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On vertex neighborhood in minimal imperfect graphs

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Abstract

Lubiw (J. Combin. Theory Ser. B 51 (1991) 24) conjectures that in a minimal imperfect Berge graph, the neighborhood graph $\mathcal{N}(v)$ of any vertex v must be connected; this conjecture implies a well known Chvátal's conjecture (Chvátal, First Workshop on Perfect Graphs, Princeton, 1993) which states that $\mathcal{N}(v)$ must contain a P_4. In this note we will prove an intermediary conjecture for some classes of minimal imperfect graphs. It is well known that a graph is P_4-free if, and only if, every induced subgraph with at least two vertices either is disconnected or its complement is disconnected; this characterization implies that P_4-free graphs can be constructed by *complete join* and *disjoint union* from isolated vertices. We propose to replace P_4-free graphs by a similar construction using bipartite graphs instead of isolated vertices.

1. Introduction

A graph is *perfect* if the vertices of any induced subgraph H can be colored in such a way that no two adjacent vertices receive the same color, with a number of colors (denoted by $\chi(H)$) not exceeding the cardinality $\omega(H)$ of a maximum clique of H. A graph G is *minimal imperfect* if all of its proper induced subgraphs are perfect but G is not.

It is an easy task to check that an odd chordless cycle of length at least five (called a *hole*), as well as its complement (called an *anti-hole*) are minimal imperfect graphs; moreover Berge [3] conjectures that they are the only minimal imperfect graphs. Nowadays, this conjecture is known as the *Strong Perfect Graph Conjecture* (SPGC), and graphs that contain neither odd hole nor odd anti-hole are called *Berge graphs*.

A set C of vertices of a graph G is called a *star-cutset* if $G - C$ is disconnected and in C there is a vertex adjacent to all other vertices of C.

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Lemma 1 (Star-Cutset Lemma — Chvátal [6]). *No minimal imperfect graph contains a star-cutset.*

For a graph G, we denote by $N_G(x)$ the set of vertices of G adjacent to vertex x of G and by $\mathcal{N}_G(x)$ the *neighborhood graph* of x induced by $N_G(x)$; when there can be no confusion we shall write $N(x)$ for $N_G(x)$ and $\mathcal{N}(x)$ for $\mathcal{N}_G(x)$. Let us remark that Lemma 1 implies that if $G=(V,E)$ is minimal imperfect then for every v in V the graph induced by $V-(\{v\}\cup N(v))$ must be connected.

We can also notice that if G is a minimal imperfect graph then for any vertex v of G, the subgraph induced by $\overline{N_G(v)}$ must be connected (otherwise $\{v\}\cup N_{\bar{G}}(v)$ forms a star-cutset in $\bar{G}$). This result is due to Gallai [9] (also see Olaru [13–15]):

Theorem 1 (Gallai [9]). *If G is a minimal imperfect Berge graph, then for every vertex v of G, $\overline{N_G(v)}$ induces a connected subgraph of G.*

It is natural to ask whether the neighborhood of any vertex satisfies a similar property.

Conjecture 1 (Lubiw [11]). If G is a minimal imperfect Berge graph, then for every vertex v of G, $N_G(v)$ induces a connected subgraph of G.

If we could verify Lubiw's conjecture, we should have shown that, in a minimal imperfect Berge graph, the neighborhood of any vertex contains a P_4. Indeed:

Theorem 2 (Seinsche [18]). *A graph is P_4-free if, and only if, every induced subgraph with at least two vertices either is disconnected or its complement is disconnected.*

Conjecture 2 (Chvátal [7]). If G is a minimal imperfect Berge graph, then, for every vertex v of G, the subgraph $\mathcal{N}_G(v)$ of G contains a P_4.

The *complete join* of two (vertex disjoint) graphs $A=(V_A,E_A)$ and $B=(V_B,E_B)$ is the graph with vertex set $V_A\cup V_B$ and edge set $E_A\cup E_B\cup\{ab\,|\,a\in V_A,\ b\in V_B\}$. Seinsche's Theorem implies that P_4-free graphs can be constructed by complete join and disjoint union from single vertices.

Let $\mathcal{B}$ be the family of bipartite graphs and let $\mathcal{B}^*$ be the family, containing $\mathcal{B}$, such that for any two graphs G_1 and G_2 in $\mathcal{B}^*$, the complete join and the disjoint union of G_1 and G_2 are in $\mathcal{B}^*$. It is useful to notice that if $G\in\mathcal{B}^*$, then every induced subgraph H of G (denoted by $H\subseteq G$) also satisfies $H\in\mathcal{B}^*$.

Proposition 1. *$G\in\mathcal{B}^*$ iff for every induced subgraph H of G either H or $\bar{H}$ is disconnected (and all the connected components are graphs in $\mathcal{B}^*$) or H is a bipartite subgraph of G.*

Lemma 2. *No minimal imperfect Berge graph has a vertex whose neighborhood induces a bipartite subgraph.*

Proof. Assume that there exists a vertex v such that $\mathscr{N}(v)$ is a bipartite subgraph of a minimal imperfect Berge graph G. We have $\omega(G) \leqslant 3$ (since, in a minimal imperfect graph, every vertex belongs to exactly ω ω-cliques, see Padberg [16]) which, since G is Berge, contradicts Tucker's Theorem [20] (SPGC holds for K_4-free graphs). □

Now, suppose that in a minimal imperfect Berge graph $G = (V, E)$ there exists a vertex v such that $\mathscr{N}_G(v) \in \mathscr{B}^*$. We know that $\mathscr{N}(v)$ is a connected subgraph of G's complement (Theorem 1) and cannot be a bipartite subgraph of G (Lemma 2); therefore, by Proposition 1, $\mathscr{N}(v)$ must be disconnected.

So, if Lubiw's Conjecture holds, for any vertex $v \in V$ we have $\mathscr{N}_G(v) \notin \mathscr{B}^*$ and therefore $\mathscr{N}_G(v)$ must contain a P_4 (otherwise $\mathscr{N}_G(v) \in \mathscr{B}^*$). So we conjecture:

Conjecture 3. If G is a minimal imperfect Berge graph, then for every vertex v of G we have $\mathscr{N}_G(v) \notin \mathscr{B}^*$.

In this note we will both prove Conjecture 3 for some classes of graphs and a weakened form of this conjecture.

2. Pretty graphs

In this part, we are interested in a weakened form of Conjecture 3. More precisely we will consider a class of graphs defined by

$$G \in \mathscr{R}^* \text{ iff } G \in \mathscr{B}^* \quad \text{and} \quad G \text{ is } 2K_2\text{-free}$$

In [10], Linhares-Sales et al. consider the class of graphs such that every induced subgraph contains a vertex whose neighborhood graph is $(P_4, 2K_2)$-free. They call those graphs *pretty graphs* and prove that SPGC holds for such graphs.

Proposition 2 (Linhares-Sales et al. [10]). *No minimal imperfect graph, different from an odd hole, has a pretty vertex (that is, a vertex v such that $\mathscr{N}(v)$ is $(P_4, 2K_2)$-free).*

Let G be a graph. A vertex v is said to be *semi-pretty* if $\mathscr{N}_G(v) \in \mathscr{R}^*$; moreover, if every induced subgraph contains a semi-pretty vertex, G is said to be *semi-pretty*. It is an easy task to check that pretty graphs are semi-pretty and we will prove:

Proposition 3. *No minimal imperfect Berge graph has a semi-pretty vertex.*

It is first useful to precise the structure of graphs from $\mathscr{R}^*$:

Lemma 3. *Let H be a subgraph of a graph $G \in \mathscr{R}^*$. Then either H is the complete join or the disjoint union of some graphs (and in this last case, at most one component of H has more than one vertex) or H is a connected bipartite ($2K_2$-free) subgraph of G.*

Lemma 4 (Tucker [21]). *If G is a minimal imperfect graph different from an odd hole, and e is an edge of G that lies in no triangle, then $G - e$ is minimal imperfect.*

Proof of Proposition 3. Let G be a minimal imperfect Berge graph and let v be a semi-pretty vertex of G. If $\mathscr{N}(v)$ is a bipartite subgraph of G, then we have $\omega \leqslant 3$ which, since G is Berge, leads to a contradiction. So, we can suppose that $\omega(G) \geqslant 4$ and we shall consider the following two cases:

Case 1: *$\mathscr{N}(v)$ is a connected subgraph of G.* Lemma 3 implies that $\mathscr{N}(v)$ either is a bipartite subgraph (which contradicts $\omega(G) \geqslant 4$) or is a disconnected subgraph of $\bar{G}$ which implies that $N(v)$ forms a star-cutset in $\bar{G}$, a contradiction.

Case 2: *$\mathscr{N}(v)$ is a disconnected subgraph of G.* Let I be the set of all isolated vertices in $N(v)$ (we know that there exists at least one such vertex by Lemma 3) and notice that, for every vertex u in I, edge uv of G lies in no triangle. Now, consider the graph G' obtained from G by removing all edges in $\{uv \mid u \in I\}$; by Lemma 4, this graph is minimal imperfect and we have $\omega(G') = \omega(G)$. Moreover, since $\mathscr{N}_{G'}(v) \in \mathscr{R}^*$, this subgraph is connected, that is either bipartite or the complete join of some graphs, a contradiction. □

Remark 1. One can show that the polynomial-time algorithm that optimally colors vertices of pretty graphs given in [10] can be easily extended to semi-pretty graphs.

3. Raspail graphs

A graph is *Raspail* (a.k.a. *short-chorded*) if every odd cycle of length at least of 5 has a short chord (a chord joining vertices distance 2 apart in the cycle). We do not know whether Raspail graphs are perfect (although they are Berge graphs) but we know the perfection of two subclasses:

- *SP* graphs [11],
- Gallai-perfect graphs [19],

where Gallai-perfect graphs, introduced by Chvátal, are defined as follows: given any graph G, define the graph $\mathrm{Gal}(G)$ by letting the vertices of $\mathrm{Gal}(G)$ be the edges of G, and making two vertices of $\mathrm{Gal}(G)$ adjacent if and only if the corresponding two edges form an induced P_3 in G; a graph is called *Gallai-perfect* if and only if $\mathrm{Gal}(G)$ contains no odd hole.

Remark 2. For any graph G, one has $\mathrm{Gal}(G)=L(G)$ iff $\omega(G)\leqslant 2$ (where $L(G)$ is the line graph of G).

Theorem 3 (Sun [19]). *Every Gallai-perfect graph is perfect.*

A graph G is an *SP graph* if it is Raspail and every induced subgraph H of G has a vertex v such that $\mathcal{N}_H(v)$ is P_4-free.

Theorem 4 (Lubiw [11]). *Every* SP *graph is perfect.*

We will first recall the sketch of Lubiw's proof since we shall use a similar method in order to prove our conjecture for Raspail graphs. Let $G=(V,E)$ be a Raspail graph and let $v\in V$ be such that $\mathcal{N}_G(v)$ is disconnected (therefore $N(v)$ can be partitioned into two subsets N_1 and N_2 such that $\forall a\in N_1$, $\forall b\in N_2$, $ab\notin E$). Then, we define G' as the graph obtained from G in which we have replaced vertex v by two new vertices v_1 (such that $N_{G'}(v_1)=N_1$) and v_2 (such that $N_{G'}(v_2)=N_2$).

Lemma 5 (Lubiw [11]). *If G is Raspail then so is G', moreover $\omega(G)=\omega(G')$ and $\chi(G)=\chi(G')$.*

Notice that Theorems 1 and 2 imply:

Lemma 6. *Let G be a minimal imperfect graph and let v be a vertex of G such that $\mathcal{N}_G(v)$ is P_4-free. Then $\mathcal{N}_G(v)$ is disconnected.*

In order to prove Theorem 4, Lubiw shows that Raspail graphs satisfy Chvátal's Conjecture (Conjecture 2):

Lemma 7 (Lubiw [11]). *If G is minimal imperfect and Raspail, then G cannot have a vertex v such that $\mathcal{N}_G(v)$ is P_4-free.*

Proof. Let G be a minimal imperfect Raspail graph with a vertex whose neighborhood graph is P_4-free. It will be proved by induction on t_G and then on p_G, where

$$t_G=\max\{|N_G(u)|: u\in V \text{ and } \mathcal{N}_G(u) \text{ is } P_4\text{-free}\},$$

$$p_G=|\{u\in V: \mathcal{N}_G(u) \text{ is } P_4\text{-free and } |N_G(u)|=t_G\}|$$

that G is perfect. Let v be a vertex of G such that $\mathcal{N}_G(v)$ is P_4-free and $|N_G(v)|=t_G$. Notice that $|N(v)|\geqslant 2$ since minimal imperfect graphs are 2-connected. So, $\mathcal{N}_G(v)$ is disconnected (Lemma 6); say $N(v)=N_1\cup N_2$ with no edges of G between N_1 and N_2, and both parts non-empty. We build a new graph G' using operation described in Lemma 5, so G' is Raspail, $\omega(G)=\omega(G')$ and $\chi(G)=\chi(G')$. Then if G' is perfect, so is G. Therefore, assume that G' is imperfect and let $H\subseteq G'$ be a minimal imperfect subgraph.

If neither v_1 (which is such that $N_{G'}(v_1) = N_1$) nor v_2 (such that $N_{G'}(v_2) = N_2$) are in $V(H)$, then H is a proper subgraph of G, hence perfect. Moreover, if we have both v_1 and v_2 in $V(H)$, they form an even-pair, that is, all chordless paths joining v_1 to v_2 have an even number of edges (otherwise, we would have an odd hole in G), a contradiction [12,4,5,8]. Therefore, suppose that $v_1 \in V(H)$; then H is Raspail and has a vertex whose neighborhood graph is P_4-free. Moreover, H is such that $t_H \leqslant t_{G'} \leqslant t_G$ and, if $t_H = t_G$, we have $p_H \leqslant p_{G'} < p_G$ since $|N_{G'}(v_1)|$ is smaller than $|N_G(v)|$. So, by the induction hypothesis, H is perfect, thus G' is perfect, and then so is G. $\square$

It is important to notice that the proof of Lemma 7 uses the neighborhood condition ($\mathcal{N}_G(v)$ is P_4-free) in Lemma 6 and the fact that G is Raspail in Lemmas 5 and 6. The proof also uses the fact that these two properties are hereditary (*i.e.* satisfied by G and all of its induced subgraphs). So, in order to prove Conjecture 3 for Raspail graphs, we only need to give a lemma analogous to Lemma 6 for our new neighborhood condition ($\mathcal{N}_G(v) \in \mathscr{B}^*$), that is:

Lemma 8. *Let G be a minimal imperfect Raspail graph and let v be a vertex of G such that $\mathcal{N}_G(v) \in \mathscr{B}^*$. Then $\mathcal{N}_G(v)$ is disconnected.*

Proof. Let G be a minimal imperfect Raspail graph and let v be a vertex of G such that $\mathcal{N}_G(v) \in \mathscr{B}^*$. Since $\mathcal{N}(v)$ cannot be bipartite (Lemma 2), Proposition 1 implies that either $\mathcal{N}_G(v)$ or $\overline{\mathcal{N}_G(v)}$ is disconnected. But if $\overline{\mathcal{N}_G(v)}$ is disconnected, then $\{v\} \cup N_{\bar{G}}(v)$ forms a star-cutset in $\bar{G}$, a contradiction; so, $\mathcal{N}_G(v)$ is disconnected. $\square$

Therefore, we have shown the following result:

Proposition 4. *A minimal imperfect Raspail graph G does not have a vertex v such that $\mathcal{N}_G(v) \in \mathscr{B}^*$.*

That is, we have proved Conjecture 3 for Raspail graphs. Since Raspail graphs have been generalized by Rusu [17], who has introduced quasi-Raspail graphs, one may consider our conjecture in respect to those graphs. We recall that a graph is called *quasi-Raspail*, if for every vertex v and every odd chordless path P in $G - v$ between two vertices x and y in $N(v)$, the cycle induced by $\{v\} \cup V(P)$ has, at least, one short chord. Notice that quasi-Raspail graphs are not Berge, since $\overline{C_7}$ is a quasi-Raspail graph. Rusu has shown a lemma analogous to Lemma 5 for quasi-Raspail graphs:

Lemma 9 (Rusu [17]). *If G is quasi-Raspail then so is G' (which is a graph built from G using the construction described in Lemma* 5), *moreover $\omega(G) = \omega(G')$ and $\chi(G) = \chi(G')$.*

Lemma 10. *If G is a minimal imperfect quasi-Raspail graph and v is a vertex of G such that $\mathcal{N}_G(v) \in \mathscr{B}^*$, then either $\mathcal{N}_G(v)$ is disconnected or $G \simeq \overline{C_7}$.*

Proof. Let G be a minimal imperfect quasi-Raspail graph and let v be a vertex such that $\mathscr{N}_G(v) \in \mathscr{B}^*$. Since $\overline{\mathscr{N}_G(v)}$ must be connected, we only need to consider the case where $\mathscr{N}_G(v) \in \mathscr{B}$ (that is, where $\mathscr{N}(v)$ is a bipartite subgraph of G). A maximal clique containing v would be of size at most 3, therefore $\omega(G) \leqslant 3$. Tucker's Theorem [20] and the fact that odd holes are not quasi-Raspail graphs imply that $G \simeq \overline{C_{2p+1}}$, $(p \geqslant 3)$; but $\omega(\overline{C_{2p+1}}) \geqslant 4$ as soon as $p \geqslant 4$, so $G \simeq \overline{C_7}$. □

Therefore, we have shown that Conjecture 3 holds for quasi-Raspail graphs:

Proposition 5. *If G is a minimal imperfect quasi-Raspail graph and $G \not\simeq \overline{C_7}$, then for every vertex v of G one has $\mathscr{N}_G(v) \notin \mathscr{B}^*$.*

3.1. Some other classes of graphs related to Conjecture 3

Conjecture 3 was also proved to hold for P_5-free minimal imperfect Berge graphs. More precisely, for those graphs, we have shown a stronger result:

Theorem 5 (Barré and Fouquet [2]). *Let G be a P_5-free minimal imperfect Berge graph and let C be a minimal cutset of G. Then the subgraph of G induced by C cannot belong to $\mathscr{B}^*$.*

Notice that the neighborhood of any vertex forms a minimal cutset. Another interesting class of graphs is those with no long hole (that is, with no induced hole on 5 or more vertices).

Proposition 6 (Barré [1]). *No minimal imperfect graph with no long hole (different from $\overline{C_7}$) has a vertex whose neighborhood induces a graph from $\mathscr{B}^*$.*

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