Multi-asset investment-consumption model with transaction costs

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Abstract

In this paper, we consider the multi-asset optimal investment-consumption model: a riskless asset and d risky assets. when the initial time is $t \geq 0$, for a proportional transaction costs and discount factors, we proof that the value function of the model is a unique viscosity solution of a Hamilton–Jacobi–Bellman (HJB) equations.

Keywords: Finance; Investment-consumption and portfolio models; HJB equation; Viscosity solution; Transaction costs; Discount factor

1. Introduction

The expected utility maximization from consumption with transaction costs was formulated by Magill and Constantinides [10]. For the investment portfolio model with a risky asset, there have some conclusions. For example, in Davis et al. [5], the author define the value function as the expected utility maximization from the terminal wealth. Using the theory of utility pricing, they discuss the problem of European option pricing with transaction costs for continuous horizon; and Shreve and Soner [11] and Tourin and Za-
riphopoulou [13] considered the utility maximization from consumption with transaction costs over an infinite horizon. They proved that the value function is a unique viscosity solution of HJB equation over \([0, \infty)\). Akian et al. [1] considers the multi-asset model. When the utility function is \(U(c) = c^\gamma / \gamma\) and the initial time is zero, they prove the value function is the unique viscosity solution of the given HJB equation.

In this paper, we consider the multi-dimension investment-consumption and portfolio models. A riskless asset is bank account and the instantaneous rate of return is \(r\). A dimension risky asset is stock whose price is driven by a Brownian motion (BM). Any movement of money between the assets incurs a transaction cost and transaction fees which are assumed to be proportional to the amount transacted are paid from the bank account. The investor consumes at a nonnegative rate \(c_t\) from the bank account. In our version of the model, the investor cannot borrow money to finance his investment in the bank account and he cannot short-sell the stock. In other words, the amount of money allocated in bond and stock must stay nonnegative. For any given initial time \(t \geq 0\), given the initial wealth \(x\) and the initial number of stock shares \(y\), the investor’s objective is to maximize the expected discount utility from consumption over an infinite horizon \([t, +\infty)\).

2. Model

We consider the \(d + 1\) dimension investment model: one riskless asset (bond or bank account) and \(d\) dimension risky asset (stock). For any given initial time \(t \geq 0\) and each \(s \geq t\), we let \(B_s\) denote the price of bond (or the amount of money in the bank account), \(r > 0\) the interest rate and \(S^i_s\) the price of the \(i\)th risky asset, \(i = 1, \ldots, d\), then \(B_s, S^i_s\) satisfy the following equations:

\[
\begin{align*}
  d B_s &= r B_s \, ds, \quad s \geq t, \\
  B_t &= x > 0, \\
  d S^i_s &= S^i_s (b^i (S^i_s) \, ds + \sigma^i (S^i_s) \, d W^i_s), \quad s \geq t, \\
  S^i_t &= S^i, \quad i = 1, \ldots, d,
\end{align*}
\]

where \(b^i (S^i_s)\) is the mean rate of return, \(\sigma^i (S^i_s)\) is the dispersion coefficient and the process \(W^i_s\), which represents the source of uncertainty in the market, is a mutual independently Brownian motion defined on the underlying probability space \((\Omega, \mathcal{F}, P)\), the corresponding natural filtration is \(\{\mathcal{F}_s\}_{s \geq t}\). It is assumed that no transaction occur in the stocks and investors trade only in the stock and the bank account. \(\lambda^i\) and \(\mu^i\) are the fraction of the traded amount in the \(i\)th stock, which the investor pays in transaction costs when buying or selling stock the \(i\)th stock respectively and satisfy: \(\lambda^i \geq 0\), \(0 \leq \mu^i < 1\), \(\lambda^i + \mu^i > 0\), \(\forall i = 1, \ldots, d\). For simplicity we assume here that the transaction costs and the consumption are deducted from the holdings of the bank account.

The investor rebalances his portfolio dynamically by choosing at any time \(s\), for \(s \geq t\). At time \(s\), for \(i = 1, \ldots, d\), the investor holds \(B_s\) dollars of the bank account, \(z^i_s\) dollars of the \(i\)th stock, and consumes at the rate \(c_s\) dollars out of the bank account. We let \(y^i_s\) denote the number of shares of the \(i\)th stock and a pair of right-continuous with left limits (RCLL), nondecreasing processes \((L^i_s, M^i_s)_{i=1,\ldots,d}\) denote the cumulative number of shares bought
or sold, respectively. We observe that purchase of \( dL_i \) units of the \( i \)th stock requires a payment of \((1 + \lambda_i) S_i dL_i \) units of bank account, while sale of \( dM_i \) units of the \( i \)th stock requires only \((1 - \mu_i) S_i dM_i \) units of bank account. By convention, \( L_i^t = M_i^t = 0 \). Thus, for \( s \geq t \), the market model equations are:

\[
\begin{align*}
\frac{dB_s}{ds} &= (r B_s - c_s) \, ds + \sum_{i=1}^d \left( -(1 + \lambda_i) S_i^s dL_i^s + (1 - \mu_i) S_i^s dM_i^s \right), \\
S_i^s &= S_i^t \left( b^i(S_i^t) \, ds + \sigma^i(S_i^t) \, dW_i^s \right), \\
\frac{dy_i^s}{ds} &= dL_i^s - dM_i^s, \\
\end{align*}
\]

with \( z_i^t = z_i^s \), we describe the evolution of the bank account and the stock holdings as

\[
\begin{align*}
B_s &= x + \int_t^s (r B_u - c_u) \, du + \sum_{i=1}^d \int_t^s \left( -(1 + \lambda_i) S_i^u dL_i^u + (1 - \mu_i) S_i^u dM_i^u \right), \\
z_i^s &= z_i^t + \int_t^s b^i(S_i^u) z_i^u \, du + \int_t^s \sigma^i(S_i^u) z_i^u dW_i^u + \int_t^s S_i^u dy_i^u, \\
\end{align*}
\]

The trading strategies \( \Lambda = (c_s, L_i^s, M_i^s: i = 1, \ldots, d) \) is said to be admissible if it is \( \mathcal{F}_s \)-progressively measurable, satisfies:

\[
a.s. \ s \geq t, \ c_s \geq 0, \quad E \int_t^s c_u \, du < \infty, \quad x + \sum_{i=1}^d C(-dy_i^u) z_i^u \geq 0, \tag{2.5}
\]

where

\[
C(y) = \begin{cases} 
1 - \mu_i, & y > 0, \\
0, & y = 0, \\
1 + \lambda_i, & y < 0.
\end{cases} \tag{2.6}
\]

We let \( \mathcal{A}(x, (y^1, \ldots, y^d)) \) denote the set of the admissible strategies. The discount factor \( \rho > 0 \) weights consumption now versus consumption later, large \( \rho \) denoting instant gratification. We assume that

\[
\rho \geq r, \quad \rho \geq b^i(S^t). \tag{2.7}
\]

\( U \) is the utility function, which is assumed to have the following properties:

(1) \( U: [0, \infty) \rightarrow [0, \infty) \) is a strictly increasing, concave \( C^2(0, +\infty) \) function.

(2) There exist constants \( M > 0 \) and \( 0 < \gamma < 1 \) such that \( U \) satisfies the growth condition

\[
U(c) \leq M (1 + c)^\gamma, \quad \forall c \geq 0. \tag{2.8}
\]

We use the following notations:

\[
y_s = (y_1^s, \ldots, y_d^s), \quad y = (y^1, \ldots, y^d), \\
S_s = (S_1^s, \ldots, S_d^s), \quad S = (S^1, \ldots, S^d), \\
L_s = (L_1^s, \ldots, L_d^s), \quad M_s = (M_1^s, \ldots, M_d^s). \tag{2.9}
\]
then $y_s$ denotes the vector of the amount for $d$ dimension stock at time $s$, $y$ denotes the vector of the amount for $d$ dimension stock at time $t$, $S_s$ denotes the vector of the price for $d$ dimension stock at time $s$, $S$ denotes the vector of the price for $d$ dimension stock at time $t$, $L_s$ denotes the vector of the amount of buying $d$ dimension stock at time $s$, $M_s$ denotes the vector of the amount of selling $d$ dimension stock at time $s$.

The investor’s objective is to maximize over all policies $\Lambda$ in $\mathcal{A}(x, y)$ the expected discounted utility of consumption. That is, we define the value function as

$$V(x, y, S, t) = \sup_{\Lambda \in \mathcal{A}(x, y)} J(x, y, S, t; \Lambda)$$

(2.10)

For the well definition of the above value function, we make the following assumptions:

1. The discount factor $\rho$ satisfies

$$\rho > r \gamma + \frac{\gamma}{1 - \gamma} \sum_{i=1}^{d} \left( \frac{b_i(S^i) - r}{\sigma^i(S^i)} \right)^2.$$  

(2.11)

2. For any $i$, $1 \leq i \leq d$, the coefficients $b^i : [0, \infty) \rightarrow [0, \infty)$ and $\sigma^i : [0, \infty) \rightarrow [0, \infty)$ in (2.2) satisfy

(i) We let $f$ denote the functions $b^i(S^i)S^i$ and $\sigma^i(S^i)S^i$, then for any $S^i, \bar{S}^i \geq 0$,

$$|f(S^i) - f(\bar{S}^i)| \leq L|S^i - \bar{S}^i|, \quad f^2(S^i) \leq L(1 + (S^i)^2),$$

(2.12)

where $L$ is a positive constant.

(ii) The function $b^i : [0, \infty) \rightarrow [0, \infty)$ satisfies

$$b^i(S^i) > r, \quad \text{for any } S^i > 0, \quad b^i(0) = r.$$  

(2.13)

(iii) The function $\sigma^i : [0, \infty) \rightarrow [0, \infty)$ satisfies

$$\sigma^i(S^i) > 0, \quad \frac{b^i(S^i) - r}{\sigma^i(S^i)} \leq G,$$

(2.14)

where $G$ is a large constant.

We define the solvency region as an open domain $D = R_+ \times R^d_+ \times R_+^d \times [0, \infty)$ and $\bar{D}$ denote the closure of the open domain $D$.

**Lemma 2.1.** The value function $V$ is a concave nondecreasing with respect to the wealth variable $x$ and the stock shares $y^i$, $i = 1, \ldots, d$.

**Proof.** For $\Lambda = (c, L, M) \in \mathcal{A}(x, y)$, $\bar{\Lambda} = (\bar{c}, \bar{L}, \bar{M}) \in \mathcal{A}(\bar{x}, \bar{y})$ and $\lambda \in [0, 1]$, we have $\lambda \Lambda + (1 - \lambda)\bar{\Lambda} \in \mathcal{A}(\lambda x + (1 - \lambda)\bar{x}, \lambda y + (1 - \lambda)\bar{y})$. That is to say, the solvency domain is concave, combining with the concavity of $U$, we have the concavity of the
value function $V$. The nondecreasing of $V$ in $x$, $y^i$ follows from the observation that $\mathcal{A}(x, y^1, \ldots, y^d) \subseteq \mathcal{A}(\tilde{x}, \tilde{y}^1, \ldots, \tilde{y}^d)$ for $x \leq \tilde{x}$, $y^i \leq \tilde{y}^i$, $i = 1, \ldots, d$. □

Using classical results from the theory of singular stochastic control (e.g., see Fleming and Soner [6, Chapter 5] and Lions [9]), we state a fundamental property of the value function known as the dynamic programming principle (DPP). We first recall the definition of stopping time. A nonnegative random variable $\theta$ is a stopping time if for each $s \geq t$ and each $t \geq 0$, $\{\theta \leq s\} \in \mathcal{F}_s$.

**Theorem 2.1.** If $\theta$ is a stopping time, then we have

$$V(x, y, S, t) = \sup_{A \in \mathcal{A}(x, y)} E \left\{ \int_t^\theta e^{-\rho s} U(c_s) \, ds + e^{-\rho(\theta - t)} V(B_\theta, y_\theta, S_\theta, \theta) \right\}. \tag{2.15}$$

3. HJB equation and viscosity solutions

We first recall the definition of viscosity solutions. The notion of viscosity solutions was introduced by Crandall and Lions [4] for first-order equations, and by Lions [9] for second-order equations. For a general overview of the theory, we refer to the user’s guide by Crandall et al. [3] and the book by Fleming and Soner [6]. Next, we recall the notion of constrained viscosity solutions which was introduced by Soner [12] and Capuzzo-Dolcetta and Lions [2] for first-order equations (see also Ishii and Lions [8]). To this end, we consider a nonlinear second-order partial differential equation of the form

$$F(x, W, DW, D^2W) = 0, \quad \text{on } \Omega \times [0, T], \tag{3.1}$$

where $F$ is a given continuous function in $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$, $\mathbb{S}^N$ is the space of symmetric $N \times N$ matrices, $\Omega$ is an open domain of $\mathbb{R}^N$ and the ellipticity of (3.1) is expressed by

$$F(x, v, p, A) \geq F(x, v, p, B) \quad \text{if } A \succeq B, \quad A, B \in \mathbb{S}^N, \quad p \in \mathbb{R}^N, \quad v \in \mathbb{R}, \quad x \in \Omega. \tag{3.2}$$

A special case of (3.1) is given by

$$F(x, v, p, X) = \max_{\eta \in U} \left\{ \sum_{i,j=1}^N a_{ij}(x, \eta) X_{ij} + \sum_{i=1}^N b_i(x, \eta) p_i - \beta(x, \eta) v + u(x, \eta) \right\}, \tag{3.3}$$

where (3.2) is satisfied when the matrix $(a_{ij}(x, \eta))_{i,j}$ is symmetric nonnegative in $\Omega \times U$.

**Definition 3.1.** A continuous function $W : \bar{\Omega} \times [0, T] \to \mathbb{R}$ is a constrained viscosity solution of (3.1) if the following two conditions hold:

(i) $W$ is a viscosity subsolution of (3.1) on $\bar{\Omega} \times [0, T]$; that is, for any $\phi \in C^{2,1}(\bar{\Omega} \times [0, T])$ and any local maximum point $X_0 \in \bar{\Omega} \times [0, T]$ of $W - \phi$, 

\[ F(X_0, W(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0. \quad (3.4) \]

(ii) \( W \) is a viscosity supersolution of (3.1) in \( \bar{\Omega} \times [0, T] \); that is, if for any \( \phi \in C^{2,1}(\bar{\Omega} \times [0, T]) \) and any local minimum point \( X_0 \in \bar{\Omega} \times [0, T] \) of \( W - \phi \),

\[ F(X_0, W(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0. \quad (3.5) \]

Now, we state the main theorem.

**Theorem 3.1.** The value function for (2.10) \( V : \bar{D} \rightarrow \mathbb{R} \) is the unique constrained viscosity solution on \( \bar{D} \) of the HJB equation:

\[
\begin{align*}
&\min \left\{ \min_{1 \leq i \leq d} \left( -\frac{\partial V}{\partial y^i} + (1 + \lambda^i) S^i \frac{\partial V}{\partial x} \right), \min_{1 \leq i \leq d} \left( \frac{\partial V}{\partial y^i} - (1 - \mu^i) S^i \frac{\partial V}{\partial x} \right) \right. \\
&
\quad \left. - \frac{\rho V}{x} + \frac{V_t}{x} + r x V_x + \sum_{i=1}^{d} b^i(S^i) S^i \frac{\partial V}{\partial S^i} + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma \left( S^i \right) S^i \right)^2 \frac{\partial^2 V}{\partial (S^i)^2} \\
&\quad + \max_{c \geq 0} \left[ -c V_x + U(c) \right] \right\} = 0.
\end{align*}
\]  

(3.6)

**Proof.** First, we prove that \( V(x, y, S, t) \) is a viscosity supersolution of (3.6) in \( D \). That is, for all smooth function \( \phi(X) \), \( X = (x, y, S, t) \in \bar{D}, X_0 = (x_0, y_0, S_0, t_0) \in D \) be a minimum of \( V - \phi \), \( y_0 = S_0 = (y_0^1, \ldots, y_0^d), S_0 = S_0 = (S_0^1, \ldots, S_0^d), \) the following inequality holds:

\[
\begin{align*}
&\min \left\{ \min_{1 \leq i \leq d} \left( -\frac{\partial \phi(X_0)}{\partial y^i} + (1 + \lambda^i) S_0^i \frac{\partial \phi(X_0)}{\partial x} \right), \min_{1 \leq i \leq d} \left( \frac{\partial \phi(X_0)}{\partial y^i} - (1 - \mu^i) S_0^i \frac{\partial \phi(X_0)}{\partial x} \right) \right. \\
&
\quad \left. - \frac{\rho \phi(X_0)}{x} + \phi_t(X_0) + r x_0 \phi_x(X_0) + \sum_{i=1}^{d} b^i(S_0^i) S_0^i \frac{\partial \phi(X_0)}{\partial S^i} \\
&\quad + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma \left( S_0^i \right) S_0^i \right)^2 \frac{\partial^2 \phi(X_0)}{\partial (S^i)^2} + \max_{c \geq 0} \left[ -c \phi_x(X_0) + U(c) \right] \right\} \geq 0.
\end{align*}
\]  

(3.7)

Without loss of generality, we assume that

\[ V(X_0) = \phi(X_0), \quad V(X) \geq \phi(X) \quad \text{on} \quad \bar{D}. \quad (3.8) \]

We prove that each minimum operator of (3.7) is nonnegative.

For the first operator, let \( L_s = L_0 = L_{0_0} = (L_{0_0}^1, \ldots, L_{0_0}^d) > 0, M_s = 0, s \geq t_0, \) for any \( i, 1 \leq i \leq d, \)

\[ V(x_0, y_0, S_0, t_0) \geq V(x_0 - (1 + \lambda^i) S_0^i L_{0_0}^i, y_0 + L_0, S_0, t_0) \quad (3.9) \]

form (3.8) the above inequality holds for \( \phi(X) \), that is
\[ \phi(x_0 - (1 + \lambda^i)S^i_0 L^i_0, y_0 + L_0, S_0, t_0) - \phi(x_0, y_0, S_0, t_0) \leq 0 \]  
\text{(3.10)}

dividing by \( L^i_0 \), and let \( L^i_0 \to 0 \), we get
\[ \frac{\partial \phi(X_0)}{\partial y^i} - (1 + \lambda^i)S^i_0 \frac{\partial \phi(X_0)}{\partial x} \leq 0 \]  
\text{(3.11)}

for the random selection of \( i \), we have
\[ \min_{1 \leq i \leq d} \left( -\frac{\partial \phi(X_0)}{\partial y^i} + (1 + \lambda^i)S^i_0 \frac{\partial \phi(X_0)}{\partial x} \right) \geq 0. \]  
\text{(3.12)}

For the second operator, let \( L_s = M = M_0 = M_0 = (M^1_0, \ldots, M^d_0) > 0 \), \( s \geq t_0 \) and for any \( i \),
\[ V(x_0, y_0, S_0, t_0) \geq V(x_0 + (1 - \mu^i)S^i_0 M^i_0, y_0 - M_0, S_0, t_0) \]  
\text{(3.13)}

the above inequality holds for \( \phi(X) \),
\[ \phi(x_0 + (1 - \mu^i)S^i_0 M^i_0, y_0 - M_0, S_0, t_0) - \phi(x_0, y_0, S_0, t_0) \leq 0 \]  
\text{(3.14)}

dividing by \( M^i_0 \), and let \( M^i_0 \to 0 \), then
\[ -\frac{\partial \phi(X_0)}{\partial y^i} + (1 - \mu^i)S^i_0 \frac{\partial \phi(X_0)}{\partial x} \leq 0 \]  
\text{(3.15)}

for the random selection of \( i \), we have
\[ \min_{1 \leq i \leq d} \left( \frac{\partial \phi(X_0)}{\partial y^i} - (1 - \mu^i)S^i_0 \frac{\partial \phi(X_0)}{\partial x} \right) \geq 0. \]  
\text{(3.16)}

Finally, for the third operator, we let \( L_s = M = 0 \), \( s \geq t_0 \), then \( B_s \) is a solution of the following equation:
\[ dB_s = (r B_s - c_s) ds, \]  
\text{(3.17)}

and \( B_{t_0} = x_0, S_{t_0} = S_0, y_{t_0} = y_0 \). From the dynamic programming principle (2.15), together with (3.8), we have
\[ EV(X_s) = EV(B_s, y_0, S_s, s) \leq V(x_0, y_0, S_0, t_0) = V(X_0), \]  
\text{(3.18)}

\[ V(X_0) \geq E \left[ \int_{t_0}^{s} e^{-\rho t} U(c_t) dt + e^{-\rho(s-t_0)} \phi(X_s) \right]. \]  
\text{(3.19)}

where \( X_s = (B_s, y_0, S_s, s) \). Applying Itô’s lemma to \( e^{-\rho t} \phi(X_s) \) and taking expectation, we have
\[ E[e^{-\rho t}\phi(X_s)] \]
\[ = e^{-\rho t_0} V(X_0) + E \left[ \int_{t_0}^{s} e^{-\rho t} \left[ -\rho \phi(X_t) + \dot{\phi}(X_t) + r X_t \phi_x(X_t) \right. \right. \]
\[ + \sum_{i=1}^{d} b^i(S^i_t) S^i_t \frac{\partial \phi(X_t)}{\partial S^i} + \left. \frac{1}{2} \sum_{i=1}^{d} \left( \sigma^i(S^i_t) S^i_t \right)^2 \frac{\partial^2 \phi(X_t)}{\partial (S^i)^2} - c_t \phi_x(X_t) \right] dt \]  
\text{(3.20)}
combining the above equality with (3.19), and using standard estimates from the theory of stochastic differential equations (see Gikhman and Skorohod [7]), we have

\[
E \int_{t_0}^{s} e^{-\rho t} \left[ -\rho \phi(X_0) + \phi_t(X_0) + r x_0 \phi_x(X_0) + \sum_{i=1}^{d} b_i^i (S_0^i) S_0^i \frac{\partial \phi(X_0)}{\partial S^i} + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma_i^i (S_0^i) S_0^i \right)^2 \frac{\partial^2 \phi(X_0)}{\partial (S^i)^2} - c_0 \phi_x(X_0) + U(c_0) \right] dt + E \int_{t_0}^{s} h(t) dt \leq 0,
\]

(3.21)

where \( h(t) = 0(t) \). Dividing both sides by \( E(s - t_0) \), and letting \( s \to t_0 \), we take the limit for left side and get

\[
-\rho \phi(X_0) + \phi_t(X_0) + r x_0 \phi_x(X_0) + \sum_{i=1}^{d} b_i^i (S_0^i) S_0^i \frac{\partial \phi(X_0)}{\partial S^i} + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma_i^i (S_0^i) S_0^i \right)^2 \frac{\partial^2 \phi(X_0)}{\partial (S^i)^2} + \max_{c \geq 0} \left[ -c \phi_x(X_0) + U(c) \right] \leq 0.
\]

(3.22)

So \( V(x, y, S, t) \) is a viscosity supersolution of (3.6).

Next, we show that \( V(x, y, S, t) \) is a viscosity subsolution of (3.6) on \( \bar{D} \). For all smooth function \( \phi(X) \), \( X = (x, y, S, t) \in \bar{D} \), let \( X_0 = (x_0, y_0, S_0, t_0) \in \bar{D} \) be a maximum point of \( V - \phi \). Without loss of generality, we may assume that

\[
V(X_0) = \phi(X_0), \quad V(X) \leq \phi(X) \quad \text{on} \ \bar{D}.
\]

(3.23)

We need to show that

\[
\min \left\{ \min_{1 \leq i \leq d} \left[ -\frac{\partial \phi(X_0)}{\partial y^i} + (1 + \lambda^i) S_0^i \frac{\partial \phi(X_0)}{\partial x} \right], \right. \]

\[
\left. \min_{1 \leq i \leq d} \left[ \frac{\partial \phi(X_0)}{\partial y^i} - (1 - \mu^i) S_0^i \frac{\partial \phi(X_0)}{\partial x} \right], \right.
\]

\[
- \left( -\rho \phi(X_0) + \phi_t(X_0) + r x_0 \phi_x(X_0) + \sum_{i=1}^{d} b_i^i (S_0^i) S_0^i \frac{\partial \phi(X_0)}{\partial S^i} + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma_i^i (S_0^i) S_0^i \right)^2 \frac{\partial^2 \phi(X_0)}{\partial (S^i)^2} + \max_{c \geq 0} \left[ -c \phi_x(X_0) + U(c) \right] \right) \leq 0.
\]

(3.24)

We prove: if the first and second operator of the above inequality satisfy

\[
\min_{1 \leq i \leq d} \left[ -\frac{\partial \phi(X_0)}{\partial y^i} + (1 + \lambda^i) S_0^i \frac{\partial \phi(X_0)}{\partial x} \right] > 0,
\]

(3.25)

\[
\min_{1 \leq i \leq d} \left[ \frac{\partial \phi(X_0)}{\partial y^i} - (1 - \mu^i) S_0^i \frac{\partial \phi(X_0)}{\partial x} \right] > 0,
\]

(3.26)
and there exist $\theta > 0$, such that the third operator satisfy

$$-\rho \phi(X_0) + \phi_t(X_0) + r x_0 \phi_x(X_0) + \sum_{i=1}^{d} b_i^i (S_i^0) S_i^0 \frac{\partial \phi(X_0)}{\partial S^i}$$

$$+ \frac{1}{2} \sum_{i=1}^{d} \left( \sigma^i (S_i^0) S_i^0 \right)^2 \frac{\partial^2 \phi(X_0)}{\partial (S^i)^2} + \max_{c \geq 0} \left[ -c \phi_x(X_0) + U(c) \right] < -\theta,$$  (3.27)

then we can deduce a contradiction.

From (3.25) and (3.26), we get, for any $i$, $1 \leq i \leq d$, the following inequalities hold:

$$\frac{\partial \phi(X_0)}{\partial y^i} - (1 + \lambda^i) S_i^0 \frac{\partial \phi(X_0)}{\partial x} < 0,$$  (3.28)

$$\frac{\partial \phi(X_0)}{\partial y^i} - (1 - \mu^i) S_i^0 \frac{\partial \phi(X_0)}{\partial x} > 0,$$  (3.29)

From the fact that $\phi$ is smooth, the above inequality holds for $\phi(X)$, where $X = (B, y, S, t) \in B(X_0)$ and $B(X_0)$ is a neighborhood of $X_0$,

$$\frac{\partial \phi(X)}{\partial y^i} - (1 + \lambda^i) S_i \frac{\partial \phi(X)}{\partial x} < 0,$$  (3.30)

$$\frac{\partial \phi(X)}{\partial y^i} - (1 - \mu^i) S_i \frac{\partial \phi(X)}{\partial x} > 0,$$  (3.31)

$$-\rho \phi(X) + \phi_t(X) + r x \phi_x(X) + \sum_{i=1}^{d} b_i^i (S^i) S_i \frac{\partial \phi(X)}{\partial S^i} + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma^i (S^i) S^i \right)^2 \frac{\partial^2 \phi(X)}{\partial (S^i)^2}$$

$$+ \max_{c \geq 0} \left[ -c \phi_x(X) + U(c) \right] < -\theta.$$  (3.32)

For $X_0$, it follows, from Zhu [14], that there exists an optimal trajectory $\tilde{X}(t) = (\tilde{x}, \tilde{y}, \tilde{S}, \tilde{t})$ with $\tilde{X}(t_0) = X_0$, that is, the value function attained the sup of (2.10) at $\tilde{X}(t)$ and the pair of processes $\tilde{A}_t = (\tilde{c}_t, \tilde{L}_t, \tilde{M}_t)$ is the corresponding optimal trading strategy. The following Lemma 3.2 shows that $\tilde{X}(t)$ has no jumps, P-a.s. at $t = t_0$, so $\tau$ defined by

$$\tau = \inf \left\{ t \geq t_0 : \tilde{X}(t) \notin B(X_0) \right\},$$  (3.33)

then $\tau$ is stopping time, and $\tau \geq t_0$, P-a.s. Let

$$I_1' = \int_{t_0}^{\tau} e^{-pt} \left( -\frac{\partial \phi(\tilde{X}(t))}{\partial y^i} + (1 + \lambda^i) \tilde{S}_i \frac{\partial \phi(\tilde{X}(t))}{\partial x} \right) d\tilde{L}_t,$$  (3.34)

$$I_2' = \int_{t_0}^{\tau} e^{-pt} \left( -\frac{\partial \phi(\tilde{X}(t))}{\partial y^i} - (1 - \mu^i) \tilde{S}_i \frac{\partial \phi(\tilde{X}(t))}{\partial x} \right) d\tilde{M}_t,$$  (3.35)
\begin{equation}
I_3 = \int_{t_0}^{\tau} e^{-\rho t} \left( -\rho \phi(\bar{X}(t)) + \phi_t(\bar{X}(t)) + r \bar{x} \phi_x(\bar{X}(t)) + \sum_{i=1}^{d} b^i(\bar{S}^i) \frac{\delta \phi(\bar{X}(t))}{\delta S^i} \right. \\
\left. + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma^i(\bar{S}^i) \right)^2 \frac{\delta^2 \phi(\bar{X}(t))}{\delta (S^i)^2} + \max_{\tilde{c}_i \geq 0} \left[ -\tilde{c}_i \phi_x(\bar{X}(t)) + U(\tilde{c}_i) \right] \right) dt,
\end{equation}

from the above assumptions, we have

\begin{equation}
\sum_{i=1}^{d} EI_{1i}^I - \sum_{i=1}^{d} EI_{2i}^I + EI_3 < -\frac{\theta}{\rho} E(e^{-\rho t_0} - e^{-\rho \tau}).
\end{equation}

Applying Itô’s lemma to \(e^{-\rho t} \phi(X)\), from (2.3) we obtain

\begin{equation}
E \left( e^{-\rho \tau} \phi(\bar{X}(\tau)) \right) = e^{-\rho t_0} \phi(\bar{X}_0) + \sum_{i=1}^{d} (EI_{1i}^I - EI_{2i}^I) + EI_3 - E \int_{t_0}^{\tau} e^{-\rho t} U(\tilde{c}_i) dt.
\end{equation}

The dynamic programming principle (2.15) together with the assumptions for the maximum of \(V - \phi\) at \(X_0\), yields

\begin{equation}
\phi(X_0) \leq E \left[ \int_{t_0}^{\tau} e^{-\rho t} U(\tilde{c}_i) dt + e^{-\rho (\tau - t_0)} \phi(\bar{X}(\tau)) \right].
\end{equation}

Combining (3.39) with (3.37) and (3.38), we have

\begin{equation}
0 \leq -\frac{\theta}{\rho} E(e^{-\rho t_0} - e^{-\rho \tau}) - E \int_{t_0}^{\tau} (1 - e^{-\rho t_0}) e^{-\rho t} U(\tilde{c}_i) dt,
\end{equation}

that is,

\begin{equation}
\frac{\theta}{\rho} E(e^{-\rho t_0} - e^{-\rho \tau}) + E \int_{t_0}^{\tau} (1 - e^{-\rho t_0}) e^{-\rho t} U(\tilde{c}_i) dt \leq 0.
\end{equation}

This is impossible because each part of the above inequality is strictly positive. So we complete the proof. \(\square\)

**Lemma 3.2.** Suppose that inequality (3.28) holds. For each \(i\), we let \(A\) denote the event that the optimal trajectory \(\bar{X}(t)\) has a jump of size at least \(\varepsilon\) along the direction \(- (1 + \lambda^i) S_{0i}\), \(\bar{c}_i, 0, t_0)\) at \(X_0\), where \(\bar{c}_i\) denotes the vector which the \(i\)th component is 1 and the else is 0, 0 denotes the zero vector, \(y_0(\varepsilon)\) denotes the \(i\)th component \(y_0^i + \varepsilon\) and the else component is the same with the vector \(y_0\). We assume that the state after the jump is \((x_0 - (1 + \lambda^i) S_{0i} \varepsilon, y_0(\varepsilon), S_0, t_0) \in B(X_0)\), then

\begin{equation}
\left( \frac{\partial \phi(X_0)}{\partial y^i} - (1 + \lambda^i) S_{0i} \frac{\partial \phi(X_0)}{\partial x} \right) P(A) \geq 0,
\end{equation}

where \(P(A)\) is the probability of event \(A\).
and hence \( P(A) = 0 \). Similarly, if the inequality (3.29) holds, then the optimal trajectory has no jumps along the direction \(((1 - \mu^i) S^i_0, -\vec{e}_i, 0, t_0)\), \( P \)-a.s. at \( X_0 \).

**Proof.** By the dynamic programming principle and for each \( i \), we have

\[
V(x_0, y_0, S_0, t_0) = \int_A V(x_0 - (1 + \lambda^i) S^i_0 \varepsilon, y_0(\varepsilon), S_0, t_0) \, dP
+ \int_{\Omega - A} V(x_0, y_0, S_0, t_0) \, dP. \tag{3.43}
\]

Hence,

\[
\int_A (V(x_0 - (1 + \lambda^i) S^i_0 \varepsilon, y_0(\varepsilon), S_0, t_0) - V(x_0, y_0, S_0, t_0)) \, dP = 0 \tag{3.44}
\]

since \( V(X_0) = \phi(X_0) \), \( V(X) \leq \phi(X) \) on \( \bar{D} \), we obtain

\[
\int_A (\phi(x_0 - (1 + \lambda^i) S^i_0 \varepsilon, y_0(\varepsilon), S_0, t_0) - \phi(x_0, y_0, S_0, t_0)) \, dP \geq 0. \tag{3.45}
\]

Let \( \varepsilon \to 0 \) and by Fatou’s lemma, the above inequality yields

\[
\int_A \limsup_{\varepsilon \to 0} \phi(x_0 - (1 + \lambda^i) S^i_0 \varepsilon, y_0(\varepsilon), S_0, t_0) - \phi(x_0, y_0, S_0, t_0) \, dP \geq 0, \tag{3.46}
\]

which, in turn, implies (3.42).

We prove the uniqueness property from the comparison theorem. In fact, suppose that \( u, v \) are viscosity solutions of (3.6) which belong to the same class of viscosity solutions that are continuous, concave and nondecreasing with respect to \( x, y^i, i = 1, \ldots, d \), then they are both subsolutions and supersolutions as Definition 3.1 requires. That is, \( u \) is a subsolution and \( v \) is a supersolution of (3.6), then the following theorem holds for \( u, v \) and \( u \leq v \). Similarly, \( u \) is also a supersolution and \( v \) a subsolution, and we have \( v \leq u \). Hence, \( u = v \) and the uniqueness of the value function follows. \( \square \)

**Theorem 3.2.** Suppose that \( u \) and \( v \) are continuous functions which are concave and nondecreasing with respect to \( x, y^i, i = 1, \ldots, d \). Let \( u \) be a bounded viscosity subsolution of (3.6) on \( \bar{D} \), let \( v \) be a bounded from below viscosity supersolution of (3.6) in \( D \), then \( u \leq v \) on \( \bar{D} \).

**Proof.** First, we construct a positive strict supersolution to the HJB equation (3.6) in \( D \). We recall the growth condition (2.9), and let the function \( h : D \to \mathbb{R}^+ \) be given by

\[
h(x, y, S, t) = N(1 + x + \sum_{i=1}^d K^i y^i S^i) + C_1 t + C_2, \]

where the constants \( N, C_1, C_2, K^i, i = 1, \ldots, d \), satisfy

\[
1 + \lambda^i > K^i > 1 - \mu^i, \quad N > M, \quad 0 < C_1 < \frac{\rho(C_2 + N)}{2} - M. \tag{3.47}
\]
Let

\[ H(X, h(X), D h(X), D^2 h(X)) \]

\[ = \min \left\{ \min_{1 \leq i \leq d} \left( -\frac{\partial h}{\partial y^i} + (1 + \lambda^i) S_i^x \frac{\partial h}{\partial x} \right), \min_{1 \leq i \leq d} \left( \frac{\partial h}{\partial y^i} - (1 - \mu^i) S_i^x \frac{\partial h}{\partial x} \right), \right. \]

\[ - \left( -\rho h + h_t + rxh_x + \sum_{i=1}^{d} b^i(S^i) S_i^x + \frac{1}{2} \sum_{i=1}^{d} (\sigma^i(S^i))^2 \frac{\partial^2 h}{S_i^x} \right) \]

\[ + \max_{c \geq 0} \left[ -ch_x + U(c) \right] \right\}. \] (3.48)

We need to show that \( h(x, y, S, t) \) is the supersolution of (3.6), that is

\[ \min_{1 \leq i \leq d} \left( NS_i \left( 1 + \lambda^i - K^i \right) \right) \]

\[ + \min_{1 \leq i \leq d} \left( NS_i \left( K^i - (1 - \mu^i) \right) \right) > 0. \] (3.49)

hence \( h \) is a strict supersolution of (3.6).

Next, we define the function \( w^\alpha = \alpha v + (1 - \alpha)h \), where \( 0 < \alpha < 1 \), then \( w^\alpha \) is a viscosity supersolution of \( H - (1 - \alpha)Z = 0 \). In fact, let \( \psi \in C^{1,2}(\overline{D}) \) and assume that \( w^\alpha - \psi \) has a minimum at \( X_0 \), and let \( \phi = \frac{\psi - (1 - \alpha)h}{\alpha} \), then \( v - \phi \) also has a minimum at \( X_0 \).

From the fact that \( v \) is a viscosity supersolution of \( H(X, v(X), Dv(X), D^2 v(X)) = 0 \) and the above inequality (3.49), we have

\[ \alpha H(X_0, \phi(X_0), D\phi(X_0), D^2 \phi(X_0)) + (1 - \alpha)H(X_0, h(X_0), Dh(X_0), D^2 h(X_0)) \]

\[ \geq (1 - \alpha)Z. \] (3.50)

Since the Hamiltonian \( H(X, p, q, A) \) is jointly concave with respect to \( (p, q, A) \), then the above inequality yields

\[ H(X_0, \psi(X_0), D\psi(X_0), D^2 \psi(X_0)) \geq (1 - \alpha)Z \] (3.51)

which in turn implies that \( w^\alpha \) is a viscosity supersolution of \( H - (1 - \alpha)Z = 0 \).

Finally, applying the comparison results of Theorem VI.5 in Ishii and Lions [8] to \( u \) and \( w^\alpha \), we get

\[ u \leq w^\alpha, \quad \text{on } \overline{D}, \] (3.52)

and letting \( \alpha \to 1 \), we obtain the conclusion.
It is worthwhile to say that if we relax the continuity assumption and allow for \( u \) and \( v \) to be upper-semicontinuous and lower-semicontinuous functions, respectively, then the above result still holds.

References


Further reading