma(y') \right|^2 \frac{dr}{r} \right)^{1/2}.
$$

(1)

Take a smooth positive function $p$ supported on the set $\{ r \in \mathbb{R}: 1/2 < |r| < 2 \}$ with $\sum_k p(a_k r) = 1$ for all $r \neq 0$. Here $\{a_k\}$ is a lacunary sequence of positive real numbers, defined by $a_k = \phi(2^k)$ for all $k \in \mathbb{Z}$. For $\rho > 0$, let $A_\rho : \mathbb{R}^m \to \mathbb{R}^m$ be the linear mapping defined by $A_\rho \xi = (\rho^2 \xi_1, \rho \xi_2, \ldots, \rho \xi_m)$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$. Define $\Delta$ on $\mathbb{R}^m$ by $\Delta(\xi) = p(|A_\rho \xi|)$ and denote $\Delta_{a_k}(\gamma) = a_k^{-m} \Delta(a_k^{-1} \gamma)$. Then $\Delta_{a_k}(\xi) = \tilde{\Delta}(a_k \xi) = p(a_k |A_\rho \xi|)$. Now define $S_k f$ by $S_k f(x, \tilde{x}) = (\Delta_{a_k} \otimes \delta_{n-m}) * f(x, \tilde{x})$, where $\delta_{n-m}$ is the Dirac distribution acting on the variable $\tilde{x} \in \mathbb{R}^{n-m}$. It is clear that $f(x, \tilde{x}) = \sum_k S_k f(x, \tilde{x})$ for any $j \in \mathbb{Z}$ (at least for a Schwartz function $f$). Thus

$$
\mathcal{A}_u f(x, \tilde{x})
$$

$$
\leq \left\{ \sum_k \left[ \sum_j \int_1^2 \int_{\mathbb{S}^{n-1}} a(y') S_k f(x - \phi(2^k r)y', \tilde{x} - \Psi(2^k r)) d\sigma(y') \right]^2 \frac{dr}{r} \right\}^{1/2}
$$

$$
\leq \left\{ \sum_k \left[ \sum_j \int_1^2 \int_{\mathbb{S}^{n-1}} a(y') S_k f(x - \phi(2^k r)y', \tilde{x} - \Psi(2^k r)) d\sigma(y') \right]^2 \frac{dr}{r} \right\}^{1/2}
$$

$$
= \sum_j \int_1^2 \int_{\mathbb{S}^{n-1}} a(y') S_k f(x - \phi(2^k r)y', \tilde{x} - \Psi(2^k r)) d\sigma(y') \frac{dr}{r}
$$

(2)

where the last two inequalities follow from Minkowski’s inequality. We now calculate the $L^2$ norm of $T_j f$. Denote

$$
F_k(x, \tilde{x}; r) = \int_{\mathbb{S}^{n-1}} a(y') S_k f(x - \phi(2^k r)y', \tilde{x} - \Psi(2^k r)) d\sigma(y').
$$

(3)
By Fubini’s and Plancherel’s theorems, we have
\[
\|T_j f\|_2^2 = \sum_{k=1}^{2} \|\hat{F}_k(\cdot, \cdot; r)\|_2^2 \frac{dr}{r},
\]
where \(\hat{F}_k\) denotes the Fourier transform with respect to the first two variables of \(F_k\), and
\[
\hat{F}_k(\xi, \eta; r) = \hat{S}_{k+j} f(\xi, \eta) \int_{S^{m-1}} a(y) e^{i\xi|\phi(2^jr)|\zeta} e^{i\eta|\Psi(2^jr)|} d\sigma(y).
\]
We may assume without loss of generality that \(\text{supp}(a) \subset B(1, \rho) \cap S^{m-1}\), where \(1 = (1, 0, 0, \ldots, 0)\). For \(\zeta \neq 0\), we choose a rotation \(\theta\) such that \(\theta(\zeta) = |\zeta|1 = |\zeta|(1, 0, 0, \ldots, 0)\), and let \(\theta^{-1}\) denote its inverse. Note that \(a(\theta^{-1}(y'))\) is again a regular \(\infty\)-atom with support in \(B(\zeta', \rho) \cap S^{m-1}\), \(\zeta' = \zeta/|\zeta|\). Let \(y' = (v, y_2', \ldots, y_m')\). We then have
\[
\hat{F}_k(\xi, \eta; r) = e^{i\eta|\Psi(2^jr)|} \hat{S}_{k+j} f(\xi, \eta) \int_{S^{m-1}} a(\theta^{-1}(y')) e^{i\xi|\phi(2^jr)|\zeta} e^{i\eta|\Psi(2^jr)|} d\sigma(y')
\]
\[
= e^{i\eta|\Psi(2^jr)|} p(a_{k+j}|A_{\rho} \zeta|) \hat{f}(\xi, \eta) \int_{S^{m-1}} e^{i\xi|\phi(2^jr)|\zeta} E_a(v, \zeta') dv,
\]
where
\[
E_a(v, \zeta') = (1 - s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{S^{m-1}} a(s, (1 - s^2)^{1/2} \tilde{y}) d\sigma(\tilde{y}).
\]
Recall that \(E_a(v, \zeta')\) (see [7, Lemma 2.1]) has support in \((\zeta'_1 - 3w, \zeta'_1 + 3w)\) and \(w \equiv w(\zeta') = |(\rho^2 \zeta'_1, \rho \zeta'_2, \rho \zeta'_3, \ldots, \rho \zeta'_m)| = |A_{\rho} \zeta'|/|\zeta|\). We now show that \(\|T_j f\|_2 \leq C_{\lambda^{-1}/|S|} \|f\|_2\) by considering two separate cases: \(j \geq 0\) and \(j < 0\).

**Case** \(j \geq 0\). By the cancellation property of \(E_a(v, \zeta')\), we obtain
\[
\left|\hat{F}_k(\xi, \eta; r)\right| \leq p(a_{k+j}|A_{\rho} \zeta|) \left|\hat{f}(\xi, \eta)\phi(2^jr)|\zeta| \int_{\zeta'_1 - 3w}^{\zeta'_1 + 3w} \left|v E_a(v, \zeta')\right| dv
\]
\[
\leq C p(a_{k+j}|A_{\rho} \zeta|) \left|\hat{f}(\xi, \eta)\phi(2^jr)|\zeta| \int_{\zeta'_1 - 3w}^{\zeta'_1 + 3w} |v| dv
\]
\[
\leq C p(a_{k+j}|A_{\rho} \zeta|) \left|\hat{f}(\xi, \eta)\phi(2^jr)|A_{\rho} \zeta|.
\]
Therefore,
\[
\|T_j f\|_2^2 = \sum_{k=1}^{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left|\hat{F}_k(\xi, \eta; r)\right|^2 d\xi d\eta \frac{dr}{r}
\]
\[
\leq C \sum_{k=1}^{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left|\hat{f}(\xi, \eta)\phi(2^jr)\right|^2 d\xi d\eta \frac{dr}{r}
\]
\[= \sum_k \int_{\mathbb{R}^n} \int_1^2 \left| \hat{F}_k(\xi, \eta; r) \right|^2 \frac{dr}{r} d\xi d\eta \]
\[\leq C \sum_k \int_{\mathbb{R}^n} \int_{D_k+j} p^2(a_k + j|A_\rho \xi|)\phi(2^{k+1})|A_\rho \xi|^2 \left| \hat{f}(\xi, \eta) \right|^2 d\xi d\eta,\]

where \(D_k+j = \{ \xi \in \mathbb{R}^m : 1/2 < a_k + j|A_\rho \xi| < 2 \} \). Recall that \(a_k = \phi(2^k) \). Because of the support \(D_k+j \) of \(p \) and the fact that \(\phi \) satisfies hypothesis A, we have \(2 > \phi(2^{k+j})|A_\rho \xi| > \lambda^{j-1}\phi(2^{k+1})|A_\rho \xi| \). Thus \(\phi(2^{k+1})|A_\rho \xi| < C\lambda^{-j}, \ j \geq 0 \). Therefore,

\[\|T_j f\|_2 \leq C\lambda^{-j} \|f\|_2, \quad j \geq 0. \tag{8}\]

Case \( j < 0 \). From Eq. (5), we have

\[\int_1^2 \left| \hat{F}_k(\xi, \eta; r) \right|^2 \frac{dr}{r} = p^2(a_k + j|A_\rho \xi|)\left| \hat{f}(\xi, \eta) \right|^2 \int_1^2 \left| \int e^{i|\phi(2^{k+1})r|v} E_{\alpha}(v, \zeta') dv \right|^2 \frac{dr}{r} \]
\[= p^2(a_k + j|A_\rho \xi|)\left| \hat{f}(\xi, \eta) \right|^2 \int_1^2 \left( \int e^{i|\phi(2^{k+1})r|v} E_{\alpha}(v, \zeta') dv \right) dv \frac{dr}{r} \]
\[= p^2(a_k + j|A_\rho \xi|)\left| \hat{f}(\xi, \eta) \right|^2 \int_1^2 \left( \int e^{i|\phi(2^{k+1})r|v} dv \right) E_{\alpha}(v, \zeta') dv \frac{dr}{r} \]
\[\leq p^2(a_k + j|A_\rho \xi|)\left| \hat{f}(\xi, \eta) \right|^2 \int_1^2 \left( \int \frac{1}{r^{d-1}} dv \right) E_{\alpha}(v, \zeta') dv \frac{dr}{r}, \tag{9}\]

where \(r(r) = \int e^{i|\phi(2^{k+1})r|v} dv, \ 1 \leq r \leq 2 \). By applying van der Corput’s lemma and by using the fact that \(\phi \) satisfies hypothesis A, we obtain \(|r(r)| \leq r(a_k|\xi||v - \bar{v}|)^{-1/2}\). Thus by integrating by parts, we have

\[\left| \int_1^2 e^{i|\phi(2^{k+1})r|v} \frac{dr}{r} \right| = \left| \int_1^2 \frac{1}{r} \tau'(r) \ dr \right| \leq C(a_k|\xi||v - \bar{v}|)^{-1/2}, \quad l \geq 1.\]

It is also obvious that

\[\left| \int_1^2 e^{i|\phi(2^{k+1})r|v} \frac{dr}{r} \right| \leq \ln 2.\]
Thus
\[
\left| \int_1^2 \psi|\phi(2^k r)|e^{i\psi(r - \tilde{\psi})} \frac{dr}{r} \right| \leq C \min \left\{ 1, (a_k \|v - \tilde{v}\|) \right\} \leq C (a_k \|v - \tilde{v}\|)^{-1/2}.
\]
\( l \geq 1. \) \tag{10}

Therefore,
\[
\int_1^2 |\tilde{F}_k(\xi, \eta; r)|^2 \frac{dr}{r} \leq C p^2 (a_k + j |A_\rho \xi|) |\hat{f}(\xi, \eta)|^2
\]
\[
\times \int \left\{ \int (a_k \|v - \tilde{v}\|)^{-1/2} |E_a(v, \tilde{v})| \frac{dv}{\tilde{v}} \right\} |\tilde{F}_a(v, \tilde{v})| \frac{dv}{\tilde{v}} \tag{11}
\]
\[
\leq C p^2 (a_k + j |A_\rho \xi|) |\hat{f}(\xi, \eta)|^2 \left( a_k \|v - \tilde{v}\| \right)^{-1/2}
\]
\[
\times \int \left[ \int |v - \tilde{v}|^{-1/2} d\tilde{v} \right] |\tilde{F}_a(v, \tilde{v})| \frac{dv}{\tilde{v}}
\]
\[
\leq C p^2 (a_k + j |A_\rho \xi|) |\hat{f}(\xi, \eta)|^2 \left( a_k \|v - \tilde{v}\| \right)^{-1/2}
\]
\[
\times \int \left[ \int |v - \tilde{v}|^{-1/2} d\tilde{v} \right] |\tilde{F}_a(v, \tilde{v})| \frac{dv}{\tilde{v}}
\]
\[
\leq C p^2 (a_k + j |A_\rho \xi|) |\hat{f}(\xi, \eta)|^2 \left( a_k \|A_\rho \xi\| \right)^{-1/2}.
\]
\tag{12}

Thus
\[
\|T_j f\|^2 = \sum_k \int_{\mathbb{R}^{n-m}} \int_{D_{k+j}} |\tilde{F}_k(\xi, \eta; r)|^2 \frac{dr}{r} d\xi d\eta
\]
\[
\leq C \sum_k \int_{\mathbb{R}^{n-m}} \int_{D_{k+j}} p^2 (a_k + j |A_\rho \xi|) |\hat{f}(\xi, \eta)|^2 \left( a_k \|A_\rho \xi\| \right)^{-1/2} d\xi d\eta.
\]

Since \( \phi \) satisfies hypothesis A and because of the support \( D_{k+j} \) of \( p \), we have for \( j < 0, 1/2 < a_{k+j} |A_\rho \xi| = \phi(2^{k+j}) |A_\rho \xi| \leq \lambda^{-j/2} |A_\rho \xi| \) and \( a_{k+j} |A_\rho \xi| |\phi(2^k) |A_\rho \xi| \leq \lambda^{-j/2} a_k |A_\rho \xi| \), whence \( (a_k |A_\rho \xi|)^{-1/2} \leq C \lambda^{-j/2} \). Hence,
\[
\|T_j f\|_2 \leq C \lambda^{-j/2} \|f\|_2, \quad j < 0. \tag{13}
\]
Combining inequalities (8) and (13), we obtain
\[ \|T_j f\|_2 \leq C\lambda^{-|j|/M} \|f\|_2 \quad \text{for all } j \in \mathbb{Z}. \] \hfill (14)

Finally, an application of Minkowski’s inequality yields
\[ \|\mathcal{I} a f\|_2 \leq \sum_j \|T_j f\|_2 \leq C\|f\|_2. \] \hfill (15)

Our next step is to obtain the $L^p$ norm of $T_j f$ for $2 < p < \infty$. Let $q = (p/2)'$, $2 < p < \infty$, and let $g \in L^q(\mathbb{R}^n)$ with $\|g\|_q \leq 1$. By Hölder’s inequality and by a change of variables, we have
\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [T_j f(x, \tilde{x})]^2 g(x, \tilde{x}) \, dx \, d\tilde{x} \right| \\
\leq \sum_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(y') |S_{k+j} f(x - \phi(2^k r)y', \tilde{x} - \psi(2^k r))| \, dx \, d\tilde{x} \, dr \\
\leq \|a\|_1 \sum_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| a(y') \right| \left| S_{k+j} f(x - \phi(2^k r)y', \tilde{x} - \psi(2^k r)) \right|^2 \, dx \, d\tilde{x} \, dr \\
= C \sum_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| a(y') \right| \left| S_{k+j} f(x, \tilde{x}) \right|^2 |g(x + \phi(2^k r)y', \tilde{x} + \psi(2^k r))| \, dx \, d\tilde{x} \, dr \\
= C \sum_k \left( \int_{\mathbb{R}^n} |S_{k+j} f(x, \tilde{x})|^2 dx \, d\tilde{x} \right) \frac{dr}{r} \\
= C \int_{\mathbb{R}^n} \sum_k |S_{k+j} f(x, \tilde{x})|^2 \\
\times \left( \int_{\mathbb{R}^n} \left| a(y') \right| |g(x + \phi(2^k r)y', \tilde{x} + \psi(2^k r))| \, dx \, d\tilde{x} \frac{dr}{r} \right) \, dx \, d\tilde{x}. \hfill (16)
Note that
\[
\int_1^2 \int_{S^{m-1}} |a(y')| g(x + \phi(2^k r)y', \tilde{x} + \Psi(2^k r)) \, d\sigma(y') \frac{dr}{r} \\
\leq \int_1^2 \int_{S^{m-1}} |a(y')| \left\{ \frac{1}{2^k} \int_{2^k}^{2^{k+1}} \left| g(x + \phi(r)y', \tilde{x} + \Psi(r)) \right| \, dr \right\} \, d\sigma(y') \\
\leq \int_1^2 \int_{S^{m-1}} |a(y')| M^r \tilde{g}(-x, -\tilde{x}) \, d\sigma(y'),
\]
where
\[
M^r \tilde{g}(-x, -\tilde{x}) = \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{2^k} \int_{2^k}^{2^{k+1}} \left| \tilde{g}(-x - \phi(r)y', -\tilde{x} - \Psi(r)) \right| \, dr \right\}
\]
and \(\tilde{g}\) is defined to be \(\tilde{g}(x, \tilde{x}) = g(-x, -\tilde{x})\). Observe that by the method of rotation and by the hypothesis of Theorem 1, \(M^r \tilde{g}\) is bounded in \(L^q\) for \(1 < p < \infty\), and the bound is independent of the vector \(y' \in S^{m-1}\). Therefore, by Minkowski’s inequality, the \(L^q\) norm of the integral above is not greater than \(C \|a\|_1 \|g\|_q\). Thus an application of Hölder’s inequality to (16) yields
\[
\left| \int_1^2 \int_{S^{m-1}} T_j f(x, \tilde{x}) \left\{ \sum_k |S_{k+j}f|^2 \right\}^{1/2} \, d\sigma(y') \right| \leq C \|f\|_p \|g\|_q,
\]
where the last inequality follows from the Littlewood–Paley theorem. Now let \(g\) run over the unit ball of \(L^q\). The inequality above implies that
\[
\|T_j f\|_p \leq C \|f\|_p \quad \text{for } 2 < p < \infty.
\]
Interpolating between (14) and (17) (see [3]) yields \(\|T_j f\|_p \leq C \lambda^{-\epsilon |j|} \|f\|_p\) for some \(\epsilon > 0\), \(2 < p < \infty\), and thus
\[
\|S_a f\|_p \leq \sum_j \|T_j f\|_p \leq C \|f\|_p, \quad 2 < p < \infty.
\]
Combining (15) and (18), we obtain
\[
\|S_a f\|_p \leq C \|f\|_p \quad \text{for } 2 < p < \infty, \quad s = 2.
\]
We now consider the case \(s = 1\). If \(f \in L^\infty(\mathbb{R}^n)\) and \(h \in L^1(\mathbb{R}^+, \frac{dr}{r})\), then
\[
\int_0^\infty h(r) \int_{S^{m-1}} a(y') f(x - \phi(r)y', \tilde{x} - \Psi(r)) \, d\sigma(y') \frac{dr}{r}.
\]
\[
\|f\|_{L^p} \leq \int_0^\infty |h(r)| \int_{S^{n-1}} |a(y')||f| \|\|_\infty \, d\sigma(y') \frac{dr}{r} \leq \|a\|_1 \|h\|_{L^1(\mathbb{R}^+, \frac{dr}{r})} \|f\|_{L^\infty}
\]

for almost every \((x, \tilde{x})\). Taking the supremum on both sides of the above inequality over all radial functions \(h\) with \(\|h\|_{L^1(\mathbb{R}^+, \frac{dr}{r})} \leq 1\) yields \(\|\mathfrak{S}_a f(x, \tilde{x})\|_{L^p} \leq C\|f\|_{L^\infty}\) for almost every \((x, \tilde{x}) \in \mathbb{R}^n\). Hence

\[
\|\mathfrak{S}_a f\|_{L^\infty} \leq C\|f\|_{L^\infty}.
\]

It remains to show the \(L^p\) boundedness of \(\mathfrak{S}_a f\) when \(1 < s < 2\). By duality,

\[
\mathfrak{S}_a f(x, \tilde{x}) = \left\| \int_{S^{n-1}} a(y') f(x - \phi(r)y', \tilde{x} - \psi(r)) \, d\sigma(y') \right\|_{L^p(\mathbb{R}^+, \frac{dr}{r})},
\]

where \(s'\) is the conjugate of \(s\). Thus

\[
\|\mathfrak{S}_a f\|_{L^p(\mathbb{R}^n)} = \left\| \int_{S^{n-1}} a(y') f(x - \phi(r)y', \tilde{x} - \psi(r)) \, d\sigma(y') \right\|_{L^p(L^s(\mathbb{R}^+, \frac{dr}{r}), \mathbb{R}^n)}
\]

where \(H : L^p(\mathbb{R}^n) \to L^p(L^s(\mathbb{R}^+, \frac{dr}{r}), \mathbb{R}^n)\) is a linear operator defined by

\[
H(f)(x, \tilde{x}, r) = \int_{S^{n-1}} a(y') f(x - \phi(r)y', \tilde{x} - \psi(r)) \, d\sigma(y').
\]

From inequalities (20) and (21), we interpret that \(\|Hf\|_{L^p(L^s(\mathbb{R}^+, \frac{dr}{r}), \mathbb{R}^n)} \leq C\|f\|_p\) for \(2 < p < \infty\) and that \(\|Hf\|_{L^\infty(L^\infty(\mathbb{R}^+, \frac{dr}{r}), \mathbb{R}^n)} \leq C\|f\|_{L^\infty}\). Applying the real interpolation theorem for Lebesgue mixed norm spaces to the above results (see [2]), we conclude that \(\|Hf\|_{L^p(L^s(\mathbb{R}^+, \frac{dr}{r}), \mathbb{R}^n)} \leq C\|f\|_p\) for \(p \geq s'\). That is, \(\|\mathfrak{S}_a f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}\) for \(s/(s-1) \leq p < \infty\), \(1 < s < 2\). Putting all the results together, we obtain \(\|\mathfrak{S}_a f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}\) for \(s/(s-1) \leq p < \infty\) when \(1 < s \leq 2\), and for \(p = \infty\) when \(s = 1\). The proof of the \(L^p\) boundedness of \(\mathfrak{S}_a\) is complete. We now show that \(\mathfrak{S}_a f\) is unbounded for some values of \(p\) and \(s\).

**Case** \(s = 1\), \(0 < p < \infty\). We pick an \(\Omega \in H^1(S^{m-1})\) such that \(\Omega\) is continuous on \(S^{m-1}\), and choose \(\phi(|y|) = |y|\). Now choose a ball \(B \subset S^{m-1}\) such that \(\Omega(y') \geq c > 0\) on \(B\), and let \(y_0\) be the center of this ball. Reduce the size of this ball by a factor of three, and let \(\epsilon\) denote the radius of this new ball, call it \(\tilde{B}\). Now let \(f(x, \tilde{x}) = |\tilde{x}|^{-1/2} \chi_1(\tilde{x}) \chi_1(x)\), where \(\chi_1\) is the characteristic function on the unit ball. It is clear that \(f \in L^p(\mathbb{R}^n)\) for \(0 < p < \infty\). Consider the integral

\[
I_p(x, \tilde{x}) = \int_{S^{n-1}} \Omega(y') f(x - \phi(r)y', \tilde{x} - \psi(r)) \, d\sigma(y')
\]

with \(f\) defined above and \(\phi(r) = r\). Observe that whenever \(|x| \geq \epsilon^{-1}, x' = x/|x| \in \tilde{B}\), and \(|x| - 1/2 \leq r \leq |x| + 1/2\), we have
Again, we choose $\omega$ of positive measure should be less than $s$ for any positive real number $p$. But note that if $\phi$ and $\Psi$ above, the maximal operator $M_g(x_1, x_2)$ is bounded in $L^p(\mathbb{R}^{n-m+1})$ for all $p > 1$. Now let $f(x, \tilde{x}) = 1/(|x|^{|m-s|}) \chi_{10}(x) \chi_1(\tilde{x})$. Then $f \in L^p(\mathbb{R}^n)$ if $p < m/(m-\alpha)$. By duality,

$$\left\{ \mathcal{M}_f(x, \tilde{x}) \right\}' = \int_0^\infty \int_{S^{m-1}} \frac{\Omega(y') f(x, \phi(r)y', \tilde{x} - \Psi(r)) d\sigma(y')}{r} dr$$

$$\geq \frac{2\alpha}{a} \int_0^\infty \int_{S^{m-1}} \frac{\Omega(y') f(x, \phi(r)y', \tilde{x} - \Psi(r)) d\sigma(y')}{r} dr$$

for any positive real number $a$. We wish to show the integral above blows up on a subset of $\mathbb{R}^n$ of positive measure. But note that if $s' \leq p < \infty$, then by applying Minkowski’s inequality twice we see that

$$\left\| \left( \int_0^\infty \int_{S^{m-1}} \frac{\Omega(y') f(x, \phi(r)y', \tilde{x} - \Psi(r)) d\sigma(y')}{r} dr \right)^{s'} \right\|_{L^p(\mathbb{R}^n)}^{1/s'}$$

$$\leq C \| \Omega \|_{L^1(S^{m-1})} \| f \|_p$$

for $s' \leq p < \infty$. This implies that the integral is finite for almost every $(x, \tilde{x}) \in \mathbb{R}^n$. Therefore, we expect that the range of $p$ for which the above integral blows up on a subset of $\mathbb{R}^n$ of positive measure should be less than $s' = s/(s - 1)$.
Now consider \((x, \tilde{x}) \in \mathbb{R}^n\) such that \(0 < |x| < 1, x' = x/|x| \in \tilde{B}, \) and \(|\tilde{x}| < 1\). Recall that \(B, \tilde{B}\) are concentric balls centered at \(y_0'\) with radii \(\epsilon\) and \(3\epsilon\), respectively, which were constructed in the previous example for the case \(s = 1\), \(0 < p < \infty\). Now let \(B_1 \subset S^{m - 1}\) be the ball centered at \(x'\) with the same radius \(\epsilon\). Then \(B_1 \subset B, \) and \(\Omega(s') \geq c\) on \(B_1\). With the choice of \(f\) above, we have

\[
\left\{\nabla_B f(x, \tilde{x})\right\}^s \geq \int_{|x|}^{2|x|} \int_{S^{m - 1}} \frac{\Omega(y') d\sigma(y')}{|x - r y'|^{m - a}} \frac{r^{s'} dr}{r}
\]

\[
= |x|^{(a - m)s'} \int_{1}^{2} \int_{S^{m - 1}} \frac{\Omega(y') d\sigma(y')}{|x' - y'|^{m - a}} \frac{r^{s'} dr}{r}
\]

\[
\geq C \int_{1}^{r_0} \int_{S^{m - 1}} \frac{\Omega(y') d\sigma(y')}{|x' - y'|^{m - a}} \frac{r^{s'} dr}{r},
\]

where \(1 < r_0 < 2\). We choose \(r_0\) to be sufficiently close to 1 so that \(1 - 1/r_0 \leq \epsilon\). Denote \(u_r = 1 - 1/r\) and \(u_{r_0} = 1 - 1/r_0\). Note that by our choice of \(r_0, u_r \leq u_{r_0} \leq \epsilon\) for \(1 \leq r \leq r_0\). Now denote the integral above by \(I_r(x, \tilde{x})\) and write

\[
I_r(x, \tilde{x}) = \int_{B_1} \frac{\Omega(y') d\sigma(y')}{|x' - y'|^{m - a}} \geq c \int_{B_1} \left(\frac{1}{r} - 1 + |x' - y'|\right)^{a - m} d\sigma(y')
\]

\[
= c \int_{B_1} \left(u_r + |x' - y'|\right)^{a - m} d\sigma(y')
\]

\[
= c \omega_m \int_{0}^{\sqrt{1 - \cos \theta}} (u_r + \sqrt{2} \sqrt{1 - \cos \theta})^{a - m} (\sin \theta)^{m - 2} d\theta,
\]

where \(\omega_m\) is a constant depending on \(m\). By a change of variable \(t = \sqrt{2} \sqrt{1 - \cos \theta}\), we have

\[
I_{r_0}(x, \tilde{x}) \geq C \epsilon \int_{0}^{\epsilon} (u_r + t)^{a - m} m - 2 \ dt \geq C \epsilon \int_{0}^{u_{r_0}} (u_r + t)^{a - m} m - 2 \ dt
\]

\[
= C \epsilon (a - 1) \int_{0}^{u_{r_0}} (1 + t)^{a - m} m - 2 \ dt \geq C \epsilon (1 + t)^{a - m} m - 2 \ dt
\]
Thus if \( r_0 \) is sufficiently close to 1, and \( \alpha \leq 1/s \), then
\[
\mathcal{M}_f(x, \tilde{x})^s \geq C \int_1^{r_0} \left( 1 - \frac{|I^{(1)}(x, \tilde{x})|}{|I^{(1)}(x, \tilde{x})|} \right)^{s'} \frac{dr}{r} \]
\[
\geq C \int_1^{r_0} (r - 1)^{(a-1)s'} dr = \infty.
\]
Combining the two inequalities \( p < m/(m - 1) \) and \( \alpha \leq 1/s \), we see that if \( p < ms/(ms - 1) \), then \( \mathcal{M}_f(x, \tilde{x}) \) is infinite on the set \((A \times F) \cap \mathbb{R}^n\) of positive measure, where
\[
A = \{ x \in \mathbb{R}^m : |x| < 1, x'/|x| \in \tilde{B} \} \quad \text{and} \quad F = \{ \tilde{x} \in \mathbb{R}^{n-m} : |	ilde{x}| < 1 \}.
\]

For the case \( p = ms/(ms - 1) \), we will get the same result by repeating the same argument above with a new function \( f \) defined by
\[
f(x, \tilde{x}) = \frac{1}{|x|^{m-1/s} \ln(100/|x|)} \chi_{10}(x) \chi_1(\tilde{x}).
\]
Consequently, \( \mathcal{M}_f \) is unbounded in \( L^p(\mathbb{R}^n) \) for \( 0 < p \leq ms/(ms - 1) \), \( 1 < s \leq 2 \).

**Case** \( s < 1, 1 \leq p \leq \infty \). Putting all the results we have obtained so far, it is obvious that \( \mathcal{M}_f \) must be unbounded in \( L^p(\mathbb{R}^n) \) for \( 1 \leq p \leq \infty \) when \( s < 1 \), for otherwise interpolation would lead to a contradiction to the case \( s = 1 \). Theorem 1 is proved. □

**Remark.**

(1) For the case \( s = \infty \), the authors in [1] showed that there is a function \( f \in L^p \) such that the maximal operator acting on \( f \) yields an identically infinite function.

(2) For the proof of the case \( m = 2 \), we apply [7, Lemma 2.2] instead of [7, Lemma 2.1] with some slight modifications. For instance, \( E_a(v, \zeta') \) in Eq. (5) should be replaced by \( e_a(v, \zeta') \), where \( e_a(v, \zeta') \) is a \( q \)-atom for some fixed \( q \) in the interval \((1, 2)\). The exponent \(-1/(2l)\) in inequality (10) should be replaced by \(-1/(2lq')\), where \( q' \) is the conjugate of \( q \). By applying Hölder’s inequality to the inner integral on the RHS of inequality (11), we will get a similar estimate as in inequality (12), with the exponent \(-1/(2l)\) being replaced by \(-1/(2lq')\).

(3) If \( \phi \) satisfies hypothesis B instead of hypothesis A, then some minor adjustments should be noted as follows: the lacunary sequence \( \{a_k\} \) should be defined by \( a_k = \phi(2^{-k}) \), \( k \in \mathbb{Z} \). Also, the factor \( 2^k \) appearing in Eq. (1) should be replaced by \( 2^{-k} \), etc.

**Proof of Corollaries 1 and 2.** It suffices to show that under the hypotheses of \( \phi \) and \( \Psi \) given in these corollaries, the maximal operator \( M_g(x_1, x_2) \) is bounded in \( L^p(\mathbb{R}^{n-m+1}) \) for all \( p > 1 \). For this proof, see [6,10]. □
Proof of Corollary 3. We must show that \( Mg(x_1, x_2) \) is bounded in \( L^p(\mathbb{R}^{n-m+1}) \) for all \( p > 1 \). To prove this, we repeatedly apply Theorem C [6]. We only consider the case that \( \phi \) satisfies hypothesis A, since the proof for the other case (hypothesis B) is essentially the same. For \( k \in \mathbb{Z} \), define the measures \( \mu_k \) and \( \mu_k^{(0)} \) by

\[
\hat{\mu}_k(\zeta, \eta) = \frac{1}{2^k} \int_{2^k} e^{i \xi \phi(r)} e^{i \eta \Psi(r)} \, dr \quad (\zeta \in \mathbb{R}, \eta \in \mathbb{R}^{n-m})
\]

and

\[
\hat{\mu}_k^{(0)}(\eta) = \hat{\mu}_k(0, \eta) = \frac{1}{2^k} \int_{2^k} e^{i \eta \Psi(r)} \, dr.
\]

Then \( \mu_k \) and \( \mu_k^{(0)} \) are finite positive Borel measures. For nonnegative Schwartz functions \( f \) on \( \mathbb{R}^{n-m+1} \) and \( g \) on \( \mathbb{R}^{n-m} \), we have

\[
\mu_k * f(x_1, x_2) = \frac{1}{2^k} \int_{2^k} f(x_1 - \phi(t), x_2 - \Psi(t)) \, dt,
\]

\[
\mu_k^{(0)} * g(x_2) = \frac{1}{2^k} \int_{2^k} g(x_2 - \Psi(t)) \, dt \quad (x_1 \in \mathbb{R}, x_2 \in \mathbb{R}^{n-m}).
\]

We need to show that

\[
|\hat{\mu}_k(\zeta, \eta) - \hat{\mu}_k(0, \eta)| \leq C|a_k + 1/\zeta|,
\]

\[
|\hat{\mu}_k(\zeta, \eta)| \leq C|a_k \zeta|^{-1/\ell}
\]

(where \( a_k = \phi(2^k) \) is a lacunary sequence of positive real numbers) and \( \sup_{k \in \mathbb{Z}} |\mu_k^{(0)} * g(x_2)| \) is a bounded operator in \( L^p(\mathbb{R}^{n-m}) \) for all \( p > 1 \). It is clear that \( |\hat{\mu}_k(\zeta, \eta) - \hat{\mu}_k(0, \eta)| \leq C|a_{k+1} \zeta| \). Denote \( \tau(r) \) by \( \tau(r) = \int_{1}^{r} e^{i \xi \phi(2^k)} \, dt \) for \( 1 \leq r \leq 2 \). Then

\[
|\hat{\mu}_k(\zeta, \eta)| = \left| \int_{1}^{2} e^{i \xi \phi(2^k)} e^{i \eta \Psi(2^k)} \, dr \right| = \left| \int_{1}^{2} \tau'(r) e^{i \eta \Psi(2^k)} \, dr \right| 
\]

\[
\leq C|a_k \zeta|^{-1/\ell} \left( 1 + \int_{2^k}^{2^{k+1}} |\eta \cdot \Psi'(r)| \, dr \right) \leq C|a_k \zeta|^{-1/\ell}.
\]

The first inequality follows from van der Corput’s lemma. The second inequality follows since for \( i = 1, 2, \ldots, n-m \), \( \int_{2^k}^{2^{k+1}} |\eta_i(\gamma_r)| \, dr \leq C\|\gamma\|_{\infty} \), and \( C \) is independent of \( \eta_i \).

It remains to show the \( L^p \) boundedness of the operator \( \sup_{k \in \mathbb{Z}} |\mu_k^{(0)} * g(x_2)| \). We prove this by induction on the dimension \( n-m \). For the sake of argument, let \( d = n-m \). When \( d = 1 \), by an easy application of [6, Theorem A] (or see [8, Corollary 1]), we see that
sup_{k \in \mathbb{Z}} |\mu_k^{(0)} \ast g(x_2)| is bounded in L^p for all p > 1. Now assume that the result is true for curves in \( \mathbb{R}^{d-1} \). For \( k \in \mathbb{Z} \) define the measures \( \nu_k \) and \( \nu_k^{(0)} \) by

\[
\hat{\nu}_k(\zeta, \zeta_d) = \frac{1}{2^k} \int_{2^k}^{2^{k+1}} e^{i(\gamma_1(t) + \cdots + \gamma_d(t))} \, dt \quad \text{and}
\]

\[
\hat{\nu}_k^{(0)}(\zeta) = \frac{1}{2^k} \int_{2^k}^{2^{k+1}} e^{i(\gamma_1(t) + \cdots + \gamma_d(t))} \, dt.
\]

(\( \hat{\nu}_k \equiv (\zeta_1, \ldots, \zeta_d) \in \mathbb{R}^{d-1}, \zeta_d \in \mathbb{R} \)).

Then \( \nu_k \) and \( \nu_k^{(0)} \) are finite positive Borel measures. For nonnegative Schwartz functions \( f \) on \( \mathbb{R}^d \) and \( g \) on \( \mathbb{R}^{d-1} \), we have

\[
\nu_k \ast f(\hat{x}, x_d) = \frac{1}{2^k} \int_{2^k}^{2^{k+1}} f(x_1 - \gamma_1(t), \ldots, x_d - \gamma_d(t)) \, dt,
\]

\[
\nu_k^{(0)} \ast g(x) = \frac{1}{2^k} \int_{2^k}^{2^{k+1}} g(x_1 - \gamma_1(t), \ldots, x_{d-1} - \gamma_{d-1}(t)) \, dt \quad (x \in \mathbb{R}^{d-1}, x_d \in \mathbb{R}).
\]

By applying [6, Theorem C], we must show that

\[
|\hat{\nu}_k(\zeta, \zeta_d) - \hat{\nu}_k(\zeta, 0)| \leq C|b_k + 1\zeta_d|,
\]

\[
|\hat{\nu}_k(\zeta, \zeta_d)| \leq C|b_k \zeta_d|^{-\varepsilon} \quad \text{for some positive } \varepsilon > 0.
\]

Here \( \{b_k\} = \{\gamma_2(2^k)\} \) is a lacunary sequence of positive real numbers. The first inequality is obvious. The proof of the second estimate for \( |\hat{\nu}_k(\zeta, \zeta_d)| \) is analogous to the proof of the second estimate of \( |\mu_k(\zeta, \eta)| \) (with \( \tau(\zeta) \)) being defined as \( \tau(\zeta) = \int_{2^k}^{2^{k+1}} e^{i(\gamma_1(t) + \cdots + \gamma_d(t))} \, dt \), etc.). Finally, we need to prove that the operator \( \sup_{k \in \mathbb{Z}} |\nu_k^{(0)} \ast g(x)| \) is a bounded operator in \( L^p(\mathbb{R}^{d-1}) \) for all \( p > 1 \). But this is true from the induction hypothesis. Thus \( \sup_{k \in \mathbb{Z}} |\nu_k \ast f(x, x_d)| \) or equivalently \( \sup_{k \in \mathbb{Z}} |\mu_k^{(0)} \ast g(x_2)| \) is bounded in \( L^p \) for all \( p > 1 \). Consequently, \( Mg(x_1, x_2) \) is bounded in \( L^p(\mathbb{R}^{n-m+1}) \) for all \( p > 1 \). Corollary 3 is proved.

**Proof of Theorem 2.** The proof of this theorem is partially an application of Theorem 1, with \( \Psi \equiv 0 \). Again, it suffices to consider the \( \infty \)-atom in place of \( \Omega \). Now for each \( f \in L^p(\mathbb{R}^m) \), the function \( \hat{f} \) defined by \( \hat{f}(\hat{x}, \tilde{x}) = f(x) \chi_1(\tilde{x}) \) (\( x \in \mathbb{R}^m, \tilde{x} \in \mathbb{R}^{n-m} \)) is clearly in \( L^p(\mathbb{R}^n) \); and \( \|\hat{f}\|_{L^p(\mathbb{R}^n)} \leq |B| \|f\|_{L^p(\mathbb{R}^m)} \) where \( |B| \) is the Lebesgue measure of the unit ball \( B \) in \( \mathbb{R}^{n-m} \). Thus for every \( f \in L^p(\mathbb{R}^m) \), \( s/(s-1) \leq p < \infty \), \( 1 < s \leq 2 \), we have \( |B| \|\hat{f}\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^m)} \) (\( C = |B| \|f\|_{L^p(\mathbb{R}^m)} \)). The first equality follows from Fubini’s theorem and the observation that \( \hat{\odot} \hat{f}(\hat{x}, \tilde{x}) = \hat{\odot} f(x) \chi_1(\tilde{x}) \); and the inequality above follows from Theorem 1. This implies that \( \|\hat{\odot} f\|_p \leq C \|f\|_p \) for all...
$f \in L^p(\mathbb{R}^m)$ with $s/(s - 1) \leq p < \infty$, $1 < s \leq 2$. The proof for the remaining cases of $p$ and $s$ are essentially the same as in the proof of Theorem 1. Theorem 2 is proved. □

Proof of Theorem 3. It suffices to consider the regular $\infty$-atom $a$ in place of $\Omega$. There is no loss of generality to assume that $\|h\|_{L^1(\mathbb{R}^+, \varphi_\epsilon)} = 1$. It is then obvious from Theorem 1 that $T_{\epsilon}f$ is bounded in $L^p(\mathbb{R}^n)$ for $s/(s - 1) \leq p < \infty$, $1 < s \leq 2$. We claim that the truncated operator $T_{\epsilon}f$ ($\epsilon > 0$) is also bounded in $L^p(\mathbb{R}^n)$ with the same ranges of $p$ and $s$ as above, and the bound is independent of $\epsilon$. To see this, write

$$T_{\epsilon}f(x, \tilde{x}) = \int_{|y| > \epsilon} \frac{h(|y|)a(y')}{|y|^m} f(x - \phi(|y|)) y' + \Psi(|y|) dy$$

where $\tilde{h}(|y|) = h(|y|) \chi_\epsilon(|y|)$ and $\chi_\epsilon(|y|)$ is the characteristic function on the set $\{ y \in \mathbb{R}^m : |y| > \epsilon \}$. Then $\|\tilde{h}\|_{L^1(\mathbb{R}^+, \varphi_\epsilon)} \leq \|h\|_{L^1(\mathbb{R}^+, \varphi_\epsilon)} = 1$ for all $\epsilon > 0$. Therefore by Theorem 1, $\|T_{\epsilon}f\|_p \leq \sup_{\epsilon} \|T_{\epsilon}f\|_p = \|T_{\epsilon}f\|_p \leq C \|f\|_p$ for $s/(s - 1) \leq p < \infty$, $1 < s \leq 2$, and $C$ is independent of $\epsilon$. By the routine duality argument, $T_{\epsilon}f$ is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < s$, $1 < s < 2$, and the bound is again independent of $\epsilon$. Passing to the limit as $\epsilon \rightarrow 0$, Fatou’s lemma gives $\|T_{\epsilon}f\|_p \leq C \|f\|_p$ for $1 < p < s$, $1 < s < 2$. Now if $s = 2$ then we are done; otherwise an application of the real interpolation theorem gives the $L^p$ bounds of $T_{\epsilon}f$ for the remaining range of $p$: $s < p < s/(s - 1)$. Finally, using density argument, we may infer that $T_{\epsilon}f$ has a bounded extension in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. It remains to prove the $L^p$ bounds for $T_{\epsilon}^\star f$; and for this proof, we need the following lemmas.

Lemma 1. Assume that $h \in L^2(\mathbb{R}^+, \frac{dz}{z}) \cap L^\infty(\mathbb{R}^+) \ (1 < s \leq 2)$. For a measurable, locally integrable function $f$ on $\mathbb{R}^n$, define a sequence of finite measures $\sigma_k$ on $\mathbb{R}^n$ by

$$\sigma_k * f(\tilde{x}) = \int_{|y| \geq 2^k} \frac{a(y') h(|y|)}{|y|^m} f (\tilde{x} - \Gamma(y)) dy, \quad y \in \mathbb{R}^m, 2 \leq m < n,$$

where $a(y')$ is an $\infty$-atom on $S^{m-1}$.

If $\phi$ satisfies hypothesis A, then for all $k \in \mathbb{Z}$,

$$|\hat{\sigma}_k(\zeta, \eta)| \leq C \min \{ |ak+1 A_p \zeta|, |ak A_p \zeta|^{-1/4} \}, \quad \text{where } a_k = \phi(2^k).$$

Here $(\zeta, \eta) \in \mathbb{R}^m$ with $\zeta \in \mathbb{R}^m$, $\eta \in \mathbb{R}^{n-m}$, and recall that $A_p \zeta = (\rho^2 \zeta_1, \rho \zeta_2, \ldots, \rho \zeta_m)$. If $\phi$ satisfies hypothesis B, then for all $k \in \mathbb{Z}$,

$$|\hat{\sigma}_k(\zeta, \eta)| \leq C \min \{ |b_{-k} A_p \zeta|, |b_{-k-1} A_p \zeta|^{-1/4} \}, \quad \text{where } b_k = \phi(2^{-k}).$$

Lemma 2. Let $|\sigma_k|$ denote the total variations of the measures $\sigma_k$, and denote $\sigma^\star f(x, \tilde{x}) = \sup_{k \in \mathbb{Z}} |\sigma_k| * f(x, \tilde{x})$, $f \in L^p(\mathbb{R}^n)$. Then $\|\sigma_k\|_1 \leq C$ for all $k \in \mathbb{Z}$ and $\|\sigma^\star f\|_p \leq C \|f\|_p$ for all $p$ with $1 < p < \infty$. 
Proof of Lemma 1. We only prove for the case \( m \geq 3 \) and the case that \( \phi \) satisfies hypothesis A, since the proofs of the remaining cases are essentially the same. By taking the Fourier transform of \( \sigma_k \ast f \), we see that

\[
\hat{\sigma}_k(\zeta, \eta) = \int_{|y| \leq 2^k} |y|^{m-1} a(y') |h(|y|)| e^{i|\zeta|\phi(|y|)^{m}'} e^{i\eta \cdot \Psi(|y|)} dy \quad (\zeta, \eta \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}).
\]

We may assume that \( \text{supp}(a) \subset B(1, \rho) \cap S^{m-1} \), where \( 1 = (1, 0, 0, \ldots, 0) \). For \( \zeta \neq 0 \), we choose a rotation \( \theta \) such that \( \theta(\zeta) = |\zeta| \cdot (1, 0, 0, \ldots, 0) \), and let \( \theta^{-1} \) denote its inverse. Let \( y' = (v, y'_2, \ldots, y'_m) \).

Then

\[
\hat{\sigma}_k(\zeta, \eta) = \int_{2^k}^{2^{k+1}} \int_{S^{m-1}} h(r)a(\theta^{-1}(y')) e^{i|\zeta|\phi(r)^{m}'} e^{i\eta \cdot \Psi(r)} d\sigma(y') \frac{dr}{r},
\]

where \( E_a(v, \zeta') \) has support in \( \{ (\rho^2 \zeta'_1, \rho \zeta'_2, \rho \zeta'_3, \ldots, \rho \zeta'_n) \} \).

By the cancellation property of \( E_a \), one easily sees that

\[
|\hat{\sigma}_k(\zeta, \eta)| \leq C \phi(2^{k+1}) |A_{\rho} \zeta| = C a_{k+1} |A_{\rho} \zeta| \cdot
\]

On the other hand, by Hölder’s inequality, we have

\[
|\hat{\sigma}_k(\zeta, \eta)|^2 \leq \left\{ \int_{2^k}^{2^{k+1}} |h(r)|^2 \frac{dr}{r} \right\}^{\frac{1}{2}} \left\{ \int_{2^k}^{2^{k+1}} e^{i|\zeta|\phi(r)^{m}'} E_a(v, \zeta') d\sigma(v) \frac{dr}{r} \right\}^{\frac{1}{2}} \cdot
\]

\[
\leq C \left\{ \int_{2^k}^{2^{k+1}} e^{i|\zeta|\phi(2^{k+1})^m} \frac{dr}{r} \right\}^{\frac{1}{2}} E_a(\tilde{v}, \zeta') E_a(\tilde{v}, \zeta') d\sigma(\tilde{v}) d\tilde{v}
\]

\[
\leq C |a_{k+1} A_{\rho} \zeta|^{-1/2}. \]

The last inequality follows by a similar calculation as in the calculation of

\[
\int_{1}^{2} |\tilde{F}_k(\zeta, \eta; r)|^2 \frac{dr}{r}
\]

in the proof of Theorem 1 (see Eqs. (9)–(12)). Lemma 1 is proved. \( \square \)

Proof of Lemma 2. It is clear that \( \|\sigma_k\|_1 \leq C \|a\|_{L^1(S^{m-1})} \|h\|_{L^1(\mathbb{R}^n, d\rho)} \leq C \), and the bound is independent of \( k \in \mathbb{Z} \). Observe that

\[
\sigma^* f(x, \tilde{x}) \leq \|h\|_{\infty} \int_{S^{m-1}} |a(v)| M^{V'} f(x, \tilde{x}) d\sigma(y'),
\]
where

\[ M^{y'} f(x, \tilde{x}) = \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x - \phi(r)y', x - \Psi(r))| \, dr \right\}. \]

Recall that by the method of rotation and by the hypothesis of Theorem 1, \( M^{y'} \) is bounded in \( L^p \) for \( 1 < p < \infty \), and the bound is independent of the vector \( y' \in S^{m-1} \). Thus by Minkowski’s inequality, we have \( \|\sigma^* f\|_p \leq C\|a\|_1 \|f\|_p \leq C\|f\|_p \) for \( 1 < p < \infty \). Lemma 2 is proved.

Now observe that

\[ T_k^p f(\tilde{x}) = \sup_{\epsilon > 0} \left| \int_{|y| > \epsilon} \frac{h(|y|)\Omega(y')}{|y|^m} f(x - \phi(|y|)y', \tilde{x} - \Psi(|y|)) \, dy \right| \]

\[ \leq \sup_{k \in \mathbb{Z}} \sum_{j=k}^{\infty} \sigma_j * f(x, \tilde{x}) + \sup_{k \in \mathbb{Z}} \|\sigma_k \| f(x, \tilde{x}) \]

\[ = \sup_{k \in \mathbb{Z}} \|T_k f(x, \tilde{x})\| + \sup_{k \in \mathbb{Z}} \|\sigma_k \| f(x, \tilde{x})\| . \]

By Lemma 2, the second term on the RHS of the inequality above is bounded in \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \). To show the \( L^p \) boundedness of \( \sup_{k \in \mathbb{Z}} |T_k f(x, \tilde{x})| \), we take a radial Schwartz function \( \kappa \) on \( \mathbb{R}^m \) such that \( \kappa(\zeta) = \kappa(|\zeta|) = 1 \) when \( |\zeta| < \lambda^{-1} \) and \( \kappa(\zeta) = 0 \) when \( |\zeta| > \lambda \). Recall that the number \( \lambda \) comes from the sequence \( \{a_k\} \) or \( \{b_k\} \) as in Lemma 2. Note that in both cases, \( \inf_{k \in \mathbb{Z}} |a_{k+1}/a_k| = \lambda = \inf_{k \in \mathbb{Z}} |b_{k+1}/b_k| \). It suffices to consider the sequence \( \{a_k\} \). Define \( \Phi_k \) on \( \mathbb{R}^m \) by \( \Phi_k(\zeta) = \kappa(a_k|A_p\zeta|) \), and let \( \delta_n, \delta_{n-m} \) be the Dirac distributions on \( \mathbb{R}^n \) and \( \mathbb{R}^{n-m} \), respectively. We then write \( T_k f \) in a similar fashion as in [6]. That is,

\[ T_k f = (\Phi_k \otimes \delta_{n-m}) * (T_k f - \sum_{j=-\infty}^{k-1} \sigma_j * f) + (\delta_n - \Phi_k \otimes \delta_{n-m}) \sum_{j=k}^\infty \sigma_j * f, \]

(22)

Note that \( |\Phi_k \otimes \delta_{n-m} * T_{\kappa} f(x, \tilde{x})| \leq C M^H_m \circ M^H_{m-1} \circ \cdots \circ M^H_1 T_r f(x, \tilde{x}) \) for all \( k \in \mathbb{Z} \). Here \( M^H_i g(x, \tilde{x}) \) denotes the Hardy–Littlewood maximal function acting on the \( i \)-th coordinate of the \( x \)-variable of \( g(x, \tilde{x}) \). Thus

\[ \|\sup_{k \in \mathbb{Z}} |\Phi_k \otimes \delta_{n-m} * T_{\kappa} f| \|_p \leq C \|M^H_m \circ M^H_{m-1} \circ \cdots \circ M^H_1 T_r f\|_p \leq C \|T_r f\|_p \leq C \|f\|_p, \quad 1 < p < \infty. \]

Meanwhile,

\[ \sup_{k \in \mathbb{Z}} |\Phi_k \otimes \delta_{n-m} * \sum_{j=-\infty}^{k-1} \sigma_j * f| \leq \sup_{k \in \mathbb{Z}} \sum_{j=1}^\infty |\sigma_{k-j} \otimes \Phi_k \otimes \delta_{n-m} \otimes f| , \]
where each summand in the sum above is bounded in $L^p$, due to the $L^p$ boundedness of $\sigma^*$ (see Lemma 2). Moreover, each term in the sum above has an $L^2$-norm of order $\lambda^{-j}$. To see this, note that \( \sup_{k \in \mathbb{Z}} |\sigma_{k-j} \ast \Phi_k \otimes \delta_{n-m} \ast f| \leq (\sum_{\infty} \sigma_{k-j} \ast \Phi_k \otimes \delta_{n-m} \ast f^2)^{1/2} \).

By Plancherel’s theorem, it is enough to show that \( \sum_{\infty} |\hat{\sigma}_{k-j} (\xi, \eta) \hat{\Phi}_k (\xi)|^2 \leq C \lambda^{-2j} \) \((j \geq 1)\). There exists an \( m \in \mathbb{Z} \) such that \( a_{m+1}^{-1} \leq |A_p \xi| \leq a_m^{-1} \) for \( \xi \neq 0 \). Using Lemma 1 and the support condition on \( \kappa \), we find that \( \sum_{\infty} |\hat{\sigma}_{k-j} (\xi, \eta) \hat{\Phi}_k (\xi)|^2 \leq C \sum_{\infty} |a_{k-j+1} a_m^{-1}|^2 \leq C \lambda^{-2j} \) for \( j \geq 1 \). Applying interpolation theory to the $L^2$-norm and the $L^{p_0}$-norm, \( p_0 > p \), we obtain a factor of $\lambda^{-\epsilon j} (\epsilon > 0)$ in the $L^p$-norm for each summand in the sum above. Finally, by applying Minkowski’s inequality, we see that the $L^p$-norm of the sum above converges. By using similar arguments as above, we see that \( \sup_{k \in \mathbb{Z}} |(\delta_n - \Phi_k \otimes \delta_{n-m}) \ast \sigma_j \ast f| \) is also bounded in $L^p(\mathbb{R}^n)$ for \( 1 < p < \infty \). Therefore, \( \sup_{k \in \mathbb{Z}} |T_k f| \), and hence $T^* f$ is bounded in $L^p$ for \( 1 < p < \infty \). Theorem 3 is proved.

**Remark.** If \( \Psi \) satisfies hypothesis B, then the proof for the $L^p$ boundedness of the maximal function $T^* f$ undergoes some slight changes. We let \( T_k f = \sum_{j=k+1}^{\infty} \sigma_j \ast f \), and instead of (22), we write

\[
T_k f = (\Phi_k \otimes \delta_{n-m}) \ast \sum_{j=k+1}^{\infty} \sigma_j \ast f + (\delta_n - \Phi_k \otimes \delta_{n-m}) \ast \left( T_f - \sum_{j=\infty}^{k} \sigma_j \ast f \right).
\]

**Proof of Theorem 4.** The idea for the proof of the $L^p$ boundedness of $T f$ is similar to the idea in the proof of Theorem 2 (with $\Psi \equiv 0$). The proof for the $L^p$ bound of $T^* f$ is essentially a repetition of the proof of $T^* f$ in Theorem 3 with a slight modification. That is, instead of (22), we write $T_k f$ (for the case $\phi$ satisfying hypothesis A) as

\[
T_k f = \Phi_k \ast \left( T_f - \sum_{j=\infty}^{k-1} \sigma_j \ast f \right) + (\delta_m - \Phi_k) \ast \sum_{j=k}^{\infty} \sigma_j \ast f.
\]

Therefore, we omit the details. Theorem 4 is proved.

**Comment.** When the radial function \( h \) is merely in $L^\infty(\mathbb{R}^n)$, the operators $T f$ and $T^* f$ in Theorem 3 are still bounded in $L^p$ for \( 1 < p < \infty \). Indeed, by mimicking the proof of Theorem 1 in [9] with some slight changes, one can prove the following theorem.

**Theorem 5.** Let $\phi$ satisfy either hypothesis A or hypothesis B, and let $h \in L^\infty(\mathbb{R}^n)$. If $\Psi$ is given as in Corollaries 2 or 3, or if $\Psi(t) = (t^1, t^2, \ldots, t^{k_{n-m}})$ with $0 < k_1 < k_2 < \cdots < k_{n-m}$, then the operators $T f$ and $T^* f$ are bounded in $L^p(\mathbb{R}^n)$ for \( 1 < p < \infty \).

**Acknowledgment**

The author expresses his gratitude to the referee for his/her helpful suggestions.
References