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Processes with State-Dependent Hitting Probabilities and Their Equivalence under Time Changes

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1. INTRODUCTION

Suppose that $\{X_t, t \geqslant 0\}$ is a standard Markov process with σ-compact metrizable state space S and σ-fields $\mathscr{M}_t$ (see [2], p. 45 for the definition of "standard Markov process"). Let Σ be the class of Borel subsets of S. Then, for each $E \in \Sigma$, there is a function H_E on $S \times \Sigma$ with the following properties:

(i) for each $x \in S$, $H_E(x, \cdot)$ is a measure on Σ,

(ii) for each $A \in \Sigma$, $H_E(x, A)$ is a Σ-measurable function of x,

and

(iii) for any stopping time τ and $A \in \Sigma$,

$$P(X_\gamma \in A \mid \mathscr{M}_\tau) = H_E(X_\tau, A) \qquad P\text{-a.e. on } \{\tau < \infty\}, \tag{1.1}$$

where $\gamma = \tau + \theta_\tau \sigma$, and $\sigma = \inf\{t: X_t \in E\}$.

In (iii) θ_τ is a shift operator. A more intuitive characterization of γ is as the first post-τ hitting time of E, that is, $\gamma = \inf\{\tau: t \geqslant \tau \text{ and } X_t \in E\}$.

Suppose that $\{X_t, t \geqslant 0\}$ (with associated σ-fields $\{\mathscr{M}_t\}$) does not have the Markov property, but that (iii) still holds. We say that such a process has state-dependent hitting probabilities.

Let $\{X_t, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ be standard Markov processes with the same hitting distribution. In [3] (see also [2]) Blumenthal, Getoor, and McKean, in a significant achievement of the theory of general Markov processes, showed that there is an additive functional for $\{X_t, t \geqslant 0\}$ whose inverse $\{\tau_t, t \geqslant 0\}$ is a time change for which $\{X_{\tau_t}, t \geqslant 0\}$ is a standard Markov process equivalent to $\{\tilde{X}_t, t \geqslant 0\}$. ("Equivalent" means having the same transition probability operator.) A question which naturally arises is whether analogous results hold for non-Markov processes. It is not true, in general, that hitting probabilities are preserved under time changes, even if the time changes are non-anticipating. (A time change $\{\tau_t, t \geqslant 0\}$ is called non-anticipating if

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$\{\tau_t \leqslant s\} \in \mathscr{M}_s$ for each $t \geqslant 0$, $s \geqslant 0$.) They are preserved under such time changes, however, if the process in question has state dependent hitting probabilities. The conjecture suggested by the Blumenthal-Getoor-McKean theorem is that if $\{X_t, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ are processes with the same state-dependent hitting probabilities H_E, then there is a non-anticipating time change $\{\tau_t\}$ for which $\{X_{\tau_t}, t \geqslant 0\}$ has the same finite-dimensional distributions as $\{\tilde{X}_t, t \geqslant 0\}$. Simple examples, however, show that this conjecture is false. For instance, if the probability space $(\Omega, \mathscr{A}, P)$ on which $\{X_t, t \geqslant 0\}$ is defined to a singleton $\Omega = \{\omega\}$, with P a unit mass on ω, then, for any time change $\{\tau_t, t \geqslant 0\}$ defined on Ω, $\{X_{\tau_t}, t \geqslant 0\}$ has only one possible sample path, so cannot possibly have the same finite-dimensional distributions as $\{\tilde{X}_t, t \geqslant 0\}$ unless the latter also has only one sample path. In order to salvage the conjecture, we allow the process $\{X_t, t \geqslant 0\}$ to be transferred to a probability space $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ with a richer structure, on which we construct a time change $\{\tau_t, t \geqslant 0\}$, non-anticipating relative to new σ-fields $\{\hat{\mathscr{M}}_t\}$, for which $\{X_{\tau_t}, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ have the same finite dimensional distributions.

We now outline the approach we use in the proof of the conjecture. Suppose that (S, Σ) is Euclidean n-space, and that the paths of both $\{X_t, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ are rectifiable curves. Suppose that $\{X_t\}$ and $\{\tilde{X}_t\}$ are defined on the probability spaces $(\Omega, \mathscr{A}, P)$ and $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ respectively. Let $\{\mathscr{M}_t\}$ be an increasing family of sub-σ-fields of $\mathscr{A}$ having the property that X_t is measurable with respect to $\mathscr{M}_t$ for each $t \geqslant 0$, and let $\{\hat{\mathscr{M}}_t\}$ be a corresponding family of sub-σ-fields of $\hat{\mathscr{A}}$. For each $\omega \in \Omega$, let $\{Y_s(\omega), s \geqslant 0\}$ be the sample path $\{X_t(\omega), t \geqslant 0\}$ reparametrized by arc-length s. It is suggestive to think of $\{Y_s(\omega)\}$ as a geometrical object, called a *trajectory*. We call $\{X_t(\omega)\}$ the *path* corresponding to ω. Different ω's in general determine different paths, but different paths may correspond to identical trajectories. If we regard $Y_t(\omega)$ as the position at time t of a particle on the trajectory $\{Y_s(\omega)\}$, we may think of particles following distinct paths with the same trajectory as travelling along that trajectory at different rates of speed. Now let $\{\tilde{Y}_s, s \geqslant 0\}$ be the trajectory process determined by $\{\tilde{X}_t, t \geqslant 0\}$. If $\{X_t, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ are processes with the same state-dependent hitting probabilities, $\{Y_t, t \geqslant 0\}$ and $\{\tilde{Y}_t, t \geqslant 0\}$ have the same finite-dimensional distributions. Suppose that the paths $\{X_t(\omega)\}$ and $\{\tilde{X}_t(\tilde{\omega})\}$ correspond to the same trajectory, in other words, that $\{Y_s(\omega)\} = \{\tilde{Y}_s(\tilde{\omega})\}$. Fix t. As t' goes from 0 to t, the successive positions $\tilde{X}_{t'}(\tilde{\omega})$ of the particle labelled by $\tilde{\omega}$ sweep out an initial portion of the trajectory $\{\tilde{Y}_s(\tilde{\omega})\}$. Let τ_t be the time it takes for the particle on the path $\{X_t(\omega)\}$ to sweep out this portion of the trajectory $\{Y_s(\omega)\} = \{\tilde{Y}_s(\tilde{\omega})\}$. Then $t \to \tau_t(\omega)$ is continuous and monotone, and $X_{\tau_t}(\omega) = \tilde{X}_t(\tilde{\omega})$. If, for each possible trajectory, there are at least as many ω's as $\tilde{\omega}$'s whose paths sweep out that trajectory, one might hope to match up the ω paths to the $\tilde{\omega}$ paths in a measurable way, and for each ω use the procedure just given to define $\tau_t(\omega)$ in terms of the ω-path $\{X_t(\omega)\}$ and a corresponding $\tilde{\omega}$-path $\{\tilde{X}_t(\tilde{\omega})\}$. The necessity of enlarging the space Ω arises if there are trajectories with fewer

ω-paths than $\tilde{\omega}$-paths. The measurability problems involved in this approach, however, are formidable, and the construction we actually use is more like the following one. First, let $\hat{\Omega} = \Omega \times \tilde{\Omega}$, and embed $\{X_t, t \geqslant 0\}$ in $\hat{\Omega}$ in the obvious way. Let $\hat{\mathcal{Q}} = \mathcal{Q} \times \tilde{\mathcal{Q}}$. We put on $\hat{\mathcal{Q}}$ not the product measure $P \times \tilde{P}$, but the measure $\hat{P}$ defined for $A \in \mathcal{Q}$ and $\tilde{A} \in \tilde{\mathcal{Q}}$ by $\hat{P}(A \times \tilde{A}) = \int_A \tilde{P}_\omega(\tilde{A})\, P(d\omega)$, where $\tilde{P}_\omega(\tilde{A})$ is the conditional probability of $\tilde{A}$ given that $\{\tilde{Y}_s\} = \{Y_s(\omega)\}$. Let $\hat{\mathcal{M}}_t = \mathcal{M}_t \times \tilde{\mathcal{Q}}$. The time change is now defined as follows. Let $(\omega, \tilde{\omega}) \in \hat{\Omega}$. If $\{Y_s(\omega)\}$ is not equal to $\{\tilde{Y}_s(\tilde{\omega})\}$, $\tau_t(\omega, \tilde{\omega})$ is undefined or set equal to ∞. The set of $(\omega, \tilde{\omega})$'s for which this happens is $\hat{P}$-null. If $\{Y_s(\omega)\}$ is equal to $\{\tilde{Y}_s(\tilde{\omega})\}$, $\tau_t(\omega, \tilde{\omega})$ is defined as above. Then $\{\tau_t, t \geqslant 0\}$ is a time change on $\hat{\Omega}$, non-anticipating relative to $\{\hat{\mathcal{M}}_t\}$, having the property that for $\hat{P}$-almost all $(\omega, \tilde{\omega}) \in \hat{\Omega}$, $X_{\tau_t(\omega,\tilde{\omega})}(\omega) = \tilde{X}_t(\tilde{\omega})$ for all $t \geqslant 0$. Therefore $\{(\omega, \tilde{\omega})\colon X_{\tau_{t_1}} \in E_1, \ldots, X_{\tau_{t_n}} \in E_n\} = \{(\omega, \tilde{\omega})\colon \tilde{X}_{t_1}(\tilde{\omega}) \in E_1, \ldots, \tilde{X}_{t_n}(\tilde{\omega}) \in E_n\}$ for any $n \geqslant 1$, $t_1, \ldots, t_n$ in $[0, \infty)$, and $E_1, \ldots, E_n$ in Σ. It follows from the definition of $\hat{P}$ and the fact that $\{Y_s, s \geqslant 0\}$ and $\{\tilde{Y}_s, s \geqslant 0\}$ have the same distribution that $\hat{P}(\{(\omega, \tilde{\omega})\colon \tilde{X}_{t_1}(\tilde{\omega}) \in E_1, \ldots, \tilde{X}_{t_n}(\tilde{\omega}) \in E_n\}) = \tilde{P}(\{\tilde{\omega}\colon \tilde{X}_{t_1}(\tilde{\omega}) \in E_1, \ldots, \tilde{X}_{t_n}(\tilde{\omega}) \in E_n\})$. Therefore $\{X_{\tau_t}, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ have the same finite dimensional distributions.

Even if the state space S is finite dimensional Euclidean space, the assumption that paths of $\{\tilde{X}_t, t \geqslant 0\}$ be rectifiable excludes the most interesting processes, like Brownian motion. In section 3 of this paper, we introduce what, in effect, is our substitute for arc-length. Whereas a particle moving along a curve can keep track of how far it has travelled only if the curve is rectifiable, we show that a particle moving along a curve can keep track of the oscillations it has undergone provided that the curve is everywhere right-continuous with left limits. The oscillation record does not yield a reparametrization of the curve with parameter set $[0, \infty)$ as does arc-length. The parameter set is instead a collection of finite sequences of non-negative integers ordered lexicographically, and the collection associated with a given function varies from function to function. The main purpose of this paper is to present this method of recording oscillations, and to develop its basic properties. We do this in section 3. Section 2 is given over to notation and other preliminary material. In section 4 we enlarge the space supporting the $\{X_t, t \geqslant 0\}$ process, and show that the resulting enlargement is "distributional" in the sense of [1] and [4]. In section 5, we establish our basic conjecture in the case that the paths of both $\{X_t, t \geqslant 0\}$ and $\{\tilde{X}_t, t \geqslant 0\}$ are right continuous with left limits, but free of intervals of constancy. In section 6, we give some examples.

2. Notation and Preliminaries

Let (S, d) be a σ-compact metric space, and Σ its Borel sets. We shall denote by D the set of all S-valued functions on $[0, \infty)$ which are right continuous and have left limits at every $t \in [0, \infty)$, and by $\mathcal{D}$ the σ-field on D generated by all sets of the form $\{f: f \in D, f(t) \in E\}$, where t ranges over $[0, \infty)$ and E over Σ. There is a metric on D relative to which D is a complete, separable metric space with $\mathscr{D}$ as its class of Borel sets ([8], [5]). For each $t \in [0, \infty)$, $\mathscr{D}_t$ denotes the σ-field generated by sets of the form $\{f: f \in D, f(s) \in E\}$ where E ranges over Σ and s ranges over $[0, t]$, and $\mathscr{D}_{t+}$ denotes $\bigcap_{s>t} \mathscr{D}_{t+}$.

Let $D_s = \{f: f \in D \wedge f(t) = f(s) \text{ if } t \geqslant s\}$. Clearly $D_s \in \mathscr{D}$. Let χ_s be the map of D into D_s which sends $f \in D$ into the function $\chi_s(f)$ whose value at t is $t \wedge s$. Clearly χ_s is $(\mathscr{D} - \mathscr{D})$ measurable, and is onto D_s.

2.1 Proposition. *Let $s \geqslant 0$ and $A \in \mathscr{D}$. Then* (i), (iii) *and* (iii) *are equivalent* (*in* (iii), *f and g are assumed to belong to D*).

(i) $A \in \mathscr{D}_s$.

(ii) $A \in \chi_s^{-1}(\mathscr{D})$.

(iii) *If $f \in A$, and if $g(t) = f(t)$ for all $t \in [0, s]$, then $g \in A$. Also, conditions* (iv) *and* (v) *are equivalent:*

(iv) $A \in \mathscr{D}_t$

(v) *If $f \in A$, and there is a $\delta > 0$ such that $g(t) = f(t)$ for all $t \in [0, s + \delta]$, then $g \in A$.*

Proof. Let $\mathscr{D}^*$ be the collection of all $A \in \mathscr{D}$ for which (iii) holds. We establish the equivalence of (i), (ii), and (iii) by showing $\mathscr{D}_s \subset \mathscr{D}^* \subset \chi_s^{-1}(\mathscr{D}) \subset \mathscr{D}_s$. It is easy to see that $\mathscr{D}^*$ is a σ-field, and that it contains all sets of the form $\{f: f(t) \in E\}$ for $E \in \Sigma$ and $t \leqslant s$. Therefore $D_s \subset \mathscr{D}^*$. Suppose $A \in \mathscr{D}^*$. Let $f \in A$. We see from (ii) that $\chi_s(f) \in A$, so $f \in \chi_s^{-1}(A)$. Conversely, if $\chi_s(f) \in A$, $f \in A$ by virtue of (ii). Thus $A = \chi_s^{-1}(A)$. Thus $\mathscr{D}^* \subset \chi_s^{-1}(\mathscr{D})$. If $A = \{f: f(t) \in E\}$, and $t \leqslant s$, $A = \chi_s^{-1}(A)$ and $A \in \mathscr{D}_s$. That $\chi_s^{-1}(\mathscr{D}) \subset \mathscr{D}_s$ now follows from the facts that $\mathscr{D}_s$ is a σ-field and that $\chi_s^{-1}(\mathscr{D})$ is generated by the χ_s^{-1} images of any class of sets generating $\mathscr{D}$.

Suppose that $A \in \mathscr{D}_{s+}$, that $f \in A$, and that $g = f$ on $[0, s + \delta]$, where $\delta > 0$. Pick n so that $1/n < \delta$. Then $g = f$ on $[0, s + (1/n)]$. Since $A \in \mathscr{D}_{s+(1/n)}$, $g \in A$ by (i) $\to$ (iii). This shows that (iv) $\to$ (v). Let $A \in \mathscr{D}$ have the property described in (v). Suppose $f \in A$, and that $g = f$ on $[0, s + (1/n)]$. Then $g \in A$ by assumption. It follows from (iii) $\to$ (i) that $A \in \mathscr{D}_{s+(1/n)}$. Since this holds for each $n = 1, 2, \ldots$, $A \in \bigcap_{n=1}^{\infty} \mathscr{D}_{s+(1/n)} = \mathscr{D}_{s+}$. This completes the proof of the proposition.

We adjoin ∞ to the half-line $[0, \infty)$ in the usual way, obtaining the extended real line $[0, \infty]$. The class of Borel subsets of $[0, \infty]$ is denoted by $\mathscr{B}$. A function σ on D into $[0, \infty]$ is called a *path-defined stopping time relative* to $\{\mathscr{D}_t\}$ (relative to $\{\mathscr{D}_{t+}\}$) if, for each $t > 0$, the set $\{f: f \in D, \sigma(f) \leqslant t\}$ belongs to $\mathscr{D}_t$ (to $\mathscr{D}_{t+}$). It is convenient for us to assign a value to $f(t)$ when $t = \infty$. To do this we adjoin to S as an isolated point a point Δ not belonging to S. We set $S^* = S \bigcap \{\Delta\}$, and Σ^* is the corresponding class of Borel subsets of S^*. We shall at times be careless about the distinction between (S^*, Σ^*) and (S, Σ). The map $f \to f(\sigma(f))$ of D into S^* is $\mathscr{D} - \Sigma^*$ measurable. To see this, let σ_n be equal to $k + 1/2^n$ on $[k/2^n \leqslant \sigma < k + 1/2^n)$, $k = 0, 1, \ldots$, and equal to ∞ on $[\sigma = \infty]$. It is easy to see that $f \to f(\sigma_n(f))$ is $\mathscr{D} - \Sigma^*$-measurable. (As a matter of fact, $[\sigma_n \leqslant s]$ belongs to $\mathscr{D}_s$ (to $\mathscr{D}_{s+}$) if σ is a stopping time relative to $\{\mathscr{D}_t\}$ (to $\{\mathscr{D}_{t+}\}$).) Because of the right continuity of f, $f(\sigma_n(f)) \to f(\sigma(f))$ as $n \to \infty$. We shall often use $f(\sigma)$ as an abbreviation for $f(\sigma(f))$. If σ is a stopping time relative to $\{\mathscr{D}_t\}$ (to $\{\mathscr{D}_{t+}\}$), $\mathscr{D}_\sigma$ denotes the class of all subsets A of $\mathscr{D}$ for which $A \cap \{\sigma \leqslant s\}$ belongs to $\mathscr{D}_s$ (to $\mathscr{D}_{s+}$) for each $s \geqslant 0$.

Let τ be a stopping time relative to $\{\mathscr{D}_{t+}\}$, and E a subset of S. The functions σ and σ^+ are defined from D into $[0, \infty]$ by

$$\begin{aligned} \sigma(f) &= \inf\{t : t \geqslant \tau(f), f(t) \in E\} \\ \sigma^+(f) &= \inf\{t : t > \tau(f), f(t) \in E\}. \end{aligned} \tag{2.1}$$

We use here and elsewhere the convention that the infimum over an empty set is ∞. Both σ and σ^+ will be referred to as *post-τ hitting times of E.*

2.2 PROPOSITION. *If E is open, both σ and σ^+ are stopping times relative to $\{\mathscr{D}_{t+}\}$.*

Proof. Assume that E is open. For each n, let $\sigma_n(f)$ be equal to $\inf\{t: t \geqslant k + 1/n, f(t) \in E\}$ on $\{k/n \leqslant \tau(f) < k + 1/n\}$, $k = 0, 1, \ldots$, and to ∞ on $[\tau(f) = \infty]$. Because of the right continuity of f and the assumption that E is open, $\{\alpha_n < s\}$ is the union of the sets $\{f(r) \in E, k/n \leqslant \tau(f) < k + 1/n\}$ as r ranges over the rational members of $[k + 1/n, s)$ and k over those non-negative integers for which $k + 1/n \leqslant s$. Thus $\{\sigma_n < s\} \in \mathscr{D}_{s+}$. Since $\{\sigma_n \leqslant s\} = \bigcap_m [\sigma < s + (1/m)]$ it follows that $\{\sigma_n \leqslant s\} \in \mathscr{D}_{s+}$. Thus σ_n is a stopping time relative to $\{\mathscr{D}_{t+}\}$. It is clear that $\sigma_n \geqslant \sigma^+$ for each n, and that $\sigma_n \searrow$. Let $\sigma_\infty = \lim_n \sigma_n$. Clearly $\sigma_\infty \geqslant \sigma^+$. Suppose that $\sigma^+(f) < \infty$, and that $t_0 > \sigma^+(f)$. Then there is a $t \in [\sigma^+(f), t_0)$ with $t > \tau(f)$ and $f(t) \in E$. Let k_n be such that $k_n/n \leqslant \tau(f) < (k_n + 1)/n$. Then $t > (k_n + 1)/n$ for all sufficiently large n. Since $f(t) \in E$, $t \geqslant \sigma_n(f)$ for these n. It follows that $t_0 > t \geqslant \sigma_\infty(f)$, so if $t_0 > \sigma^+(f)$, then $t_0 > \sigma_\infty(f)$. Therefore $\sigma^+ \geqslant \sigma_\infty$, so $\sigma^+ = \sigma_\infty$. The non-decreasing limit of stopping times is a stopping time ([2], page 33), so σ^+ is a stopping time relative to $\{\mathscr{D}_{t+}\}$. We can express σ in terms of σ^+ as follows: $\sigma = \tau$ on $\{f(\tau) \in E\}$, and $\sigma = \sigma^+$ on $\{f(\tau) \notin E\}$. However, $\{f(\tau) \in E\} \in \mathscr{D}_\tau$([2], page 34), so $\{f(\tau) \in E\} \cap \{\tau \leqslant S\} \in \mathscr{D}_{s+}$. Since $\tau \leqslant \sigma^+$, $\mathscr{D}_\tau \subset \mathscr{D}_{\sigma+}$ ([2], page), so $\{f(\tau) \in E\} \cap \{\sigma^+ \leqslant s\} \in \mathscr{D}_{s+}$. This show that σ is also a stopping time relative to $\{\mathscr{D}_{t+}\}$, which completes the proof of the proposition.

2.3 PROPOSITION. *Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $(D, \mathscr{D})$ a measurable space. Let $\mathscr{C}$ be a sub-σ-field of $\mathfrak{A}$. Let X be a $(\mathfrak{A} - \mathscr{D})$ measurable map of Ω into D*

and F a $\mathscr{D}$-measurable map of D into $[0, \infty)$. *Let* $\pi = P \circ X^{-1}$ (*that is*: $\pi(A) = P(X^{-1}(A))$, $A \in \mathscr{D}$). *Then*

$$E_p(F \circ X \mid X^{-1}(\mathscr{C})) = E_\pi(F \mid \mathscr{C}) \circ X \tag{2.2}$$

Proof. By E_p and E_π we mean the expectation operators on $(\Omega, \mathscr{A}, P)$ and $(D, \mathscr{D}, \pi)$ respectively. For the definition and basic properties of conditional expectation, we refer the reader to any of the standard treatises, for example, [6]. To prove the proposition, we must show that for any $C \in \mathscr{C}$, the integral over $X^{-1}(C)$ of the right hand side of (2.2) with respect to P is equal to $\int_{X^{-1}(C)} F \circ X\, dP$. We require

$$\int_{X^{-1}(C)} F \circ X\, dP = \int_C F\, d\pi, \qquad C \in \mathscr{C}. \tag{2.3}$$

To prove (2.3), it is enough to establish that the equality holds for $F = 1_A$, $A \in \mathscr{D}$. Then $F \circ X = 1_B \circ X = 1_{X^{-1}(B)}$, and the left hand side is $P(X^{-1}(B) \cap X^{-1}(C)) = P(X^{-1}(B \cap C))$. The right hand side is $\pi(B \cap C) = P(X^{-1}(B \cap C))$. This proves (2.3). Now fix $C \in \mathscr{C}$. Using (2.3) twice, the first time with F replaced by $E_\pi(F \mid \mathscr{C})$, we obtain

$$\begin{aligned} \int_{X^{-1}(C)} E_\pi(F \mid \mathscr{C}) \circ X\, dP &= \int_C E_\pi(F \mid \mathscr{C})\, d\pi \\ &= \int_C F\, d\pi \\ &= \int_{X^{-1}(C))} F \circ X\, dP. \end{aligned}$$

This completes the proof of the proposition.

Suppose that $\mathscr{E}$ is a σ-field over a set E which is countably generated, that is, there is a countable $\mathscr{E}_0 \subset \mathscr{E}$ such that $\mathscr{E}$ is the σ-field generated by $\mathscr{E}$. We may assume without loss of generality that $\mathscr{E}_0$ is a field. For each $x \in E$, let $E_x = \bigcap \{F : F \in \mathscr{E}_0, x \in F\}$. Clearly $E_x \in \mathscr{E}$, and it is easy to see that either $E_x = E_y$ or $E_x \cap E_y = \varnothing$. Let $\mathscr{E}_1$ be the class of all members F of $\mathscr{E}$ for which $F = \bigcup \{E(x) : x \in F\}$. It is clear that $\mathscr{E}_0 \subset \mathscr{E}_1$. In particular $E \in \mathscr{E}_1$, and it follows that $\mathscr{E}_1$ is closed under complements. It is clearly closed under unions, so it is a σ-field. Hence $\mathscr{E}_1 = \mathscr{E}$, so every member of $\mathscr{E}$ is a union of E_x's. The E_x's are called the *fibres* of $\mathscr{E}$.

2.4 THEOREM (Disintegration of measures). *Let D be a complete separable metric space, with $\mathscr{D}$ its class of Borel sets. Let π be a probability measure on $\mathscr{D}$, and let $\mathscr{E}$ be a countably generated sub-σ-field of $\mathscr{D}$. There is a family $\{\pi_x\}_{x \in D}$ of measures on $\mathscr{D}$ satisfying the following conditions.*

(i) *for each* $A \in \mathscr{D}$, $\pi_x(A)$ *is an* $\mathscr{E}$*-measurable function of* x,

(ii) *for each* $A \in \mathscr{D}$ *and* $E \in \mathscr{E}$,

$$\pi(A \cap E) = \int_E \pi_x(A)\, \pi(dx),$$

and

(iii) *for* π*-almost all* $x \in D$, π_x *is a probability measure with* $\pi_x(E_x) = 1$.

Proof. The proof of this theorem is adapted from the proof of a more general disintegration of measures theorem presented in a measure theory course taught by Professor M. Sion, and included in a set of course notes written up by Mr. Faulkner, a student in the course. We take this opportunity to thank both Professor Sion and Mr. Faulkner for making these notes available to use and for their most helpful discussions.

Assume the hypotheses of the theorem. For each $A \in \mathscr{D}$, let $\pi(x, A)$ be a function of x which is an $\mathscr{E}$-measurable version of the conditional probability of A given $\mathscr{E}$. Thus (i) and (ii) are satisfied for each $A \in \mathscr{D}$ if we substitute $\pi(x, A)$ for $\pi_x(A)$. $\mathscr{D}$ is countably generated ([7], page 5). Let $\mathscr{D}_0$ be a countable generating sub-field of $\mathscr{D}$. It is a consequence of the usual properties of conditional probabilities that there is a set $N_1 \in \mathscr{D}$, with $\pi(N_1) = 0$, such that, if $x \notin N_1$, then $\pi(x, \cdot)$ is a finitely additive measure on $\mathscr{D}_0$ with $\pi(x, D) = 1$.

For each $A \in \mathscr{D}$ and $\epsilon > 0$ there is a compact set $C \subset D$ with $C \subset A$ and $\pi(AC) < \epsilon$ ([7], page 29). It easily follows that there is a countable family $\mathscr{C}_0$ of compact subsets of D such that $\pi(A) = \sup\{\pi(C) : C \in \mathscr{C}_0, C \subset A\}$ for each $A \in \mathscr{D}_0$. For each $x \in D$ and $A \in \mathscr{D}_0$, let

$$\bar{\pi}_x(A) = \sup\{\pi(x, C) : C \in \mathscr{C}_0, C \subset A\}$$

Since $\mathscr{C}_0$ is countable, $\bar{\pi}_x(A)$ is an $\mathscr{E}$-measurable function of x. Given any E and F in $\mathscr{D}$, with $E \subset F$, $\pi_x(E) \leqslant \pi_x(F)$ for π-almost all x. Therefore there is a $N_2 \in \mathscr{D}$, with $\pi(N_2) = 0$ and for which $\pi(x, C) \leqslant \pi(x, A)$ provided $x \notin N_2$, $C \in \mathscr{C}_0$, and $A \in \mathscr{D}_0$ and $C \subset A$. If $x \notin N_2$, then $\bar{\pi}_x(A) \leqslant \pi(x, A)$ for each $A \in \mathscr{D}_0$. Let $A \in \mathscr{D}_1$. Suppose $\epsilon > 0$. There is a $C \in \mathscr{C}_0$ with $C \subset A$ and $\pi(A) \leqslant \pi(C) + \epsilon$. Then, by property (ii),

$$\pi(A) - \epsilon \leqslant \pi(C) = \int \pi(x, C)\, \pi(dx) \leqslant \int \bar{\pi}_x(A)\, \pi(dx).$$

Since $\epsilon > 0$ is arbitrary, $\pi(A) \leqslant \int \bar{\pi}_x(A)\, \pi(dx)$. But $\pi(A) = \int \pi(x, A)\, \pi(dx)$ by virture of (ii). Since $\bar{\pi}_x(A) \leqslant \pi(x, A)$ π-almost everywhere, this is possible only if $\bar{\pi}_x(A) = \pi(x, A)$ for π-almost all x. Therefore there is an $N_3 \in \mathscr{D}$ having π-measure zero with $\pi_x(A) = \pi(x, A)$ for all $A \in \mathscr{D}_0$ and $x \notin N_3$. It follows that, if $x \notin N = N_1 \cup N_3$, then $\pi(x, \cdot)$ is a finitely additive measure on $\mathscr{D}_0$ for which

$$\pi(x, A) = \sup\{\pi(x, C) : C \in \mathscr{C}_0, C \subset A\}, \qquad A \in \mathscr{D}_0.$$

We claim that $\pi(x, \cdot)$ is countably additive on $\mathscr{D}_0$ if $x \notin N$. Suppose $x \notin N$. Denote $\pi(x, \cdot)$ by λ. To show that λ is countably additive, it suffices to show that if if $\{A_n\}$ is a descending sequence in $\mathscr{D}_0$ with $\lim_n \lambda(A_n) > 0$, then $\bigcap_n A_n \neq \varnothing$. So assume $\{A_n\}$ is such a sequence, and that $\lambda(A_n) \geqslant \delta > 0$ for all n. Let $\epsilon_n > 0$, $n = 1, 2,..., \sum_{n=1}^{\infty} \epsilon_n < \delta$. For each n there is a $C_n \in \mathscr{C}_0$ with $C_n \subset A_n$ and $\lambda(A_n \backslash C_n) \leqslant \epsilon_n$. Note that $\bigcap_{k=1}^{n} C_k \subset A_n$, and that $A_n \backslash C_n \subset A_k \backslash C_n$, $k = 1,..., n$. We have

$$\begin{aligned}
\lambda\left(\bigcap_{k=1}^{n} C_k\right) &= \lambda(A_n) - \lambda\left(A_n \big/ \bigcap_{k=1}^{n} C_k\right) \\
&= \lambda(A_n) - \lambda\left(\bigcup_{k=1}^{n} (A_n \backslash C_k)\right) \\
&\geqslant \lambda(A_n) - \sum_{k=1}^{n} \lambda(A_k \backslash C_k) \\
&\geqslant \delta - \sum_{k=1}^{n} \epsilon_n > 0.
\end{aligned}$$

Therefore $\bigcap_{k=1}^{n} C_k \neq \varnothing$. Since this is true for each n, and each C_n is compact, $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$. Thus $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$, as was to be shown, so $\lambda = \pi(x, \cdot)$ is countably additive on $\mathscr{D}_0$.

For each $x \notin N$, $\pi(x, \cdot)$ extends to a countably additive measure π_x on $\mathscr{D}$. Define $\pi_x(A) = 0$ for all $A \in \mathscr{D}$, $x \in N$. The family $\{\pi_x\}_{x \in D}$ of measures satisfies (i) and (ii). Let $\mathscr{E}_0$ be a countable subclass of $\mathscr{E}$ which generates $\mathscr{E}$. If $E \in \mathscr{E}_0$, then

$$\int_{E^c} \pi_x(E)\, \pi(dx) = \pi(E \cap E^c) = 0$$

by (iii). Since $\pi_x(D) = 1$ for π-almost all x, this implies that $\pi(D_E) = 1$, where $D_E = \{x: \pi_x(E) = 1_E(x)\}$. Let $D_0 = \bigcap_{E \in \mathscr{E}_0} D_E$. Then $\pi(D_0) = 1$. If $x \in D_0$, then $\pi_x(E) = 1$ for all $E \in \mathscr{E}_0$ with xE, so $\pi_x(E_x) = 1$. The family $\{\pi_x\}_{x \in D}$ therefore also satisfies (iii). This completes the proof of the theorem.

Suppose $(\Omega, \mathscr{A}, P)$ is a probability space, that $\{\mathscr{M}_t\}_{t \geqslant 0}$ is a nondecreasing family of sub-σ-fields of $\mathscr{A}$, and that, for each $t \geqslant 0$, X_t is a $(\mathscr{M}_t - \Sigma)$ measurable map of Ω into S. Then $X = (\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ is called a *stochastic process*. We shall assume that $(\Omega, \mathscr{A}, P)$ is a complete measure space, and that $\{\mathscr{M}_t\}$ is right continuous, that is, $\mathscr{M}_t = \mathscr{M}_{t+}$, where $\mathscr{M}_{t+} = \bigcap_{s>t} \mathscr{M}_s$. We shall also assume that the sample paths of X_t are right-continuous and have left limits: in other words, for each $\omega \in \Omega$, the map $t \to X_t(\omega)$ of $[0, \infty)$ into S belongs to D. It will cause no confusion if we use the letter "X" to denote not only the stochastic process but the associated map of Ω into D. Thus, X is defined to be the function on Ω whose value $X(\omega)$ is that member of D whose value at t is given by $X_t(\omega)$. It is clear that X is $(\mathcal{O}\mathcal{l} - \mathcal{D})$ measurable. We will use π to denote the image of P under the map X, that is, $\pi = P \circ X^{-1}$. In terms of finite dimensional cylinder sets, $\pi(\{f: f \in D, f(t_1) \in E_1, ..., f(t_n) \in E_n\} = P(X_{t_1} \in E_1, ..., X_{t_n} \in E_n)$. A function τ on Ω into $[0, \infty]$ is called a *stopping time for the process* $X = (\Omega, \mathcal{O}\mathcal{l}, \mathcal{M}_t, X_t, P)$ if $\{\tau \leqslant t\} \in \mathcal{M}_t$ for each $t \in [0, \infty)$. We use the symbol $\mathcal{M}_t$ to denote the sub-σ-field of $\mathcal{O}\mathcal{l}$ consisting of all subsets A of Ω for which $A \cap \{\tau \leqslant t\} \in \mathcal{M}_t$ for each $t \geqslant 0$. Then X_τ is $\mathcal{M}_t$-measurable (see [2], theorem 6.11 and the remark preceding it). It is clear that if σ is a path-defined stopping time on D relative to $\{\mathcal{D}_t\}$, then the function $\tau = \sigma \circ X$ is a stopping time for the process $(\Omega, \mathcal{O}\mathcal{l}, \mathcal{M}_t, X_t, P)$, and that $X^{-1}(\mathcal{D}_\sigma) \subset \mathcal{M}_\tau$. Because of the right continuity of $\{\mathcal{M}_t\}$, the same is true if σ is a stopping time relative to $\{\mathcal{D}_{t+}\}$. If τ is a stopping time for the process X and E a subset of S, the *post-τ hitting time of E* is defined in the same way it was for τ a stopping time defined on D, and is a stopping time for the process X.

2.5 Definition. Let $X = (\Omega, \mathcal{O}\mathcal{l}, \mathcal{M}_t, X_t, P)$ be a stochastic process. We say that X has *state-dependent hitting probabilities* if the following is true. Let E be open in S, and $A \in \Sigma$. Then there is a non-negative Σ-measurable function g on S having the property that for any stopping time τ,

$$P(X_\gamma \in A \mid \mathcal{M}_\tau) = g(X_\tau) \qquad P \text{ a.s.}, \tag{2.4}$$

where γ is the post-τ hitting time of E.

We emphasize that our assumptions that $\{\mathcal{M}_t\}$ is right continuous and that the sample paths of X are everywhere right continuous with left limits are implicit in the statements of all definitions, propositions, lemmas, and theorems. If τ is a path-defined stopping time, and σ the post-τ hitting time of E, then (2.4) together with proposition 2.2 imply that

$$\pi(f(\sigma) \in A \mid \mathcal{D}_\tau) = g(f(\sigma)) \qquad \pi \text{ a.e.}. \tag{2.5}$$

2.6 Definition. Let $X = (\Omega, \mathcal{O}\mathcal{l}, \mathcal{M}_t, X_t, P)$ and $\tilde{X} = (\Omega, \tilde{\mathcal{O}\mathcal{l}}, \tilde{\mathcal{M}}_t, \tilde{X}_t, \tilde{P})$ be stochastic processes. We say that *X and $\tilde{X}$ have the same state-dependent hitting probabilities* if, for any open E in S and $A \in \Sigma$, there is a non-negative Σ-measurable function g for which the following is true. For any stopping times τ and τ relative to $\{\mathcal{M}_t\}$ and $\{\tilde{\mathcal{M}}_\tau\}$,

$$\begin{aligned} P(X_\sigma \in A \mid \mathcal{M}_\tau) &= g(X_\tau) \qquad P\text{-a.s.} \\ P(\tilde{X}_{\tilde{\sigma}} \in A \mid \tilde{\mathcal{M}}_{\tilde{\tau}}) &= g(\tilde{X}_{\tilde{\tau}}) \qquad \tilde{P}\text{-a.s.}, \end{aligned} \tag{2.6}$$

where σ and $\tilde{\sigma}$ are, respectively, the post-τ and post-$\tilde{\tau}$ hitting times of E.

It follows from (2.6) and proposition 2.2 that if τ is a path defined stopping time on D (relative to $\{\mathscr{D}_{t+}\}$), and σ is the post-τ hitting time of E, then

$$\begin{aligned} \pi(f(\sigma) \in A \mid \mathscr{D}_\tau) &= g(f(\tau)) \qquad \pi\text{-a.e.}, \\ \tilde{\pi}(f(\sigma) \in A \mid \mathscr{D}_\tau) &= g(f(\tau)) \qquad \tilde{\pi}\text{-a.e.}. \end{aligned} \tag{2.7}$$

Here and elsewhere $\tilde{\pi} = \pi \circ \tilde{X}^{-1}$.

Let $X = (\Omega, \mathcal{O}\!\ell, \mathscr{M}_t, X_t, \dot{P})$ be a stochastic process. We say that $\{\tau_t, t \geqslant 0\}$ is a *time change relative to* $\{\mathscr{M}_t\}$ if, for each $t \in [0, \infty)$, τ_t is a stopping time relative to $\{\mathscr{M}_s\}$, and for each $\omega \in \Omega$, $\tau_t(\omega)$ is a non-decreasing and right continuous function of t.

2.7 Theorem. *Let $X = (\Omega, \mathcal{O}\!\ell, \mathscr{M}_t, X_t, P)$ be a stochastic process with state dependent hitting probabilities. Let $\{\sigma_t, t \geqslant 0\}$ be a time change relative to $\{\mathscr{M}_t\}$ such that, for each $\omega \in \Omega$, $\sigma_t(\omega)$ is a strictly increasing, continuous, finite-valued function of t with $\sigma_t(\omega) \uparrow \infty$ as $t \to \infty$. Assume also that $\sigma_0(\omega) = 0$ for all $\omega \in \Omega$. For each t, let $Y_t = X_{\sigma_t}$ and $\mathscr{N}_t = \mathscr{M}_{\sigma_t}$. Then $Y = (\Omega, \mathcal{O}\!\ell, \mathscr{N}_t, Y_t, P)$ is a process with the same state-dependent hitting probabilities as X.*

Proof. We note that X_{σ_t} is measurable with respect to $\mathscr{M}_{\sigma_t}$, so Y is a stochastic process. Let E be an open subset of S. Let τ be a stopping time relative to $\{\mathscr{N}_t\}$. Let σ_τ be defined by $\sigma_\tau(\omega) = \sigma_{\tau(\omega)}(\omega)$, $\omega \in \Omega$.

(a) *σ_τ is a stopping time relative to $\{\mathscr{M}_\tau\}$.*

Proof of (a). Suppose τ assumes only the values $t_1, t_2, \ldots$. Then $\{\sigma_\tau \leqslant s\} = \bigcup_{i=1}^{\infty} \{\sigma_{t_k} \leqslant s, \tau = t_k\}$. But $\{\tau = t_k\} \in \mathscr{N}_{t_k} \subset \mathscr{M}_{\sigma_{t_k}}$ so $\{\sigma_{t_k} \leqslant s, \tau = t_k\} = \{\tau = t_k\} \cap \{\sigma_{t_k} \leqslant s\} \in \mathscr{M}_s$. It follows that $\{\sigma_\tau \leqslant s\} \in \mathscr{M}_s$ for each s, so σ_τ is a stopping time relative to $\{\mathscr{M}_t\}$. If τ is a stopping time relative to $\{\mathscr{N}_t\}$, let $\tau^{(n)} = (k+1)/2^n$ on $\{k/2^n \leqslant \tau < (k+1)/2^n\}$. By virtue of the right continuity of $\{\sigma_t\}$, $\sigma_{\tau^{(n)}} \searrow \sigma_\tau$ as $n \to \infty$. It follows that σ_τ is a stopping time relative to $\{\mathscr{M}_t\}$ ([2], page 32).

Let γ be the post-τ hitting time of E: σ_γ is also a stopping time relative to $\{\mathscr{M}_t\}$.

(b) *σ_γ is the post-σ_τ hitting time of E.*

Proof of (b). First, we claim that for any $S \subset [0, \infty]$, $\sigma_{\inf S} = \inf\{\sigma_t : t \in S\}$. This is an easy consequence of the continuity of σ_t. By definition, $\gamma = \inf\{t : t \geqslant \tau, Y_t \in E\} = \inf\{t : t \geqslant \tau, X_{\sigma_t} \in E\}$. Note that $\sigma_\gamma = \inf\{\sigma_t : t \geqslant \tau, X_{\sigma_t} \in E\} = \inf\{\sigma_t : \sigma_t \geqslant \sigma_\tau, X\sigma_t \in E\} = \inf\{s : s \geqslant \sigma_\tau, X_s \in E\}$. (We have used the fact that $t \to \sigma_t(\omega)$ is a strictly increasing one-one map of $[0, \infty]$ onto itself). Thus σ_γ is the post σ_τ hitting time of E, which proves (b).

(c) $\mathscr{N}_\tau \subset \mathscr{M}_{\sigma_\tau}$.

Proof of (c). Again we begin by assuming that τ has only a countable number of values, namely $t_1, t_2, \ldots$. Suppose $A \in \mathscr{N}_\tau$. Then $A \cap \{\sigma_\tau \leqslant s\} = \bigcup_{k=1}^\infty A \cap \{\tau = t_k, \sigma_{t_k} \leqslant s\}$. But $B = A \cap \{\tau = t_k\} \in \mathscr{N}_{t_k} = \mathscr{M}_{\sigma_{t_k}}$, so $A \cap \{\tau = t_k, \sigma_{t_k} \leqslant s\} = B \cap \{\sigma_{t_k} \leqslant s\} \in \mathscr{M}_s$. If τ is an arbitrary stopping time relative to $\{\mathscr{N}_t\}$, let $\tau^{(n)}$ be defined as in the proof of (a). By what we have just shown, $\mathscr{N}_\tau \subset \mathscr{N}_{\tau^{(n)}} \subset \mathscr{M}_{\sigma_{\tau^{(n)}}}$ for each $n = 1, 2, \ldots$, so $\mathscr{N}_\tau \subset \bigcap_{n=1}^\infty \mathscr{M}_{\sigma_{\tau^{(n)}}}$. Since $\{\mathscr{M}_t\}$ is right continuous, and since $\sigma_{\tau^{(n)}} \searrow \sigma_\tau$, it follows from (6.7) on page 33 of [2] that $\bigcap_{n=1}^\infty \mathscr{M}_{\sigma_{\tau^{(n)}}} = \mathscr{M}_{\sigma_\tau}$. This completes the proof of (c). (The reference cited also establishes the right continuity of $\{\mathscr{N}_t\} = \{\mathscr{M}_{\sigma_\tau}\}$.)

Now suppose that $A \in \Sigma$. Then

$$\begin{aligned} P(Y_\gamma \in A \mid \mathscr{N}_\tau) &= P(X_{\sigma_\gamma} \in A \mid \mathscr{N}_\tau) \\ &= E(P(X_{\sigma_\gamma} \in A \mid \mathscr{M}_{\sigma_\tau}) \mid \mathscr{N}_\tau) \\ &= E(g(X_{\sigma_\tau}) \mid \mathscr{N}_\tau) \\ &= E(g(Y_\tau) \mid \mathscr{N}_\tau) \\ &= g(Y_\tau), \end{aligned}$$

where g is as in the statement of Definition 2.5. This completes the proof of the theorem.

Let $(X_t, \mathscr{M}_t, P_x)$ be a strong Markov process (see page 37 of [2]) (We assume that $\xi = \infty$, where ξ is the killing time, and that the sample paths all belong to D.) Let $\{X_\tau, t \geqslant 0\}$ be the process determined by selecting a distribution for X_0. Then $X = (\Omega, \mathscr{Q}, \mathscr{M}_\tau, X_t, P)$ is a stochastic process with state-dependent hitting probabilities. The last theorem thus implies that, with some restrictions, non-anticipating time changes of a strong Markov process yield processes with state-dependent hitting probabilities. Since non-anticipation alone does not guarantee that the resulting process is Markov, this shows that the class of process with state-dependent hitting probabilities which are not Markov is quite extensive.

3. Keeping Track of Oscillations

Given an $f \in D$, we associate with it certain finite sequences as records of oscillations undergone by $f(t)$ as t varies over $[0, \infty)$. Fix f, and consider, for example, the sequence $\mathfrak{s} = (2, 7, 3, 1)$. We associate $\mathfrak{s}$ with f if the following four conditions are met. (1) $f(t)$ executes at least two oscillations of size 1. (2) After its second oscillation of size 1, it executes at least seven oscillations of size 1/2 before executing a third oscillation of size 1. (It may or may not execute a third oscillation of size 1. The association of (2, 7, 3, 1) with f does not provide that information.) (3) After this seventh oscillation of size 1/2, and before either its third oscillation of size 1 or a possible eighth osillation of size 1/2 following the seven mentioned in (2), it executes at least 3 oscillations of size 1/4. (4) After this third oscillation of size 1/4, and before either its third oscillation of size 1, the possible eighth oscillation of size 1/2 mentioned in (3), or a possible fourth oscillation of size 1/4 following the three mentioned in (3), it executes at least one oscillation of size 1/8. If (1) doesn't hold, or (1) holds but (2) doesn't, or (1) and (2) hold but (3) doesn't, or (1), (2), and (3) hold but (4) doesn't, we do not associate (2, 7, 3, 1) with f. Assuming that (2, 7, 3, 1) is associated with f, $\rho(2, 7, 3, 1)$ is the time f undergoes that first oscillation of size 1/8 figuring in condition (4), and $f(\rho(2, 7, 3, 1))$ is denoted by $x(2, 7, 3, 1)$. The general difinitions proceed by induction, as do the proofs of certain basic properties of ρ and $\mathscr{T}(f)$, the set of finite sequences associated with f.

We introduce some notation. For each $n = 1, 2,\ldots, A_n$ is the set of all ordered n-tuples of non-negative integers. We denote by the lexicographical ordering on $A_n : (i_1,\ldots, i_n) \prec (j_1,\ldots,j_n)$ if $i_1 < j_1$ or $i_1 = j_1$ and $i_2 < j_2$ or $\cdots$ $i_1 = j_1,\ldots, i_{n-1} = j_{n-1}$ and $i_n < j_n$. Let $\mathscr{T} = \bigcup_{n=1}^{\infty} \mathscr{T}_n$. We extend $\prec$ to $\mathscr{T}$ by identifying $(i_1,\ldots, i_m)$ with the n-tuple $(i_1,\ldots, i_m, 0,\ldots, 0)$ when comparing $(i_1,\ldots, i_m)$ with $(j_1,\ldots, j_n)$, $m < n$. The symbols $\succ$ and $\preccurlyeq$ have their obvious meaning, as do phrases like "$\prec$-precedes", "$\prec$-next", "$\prec$-predecessor", etc. The letters, s, t,..., with or without affices will designate members of $\mathscr{T}$.

We will denote the set of $t \in \mathscr{T}_n$ associated with f as indicated above by $\mathscr{T}_n(f)$. We proceed with the formal definition of $\mathscr{T}_n(f)$ and the function $\rho_n(f, \cdot)$ defined on $\mathscr{T}_n(f)$. It will turn out that, if $m < n$, then the n-tuple $t = (i_1,\ldots, i_m, 0,\ldots, 0)$ belongs to $\mathscr{T}_n(f)$ if and only if $s = (i_1,\ldots, i_m) \in \mathscr{T}_m(f)$, in which case $\rho_n(t, f) = \rho_m(s, f)$. Thus $\mathscr{T}_n(f)$ and $\rho_n(f, \cdot)$ are, in a natural way, extensions of $A_m(f)$ and $\rho_m(f, \cdot)$ respectively. We drop the subscript n from $\rho_n(f, \cdot)$, denoting it by $\rho(f, \cdot)$, and regard it as a function on $\mathscr{T}(f) = \bigcup_{n=1}^{\infty} \mathscr{T}_n(f)$. We occasionally denote $f(\rho(f, t))$ by $x(f, t)$. If f is fixed or specified unambiguously by the context, we often write $\rho(t)$ and $x(t)$ for $\rho(f, t)$ and $x(f, t)$ respectively. We define $\mathscr{T}_1(f)$ and $\rho(f, t)$ on $\mathscr{T}_1(f, \cdot)$ as follows. First, $0 \in \mathscr{T}_1(f)$ and $\rho(0) = 0$. Suppose that $i \in \mathscr{T}_1(f)$, that $\rho(f, i)$ has been defined, and that $\rho(f, i) \in [0, \infty)$. Let

$$\tau = \inf\{t: t \geqslant \rho(i) \text{ and } d(f(t), x(i)) > 1\}. \tag{3.1}$$

If $\tau < \infty$, then $i + 1 \in \mathscr{T}(f)$ and $\rho(i + 1) = \tau$. If $\tau = \infty$, $i + 1 \notin \mathscr{T}(f)$, and no integer larger than i belongs to $\mathscr{T}(f)$. Note that, as we have defined it, $\mathscr{T}_1(f)$ is either the set of all non-negative integers or else is $\{0, 1,\ldots, i\}$ for some non-negative integer i. That ρ is strictly monotone on $\mathscr{T}_1(f)$ is a consequence of (3.1) and the right continuity of f.

In defining and describing $\mathscr{T}_n(f)$ and ρ on $\mathscr{T}_n(f)$ for values of n larger than 1 we need certain basic properties.

3.1 Theorem (Basic properties of $\mathscr{T}(f)$ and ρ).

(a) *Suppose* $j < k$. *Let* $\mathfrak{s} = (i_1, ..., i_j)$. *Let* $\mathfrak{t} = (i_1, ..., i_j, 0, ..., 0) \in \mathscr{T}_k$. *Then* $\mathfrak{s} \in \mathscr{T}_j(f) \rightarrow \mathfrak{t} \in \mathscr{T}_k(f)$, *in which case* $\rho(\mathfrak{s}) = \rho(\mathfrak{t})$.

(b) (*strict monotonicity*) *Suppose* $\mathfrak{s}$ *and* $\mathfrak{t}$ *belong to* $\bigcup_{j=1}^{k} \mathscr{T}_j(f)$. *Then* $\mathfrak{s} \prec t \rightarrow \rho(\mathfrak{s}) < \rho(\mathfrak{t})$.

(c) (*reduction of last component*). *Suppose* $(i_1, ..., i_{k-1}, i) \in \mathscr{T}_k(f)$. *Then, if* $0 \leqslant l < i$, $(i_1, ..., i_{k-1}, l) \in \mathscr{T}_k(f)$.

(d) (*truncation*). *Suppose* $j < k$. *If* $(i_1, ..., i_k) \in \mathscr{T}_k(f)$, *then* $(i_1, ..., i_j) \in \mathscr{T}_j(f)$.

(e) *Under the hypothesis of* (d), $d(f(t), x(i_1, ..., i_j)) \leqslant 1/2^{i-1}$ *for all* $t \in [\rho(i_1, ..., i_j), \rho(i_1, ..., i_k))$.

(f) (*order*) *Let* $\mathfrak{t} = (i_1, ..., i_k) \in \mathscr{T}_k(f)$. *Suppose that* $\mathfrak{t}$ *is not the* $\prec$*-last member of* $\mathscr{T}_k(f)$. *Then the* $\prec$*-next member of* $\mathscr{T}_k(f)$ (*that is, the* $\prec$*-smallest member of* $\mathscr{T}_k(f)$ *which* $\prec$*-follows* $\mathscr{T}_k(f)$) *is one of* $\mathfrak{t}_1 = i_1 + 1$, $\mathfrak{t}_2 = (i_1, i_2 + 1)$, ..., $\mathfrak{t}_k = (i_1, ..., i_{k-1}, i_k + 1)$. *Let*

$$\tau_l = \inf\{t: t \geqslant \rho(\mathfrak{t}), d(f(t), x(i_1, ..., i_l)) > 1/2^{l-1}\}, \tag{3.2}$$

$l = 1, ..., k$. *Then* $\mathfrak{t}$ *is the* $\prec$*-last member of* $\mathscr{T}_k(f)$ *if and only if* $\tau_1 = \cdots = \tau_k = \infty$. *Suppose* $\tau_1 \wedge \cdots \wedge \tau_k < \infty$ *for each* $l = 1, ..., k$ *the following are equivalent.*

(i) *the* $\prec$*-next member of* $A_k(f)$ *is* t_l

(ii) $\tau_2 = \tau_1 \wedge \cdots \wedge \tau_k$, *and* $l = \inf\{m: \tau_m = \tau_1 \wedge \cdots \wedge \tau_k\}$

If $\mathfrak{t}_l$ *is the* $<$*-next member of* $\mathscr{T}_k(f)$, *then* $\rho(\mathfrak{t}_l) = \tau_l$.

(g) (*structure*). *Let* $k > 1$. *Assume that* $\mathfrak{s} = (i_1, ..., i_{k-1}) \in \mathscr{T}_{k-1}(f)$. *If* $\mathfrak{s}$ *is not the* $\prec$*-last member of* $\mathscr{T}_{k-1}(f)$, *let* $\mathfrak{t}$ *be the* $\prec$*-next member of* $\mathscr{T}_{k-1}(f)$, *and let* $T(\mathfrak{s}, \mathfrak{t})$ *be the set of members of* $\mathscr{T}_k(f)$ *which are* $\prec$*-between* $\mathfrak{s}$ *and* $\mathfrak{t}$ *and which are not of the form* $(j_1, ..., j_l, 0, ..., 0)$ *for some* $l < k$. *Then either* $T(\mathfrak{s}, \mathfrak{t})$ *is empty, or else there is an* m *for which*

$$T(\mathfrak{s}, \mathfrak{t}) = \{(i_1, ..., i_{k-1}, i): 1 \leqslant i \leqslant m\}.$$

If $\mathfrak{s}$ *is the* $\prec$*-last member of* $\mathscr{T}_{k-1}(f)$, *let* $T(\mathfrak{s})$ *be the set of members of* $\mathscr{T}_k(f)$ *which are* $\prec$*-greater than* $\mathfrak{s}$. *Then either* $T(\mathfrak{s})$ *is empty, or there is an* m *for which*

$$T(\mathfrak{s}) = \{(i_1, ..., i_{k-1}, i): 1 \leqslant i \leqslant m\},$$

or else

$$T(\mathfrak{s}) = \{(i_1, ..., i_{k-1}, i): i = 1, 2, ...\}.$$

(i) *In the notation of* (*g*)

$$T(\mathfrak{s}, \mathfrak{t}) = \varnothing \rightarrow d(f(t)), x(\mathfrak{s})) \leqslant 1/2^{k-1} \quad \textit{for all} \quad t \in [\rho(\mathfrak{s}), \rho(t)), \tag{3.3}$$

$$T(\mathfrak{s}) = \varnothing \rightarrow d(f(l), x(\mathfrak{s})) \leqslant 1/2^{k-1} \quad \textit{for all} \quad t \in [\rho(\mathfrak{s}), \infty). \tag{3.4}$$

In our inductive definitions and proofs, we will be making repeated use of the following immediate consequence of basic property (e).

3.2 PROPOSITION. *Suppose* $(i_1, ..., i_k) \in \mathscr{T}_k(f)$ *and that* $j < k$. *Then* $\inf\{t: t \geqslant \rho(i_1, ..., i_l),\ d(f(t),\ x(i_1, ..., i_j)) > 1/2^{j-1}\}$ *is the same for each* l *in* $\{j, j+1, ..., k\}$.

Let $P(n)$ be the assertion that the statements of 3.1 all hold provided $k \leqslant n$. $P(1)$ follows from the definitions of $\mathscr{T}_1(f)$ and $\rho(\ell)$ for $\ell \in \mathscr{T}_1(f)$. Assume, then, that $\mathscr{T}_n(f)$ has been defined and that $\rho(\ell)$ has been defined for $\ell \in \mathscr{T}_n(f)$. Also assume $P(n)$. We now define $\mathscr{T}_{n+1}(f)$, $\rho(\ell)$ for $\ell \in \mathscr{T}_{n+1}(f)$, and prove $P(n+1)$. First, if $(i_1, ..., i_n) \in \mathscr{T}_n(f)$, then $(i_1, ..., i_n, 0) \in \mathscr{T}_{n+1}(f)$ and $\rho(i_1, ..., i_n, 0) = \rho(i_1, ..., i_n)$. Fix $(i_1, ..., i_n) = \mathfrak{s}$. Suppose both that $(i_1, ..., i_n, l) \in \mathscr{T}_{n+1}(f)$ and that $\rho(i_1, ..., i_n, l)$ has been defined for $l = 0, ..., i$, and that either $l = 0$ or $\rho(i_1, ..., i_n, 0) < \cdots < \rho(i_1, ..., i_n, i)$. Let

$$t_{il} = \inf\left\{t : t \geqslant \rho(i_1, ..., i_n, i),\ d(f(t), x(i_1, ..., i_l)) > \frac{1}{2^{l-1}}\right\}, \qquad (3.5)$$

$l = 1, ..., n+1$ (with $(i_1, ..., i_l) = (i_1, ..., i_n, i)$ if $l = n+1$). If $t_{i,n+1} < t_{il}$ for each $l = 1, 2, ..., n$, then $(i_1, ..., i_n, i+1) \in \mathscr{T}_{n+1}(f)$, and we set $\rho(i_1, ..., i_n, i+1) = t_{i,n+1}$: clearly $\rho(i_1, ..., i_n, i) < \rho(i_1, ..., i_n, i+1)$. Otherwise $(i_1, ..., i_n, i+1) \notin \mathscr{T}_{n+1}(f)$; indeed $(i_1, ..., i_n, j) \notin \mathscr{T}_{n+1}(f)$ for all $j > i$. Assume that this procedure has been carried out for all $\mathfrak{s} = (i_1, ..., i_n) \in \mathscr{T}_n(f)$, producing for each such $\mathfrak{s}$ either a finite sequence $(i_1, ..., i_n, 0), ..., (i_1, ..., i_n, i)$ or the infinite sequence $\{(i_1, ..., i_n, i)\}_{i=1}^{\infty}$ in $\mathscr{T}_{n+1}(f)$, with ρ defined and strictly increasing thereon. We next show that 3.1(e) holds for all $k \leqslant n+1$. Since we are assuming $P(n)$, this amounts to showing that it holds for $k = n+1$. Suppose $j \leqslant n+1$. We must show that if $(i_1, ..., i_n, i) \in \mathscr{T}_{n+1}(f)$, then

$$d(f(t), x(i_1, ..., i_j)) \leqslant \frac{1}{2^{j-1}} \qquad (3.6)$$

for each $t \in [\rho(i_1, ..., i_j),\ \rho(i_1, ..., i_n, i))$. We do this by induction on i. First, suppose that $j < n$. Then this holds for $i = 0$ by virtue of $P(n)$. Suppose it holds for $i = 0, ..., l$ and that $(i_1, ..., i_n, l+1) \in \mathscr{T}_{n+1}(f)$. Then (3.6) holds for each $t \in [\rho(i_1, ..., i_j), \rho(i_1, ..., i_n, l)$. If there is a $t \in [\rho(i_1, ..., i_n, l),\ \rho(i_1, ..., i_n, l+1))$ for which (3.6) does not hold, then $t_{l,n+1} < \rho(i_1, ..., i_n, l+1)$ by virtue of (3.5). This contradicts the definition of $\rho(i_1, ..., i_n, l+1)$. This takes care of the case $j < n$. Again proceed by induction on i. For $i = 0$, what we want to prove is vacuously true. Assume it holds for $i = 0, ..., l$, and that $(i_1, ..., i_n, l+1) \in \mathscr{T}_{n+1}(f)$. Then (3.6) holds for all $t \in [\rho(i_1, ..., i_n),\ \rho(i_1, ..., i_n, l))$ by hypothesis, and the existence of a t in $[\rho(i_1, ..., i_n, l),\ \rho(i_1, ..., i_n, l+1))$ for which 3.6 doesn't hold contradicts the definition of $\rho(i_1, ..., i_n, l+1)$. This takes care of the case $j = n$, and completes the proof of 3.1(e) for $k \leqslant n+1$.

We now complete the proof of $P(n+1)$. Suppose that $\mathfrak{s} = (i_1, ..., i_n) \in \mathscr{T}_n(f)$, and that $\mathfrak{s}$ is not the $\prec$-largest member of $\mathscr{T}_n(f)$. By $P(n)$, more precisely 3.1(f) with $k = n$, the $\prec$-smallest member of $\mathscr{T}_n(f)$ strictly $\prec$-greater than $\jmath$ is $\ell = (i_1, ..., i_{l-1}, i_l + 1)$ for some $l = 1, ..., n$. We claim that for each $\ell' \in \mathscr{T}_{n+1}(f)$ for which $\jmath \preccurlyeq \ell' \prec \ell$, we have $\rho(\jmath) < \rho(\ell') < \rho(\ell)$. We have already shown that $\rho(\jmath) < \rho(\ell^1)$ for such ℓ'. We have $\rho(\jmath) < \rho(t)$, for ρ is strictly monotone on $\mathscr{T}_n(f)$ by $P(n)$. Therefore $\rho(i_1, ..., i_n, 0) = \rho(i_1, ..., i_n) < \rho(\ell)$. Suppose $\rho(i_1, ..., i_n, i) < \rho(\ell)$, and that $(i_1, ..., i_n, i + 1) \in \mathscr{T}_{n+1}(f)$. Then $\rho(i_1, ..., i_n, i + 1) = t_{i,n+1} < t_{i,l}$. But

$$t_{i,l} = \inf \left\{ t : t \geqslant \rho(i_1, ..., i_n, i), d(f(t), x(i_1, ..., i_l) > \frac{1}{2^{l-1}} \right\}$$

$$= \inf \left\{ t : t \geqslant \rho(i_1, ..., i_l), d(f(t), x(i_1, ..., i_l)) > \frac{1}{2^{l-1}} \right\}$$

(proposition 3.2)

$$= \rho(i_1, ..., i_{l-1}, i_l + 1).$$

Thus $\rho(i_1, ..., i_n, i + 1) < \rho(\ell)$, establishing our claim by induction. Since $d(f(\rho(i_1, ..., i_n, i + 1)), f(\rho(i_1, ..., i_n, i)) \geqslant 1/2^{n-1}$ by the right continuity of f, the set $\{\ell' : \ell' \in \mathscr{T}_{n+1}(f), \jmath < \ell' < \ell\}$ is finite by virtue of f having a left limit everywhere. Suppose $(i_1, ..., i_n)$ is the $\prec$-largest member of $\mathscr{T}_n(f)$. We have already shown that $\rho(i_1, ..., i_n) < \rho(i_1, ..., i_n, 1) < \cdots$. Since the only members of $\mathscr{T}_{n+1}(f)$ which are $\prec$-larger than $(i_1, ..., i_n)$ (assuming there are any) are of the form $(i_1, ..., i_n, i)$, we have established the monotonicity property 3.1(b). We have established the structure property 3.1(h) for $k = n + 1$. Property (1) of 3.1, for $k = n + 1$, is an immediate consequence of our definition. So is property (c), and property (d) is obtained by iterating property (c). We now establish the order property 3.1(f). Suppose first that $(i_1, ..., i_k) \in \mathscr{T}_k(f)$, and that $\tau_1 = \cdots = \tau_k = \infty$. Suppose $(j_1, ..., k_k) \in \mathscr{T}_k(f)$, and that $(i_1, ..., i_k) \prec (j_1, ..., j_k)$. It is an easy consequence of the definition of $\prec$ together with the truncation and reduction-of-last component properties that $(i_1, ..., i_{m-1} + 1) \in \mathscr{T}_k(f)$ for some $m = 1, ..., k$, whence $\rho(i_1, ..., i_{m-1}, i_m + 1) < \infty$. But

$$\rho(i_1, ..., i_{m-1}, i_m + 1) = \inf \left\{ t : t \geqslant \rho(i_1, ..., i_m), d(f(t), x(i_1, ..., i_m)) > \frac{1}{2^{m-1}} \right\}$$

$$= \inf \left\{ t : t \geqslant \rho(i_1, ..., i_n), d(f(t), x(i_1, ..., i_m) > \frac{1}{2^{m-1}} \right\},$$

the last equality being a consequence of proposition 3.2. But this last expression is equal to τ_m, which we have assumed is equal to ∞. Contradiction. This shows that if $\tau_1 = \cdots = \tau_k = \infty$, then $(i_1, ..., i_k)$ is the $\prec$-last element of $\mathscr{T}_k(f)$. Suppose, then, that one of $\tau_1, ..., \tau_k$ is finite. Assume that $\tau_l = \tau_1 \wedge \cdots \wedge \tau_k$ and that $j < l \rightarrow \tau_j < \tau_l$. We must show that $(i_1, ..., i_{l-1}, i_l + 1) \in \mathscr{T}_l(f)$, that $\tau_l = \rho(i_1, ..., i_{l-1}, i_l + 1)$, and that no member of $\mathscr{T}_k(f)$ is $\prec$-between $(i_1, ..., i_k)$

and $(i_1, ..., i_{l-1}, i_l + 1)$. This is obvious from what we have already shown in $l = k$, so assume $l < k$. We have

$$\tau_l = \inf \left\{t : t \geq \rho(i_1, ..., i_l), d(f(t), x(i_1, ..., i_l)) > \frac{1}{2^{l-1}}\right\} \tag{3.7}$$

by virtue of proposition 3.2. Suppose there is a $j = 1, ..., l - 1$ with

$$\inf \left\{t : t \geq \rho(i_1, ..., i_l), d(f(t), x(i_1, ..., i_j) > \frac{1}{2^{j-1}}\right\} \leq \tau_l .$$

Using proposition 3.2 again, we obtain

$$\tau_j = \inf \left\{t : t \geq \rho(i_1, ..., i_k), d(f(t), x(i_1, ..., i_j)) > \frac{1}{2^{j-1}}\right\} \leq \tau_l ,$$

which contradicts $j < l \to \tau_l < \tau_j$. Therefore $(i_1, ..., i_{l-1}, i_l + 1) \in \mathscr{T}_l(f)$, with $\tau_l = \rho(i_1, ..., i_{l-1}, i_l + 1)$ by virtue of (3.7) and the definition of $\rho(i_1, ..., i_{l-1}, i_l + 1)$. Suppose there is a member ℓ of $\mathscr{T}_k(f)$ $\prec$-between $(i_1, ..., i_k)$ and $(i_1, ..., i_{l-1}, i_l + 1)$. Then $\ell = (i_1, ..., i_l, j_{l+1}, ..., j_k)$, with $j_{l+1} \geq i_{l+1}, ..., j_k \geq i_k$. The inequality must be strict in at least one of these last inequalities, or $\ell = (i_1, ..., i_k)$. Let m be the smallest of $l + 1, ..., k$ for which $j_m > i_m$. Then $(i_1, ..., i_{m-1}, j_m) \in \mathscr{T}(f)$ by truncation, so $(i_1, ..., i_{m-1}, i_m + 1) \in \mathscr{T}(f)$ by reduction of last component. Let

$$\tau_j' = \inf \left\{t : t \geq \rho(i_1, ..., i_m), d(f(t), x(i_1, ..., i_j) > \frac{1}{2^{j-1}}\right\},$$

$j = 1, ..., m$. Since $(i_1, ..., i_{m-1}, i_m + 1) \in \mathscr{T}(f)$, $\tau_m' < \tau_j'$, $j = 1, ..., m$. But proposition 3.2 implies that $\tau_j' = \tau_j$, $j = 1, ..., l$, so $\tau_m < \tau_j$, $j = 1, ..., l$. But $\tau_l = \tau_1 \wedge \cdots \wedge \tau_k$, and so $\tau_l \leq \tau_m$. Contradiction. No member of $\mathscr{T}_k(f)$ is between $(i_1, ..., i_k)$ and $(i_1, ..., i_{l-1}, i_l + 1)$. This completes the proof of 3.1(f). Thus $P(n + 1)$ holds. This completes the proof of the induction step, and therefore the proof of theorem 3.1.

3.3 COROLLARY. *Suppose that $\mathscr{s} \in \mathscr{T}_n(f)$. If $\mathscr{s}$ is the $\prec$-last member of $\mathscr{T}_n(f)$, then*

$$d(f(t), f(\rho(\mathscr{s}))) \leq 1/2^{n-1} \tag{3.8}$$

for all $t \geq \rho(\mathscr{s})$. If ℓ is the $\prec$-next member of $\mathscr{T}_n(f)$, then (3.8) *holds for all $t \in [\rho(\mathscr{s}), \rho(\ell))$.*

Proof. If $\mathscr{s}$ is the $\prec$-last member of $\mathscr{T}_n(f)$, then $\inf\{t: t \geq \rho(\mathscr{s}), d(f(t), f(\rho(\mathscr{s}))) > 1/2^{n-1}\} = \infty$ by 3.1(f) (with $k = n$ and $l = n$ in 3.2.) Thus (3.8) holds for all $t \geq \rho(\mathscr{s})$. If ℓ is the $\prec$-next member of $\mathscr{T}_n(f)$, then $\rho(\ell) \leq \inf\{t: t \geq \rho(\mathscr{s}), d(f(t), f(\rho(\mathscr{s}))) > 1/2^{n-1}\}$, again by 3.1(f).

3.4 COROLLARY. *Let* $t = (i_1, ..., i_m) \in \mathscr{T}(f)$. *For each* $j = 1, ..., m$, *let* S_j *be the closed sphere of radius* $1/2^{j-1}$ *and center* $x(i_1, ..., i_j)$. *Then* $x(t) \in S_j$, $j = 1, ..., m$, *in fact, there is a* $\delta_j > 0$ *such that* $f(t) \in S_j$ *for all* $t \in [x(t), x(t) + \delta_j)$.

Proof. The proof is by induction on m. The corollary is trivial for $m = 1$. Suppose it holds for $1, ..., m-1$. Then $x(i_1, ..., i_{m-1}) \in S_j$, $j = 1, ..., m-1$. We continue by induction on the value k of $m-1$. Since $x(i_1, ..., i_{m-1}, 0)$ is clearly an interior point of the sphere of radius $1/2^{m-1}$ centered at itself, $x(i_1, ..., i_{m-1}, 0) \in S_m$, and $x(i_1, ..., i_{m-1}, 0) = x(i_1, ..., i_m) \in S_j$, $j = 1, ..., m-1$ by the original induction hypothesis. Suppose $(i_1, ..., i_{m-1}) \in \mathscr{T}(f)$ and $x(i_1, ..., i_{m-1}, k) \in S_j$ for each $j = 1, ..., m$. If $(i_1, ..., i_{m-1}, k+1) \in \mathscr{T}(f)$, it follows from 3.1(f) that $\rho(i_1, ..., i_{m-1}, k+1)$ is strictly less than $\inf\{t: t \geqslant \rho(i_1, ..., i_m, k),\ d(f(t), x(i_1, ..., i_j)) > 1/2^{j-1}\}$ for each $j = 1, ..., m-1$. The conclusion thus follows for these values of j, and it follows for $j = m$ from the right continuity of f.

3.5 COROLLARY. *Suppose* $0 \leqslant s < t$, *and that* $f(s) \neq f(t)$. *Then there is an* $\jmath \in \mathscr{T}(f)$ *with* $s < \rho(\jmath) \leqslant t$.

Proof. Let $\Delta = d(f(s), f(t))$. Assume $\Delta > 0$. Choose n with $1/2^{n-2} < \Delta$. Let $\jmath \in \mathscr{T}_n(f)$ be such that $\rho(\jmath) \leqslant s$, and $\rho(\ell) > s$ if $\ell \in \mathscr{T}_n(f)$ with $\ell \succ \jmath$. Let $r = \rho(\jmath)$. By the triangle inequality $\Delta \leqslant d(f(s), f(r)) + d(f(t), f(r))$, so either $d(f(s), f(r)) > 1/2^{n-1}$ or $d(f(t), f(r)) > 1/2^{n-1}$. If $d(f(s), f(r)) > 1/2^{n-1}$, then, by the preceding corollary, $\jmath$ is not the $\prec$-last member of $\mathscr{T}_n(f)$, and, furthermore, if ℓ is the $\prec$-next member of $\mathscr{T}_n(f)$, then $\rho(t) \leqslant \rho(\jmath)$. This contradicts the definition of $\jmath$, so $d(f(t), f(r)) > 1/2^{n-1}$. We again invoke the preceding corollary, this time to conclude that there is a $\prec$-next member ℓ in $\mathscr{T}_n(f)$ and that $r < \rho(\ell) \leqslant t$. Since $\ell > \jmath$, $\rho(\ell) > s$, so $\rho(\ell) \in (s, t]$.

3.6 COROLLARY. *If* f *is discontinuous at* t, $t = \rho(\jmath)$ *for some* $\jmath \in \mathscr{T}(f)$.

Proof. Suppose $d(f(t), f(t-0)) = \delta > 0$. Choose n so that $\delta > 1/2^{n-2}$. Let $\jmath$ be the $\prec$-largest member of $\mathscr{T}_n(f)$ with $\rho(\jmath) \leqslant t$. If $\rho(\jmath) = t$, we are done. Otherwise, either $\jmath$ is the $\prec$-last member of $\mathscr{T}_n(f)$ or else the $\prec$-next member ℓ of $\mathscr{T}_n(f)$ satisfies $\rho(\ell) > t$: in either case it follows from corollary 3.3 that $d(f(s), f(\rho(\jmath))) \leqslant 1/2^{n-1}$ for $s \in [\rho(\jmath), t)$. Thus $d(f(\rho(\jmath), f(t-0)) \leqslant 1/2^{n-1}$. Therefore

$$d(f(t), f(\rho(\jmath))) \geqslant d(f(t), f(t-0)) - d(f(\rho(\jmath)), f(t-0)) \geqslant \delta - \frac{1}{2^{n-1}}.$$

But $\delta > 1/2^{n-2}$, so $d(f(t), f(\rho(\jmath))) > 1/2^{n-1}$. Contradiction: $\rho(\jmath)$ must equal t. This finishes the proof.

3.7 COROLLARY. *The set* $\{\rho(f, \jmath): \jmath \in \mathscr{T}_n(f)\}$ *has no finite limit points.*

Proof. We prove this by induction on n. $\mathcal{T}_1(f)$ is either the set of all non-negative integers, or else a set $\{0, 1, ..., k\}$. It is clear from the definition of $\rho(s)$ for $s \in \mathcal{T}_1(f)$ that $d(f(\rho(k)), d(f(\rho(k+1)) \geqslant 1$, so $\{\rho(k)\}$ must be unbounded or finite (since f has left limits everywhere). Suppose that the corollary is true for $\mathcal{T}_{n-1}(f)$. From this assumption together with 3.1(g) we see that $\prec$-between no two successive members of $\mathcal{T}_{n-1}(f)$ is there an infinite number of members of $\mathcal{T}_n(f)$. Therefore if $\mathcal{T}_n(f)$ is to have a finite limit point, $\mathcal{T}_{n-1}(f)$ must have a $\prec$-last member $(i_1, ..., i_{n-1})$. It then follows from 3.1(g) that the members of $\mathcal{T}_n(f)$ $\prec$-following $(i_1, ..., i_{n-1})$ consist either of the entire sequence $(i_1, ..., i_{n-1}, 1), (i_1, ..., i_{n-1}, 2), ...$ or an initial segment of it. But if s and t are consecutive members of this sequence, $d(f(\rho(s)), f(\rho(t) \geqslant 1/2^{n-1}$ by virtue of 3.1(f) [use (3.2) with $k = l = n$] and the right continuity of f, so even if $(i_1, ..., i_n, i) \in \mathcal{T}_n(f)$ for all i, the fact that f has left limits everywhere ensures that the corresponding ρ-values do not converge to a finite-limit. Thus the statement of the corollary holds for n, and the proof by induction is complete.

3.8 COROLLARY. *Let $t = (i_1, ..., i_m) \in \mathcal{T}(f)$. For each $n > m$, let s_n be the n-tuple $(i_1, ..., i_m, 0, ..., 0, 1)$. Suppose $\rho(t)$ is not the left hand endpoint of an interval of constancy for f. Then*

(a) *there is a sequence $n_k \uparrow \infty$ with $s_{n_k} \in \mathcal{T}(f)$,*

(b) *if $s_{n_k} \in \mathcal{T}(f)$ for $n_k \to \infty$, then $\rho(s_{n_k}) \to \rho(t)$.*

Proof. To prove (a), we need only establish that there is an $n > m$ for which the n-tuple $(i_1, ..., i_m, 0, ..., 0, 1) \in \mathcal{T}(f)$, apply this result to the n-tuple $(i_1, ..., i_m, 0, ..., 0)$ in place of $(i_1, ..., i_m)$, and repeat this process again and again. By virtue of corollary 3.3, there is a $\delta > 0$ such that $d(f(t), x(i_1, ..., i_j)) \leqslant 1/2^{j-1}$ for all $t \in [\rho(t), \rho(t) + \delta]$. Let $\Delta = \sup\{d(f(t), x(t)) : t \in [\rho(t), \rho(t) + \delta/2]\}$. Since $\rho(t)$ is not the left-hand endpoint of an interval of constancy, $\Delta > 0$. Suppose k satisfies $1/2^{k-1} < \Delta$ and $k > m$. Then $\inf\{t: t \geqslant \rho(t), d(f(t), x(t)) > 1/2^{n-1}\} \geqslant \rho(t) + \delta/2$, while $\inf\{t: t \geqslant \rho(t), d(f(t), x(i_1, ..., i_j)) > 1/2^{j-1}\} \geqslant \rho(t) + \delta$, $j = 1, ..., m$. It follows from 3.1(f) that one of $(i_1, ..., i_m, 1), ..., (i_1, ..., i_m, 0, ..., 0, 1)$ (the last one listed being an l-tuple) is the $\prec$-next member of $\mathcal{T}_l(f)$ following the l-tuple $(i_1, ..., i_m, 0, ..., 0)$. This $\prec$-next member is the sought after n-tuple.

To prove (b), since $\rho(s_{n_k})$ is non-increasing, it converges to a limit as $k \to \infty$. This limit is clearly no smaller than $\rho(t)$. Suppose it is equal to $\rho(t) + \delta$, where $\delta > 0$. We apply corollary 3.3 to obtain $d(f(t), x(t)) \leqslant 1/2^{n_k-1}$ for each $t \in [\rho(t), \rho(t) + (\delta/2)]$ and $k = 1, 2, ...$. This is possible only if $f(t) = x(t)$ for $t \in [\rho(t), \rho(t) + \delta/2]$, violating the assumption that $\rho(t)$ is not the left-hand endpoint of an interval of constancy. Therefore $\rho(s_{n_k}) \to \rho(t)$. This completes the proof of (b).

So far, we have, for a fixed $f \in D$, defined $\mathcal{T}(f)$ and $\rho(f, s)$ for $s \in \mathcal{T}$, investigated the structure of $\mathcal{T}(f)$, and listed some of the properties of $\rho(f, s)$ as a function of $\jmath \in \mathscr{T}$. We cannot, however, use these notions in a theory of stochastic processes (with sample paths in D) without certain facts about the measurability of $\rho(f, \jmath)$ as a function of $f \in D$. We have not even established that $\{f: \jmath \in \mathscr{T}(f)\}$ belongs to $\mathscr{D}$, much less what we require, namely that $\{f: \jmath \in \mathscr{T}(f)$ and $\rho(f, \jmath) \leqslant s\} \in \mathscr{D}_{s+}$. To verify this fact requires some preliminary definitions, notation, and results which we now present.

Let $\jmath \in \mathscr{T}$. Suppose $\jmath = (i_1, ..., i_n)$. For each $j = 1, ..., n$, let $\jmath_j$ be the n-tuple $(i_1, ..., i_j, 0, ..., 0)$. We call $\jmath_1, ..., \jmath_{n-1}$ the *natural predecessors of* $\jmath$. Fix $f \in \mathscr{T}(\jmath)$. Let $x_j = x(\jmath_j) = f(\rho(\jmath_j))$, and $R_j(\jmath, f) = \{y: d(x_j, y) > 1/2^{j-1}\}$, $j = 1, ..., n$. Let $R(\jmath, f) = \bigcup_{j=1}^{n} R_j(\jmath, f)$. Define

$$\begin{aligned} \sigma(\jmath, f) &= \inf\{t: t \geqslant \rho(f, \jmath), f(t) \in R(\jmath, f)\}, & \jmath \in \mathscr{T}(f) \\ &= \infty & \jmath \notin \mathscr{T}(f) \end{aligned}$$

Denoting $R_j(\jmath, f)$ by R_j, let $Q_1(\jmath, f) = R_1$, and $Q_j(\jmath, f) = R^c \cap \cdots \cap R^c_{j-1} \cap R_j$, $j = 2, ..., n$. Consider the n-tuples $\ell_1 = (i_1 + 1, 0, ..., 0)$, $\ell_2 = (i_1, i_2 + 1, 0, ..., 0), ..., \ell_n = (i_1, i_2, ..., i_{n-1}, i_n + 1)$. We call $\ell_1, ..., \ell_n$ the *natural successors of* $\jmath$.

3.9 Proposition. *Suppose $\jmath \in \mathscr{T}_n(f)$. Then $\jmath$ is the $\prec$-last member of $\mathscr{T}_n(f)$ if and only if $\sigma(\jmath, f) = \infty$.* Suppose $\sigma(\jmath, f) < \infty$. *Then, for each $j = 1, ..., n$, the $\prec$-next member of $\mathscr{T}_n(f)$ is ℓ_j if and only if $f(\sigma(\jmath, f) \in Q_j(\jmath, f)$.*

Proof. The proposition is an immediate consequence of the definitions of σ, Q_j and the order property 3.1(f).

For each $m = 1, 2, ...$, there is a partition $\mathscr{E}^{(m)} = \{E_{mj} : j = 1, 2, ...\}$ of S into Σ-measurable sets of diameter no greater than $1/m$. We shall assume, without loss of generality, that $\mathscr{E}^{(m+1)}$ is a refinement of $\mathscr{E}^{(m)}$. (In what follows we shall use n-tuples $(j_1, ..., j_n)$ of positive integers as indices, with the lth component j_l referring to a partition element E_{m,j_l}. We apologize for and warn against the possible confusion of the indices $(j_1, ..., j_n)$ with the fixed member $\jmath = (i_1, ..., i_n)$ of $\mathscr{T}_n$). For each n-tuple $(j_1, ..., j_n)$ of positive integers, and each $l = 1, ..., n$ let $R_m(j_1, ..., j_n, l) = \{y: d(y, E_{m,j_l}) > 1/2^{l-1}\}$, $l = 1, 2, ..., n$ and $R_m(j_1, ..., j_n) = \bigcup_{l=1}^{n} R(j_1, ..., j_n, l)$. Set

$$\begin{aligned} \gamma^{(m)}(j_1, ..., j_n, f) &= \inf\{t : t \geqslant \rho(f, \jmath), f(t) \in R(j_1, ..., j_n)\}, & \jmath \in \mathscr{T}(f) \\ &= \infty & \jmath \notin \mathscr{T}(f) \end{aligned} \tag{3.9}$$

Let $C_m(j_1, ..., j_n) = \bigcap_{l=1}^{n} \{f: \jmath \in \mathscr{T}(f), f(\rho(\jmath_l)) \in E_{m,j_l}\}$. The definition of $C_m(j_1, ..., j_n)$ requires the observation that if $\jmath \in \mathscr{T}(f)$, each of the natural predecessors of $\jmath$ is also in $\mathscr{T}(f)$. This follows from 3.1(a) and the truncation property 3.1(d). Let $\sigma_m(f) = \gamma^{(m)}(j_1, ..., j_n, f)$ for $f \in C_m(j_1, ..., j_n)$

3.10 Proposition. *If $\jmath \in \mathscr{T}(f)$, then $\sigma_m(f) \searrow \sigma(f)$ as $m \to \infty$.*

Proof. Fix f with $\mathfrak{s} \in \mathscr{T}(f)$. For each $m = 1, 2,\ldots$, let $\alpha(m)$ be that $(j_1, \ldots, j_n)$ for which $f \in C(j_1, \ldots, j_n)$, that is, for which $f(\rho(\mathfrak{s}_l)) \in E_{m,j_l}$, $l = 1,\ldots, n$. Let $R_m(l)$, R_m denote $R_m(\alpha(m), l)$ and $R_m(\alpha(m))$ respectively. It is clear that $\sigma_m(f) \geqslant \sigma(f)$. Since $R_m(l) \uparrow$ in m for each $l = 1,\ldots, j$, it is also clear that $\sigma_m(f) \searrow$ in m. Assume that $\sigma(f) < \infty$. Let $t^* = \lim \sigma_m(f)$. Suppose $t^* > \sigma = \sigma_m(f)$. Then there is a $t' \in [\sigma, t^*)$ with $f(t') \in R(\mathfrak{s}, f)$. Suppose $f(t') \in R_l(\mathfrak{s}, f)$. Then $d(f(t'), x_l) > 1/2^{l-1} + \delta$, where $\delta > 0$. Let j_l be the lth component of $\alpha(m)$, and $E = E_{m,j_l}$. Since $\rho(f, \mathfrak{s}) \leqslant t' < t^* \leqslant \sigma_m(f)$, $d(f(t'), E) \leqslant 1/2^{l-1}$. Suppose $x \in E$. Then $d(x, x_l) \leqslant 1/m$, so

$$\begin{aligned} d(f(t'), x) &\geqslant d(f(t'), x_l) - d(x_l, x) \\ &> \left(\frac{1}{2^{l-1}} + \delta\right) - \frac{1}{m}. \end{aligned}$$

Since this is true for all $x \in E$, $d(f(t'), E) \geqslant 1/2^{l-1} + \delta - (1/m)$. But $d(f(t'), E) \leqslant 1/2^{l-1}$, and δ does not depend on m, so by choosing $m > 1/\delta$, we arrive at a contradiction. Therefore $t^* = \sigma_m(f)$, and (b) is proved.

3.11 Lemma. *For each* $\mathfrak{s} \in \mathscr{T}$ *and* $s \geqslant 0$,

$$\{f\colon \mathfrak{s} \in \mathscr{T}(f) \text{ and } \rho(f, \mathfrak{s}) \leqslant s\} \in \mathscr{D}_{s+}.$$

Proof. $\mathscr{T} = \bigcup_{n=1}^{\infty} \mathscr{T}_n$, and the proof is by induction on n. We consider first the case $n = 1$. $\mathscr{T}_1$ is the sequence of non-negative integers, and our proof is by induction. If $\mathfrak{s} = 0$, $\{f\colon \mathfrak{s} \in \mathscr{T}(f)$ and $\rho(f, \mathfrak{s}) \leqslant s\} = D \in \mathscr{D}_{s+}$. The induction step, in which we assume the validity of the statement of the lemma for $\mathfrak{s} = k$ and show that it is valid for $\mathfrak{s} = k + 1$, is no easier for $n = 1$ than it is for general n. Assume, then, that the statement of the lemma holds for each $\mathfrak{s} \in \mathscr{T}_{n-1}$. Suppose that $\mathfrak{t} = (i_1, \ldots, i_n) \in \mathscr{T}_n$. By 3.1(a) and the induction hypothesis, the lemma holds for $(i_1, \ldots, i_{n-1}, 0)$. We now show that, for any value of $k = 0, 1,\ldots$, it holds for $(i_1, \ldots, i_{n-1}, k)$. Then, in particular, it holds for $\mathfrak{t}$. The proof is by induction on k. The value $k = 0$ is already taken care of, so assume that the lemma holds for $\mathfrak{s} = (i_1, \ldots, i_{n-1}, k)$. We want to show that it holds if $\mathfrak{s}$ is taken to be $(i_1, \ldots, i_{n-1}, k + 1)$. Extend $\rho(\cdot, \mathfrak{s})$ to $f \in D$ for which $\mathfrak{s} \notin \mathscr{T}(f)$ by setting $\rho(f, \mathfrak{s}) = \infty$ for such f. Then $\{f\colon \rho(f, \mathfrak{s}) \leqslant s\} = \{f\colon \mathfrak{s} \in \mathscr{T}(f) \wedge \rho(f, \mathfrak{s}) \leqslant s\} \in \mathscr{D}_{s+}$, so $\rho(\cdot, \mathfrak{s})$ as so extended is a stopping time relative to $\{\mathscr{D}_{t+}\}$. Now fix m, and consider the partition $\mathscr{E}^{(m)}$ of S considered earlier. Set $i_n = k$: we now use notation defined earlier in terms of a fixed $(i_1, \ldots, i_n) \in \mathscr{T}$. For each n-tuple $(j_1, \ldots, j_n)$ of positive integers, the function $\gamma^{(m)}(j_1, \ldots, j_n, \cdot)$ on D is the first $\rho(\cdot, \mathfrak{s})$-hitting time of the open set $R(j_1, \ldots, j_n)$. Therefore $\gamma^{(m)}(j_1, \ldots, j_n, \cdot)$ is a stopping time relative to $\{\mathscr{D}_{t+}\}$ by virtue of proposition 2.2. We claim that, for each $l = 1,\ldots, n$ $A_l = \{f\colon \mathfrak{s} \in \mathscr{T}(f) \wedge f(\rho(\mathfrak{s}_l) \in E_{m,j_l}\} \in \mathscr{D}_{\rho(\mathfrak{s})}$ (here $\rho(\mathfrak{s})$ is not an abbreviation for the number $\rho(f, \mathfrak{s})$ but stands for the function $\rho(\cdot, \mathfrak{s})$). To prove this, we argue as follows. First $\{f\colon \mathfrak{s} \in \mathscr{T}(f)\} = \bigcup_{i=1}^{\infty} \{f\colon \mathfrak{s} \in \mathscr{T}(f) \wedge \rho(f, \mathfrak{s}) \leqslant i\}$, and since the statement of the lemma holds for $\mathfrak{j}$, $\{f\colon \mathfrak{j} \in \mathscr{T}(f)\} \in \mathscr{D}$. Second, for each $l = 1,\ldots, n - 1$, the statement of the lemma holds by virtue of the induction hypothesis and 3.1(a) if $\mathfrak{j}$ is replaced by $\mathfrak{j}_l$. Consequently the extension of $\rho(\cdot, \mathfrak{j}_l)$ to all of D obtained by setting $\rho(\cdot, \mathfrak{j}_l) = \infty$ if $\mathfrak{j}_l \notin \mathscr{T}(f)$ is a $\{\mathscr{D}_{t+}\}$-stopping time. Therefore $\{f\colon f(\rho(\mathfrak{j}_l)) \in E_{m,j_l}\} \in \mathscr{D}_{\rho(\mathfrak{j}_l)} \subset \mathscr{D}$. We use the statement of the lemma again to obtain this last for $l = n$ (recall $\mathfrak{j}_n = \mathfrak{j}$). Since $A_l = \{f\colon \mathfrak{j} \in \mathscr{T}(f)\} \cap \{f\colon f(\rho(\mathfrak{j}_l)) \in E_{m,j_l}\}$, it follows that $A_l \in \mathscr{D}$, $l = 1,\ldots, n$. To show that $A_l \in \mathscr{D}_{\rho(\mathfrak{j})}$ requires that we show that $A_l \cap \{f\colon \rho(f, \mathfrak{j})\} \in \mathscr{D}_{s+}$ for each s. We apply proposition 2.1. Suppose that $f \in A_l \cap \{f\colon \rho(f, \mathfrak{j}) \leqslant s\}$, and let $g \in D$, $g = f$ on $[0, s + \delta]$ for some $\delta > 0$. Since $\{\rho(\cdot, \mathfrak{j}) \leqslant s\} \in \mathscr{D}_{s+}$, $\rho(g, \mathfrak{j}) \leqslant s$, so $\mathfrak{j} \in \mathscr{T}(g)$. Since $\mathfrak{j}_l \in \mathscr{T}(g)$ if $\mathfrak{j} \in \mathscr{T}(g)$, we have from 3.1(b) $\rho(g, \mathfrak{j}_0) < \rho(g, \mathfrak{j}) \leqslant s$, and since $\{\rho(\cdot, \mathfrak{j}_0) \leqslant s\} \in \mathscr{D}_{s+}$, $\rho(g, \mathfrak{j}_0) = \rho(f, \mathfrak{j}_0)$, so $g((\rho(\mathfrak{j}_0)) = f(\rho(f, \mathfrak{j}_0)) \in E_{m,j_l}$. Therefore $g \in A_l \cap \{f\colon \rho(f, \mathfrak{j}) \leqslant s\}$. It follows from proposition 2.1 that $A_l \cap \{f\colon \rho(f, \mathfrak{j}) \leqslant s\} \in \mathscr{D}_{s+}$. Since s is arbitrary, $A_l \in \mathscr{D}_{\rho(\mathfrak{j})}$. It follows that $C_m(j_1, \ldots, j_n) = \bigcap_{l=1}^{n} A_l$ belongs to $\mathscr{D}_{\rho(\mathfrak{j})}$. Recall that $\sigma^{(m)}$ is defined to be equal to $\gamma^{(m)}(j_1, \ldots, j_n, \cdot)$ on $C(j_1, \ldots, j_n)$. It follows easily that $\{f\colon \sigma^{(m)}(f) \leqslant s\} \in \mathscr{D}_{s+}$ for each s. Therefore $\sigma^{(m)}$ is a $\{\mathscr{D}_{t+}\}$ stopping time. This implies that σ is a $\{\mathscr{D}_{t+}\}$ stopping time. By 3.1(c) $\{f\colon \ell \in \mathscr{T}(f) \wedge \rho(f, \ell) \leqslant s\} = \{f\colon \mathfrak{j} \in \mathscr{T}(f) \wedge \ell \in \mathscr{T}(f) \wedge \rho(f, \ell) \leqslant s\}$. By proposition 3.9, this last set is equal to $\{f\colon \sigma(f) \leqslant s \wedge f(\sigma(f)) \in Q_n(\mathfrak{j}, f) = \{f\colon \sigma(f) \leqslant s\} \cap \bigcap_{l=1}^{n-1} B(l)$, where $B(l) = \{f\colon d(f(\sigma), f(\rho(\mathfrak{j}_l)) \leqslant 1/2^{l-1}\}$. Since both $f(\rho(\cdot, \mathfrak{j}_l))$ and $f(\sigma)$ are $\mathscr{D}_\sigma$-measurable, the same is true of $B(l)$, hence of $\bigcap_{l=1}^{n-1} B(l)$. But then $\bigcap_{l=1}^{n-1} B(l) \cap \{f\colon \sigma(f) \leqslant s\} \in \mathscr{D}_{s+}$ by definition of $\mathscr{D}_\sigma$. Therefore the statement of the lemma holds if $\mathfrak{j} = (i_1, \ldots, i_{n-1}, k)$ is replaced by $(i_1, \ldots, i_{n-1}, k + 1)$. It follows by induction that it holds for all $\mathfrak{j} \in \mathscr{T}_n$. It now follows by induction that it holds for all $\mathfrak{j} \in \mathscr{T} = \bigcup_{n=1}^{\infty} \mathscr{T}_n$. This completes the proof of the lemma.

3.12 Definition. For each $\mathfrak{j} \in \mathscr{T}$ we denote by $\mathscr{D}_\mathfrak{j}$ the σ-field generated by sets of the form $A = \{f\colon \ell \in \mathscr{T}(f) \wedge \rho(f, \ell) \leqslant t \wedge f(u \wedge \rho(\ell)) \in E \wedge f(\rho(\ell)) \in F\}$ as t and u range over $[0, \infty)$, E on F over Z, and ℓ over those members of $\mathscr{T}$ for which $\ell \prec \mathfrak{j}$.

It is clear that if $\ell \prec \mathfrak{j}$ then $\mathscr{D}_\ell \subset \mathscr{D}_\mathfrak{j}$.

3.13 Proposition. $\mathscr{D}_\mathfrak{j} \subset \mathscr{D}_{\rho(\mathfrak{j})}$

Proof. We must show that if $A \in \mathscr{D}_\mathfrak{j}$, then $A \cap \{f\colon \rho(f, \mathfrak{j}) \leqslant s\} \in \mathscr{D}_{s+}$ for each $s \geqslant 0$. It is enough to show this for the class of sets A specified in the definition of $\mathscr{D}_\mathfrak{j}$. Let A be such a set. Then $A \in \mathscr{D}$ by virtue of 3.11 and the general theory of stopping times, so $A \cap \{f\colon \rho(f, \mathfrak{j}) \leqslant s\} \in \mathscr{D}$. To show that this last set belongs to $\mathscr{D}_{s+}$ it suffices by virtue of proposition 2.1 to show that if f belongs to it, and if $g \in D$ with $g = f$ on $[0, s + \delta]$ for some $\delta > 0$, then g belongs to it. But this is an almost immediate consequence of the monotonicity property 3.1(b).

The reader may be wondering why we have introduced the fields $\mathscr{D}_s$ when we have available the fields $\mathscr{D}_{\rho(s)}$. First, we note that it is not true that if $t < s$, then $\mathscr{D}_{\rho(t)} \subset \mathscr{D}_{\rho(s)}$. Second, and more important, it is easy to see that $\mathscr{D}_s$ is countably generated, so the disintegration of measures theorem applies when $\mathscr{D}_s$ is used as a conditioning sub-σ-field of $\mathscr{D}$. We do not know whether or not $\mathscr{D}_{\rho(s)}$ is countably generated.

3.14 Definition. Let $s \in \mathscr{T}$. $\mathscr{O}_s$ is the sub-σ-field of $\mathscr{D}$ generated by sets of the form $\{f\colon t \in \mathscr{T}(f) \wedge f(\rho(t)) \in E\}$ as t ranges over all $t \in \mathscr{T}$ with $t \prec s$, and E over Σ. $\mathscr{P}_s$ is the sub-σ-field of $\mathscr{D}$ generated by sets of the same form, but with t ranging instead over all $t \in \mathscr{T}$ with $t \succ s$. $\mathscr{O}$ is the σ-field generated by sets of the same form, but with t ranging over all members of $\mathscr{T}$: thus $\mathscr{O} = \mathscr{O}_s \wedge \mathscr{P}_s$.

It is clear that $\mathscr{O}_s$ and $\mathscr{P}_s$, hence of course $\mathscr{O}$, are countably generated.

3.15 Theorem. *If X and $\tilde{X}$ are stochastic processes with the same state-dependent hitting probabilities, then the restrictions of π and $\tilde{\pi}$ to $\mathscr{O}$ are identical.*

Proof. Let $\mathscr{O}_0$ be the collection of all sets of the form

$$\{f\colon s_i \in \mathscr{T}(f) \wedge \rho(f, s_i) \in E_i\,,\ i = 1,\dots, l,\ t_j \notin \mathscr{T}(f), j = 1,\dots, m\}, \tag{3.10}$$

where m and n are arbitrary non-negative integers, $s_1, \dots, s_l$, $t_1, \dots, t_m$ arbitrary members of $\mathscr{T}$ and $E_1, \dots, E_l$ arbitrary members of Σ. $\mathscr{O}_0$ is closed under finite intersections. Note that $\{f\colon s \in \mathscr{T}(f) \wedge \rho(f, s) \in E\}^c = \{f\colon s \notin \mathscr{T}(f)\} \cap \{f\colon s \in \mathscr{T}(f) \wedge f(\rho(s)) \in E^c\}$, and $\{f\colon s \notin \mathscr{T}(f)\}^c = \{f\colon s \in \mathscr{T}(f) \wedge f(\rho(s)) \in S\}$. It follows that the complement of a member of $\mathscr{O}_0$ is a finite disjoint union of members of $\mathscr{O}_0$, and that any finite union of members of $\mathscr{O}_0$ can be expressed as a finite disjoint union of such members. We conclude that the class of finite disjoint unions of members of $\mathscr{O}_0$ is a field which generates $\mathscr{O}$. Therefore to prove the theorem it suffices to show that the restrictions of π and $\tilde{\pi}$ to $\mathscr{O}_0$ are identical.

We call a sequence $C = (u_1, \dots, u_m)$ of members of $\mathscr{T}_m$ an *n-chain* if

1. $u_1 = (0,\dots, 0)$,
2. $u_1 < u_2 < \cdots < u_m$
3. if $s \in C$ and $s \neq u_1$, then each of the natural predecessors of s belongs to C,
4. for each $i = 1,\dots, m - 1$, u_{i+1} is one of the natural successors of u_i,
5. if $(i_1, \dots, i_n) \in C$, and $i_n > 0$, then $(i_1, \dots, i_{n+1}, i_n - 1) \in C$.

Note that if $(u_1, \dots, u_n)$ is an n-chain, so are $u_1(u_1, u_2), \dots, (u_1, \dots, u_{m-1})$. It is clear from Theorem 3.1 that if we list, in $\prec$-order, the first m members of $\mathscr{T}_n(f)$, the result is an n-chain. An immediate consequence of this is that the members $s_1, \dots, s_m$ of $\mathscr{T}$ of (3.10) can be embedded in at least one n-chain where we identify

$(i_1, \ldots, i_k)$ with the n-tuple $(i_1, \ldots, 0_k, 0, \ldots, 0)$ if $k < n$ and n is the least integer for which each of $\mathscr{s}_1, \ldots, \mathscr{s}_l, \mathscr{t}_1, \ldots, \mathscr{t}_m$ is a k-tuple with $k \leqslant n$). We say that an n-chain $C = (\mathscr{u}_1, \ldots, \mathscr{u}_k)$ is ***maximally*** determining for $(\mathscr{s}_1, \ldots, \mathscr{s}_l)$ and $(\mathscr{t}_1, \ldots, \mathscr{t}_m)$ if (i) each of $\mathscr{s}_1, \ldots, \mathscr{s}_l$ belongs to C, (ii) none of $\mathscr{t}_1, \ldots, \mathscr{t}_m$ belongs to C, (iii) there is no n-chain with $\mathscr{u}_1, \ldots, \mathscr{u}_k$ as its first k elements containing any of $\mathscr{t}_1, \ldots, \mathscr{t}_m$, and (iv) if $j < k$ then at least one of (i), (ii), and (iii) does not hold if $(\mathscr{u}_1, \ldots, \mathscr{u}_k)$ is replaced by $(\mathscr{u}_1, \ldots, \mathscr{u}_j)$. It is not hard to see that the set (3.10) is a countable disjoint union of sets of the form

$$\{f: \mathscr{u}_i \in A_n(f) \wedge f(\rho(\mathscr{u}_i)) \in F_i, \; i = 1, \ldots, k\}, \tag{3.11}$$

where $(\mathscr{u}_1, \ldots, \mathscr{u}_m)$ ranges over the n-chains which are maximally determining for $\mathscr{s}_1, \ldots, \mathscr{s}_l$ and $\mathscr{t}_1, \ldots, \mathscr{t}_m$, and where $F_i = S$ unless $\mathscr{u}_i = \mathscr{s}_j$ for some $j = 1, \ldots, m$, in which case $F_i = E_j$. From this it follows that we need only prove that sets of the form (3.11) are assigned the same measure by both π and $\tilde{\pi}$. We now proceed to do this. The proof will be by induction on the length m of the n-chain $(\mathscr{u}_1, \ldots, \mathscr{u}_m)$.

For $m = 1$, (3.11) reduces to $\{f: f(0) \in F\}$, and $\pi(\{f: f(0) \in F\}) = \tilde{\pi}(\{f: f(0) \in F\})$ by virtue of the assumption that X_0 and $\tilde{X}_0$ have the same distribution. Assume, then, that π and $\tilde{\pi}$ assign the same measure to all sets of the form (3.11) for any n-chain $(\mathscr{u}_1, \ldots, \mathscr{u}_m)$ and any choice of $F_1, \ldots, F_m$ in Σ, and suppose that $(\mathscr{u}_1, \ldots, \mathscr{u}_{m+1})$ is an n-chain and that F_{m+1} is in Σ. Let $\mathscr{s} = \mathscr{u}_m$. Since $\{f: \mathscr{s} \in \mathscr{T}(f)$ and $\rho(f, \mathscr{s}) \leqslant s\} \in \mathscr{D}_{s+}$ for each s, $\rho(\mathscr{s})$, extended to all of D by defining it to be $+\infty$ on $\{f: \mathscr{s} \in \mathscr{T}(f)\}$, is a stopping time relative to $\{\mathscr{D}_{t+}\}$. We shall use the symbolism introduced just before the statement of lemma 3.11, using "k" instead of the "m" referred to here. For each n-tuple $(j_1, \ldots, j_n)$ of positive integers, the function $\gamma^{(k)}(j_1, \ldots, j_n, \cdot)$ on D is the post-$\rho(\mathscr{s})$ hitting time of the open set $R(j_1, \ldots, j_n)$. Also, as we showed in the proof of lemma 3.11, $C_k(j_1, \ldots, j_n) \in \mathscr{D}_{\rho(\mathscr{s})}$. Letting $\alpha = (j_1, \ldots, j_n)$, it follows from definition 2.6 that for each $A \in \Sigma$ there is a $(\Sigma - \mathscr{B})$ measurable function g_A on S into R for which

$$\begin{aligned} \pi(f(\gamma^{(k)}(\alpha) \in A \mid \mathscr{D}_{\rho(\mathscr{s})}) &= g_A(f(\rho(\mathscr{s})) \qquad \pi\text{-a.e.} \\ \tilde{\pi}(f(\gamma^{(k)}(\alpha) \in A \mid \mathscr{D}_{\rho(\mathscr{s})}) &= g_A(f(\rho(\mathscr{s})) \qquad \tilde{\pi}\text{-a.e.} \end{aligned} \tag{3.12}$$

It follows from (3.12) that

$$\begin{aligned} \pi(B \cap \{f: f(\gamma^{(k)}(\alpha)) \in A\}) &= \int_B g_A(f(\rho(\mathscr{s}))\, \pi(df)) \\ \tilde{\pi}(B \cap \{f: f(\gamma^{(k)}(\alpha)) \in A\}) &= \int_B g_A(f(\rho(\mathscr{s}))\, \tilde{\pi}(df)) \end{aligned} \tag{3.13}$$

for each $B \in \mathscr{D}_{\rho(\mathscr{s})}$. Let $\mathscr{C}$ be the σ-field generated by all sets of the form $\{f: \mathscr{u}_i \in \mathscr{T}(f), f(\rho(\mathscr{u}_i)) \in F_i, \; i = 1, \ldots, m\}$, where $F_1, \ldots, F_m$ range over members of Σ. Clearly $\mathscr{C} \subset \mathscr{D}_{\rho(\sigma)}$. Furthermore, by the induction hypothesis, $\pi(C) = \tilde{\pi}(C)$ for all $C \in \mathscr{C}$. Since the map $f \to g_A(f(\rho(\sigma)))$ is $\mathscr{C}$-measurable, the integrals on the right hand side of each equation of (3.13) are equal. Hence

$$\pi(C \cap \{f : f(\gamma^{(k)}(\alpha)) \in A\}) = \tilde{\pi}(C\{f : f(\gamma^{(k)}(\alpha) \in A\}) \tag{3.14}$$

for each $A \in \Sigma$ and $C \in \mathscr{C}$. Clearly, for each n-tuple α of positive integers, $C_k(\alpha) \in \mathscr{C}$. If we substitute $C \cap C_k(\alpha)$ for C in each side of (3.14) and sum the result over all such α, we obtain, for each $C \in \mathscr{C}$ and $A \in \Sigma$,

$$\pi(C \cap \{f : f(\sigma^{(k)}) \in A\}) = \tilde{\pi}(C \cap \{f : f(\sigma^{(k)}(\alpha) \in A\}). \tag{3.15}$$

By the usual sort of argument we go from (3.15) to

$$\int_C (g \circ f)(\sigma^{(k)})\, \pi(df) = \int_C (g \circ f)(\sigma^{(k)})\, \tilde{\pi}(df), \tag{3.16}$$

for each non-negative Σ-measurable g on S and $C \in \mathscr{C}$. Take g to be continuous with compact support on S. If we let $k \to \infty$ in (3.16) and apply proposition 3.10 and the continuity of $g \circ f$, we obtain

$$\int_C (g \circ f)(\sigma)\, \pi(df) = \int_C (g \circ f)(\sigma)\, \tilde{\pi}(df), \qquad C \in \mathscr{C}. \tag{3.17}$$

Using the usual sort of argument, one obtains (3.17) for non-negative Σ-measurable g on S. (3.17) also implies that, if Z is any non-negative $\mathscr{C}$-measurable function on D, then

$$\int_C Z(f) \cdot (g \circ f)(\sigma)\, \pi(df) = \int_C Z(f) \cdot (g \circ f)(\sigma)\, \tilde{\pi}(df), \qquad C \in \mathscr{C}. \tag{3.18}$$

Let h be any non-negative function on the $(n + 1)$-fold product of D with itself, measurable with respect to the corresponding product σ-field Σ^{n+1}. Since each of the natural predecessors $\sigma_1, \ldots, \sigma_{n-1}$ of σ is in the n-chain $u_1, \ldots, u_m$ by virute of property (3) of n-chains, each of $f(\rho(\sigma_1)), \ldots, f(\rho(\sigma_n))$ is $\mathscr{C}$-measurable, where $\sigma_n = \sigma$. From the fact that (3.18) holds for all Z and g as specified above we obtain

$$\int_C h(f(\rho(\sigma_1)), \ldots, f(\rho(\sigma_n)), f(\sigma))\, \pi(df)$$
$$= \int h(f(\rho(\sigma_1)), \ldots, f(\rho(\sigma_n)), f(\sigma))\, \tilde{\pi}(df) \tag{3.19}$$

for each $C \in \mathscr{C}$. Because of property (4) of n-chains, u_{m+1} is one of the natural successors $t_1, \ldots, t_n$ of σ. Suppose $u_{m+1} = t_j$. In (3.17), take $C = \{f : u_i \in \mathscr{T}(f) \wedge f(\rho(u_i)) \in F_i,\ i = 1, \ldots, m\}$. By proposition 3.9, $t_j \in \mathscr{T}(f)$ if and only if $f(\sigma) \in$

$Q_j(u_m, f)$. But the indicator of the set of $f \in D$ for which this holds can be written as a Borel function of $f(\rho(\jmath_1)), \ldots, f(\rho(\jmath_n))$, and $f(\sigma)$, and so the same is true of the indicator of the set $\{f: \ell_j \in \mathscr{T}(f)$ and $f(\rho(\ell_j)) \in F_{m+1}\}$. Let the h in (3.19) be such a Borel function. Since $\ell_j = u_{m+1}$, (3.17) then yields

$$\begin{aligned}\pi(\{f : u_i \in \mathscr{T}(f), f(\rho(u_i)) \in F_i,\ i = 1, \ldots, m+1\}) \\ = \pi(\{f : u_i \in \mathscr{T}(f), f(\rho(u_i)) \in F_i,\ i = 1, \ldots, m+1\}).\end{aligned} \tag{3.20}$$

This completes the induction step, hence the proof of the theorem.

3.16 Lemma. *Let X be a stochastic process with state-dependent hitting probabilities. Let $\jmath \in \mathscr{T}$, and $A \in \mathscr{P}_\jmath$. Then*

$$\pi(A \mid \mathscr{D}_\jmath) = \pi(A \mid \mathscr{C}_\jmath). \tag{3.21}$$

Proof. It suffices to prove (3.21) for A of the form $\{f: \ell_i \in \mathscr{T}(f) \wedge f(\rho(\ell_i)) \in E_i,\ i = 1, \ldots, m\}$, where $\ell_i \in \mathscr{T}, i = 1, \ldots, m$, and $\jmath < \ell_1 < \cdots < \ell_m$. It is not hard to see that no loss of generality is involved if we assume that ℓ_1 is a natural successor of $\jmath$ and that ℓ_{i+1} is a natural successor of ℓ_i for $i = 1, \ldots, m-1$. The proof is by induction on m. For $m = 1$, what we are to prove reduces to

$$\pi(\ell \in \mathscr{T}(f), f(\rho(\ell) \in E \mid \mathscr{D}_\jmath) = \pi(\ell \in \mathscr{T}(f), f(\rho(\ell)) \in E \mid \mathscr{C}_\jmath), \tag{3.22}$$

where ℓ is a natural successor of $\jmath$, and $E \in \Sigma$. The complete proof of (3.22) is not short. The basic ideas and techniques, however, are so similar to the proof of the preceding theorem that a great deal of repetition would be involved were we to present the proof in detail. Therefore, we leave these details to the reader, and proceed to the induction step. We assume that (3.21) holds for sets A as specified above, and try to prove

$$\begin{aligned}\pi(\ell_i \in \mathscr{T}(f) \wedge f(\rho(\ell_i)) \in E_i,\ i = 1, \ldots, m+1 \mid \mathscr{D}_\jmath) \\ = \pi(\ell_i \in \mathscr{T}(f) \wedge f(\rho(\ell_i)) \in E_i,\ i = 1, \ldots, m+1 \mid \mathscr{C}_\jmath),\end{aligned} \tag{3.23}$$

where $\ell_i \in \mathscr{T}, i = 1, \ldots, m+1$, $\ell_1 < \cdots < \ell_{m+1}$, and $E_i \in \Sigma$, $i = 1, \ldots, m+1$. Let G be the indicator of $\{f: \ell_1 \in \mathscr{T}(f) \wedge f(\rho(\ell_i)) \in E_i,\ i = 2, \ldots, m+1\}$. Let $\mathscr{G}$ be the σ-field generated by $\{f: \ell_1 \in \mathscr{T}(f) \wedge f(\rho(\ell_1)) \in E\}$ as E ranges over all $E \in \Sigma$. Now (3.23) becomes

$$E_\pi(GH \mid \mathscr{D}_\jmath) = E_\pi(GH \mid \mathscr{C}_\jmath). \tag{3.24}$$

But

$$\begin{aligned}E_\pi(GH \mid \mathscr{D}_\jmath) &= E_\pi(E_\pi(GH \mid \mathscr{D}_{\ell_1}) \mid \mathscr{D}_\jmath) \\ &= E_\pi(GE_\pi(H \mid \mathscr{D}_{\ell_1}) \mid \mathscr{D}_\jmath) \\ &= E_\pi(GE_\pi(H \mid \mathscr{C}_{\ell_1}) \mid \mathscr{D}_\jmath),\end{aligned} \tag{3.25}$$

the last equality being a consequence of our induction hypothesis. Note that $GE_\pi(H \mid \mathcal{O}_{t_1})$ is measurable with respect to $\mathcal{O}_{t_2} = \mathcal{O}_s \wedge \mathcal{G}$. From 3.22 it follows that

$$\pi(K \mid \mathcal{D}_s) = \pi(K \mid \mathcal{O}_s) \tag{3.26}$$

for all non-negative $\mathcal{G}$-measurable functions K on D, from which in turn follows

$$\pi(KL \mid \mathcal{D}_s) = \pi(KL \mid \mathcal{O}_s) \tag{3.27}$$

for $\mathcal{G}$-measurable K and $\mathcal{O}_s$-measurable L (here we use $\mathcal{O}_s \subset \mathcal{D}_s$). But (3.27) yields

$$\pi(M \mid \mathcal{D}_s) = \pi(M \mid \mathcal{O}_s) \tag{3.28}$$

for $\mathcal{O}_s \wedge \mathcal{G}$-measurable M. Taking $M = GE_\pi(H \mid \mathcal{O}_{t_1})$, we continue the chain of equalities in (3.25) to obtain

$$\begin{aligned} E_\pi(GH \mid \mathcal{D}_s) &= E_\pi(GE_\pi(H \mid \mathcal{O}_{t_1}) \mid \mathcal{O}_s) \\ &= E_\pi(E_\pi(GH \mid \mathcal{O}_{t_1}) \mid \mathcal{O}_s) \\ &= E_\pi(GH \mid \mathcal{O}_s). \end{aligned} \tag{3.29}$$

This establishes (3.24), hence (3.23), and completes the proof of the induction step. The lemma is now proved.

We refer the reader to [6] for the notion of conditional independence.

3.17 COROLLARY. *Let X be a stochastic process with state-dependent hitting probabilities. Then, for each $s \in \mathcal{T}$, $\mathcal{O}$ and $\mathcal{D}_s$ are conditionally independent given $\mathcal{O}$. (relative to the measure π on $\mathcal{D}$).*

Proof. Suppose $E \in \mathcal{P}_s$. Then (3.21) holds. If $O \in O_s$, then O is also in $\mathcal{D}_s$ (clearly $\mathcal{O}_s \subset \mathcal{D}_s$), so from (3.21) follows

$$\pi(OE \mid \mathcal{D}_s) = \pi(OE \mid \mathcal{O}_s). \tag{3.30}$$

The fact that (3.30) holds for each $O \in \mathcal{O}_s$ and $E \in \mathcal{P}_s$ implies that

$$\pi(F \mid \mathcal{D}_s) = \pi(F \mid \mathcal{O}_s) \tag{3.31}$$

holds for each $F \in \mathcal{O}_s \vee \mathcal{P}_s$. But $\mathcal{O}_s \vee \mathcal{P}_s = \mathcal{O}$, and $\mathcal{D}_s = \mathcal{O}_s \vee \mathcal{D}_s$ (again because $\mathcal{O}_s \subset \mathcal{D}_s$). Therefore

$$\pi(F \mid \mathcal{O}_s \vee \mathcal{D}_s) = \pi(F \mid \mathcal{O}_s) \tag{3.32}$$

holds for each $F \in \mathcal{O}$. But this is equivalent to $\mathcal{O}$ being independent of $\mathcal{D}_s$ given $\mathcal{O}_s$ (relative to the measure π on $\mathcal{D}$), which proves the corollary.

Suppose $f \in \mathcal{D}$ and $g \in \mathcal{D}$. We write $f \equiv g (\text{mod } \mathcal{O})$ if, for each $s \in \mathcal{T}$, $s \in \mathcal{T}(f)$ if and only if $s \in \mathcal{T}(g)$, and then $f(\rho(f, s)) \equiv f(\rho(g, s))$. It is clear that $\equiv (\text{mod } \mathcal{O})$

is an equivalence relation, and that the corresponding equivalence classes are the fibres of the σ-field $\mathcal{O}$. It is not difficult to verify that, if we set $B_0 = \{(f, g): f = g(\text{mod}\, \mathcal{O})\}$, then $B_0 \in \mathcal{O} \times \mathcal{O}$.

4. The Enlargement

Let $X = (\Omega, \mathcal{U}, \mathcal{M}_t, X_t, P)$ and $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{U}}, \tilde{\mathcal{M}}_t, \tilde{X}_t, \tilde{P})$ be stochastic processes with the same state-dependent hitting probabilities. The σ-field $\mathcal{O}$ is countably generated. This permits the application of the disintegration of measures theorem 2.4 to the measure $\tilde{\pi} = \tilde{P} \circ \tilde{X}^{-1}$ on $(D, \mathcal{D})$ to obtain a family $(\tilde{\pi}_f)_{f \in D}$ of non-negative measures on $(D, \mathcal{D})$ satisfying the conditions of the theorem (with $\mathcal{E}$ taken equal to $\mathcal{O}$). Let $\hat{\Omega} = \Omega \times D$, and $\hat{\mathcal{U}} = \mathcal{U} \times \mathcal{D}$. We now define a probability measure $\hat{P}$ on $\hat{\mathcal{U}}$. Suppose $B = A \times C$, where $A \in \mathcal{U}$ and $C \in \mathcal{D}$. The equation

$$\hat{P}(B) = \int_A \tilde{\pi}_{X(\omega)}(C)\, P(d\omega) \tag{4.1}$$

extends first to a measure on the field consisting of finite disjoint unions of such sets B, and then to a measure on the σ-field generated by this field, namely $\hat{\mathcal{U}}$. We denote this measure on $\hat{\mathcal{U}}$ by $\hat{P}$, and note that $\hat{P}$ satisfies

$$\hat{P}(B) = \int \tilde{\pi}_{X(\omega)}(B_\omega)\, P(d\omega), \qquad B \in \hat{\mathcal{U}}, \tag{4.2}$$

where $B_\omega = \{f: (\omega, f) \in B\}$. (This last equation could also have been used to define $\hat{P}$).

Let Y be the map of $\hat{\Omega}$ into $D \times D$ defined by $Y(\omega, g) = (X(\omega), g)$, $(w, g) \in \hat{\Omega}$. Let $\hat{\pi} = P \circ Y^{-1}$. Then, for each C, C' in $\mathcal{D}$, $\hat{\pi}$ satisfies

$$\hat{\pi}(C \times C') = \int_C \tilde{\pi}_f(C)\, \pi(df). \tag{4.3}$$

Since $\pi_f(D)$ is $\mathcal{O}$-measurable in f, and since $\pi = \tilde{\pi}$ on $\mathcal{O}$, $\tilde{\pi}_f(D) = 1$ for π-almost all $f \in D$. It follows that $\tilde{\pi}_{X(\omega)}(D) = 1$ for P-almost all $\omega \in \Omega$. Let $B_0 = \{(f, g): f \in D, g \in D, f = g(\text{mod}\, \mathcal{O})\}$. We have observed that $B_0 \in \mathcal{D} \times \mathcal{D}$. For $\tilde{\pi}$-almost all f, $\tilde{\pi}_f$ assigns measure zero to the complement of $\{g: g = f(\text{mod}\, \mathcal{O})\}$. Thus $\pi_f((B_0)_f) = 1$ for $\tilde{\pi}$-almost all f. From the fact that $\pi_f(A)$ is $\mathcal{O}$-measurable in f for all $A \in \mathcal{D}$ it follows that $\pi_f(B_f)$ is $\mathcal{O}$-measurable in f for all $B \in \mathcal{O} \times \mathcal{O}$, in particular for $B = B_0$. It follows that $\pi_f((B_0)_f) = 1$ for π-almost all f, hence that $\hat{\pi}(B_0) = 1$.

For each $t \geq 0$, let $\hat{X}_t(\omega, f) = X_t(\omega)$, and $A_t = \mathcal{M}_t \times D = \{M \times D: M \in \mathcal{M}_t\}$. It is clear that $\hat{X}_t$ is A_t-measurable. We regard $\hat{X}_t$ as an extension of X_t to $\hat{\Omega}$, and often denote it by X_t. Let $\mathcal{M} = \vee_t \mathcal{M}_t$, and $\mathcal{F} = \mathcal{M} \times D$. A non-decreasing family $\{\hat{\mathscr{M}}_t\}$ of sub σ-fields of $\hat{\mathscr{A}}$ is called an *enlargement* of $\{\mathscr{M}_t\}$ if $\hat{\mathscr{M}}_t \supset A_t$ for each $t \geqslant 0$, in which case $X = (\hat{\Omega}, \hat{\mathscr{A}}, \hat{\mathscr{M}}_t, \hat{X}_t, \hat{P})$ is a stochastic process. An enlargement $\{\hat{\mathscr{M}}_t\}$ is said to be a *distributional enlargement* if it is right continuous, and if, for each $t \geqslant 0$,

$$\hat{P}(B \mid \mathscr{F}_t) = \hat{P}(B \mid \mathscr{F}), \qquad B \in \hat{\mathscr{M}}_t. \tag{4.4}$$

The concept of distributional enlargement is introduced in [1]. For the rationale behind the concept, we refer the reader to [1] and [4]. Suppose $\{\hat{\mathscr{M}}_t^0\}$ is an increasing family of sub σ-fields of $\hat{\mathscr{A}}$ with $\hat{\mathscr{M}}_t^0 \supset \mathscr{F}_t$ such that (4.3) is satisfied for each $B \in \hat{\mathscr{M}}_{t+}^0$. Thus if we set $\hat{\mathscr{M}}_t = \hat{\mathscr{M}}_{t+}^0$ for each t, $\{\hat{\mathscr{M}}_t\}$ is a distributional enlargement of $\{\mathscr{M}_t\}$.

Let $\hat{\mathscr{M}}_t^*$ be the σ-field generated by all sets of the form

$$\{(\omega, f)\colon \tau_s(\omega) \leqslant t,\ \tau_s(\omega) \in B,\ \rho(f, s) \in C\} \tag{4.5}$$

with s an arbitrary member of $\mathscr{F}$, and B, C arbitrary Borel subsets of $[0, \infty]$. Let $\hat{\mathscr{M}}_t^0 = \mathscr{F}_t \vee \hat{\mathscr{M}}_t^*$, and $\hat{\mathscr{M}}_t = \hat{\mathscr{M}}_{t+}^0$. Finally we complete $\hat{\mathscr{M}}_t$ with respect to $\hat{P}$, still denoting it by $\hat{\mathscr{M}}_t$.

4.1 THEOREM. *$\{\hat{\mathscr{M}}_t\}$ is a distributional enlargement of $\{\mathscr{M}_t\}$.*

Proof. It is clear that $\{\hat{\mathscr{M}}_t\}$ is an enlargement of $\{\mathscr{M}_t\}$. To prove the theorem, it suffices to show that (4.4) holds for all $B \in \hat{\mathscr{M}}_t^0$. Let m and n be non-negative integers, s_i and ℓ_j in $\mathscr{T}$ for $i = 1,..., m$ and $j = 1,..., n$, and B_i, C_i Borel subsets of $[0, \infty)$ for $i = 1,..., m$. We assume that $s_1 < \cdots < s_m$. Let

$$\begin{aligned} U &= \{(\omega, f) : \tau_{s_i}(\omega) \leqslant t,\ i = 1,..., m\} \\ V &= \{(\omega, f) : \tau_{s_i}(\omega) \in B_i,\ \rho(f, s_i) \in C_i,\ i = 1,..., m\} \\ W &= \{(\omega, f) : \tau_{\ell_j}(\omega) > t,\ j = 1,..., n\}. \end{aligned} \tag{4.6}$$

Let $\mathscr{H}$ be the class of all sets $U \cap V \cap W$ which arise from any such choice of m, n, $\{s_i\}$, $\{\ell_j\}$, $\{B_i\}$, and $\{C_i\}$. It is clear that $\mathscr{H}$ is closed under finite intersections. The complement $(U \cap V \cap W)^c$ of a member of $\mathscr{H}$ is the disjoint union of U^c, $U \cap V^c$, and $U \cap V \cap W^c$. It is not hard to see that each of these sets can in turn be expressed as a finite disjoint union of sets of the same form $U \cap V' \cap W$. It follows that the class $\mathscr{K}$ of all finite disjoint unions of members of $\mathscr{H}$ is a field. It is clear that $\mathscr{H}$ contains all sets of the form (4.6), so $\mathscr{K}$ is a field generating $\hat{\mathscr{M}}_t^*$. To show that (4.4) holds for all $B \in \hat{\mathscr{M}}_t^*$, it suffices to show that it holds for all $B \in \mathscr{H}$. Let $B \in \mathscr{H}$. Then $B = U \cap V \cap W$, where U, V, W are as in (4.6) We see that $B = E \times F$, where

$$\begin{aligned} E &= \{\omega : \tau_{s_i}(\omega) \leqslant t,\ \tau_{s_i}(\omega) \in B_i,\ i = 1, .., m,\ \tau_{\ell_j}(\omega) > t,\ j = 1,..., n\}, \\ F &= \{f : \rho(f, s_i) \in C_i,\ i = 1,..., m\}. \end{aligned} \tag{4.7}$$

Let

$$O = \{f: \rho(f, \vartheta_i) \leqslant t, \rho(f, \vartheta_i) \in B_i, i = 1, .., m, \rho(f, \ell_j) > t, j = 1,...,\}.$$

Clearly $E = X^{-1}[O]$.

1. *Let X be a bounded $(\mathscr{D} - \mathscr{B})$ measurable function of D into R. Then*

$$\int_{E\times D} X(f)\, d\hat{\pi}(f, g) = \int_E X\, d\pi \qquad E \in \mathscr{D}.$$

Proof of 1. This is an almost immediate consequence of (4.3), the only subtle point being that $\tilde{\pi}_f(D) = 1$ for π-almost all f as well as for $\tilde{\pi}$-almost all f.

Let $\mathscr{G}_s = \mathscr{D}_s \times D = \{C \times D: C \in \mathscr{D}_s\}$, and $\mathscr{G} = \mathscr{D} \times D$.

2. *The function*

$$(f, g) \to I_0(f)\, \tilde{\pi}_f(F) \qquad (f, g) \in D \times D$$

is a version of $\hat{\pi}(O \times F \mid \mathscr{G})$.

Proof of 2. Let $E \in \mathscr{D}$. Then

$$\begin{aligned}\int_{E\times D} I_{O\times F}\, d\hat{\pi} &- \hat{\pi}((O \cap E) \times F)\\ &= \int_{O\cap E} \tilde{\pi}_f(F)\, \pi(df)\\ &= \int_E I_O(f)\, \tilde{\pi}_f(F)\, \pi(df),\end{aligned}$$

where the last equality results from (1) with $X(f) = I_O(f)\, \tilde{\pi}_f(F)$.

$$= \int_{E\times D} I_O(f)\, \tilde{\pi}_f(F)\, \pi(df)$$

the last equality following from (1) with $X(f) = I_0(f)\, \tilde{\pi}_f(F)$.

3. $g \to \tilde{\pi}_g(F)$ *is* π-a.s. O_{ϑ_m}*-measurable.*

Proof of 3. We have shown (corollary 3.17) that, given $\mathscr{O}_{\vartheta_m}$, $\mathscr{O}$ and $\mathscr{D}_{\vartheta_m}$ are conditionally independent (relative to the measure $\tilde{\pi}$). Clearly $F \in \mathscr{D}_{\vartheta_m}$. This, together with the fact that $\tilde{\pi}_g(F)$ is a version of the conditional $\tilde{\pi}$-measure of F given $\mathscr{O}$, implies that $\tilde{\pi}_g(F)$ is $\tilde{\pi}$-essentially constant on $\tilde{\pi}$-almost all fibres of $\mathscr{O}_{\vartheta_m}$, hence π-essentially constant on π-almost all fibres of $\mathscr{O}_{\vartheta_m}$. Thus $\tilde{\pi}_f(F)$ is π-a.e. $\mathscr{O}_{\vartheta_m}$-measurable in f.

4. *Suppose $t' > t$. If $L \in \mathscr{O}_{\vartheta_n}$ then $L \cap O \in \mathscr{D}_{t'}$.*

Proof of 4. The class of L's in $\mathcal{O}_{\sigma_n}$ for which $L \cap O \in \mathcal{D}_{t'}$ is closed under countable intersections and monotone limits. It therefore suffices to prove $L \cap O \in \mathcal{D}_{t'}$ for a subclass of L's in $\mathcal{O}_{\sigma_n}$ which generates $\mathcal{O}_{\sigma_n}$ and is closed under complements. Such a subclass consists of L's of the form

$$L = \{f \colon \rho(f, \sigma_n) \in B \wedge f(t \wedge \rho(\sigma_n)) \in C\},$$

with $B \in \mathcal{B}$ and $C \in \Sigma$. But then

$$L \cap O = \{f \colon \rho(f, \sigma_n) \in B, \rho(f, \sigma_i) \in B \cap [0, t], i = 1,..., m, \rho(f, \ell_j) > t,$$
$$j = 1,..., n, f(t \wedge \rho(f, \sigma_n)) \in C\},$$

which clearly belongs to $\mathcal{D}_t$ if $t' > t$.

5. *If* $t' > t$, *then* $\tilde{\pi}(O \times F \mid \mathcal{G}_{t'}) = \tilde{\pi}(O \times F \mid \mathcal{G})$.

Proof of 5. By virtue of (2), it suffices to show that $I_O(f)\, \tilde{\pi}_f(F)$ is π-a.s. $\mathcal{D}_{t'}$-measurable in f. By virtue of (4), a set L which is $\mathcal{O}_{\sigma_n}$-measurable has the property that $L \cap O \in \mathcal{D}_{t'}$. This is also true for functions. Since $\tilde{\pi}_f(F)$ is π-a.s. $\mathcal{O}_{\sigma_n}$-measurable in f by (3), it follows that $I_O(f)\, \tilde{\pi}_f(F)$ is π-a.s. $\mathcal{D}_{t'}$-measurable.

6. $\tilde{\pi}(O \times F \mid \mathcal{G}_{t+}) = \tilde{\pi}(O \times F \mid \mathcal{G})$

This is an immediate consequence of (5).

7. $P(E \times F \mid Y^{-1}[\mathcal{G}_{t+}]) = P(E \times F \mid Y^{-1}[\mathcal{G}])$.

Since $E = X^{-1}[O]$ and $\tilde{\pi} = \hat{P} \circ Y^{-1}$, this is an immediate consequence of (6).

8. $(\omega, g) \to I_E(\omega)\, \tilde{\pi}_{X(\omega)}(F)$ *is a version of* $\hat{P}(E \times F \mid \mathcal{F})$.

This follows from (4.1) in the same way that (2) followed from (4.3).

9. $P(E \times F \mid \mathcal{F}) = P(E \times F \mid Y^{-1}[\mathcal{G}])$.

This follows from (8) and the fact that $I_E(\omega)\, \tilde{\pi}_{X(\omega)}(F)$ is $X^{-1}[\mathcal{D}]$-measurable in ω.

10. $P(E \times F \mid \mathcal{F}) = P(E \times F \mid Y^{-1}[\mathcal{G}_{t+}])$

This follows from 7 and 9.

11. $P(E \times F \mid \mathcal{F}_t) = P(E \times F \mid \mathcal{F})$

Proof of 11. Since $X^{-1}[\mathcal{D}_s] \subset \mathcal{M}_s$ for each s, it follows from the right continuity of $\{\mathcal{M}_t\}$ that $X^{-1}[\mathcal{D}_{t+}] \subset \mathcal{M}_t$, from which it follows in turn that $Y^{-1}[\mathcal{G}_{t+}] \subset A_t$. Now (11) follows from (10).

With (11), we have (4.4) holding for each $B \in \mathcal{H}$, hence for all $B \in \hat{\mathcal{M}}^*$. It then follows that (4.4) holds for all $B = G \cap B_O$ with $G \in \mathcal{F}_t$ and $B_O \in \hat{\mathcal{M}}_t^*$. This yields (4.4) for all $B \in \hat{\mathcal{M}}_t^0$, completing the proof of the theorem.

5. THE TIME CHANGE $\{\tau_t\}$

In this section X and $\tilde{X}$ are assumed to be stochastic processes with the same state-dependent hitting probabilities. We continue to assume that the sample paths of both processes are in D. In addition, we assume that the sample paths of both processes have no intervals of constancy. The process $\{X_t\}$ can be defined on the enlarged space $(\hat{\Omega}, \hat{\mathcal{O}}, \hat{P})$ defined in the preceding section, and adapted to the family $\{\hat{\mathscr{M}}_t\}$ of enlarged σ-fields also defined in the last section. In this section we construct a time change $\{\tau_t, t \geqslant 0\}$ relative to $\{\hat{\mathscr{M}}_t\}$ for which the process $\{X_{\tau_t}\}$ has the same finite-dimensional distributions as $\{\tilde{X}_t\}$. We show also that $\sigma(\omega, \cdot)$ is continuous and strictly increasing for $\hat{P}$-almost all $\omega \in \hat{\Omega}$.

For each pair (f, g) in $D \times D$ for which $f \equiv g \pmod{\mathscr{O}}$, let $\Delta(f, g) = \{\rho(g, s)\colon s \in \mathscr{T}(f)\}$. Since g has no intervals of constancy, it follows from corollary 3.5 that $\Delta(f, g)$ is dense in $[0, \infty)$. We define σ_0 as follows. If f and g are in D, but $f \not\equiv g \pmod{\mathscr{O}}$, then $\sigma_0(f, g, t)$ is defined for $t \in \Delta(f, g)$ by

$$\sigma_0(f, g, \rho(g, s)) = \rho(f, s). \tag{5.1}$$

5.1 LEMMA. *If $f \equiv g \pmod{\mathscr{O}}$, then $\sigma_0(f, g, \cdot)$ is a strictly increasing and right continuous function on $\Delta(f, g)$.*

Proof. If $\rho(g, s) = \rho(g, \ell)$, then $s = \ell$ since $\rho(g, \cdot)$ is strictly increasing (relative to the order $\prec$ on $\mathscr{T}(g)$). This shows that σ_0 is a well-defined function. Since $\rho(f, \cdot)$ is also strictly increasing on $\mathscr{T}(g) = \mathscr{T}(f)$, σ_0 is strictly increasing on $\Delta(f, g)$.

Let $\ell = (i_1, \ldots, i_m) \in \mathscr{T}(g)$, and suppose that $\rho(g, \ell_n) \searrow \rho(g, \ell)$, where $\ell_n \in \mathscr{T}(g)$, $n = 1, 2, \ldots$. For each $n > m$, let s_n be the ordered n-tuple $(i_1, \ldots, i_m, 0, \ldots, 0, 1)$. Let $N = \{n\colon s_n \in \mathscr{T}(g)\}$. Suppose $n \in N$. Since $s_n > \ell$, $\rho(g, s_n) > \rho(g, \ell)$. Therefore for all sufficiently large values of k, $\rho(g, \ell) < \rho(g, \ell_k) < \rho(g, s_n)$ from which it follows that

$$\ell \prec \ell_k \prec s_n \tag{5.2}$$

for all sufficiently large values of k. By virtue of part (a) of corollary 3.8, N is an infinite set. Let $\{n_j\}_{j=1}^{\infty}$ be an enumeration of N in increasing order. Then, since $\mathscr{T}(f) = \mathscr{T}(g)$, $\rho(f, s_{n_j}) \searrow \rho(f, \ell)$ as $j \to \infty$ by virtue of part (b) of corollary 3.8. But from (5.2) we obtain

$$\rho(f, \ell) < \rho(f, \ell_k) < \rho(f, s_{n_j}) \tag{5.3}$$

for each j and all $k \geqslant k_j$. Thus $\rho(f, \ell_k) \uparrow \rho(f, \ell)$ as $k \to \infty$, completing the proof.

Suppose $f \equiv g \pmod{\mathscr{O}}$. We extend $\sigma_0(f, g, \cdot)$ on $\Delta(f, g)$ to a function $\sigma(f, g, \cdot)$ on $[0, \infty)$ by right continuity. That is, we define $\sigma(f, g, t)$ to be equal to $\lim_k \sigma_0(f, g, \rho(g, s_k))$, where $\rho(g, s_k) \downarrow t$. It follows from the lemma just proved not only that this definition is valid, but that $\sigma(f, g, \cdot)$ is strictly increasing and right continuous on $[0, \infty)$.

5.2 LEMMA. *If* $f \equiv g \pmod{\mathcal{O}}$, *then* $\sigma(f, g, \cdot)$ *is continuous and strictly increasing on* $[0, \infty)$, *and* $f \circ \sigma(f, g, \cdot) = g$.

Proof. Suppose $f \equiv g \pmod{\mathcal{O}}$. We have already observed that σ is strictly increasing. Since $\sigma(f, g, \rho(g, \mathfrak{s})) = \rho(f, \mathfrak{s})$, and since $f(\rho(f, \mathfrak{s})) = g(\rho(g, \mathfrak{s}))$ for $\mathfrak{s} \in \mathscr{T}(f) = \mathscr{T}(g)$, it is clear that $f \circ \sigma(f, g, \cdot) = g$ on $\Delta(f, g)$. Since f, g, and $\sigma(f, g, \cdot)$ are right continuous, and since $\Delta(f, g)$ is dense in $[0, \infty)$, we have $f \circ \sigma(f, g, \cdot) = g$ on $[0, \infty)$. Now interchange f and g: $\sigma(g, f, \cdot)$ is strictly increasing and right continuous on $[0, \infty)$. Let $\gamma = \sigma(g, f, \cdot) \circ \sigma(f, g, \cdot)$. Clearly γ is right continuous. On $\Delta(f, g)$, γ is the identity function, and it follows from right continuity and the density of $\Delta(f, g)$ that γ is the identity function on $[0, \infty)$. The continuity of $\sigma(f, g, \cdot)$ is now a consequence of the following proposition.

5.3 PROPOSITION. *Suppose that* α *and* β *are strictly increasing, right continuous functions on and into* $[0, \infty)$, *and that* $\alpha \circ \beta$ *is the identity function on* $[0, \infty)$. *Then* α *and* β *are continuous.*

Proof. Assume the hypotheses, and suppose that α is discontinuous at some $t \in [0, \infty)$. Then $\alpha(t - 0) = s_0 < s = \alpha(t)$. The range of $\alpha \circ \beta$ omits (s_0, s). If β is discontinuous at some point, the range of β omits an interval I, so the range of $\alpha \circ \beta$ omits the interval $\alpha[I]$. In either case, the assumption that $\alpha \circ \beta$ is the identity is contradicted. This completes the proof of the proposition.

Let D_0 be the set of all members of D without intervals of constancy. It is easy to see that $D_0 \in \mathscr{D}$. Recall that $B_0 = \{(f, g): f \equiv g \pmod{\mathcal{O}}\}$.

5.5 PROPOSITION. *Let* $C_0 = B_0 \cap (D_0 \times D_0)$, *and let*

$$A = \bigcap_n \bigcup_{\mathfrak{s} \in \mathscr{T}} \left\{(f, g) : \rho(f, \mathfrak{s}) \leqslant t + \frac{1}{n} \wedge \rho(g, \mathfrak{s}) \in \left[s, s + \frac{1}{n}\right)\right\}.$$

Then $A \cap C_0 = \{(f, g): (f, g) \in C_0, \sigma(f, g, s) \leqslant t\}$.

Proof. This is an immediate consequence of the definition and right continuity of $\sigma(f, g, \cdot)$ for $(f, g) \in C_0$.

Now consider the stochastic process $\tilde{X} = (\hat{\Omega}, \hat{\mathscr{A}}, \mathscr{M}_t, \tilde{X}_t, \hat{P})$. Define τ on $\hat{\Omega} \times [0, \infty)$ by

$$\tau((\omega, g), t) = \sigma(X(\omega), g, t), \qquad (\omega, g) \in \hat{\Omega}. \tag{5.4}$$

We use τ_t to denote $\tau(\cdot\,, t)$.

5.6 THEOREM. $\{\tau_t, t \geqslant 0\}$ *is a P-almost surely continuous time change relative to* $\{\mathscr{M}_t\}$. $\{X_{\tau_t}\}$ *and* $\{\tilde{X}_t\}$ *have the same finite-dimensional distributions.*

Proof. That $\tau((\omega, g), t)$ is strictly monotone and continuous in t for P-almost all $(\omega, g) \in \hat{\Omega}$ follows from the corresponding properties of $t \to \sigma((f, g)\, t)$ for $(f, g) \in B_0$, stated in Lemma 5.3, together with the fact that $\hat{\pi}(B_0) = 1$. The process $\{X_t\}$ which is time changed into $\{X_{\tau_t}\}$ is actually the process $\{\tilde{X}_t\}$: recall that $X_t(\omega, g) = X_t(\omega)$. Thus $X_{\tau_t}(\omega, g) = \tilde{X}_{\tau(\omega,g,t)}(\omega, g)$. We shall show that, for each $C \in \mathscr{D}$, $\hat{P}(X_{\tau_t} \in C) = \tilde{\pi}(C)$, which, of course, shows that $\{X_{\tau_t}\}$ and $\{\tilde{X}_t\}$ have the same finite-dimensional distributions. Let $(\omega, g) \in \hat{\Omega}$. Then $\hat{X}_{\tau(\omega,g,t)}(\omega, g) = X_{\tau(\omega,g,t)}(\omega) =$ the value of $X(\omega)$ at $\tau(\omega, g, t)$, which is the value of $X(\omega)$ at $\sigma(X(\omega), g, t)$. But $\hat{\pi}$ assigns all its mass to pairs (f, g) for which f and g have no intervals of constancy and for which $f \circ \sigma(f, g) = g$ (lemma 5.2). It follows that $\hat{P}$ assigns all its mass to pairs (ω, g) for which $X(\omega) \circ \sigma(X(\omega), g) = g$, in other words, for which the value of $X(\omega)$ at $\sigma(X(\omega), g, t)$ is simply $g(t)$. Thus $\hat{P}(\{\omega, g): X_{\tau(\omega,g,t)}(\omega, g) \in C\} = \hat{P}(\{\omega, g): g(t) \in C\}) = \hat{\pi}(D \times C) = \tilde{\pi}(C)$ by virtue of (4.3).

To complete the proof of the theorem we must show that $\{(\omega, g): \tau(\omega, g, s) \leqslant t\} \in \hat{\mathscr{M}}_t$ for each $s \geqslant 0$, $t \geqslant 0$. Let $\hat{\mathscr{D}}_t$ be the sub-σ-field of $\mathscr{D} \times \mathscr{D}$ generated by all sets of the form $\{(f, g): \rho(f, \vartheta) \leqslant t, \rho(f, \vartheta) \in B, \rho(g, \vartheta) \in C\}$ as ϑ ranges over $\mathscr{T}$, and B, C over the Borel subsets of $[0, \infty]$. We see from 4.5 that $Y^{-1}(\hat{\mathscr{D}}_t) = \hat{\mathscr{M}}_t^* \subset \hat{\mathscr{M}}_t^0$ (recall that $Y(\omega, g) = (X(\omega), g)$). It follows that $Y^{-1}(\hat{\mathscr{D}}_{t+}) = \hat{\mathscr{M}}_{t+}^0$. From proposition 5.5 it follows that $\{(f, g): (f, g) \in C_0, \sigma(f, g, s) \leqslant t\}$ belongs to $\hat{\mathscr{D}}_{t+1/n}$ for each n, hence to $\hat{\mathscr{D}}_{t+}$. Therefore $\{(\omega, g): (X(\omega), g) \in C_0, \tau(\omega, g, s) \leqslant t\} = Y^{-1}(\{(f, g): (f, g) \in C_0, \sigma(f, g, s) \leqslant t\}) \in \hat{\mathscr{M}}_{t+}^0 = \hat{\mathscr{M}}_t$. But $\{(\omega, g): \tau(\omega, g, s) \leqslant t\}$ differs from $\{(\omega, g): \tau(\omega, g, s) \leqslant t\}$ by a subset of the $\hat{P}$-null set $\hat{\Omega} \backslash Y^{-1}(C_0)$. Since $\hat{\mathscr{M}}_t$ is $\hat{P}$-complete, this shows that $\{(\omega, g): \tau(\omega, g, s) \leqslant t\} \in \hat{\mathscr{M}}_t$, which completes the proof of the theorem.

6. Examples

Let $S = [0, \infty)$, and let Σ be the Borel subsets of $[0, \infty)$. Let $\{X_t, t \geqslant 0\}$ be a stochastic process on a probability space $(\Omega, \mathscr{A}, P)$ with values in S, and suppose that for each $\omega \in \Omega$, $X(\omega, t) = X_t(\omega)$ is a continuous strictly increasing function of t with $X(\omega, 0) = 0$ and $\lim_{t \to \infty} X(\omega, t) = \infty$. For each t, let $\mathscr{M}_t$ be the σ-field generated by X_s as s ranges over $[0, t]$. Then $X = (\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ is a stochastic process with state dependent hitting probabilities. Suppose that $\tilde{X} = (\hat{\Omega}, \hat{\mathscr{A}}, \hat{\mathscr{M}}_t, \tilde{X}_t, \tilde{P})$ is another such process, that is, one with (S, Σ) as state space, and whose sample paths are also strictly increasing continuous functions equal to 0 when $t = 0$ and increasing to ∞. Then X and $\tilde{X}$ have the same state-dependent hitting probabilities. Since we have excluded the possibility of intervals of constancy, the theorem of the last section applies, and it is possible to define $\{X_t\}$ on an enlarged probability space $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ in such a way that there is a time change $\{\tau_t\}$ (relative to enlarged σ-fields $\{\hat{\mathscr{M}}_t\}$) for which $\{X_{\tau_t}\}$ and $\{X_t\}$ have the same finite-dimensional distribution. In this special case, moreover, we can actually exhibit the time change, and we now proceed to do so.

Let $\hat{\Omega} = \Omega \times \tilde{\Omega}$, $\hat{\mathcal{A}} = \mathcal{A} \times \tilde{\mathcal{A}}$, and $\hat{P} = P \times \tilde{P}$. For $(\omega, \tilde{\omega}) \in \hat{\Omega}$, let $X_t(\omega, \tilde{\omega}) = X_t(\omega)$, $t \in [0, \infty)$. For each $t \in [0, \infty)$ and $(\omega, \tilde{\omega}) \in \hat{\Omega}$, let $\tau_t(\omega, \tilde{\omega})$ be that value of s for which $X_s(\omega) = \tilde{X}_t(\tilde{\omega})$. Clearly $\tau_t(\omega, \tilde{\omega})$ is a strictly increasing, continuous function of t. Let $\hat{\mathcal{M}}_t = \mathcal{M}_t \times \mathcal{A}$, $t \geqslant 0$. It is easy to see that $\{\tau_t \leqslant s\} \in \hat{\mathcal{M}}_s$ for each t and s in $[0, \infty)$, so $\{\tau_t\}$ is a time change relative to $\{\hat{\mathcal{M}}_t\}$. Since $X_{\tau_t}(\omega, \tilde{\omega}) = \tilde{X}_t(\tilde{\omega})$, and since $\hat{P} = P \times \tilde{P}$, $\{X_{\tau_t}\}$ and $\{\tilde{X}_t\}$ have the same finite-dimensional distribution. We now show that the enlargement is a distributional enlargement. For each $t \geqslant 0$, let $A_t = \{M \times \tilde{\Omega}; M \in \mathcal{M}_t\}$. Fix s and t, with $0 \leqslant s \leqslant t$. Let $M \in \mathcal{M}_s$ and $A \in \tilde{\mathcal{A}}$. It is trivial to verify that the function$(\omega, \tilde{\omega}) \to 1_M(\omega) \hat{P}(A)$ is a version of $\hat{P}(M \times A \mid \mathcal{F}_t)$. Thus $\hat{P}(M \times A \mid \mathcal{F}_s) = \hat{P}(M \times A \mid \mathcal{F}_t)$ for each $t \geqslant s$, and it follows that $\{\hat{\mathcal{M}}_t\}$ is a distributional enlargement of $\{\mathcal{M}_t\}$. (This is hardly surprising, for $P \times \hat{P}$ is the product measure, and the notion of distributional enlargment represents an effort to isolate the salient feature of the product-space enlargements with product measures used in defining so-called "random" stopping times and time changes. We again refer the reader to [1] and [4] for more extensive discussions of this point.)

Special cases of this example make it clear that some sort of randomization is, in general, necessary. Suppose, for example, at Ω has just one member ω_0, and that $X_t(\omega_0) = t$. Unless $\{\tilde{X}_t, t \geqslant 0\}$ is also deterministic, (that is, unless the distribution of $\tilde{X}_t$ degenerates for each t), there is no time change τ_t defined on Ω for which $\{X_{\tau_t}\}$ has the same finite-dimensional distributions as $\{\tilde{X}_t\}$. Suppose for example, $\tilde{X}_t$ moves uniformly to the right at the rate of 2 units per second with probability 1/2, and at the rate of 3 units per second with probability 1/2. Then the time change with we apply to $\{X_t\}$ amounts to flipping a coin, and speeding X_t up by a factor of 2 if the coin comes up heads, and by a factor of 3 if it comes up tales. We do not claim, however, that our method does not sometimes involve more randomization than necessary. Suppose, for example, that $\tilde{X}$ is the process just described, and that X is a process in which X_t moves to the right with unit velocity with probability 1/2, and at the rate of 6 units per second with probability 1/2. If we doubled the speed of the unit velocity path, and halved the speed of the other path, we would have a time change on the original space $(\Omega, \mathcal{A}, P)$ which takes X_t into a process equivalent to $\tilde{X}_t$. The time change we construct, however, depends not only on ω, but, also on what amounts to the toss of a fair coin. Given that the X_t process is moving to the right with unit velocity, the time change speeds it up by a factor of 2 with probability 1/2 and by a factor of 3 with probability 1/2. Given that X_t is moving to the right with a velocity of 6 units per second, the time change slows it down by a factor of 1/6 with probability 1/2, and by a factor of 1/2 with probability 1/2. One of the open problems in the general case is, in constructing the time change, to not over-enlarge, but to introduce randomness only when it is required. In work now under preparation, we show how to do this when the $\tilde{X}$ process is Markov.

We next consider an example with a more complicated state space. Consider the set of all lattice points (i, j) in the right half plane: that is, for which

$i = 0, 1, 2, \ldots$ and $j = 0, +1, +2, \ldots$. For each such i and j, connect (i, j) to $(i + 1, j + 1)$ by one line segment and (i, j) to $(i + 1, j - 1)$ by another. Let S be the resulting chicken-wire-fence-like structure. Endow S with the obvious topology—the one whose restriction to each of the connecting line segments is Euclidean—and let Σ be the corresponding Borel field. For each (i, j), let $p(i, j)$ and $q(i, j)$ be a pair of non-negative numbers adding up to one. Now consider a particle moving as follows. At time $t = 0$, it is at one of the lattice points $(0, j)$, and immediately starts moving along either the segment connecting $(0, j)$ to $(1, j + 1)$ or the segment connecting $(0, j)$ to $(1, j - 1)$, the first segment being chosen with probability $p(0, j)$, the second with probability $q(0, j)$. Its journey along the chosen segment may be either deterministic or non-deterministic, subject only to the proviso that it move always to the right, and never stands still. When it reaches, say $(1, j + 1)$, it switches to the segment connecting $(1, j + 1)$ to $(2, j + 2)$ with probability $p(1, j + 1)$ and to the one connecting $(1, j + 1)$ to $(2, j)$ with probability $q(1, j + 1)$. The particle continues to proceed in this zig-zag fashion. It is clear that the resulting process has state-dependent hitting probabilities. If we consider any other such process, it has the same state-dependent hitting probabilities as the first, provided that the choice of segments upon reaching (i, j) is made in accordance with the same probabilities $p(i, j)$ and $q(i, j)$ as for the first process. Two paths are equivalent mod $\mathscr{C}$ if and only if they traverse the same vertices (i, j). In this respect the vertices (i, j) play the role of the positions $x(\omega) = f(\rho(\omega))$ for $\omega \in \mathscr{T}(f)$. If the motion along the segments is deterministic, the process is a Markov process. Note that then there is only one path with given vertices, i.e., each equivalence class mod $\mathscr{C}$ has only one member. In work under preparation we show that this is characteristic of strong Markov processes whose paths have no intervals of constancy.

References

1. J. R. Baxter and R. V. Chacon, Enlargement of σ-algebras and compactness of time changes, *Canad. J. Math.* **29** (1977), 1055–1065.
2. R. M. Blumenthal, R. K. Getoor, and H. P. McKean, Jr., Markov processes with identical hitting distributions, *Illinois J. Math.* **6** (1962), 402–430.
3. R. M. Blumenthal and R. K. Getoor, "Markov Processes and Potential Theory," Academic Press, New York, 1968.
4. R. V. Chacon, Distributional enlargements, preprint.
5. B. P. Kharlamov, On convergence of semi-Markov walks to a continuous semi-Markov process, *Theory Probability Appl.* **21** (1976), 482–498.
6. M. Loève, "Probability Theory," 3rd ed., Van Nostrand, Princeton, N. J., 1963.
7. K. R. Parthasarathy, "Probability Measure on Metric Spaces," Academic Press, New York, 1967.
8. C. Stone, Weak convergence of stochastic processes defined on a semi-infinite time interval, *Proc. Amer. Math. Soc.* **14** (1963), 694–696.