Generating limit cycles from a nilpotent critical point via normal forms

M.J. Álvarez a,*, A. Gasull b

a Departament de Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca, Spain
b Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

Received 17 January 2005
Available online 20 June 2005
Submitted by H.W. Broer

Abstract

It is well known that the normal form theory can be applied to solve the center–focus problem for monodromic planar nilpotent singularities. In this paper we see how this theory can also be applied to generate limit cycles from this type of singularities.

Keywords: Planar vector filed; Nilpotent singularity; Normal form theory

1. Introduction and main results

Consider an autonomous planar ordinary differential equation having a nilpotent critical point. In a suitable coordinate system this differential equation can be written as

\[
\begin{align*}
\dot{x} &= -y + X_2(x, y), \\
\dot{y} &= Y_2(x, y),
\end{align*}
\]

(1.1)
where $X_2$, $Y_2$ are analytic functions with power series beginning, at least, in terms of degree two. The main goal of this paper is to use the normal form theory to generate limit cycles from the origin of the above system.

Before describing in detail our approach we give a short overview of how this is usually done when considering elementary critical points, instead of nilpotent ones. Indeed this problem is strongly related with the celebrated center–focus problem. Recall that it consists in distinguishing when a monodromic critical point (i.e., a critical point for which there are not orbits tending to it with a definite slope, in positive or negative time) is a focus (i.e., a critical point with a neighborhood where the orbits spiral toward or backward it) or a center (i.e., a critical point with a neighborhood where all the orbits are closed and periodic).

The center–focus problem for a non-degenerate critical point was theoretically solved by Lyapunov at the end of the XIX century, see [10]. In his work the author introduced the concept of the functions now known as Lyapunov functions, as well as the Lyapunov constants that give the stability of the point. In particular the center case is characterized by the vanishing of all the Lyapunov constants. Nevertheless, despite of the strong computers and the big efforts done in the last years with the appearance of new algorithms for its computation, there are still big difficulties in the complete solution of the problem when a particular family of differential equations is given. Even in the case of polynomial systems of a given degree, for which the Hilbert Basis Theorem asserts that the number of needed Lyapunov constant is finite, it is neither known which is this number.

For a given family of differential equations, the number of Lyapunov constants needed to solve the center–focus problem is also related with the so called cyclicity of the point (i.e., the number of limit cycles that appear from it by small perturbations of the coefficients of the given differential equation inside the family considered). In fact in the simplest and “ideal”\(^1\) case (see for instance [12] or [13]), the cyclicity is one less that the number of significative constants needed to solve the center–focus problem.

In any case, the Lyapunov constants are always useful to get lower bounds of the cyclicity of the critical point.

As far as we know there are essentially three different ways of obtaining the Lyapunov constants: by using normal form theory [7], by computing the Poincaré return map [4] or by using Lyapunov functions [13].

To our knowledge the three tools explained above have been also used to study the center–focus problem for nilpotent critical points, see for instance [1,5,11], respectively. On the other hand, just the second and the third one have been used to generate limit cycles from the critical point, see for instance [1,3], respectively. The aim of this paper is to use the first one, i.e., the normal form theory, in order to compute what will be called generalized Lyapunov constants (see Section 2) for a nilpotent critical point and to apply them to give lower bounds for its cyclicity. As a byproduct, in the last section we can also solve the center–focus problem for several families of planar vector fields.

Before giving our main result, let us recall some known results about the normal form of nilpotent critical points. Takens proved in [15] that a system with nilpotent linear part,

\(^1\) We do not enter here in the details, but roughly speaking this is the case where all the Lyapunov constants, until the last one needed to solve the center–focus problem, are independent and generate a radical ideal in the space of polynomials having as variables the coefficients of the family of systems considered.
i.e., a system of the form \((1.1)\), can be formally transformed into a generalized Liénard system

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= a(x) + y\tilde{b}(x),
\end{align*}
\]  

(1.2)

where \(a(x) = a_{s-1}x^{s-1}(1 + O(x))\), \(s \geq 3\), and \(\tilde{b}(x)\), with \(b(0) = 0\), are formal power series. Recently Stróźyna and Żoładek proved in [14] that indeed this normal form can be achieved through an analytic change of variables. In Lemma 2.3 (which essentially appears in [8,14]) a reparametrization of the time is used in order to simplify even more the above normal form. Indeed it holds that, to study monodromic critical points, we can reduce our attention to the study of the vector fields

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^{2n-1} + yb(x),
\end{align*}
\]  

(1.3)

where \(b(x)\) is an analytic function obtained from \(a(x)\) and \(\tilde{b}(x)\).

From expression (1.3) it is not difficult to characterize the centers of monodromic nilpotent singularities—the condition is that \(b(x)\) has to be an odd function, see Proposition 2.5—and thus to prove that all them are reversible. This result has been already obtained by Moussu without using the analyticity of the change giving rise to the normal form, see [11].

It is intuitively clear that the center conditions obtained by imposing that \(b(x)\) has to be an odd function must have a strong relation with the generalized Lyapunov constants. The second part of Theorem A is the main result of the paper and gives the relation between the first non-zero term of the even part of \(b(x)\) and the first non-zero generalized Lyapunov constant. It is the version for nilpotent critical points of the following well-known result, essentially due to Poincaré.

**Theorem 1.1.** Consider an analytic planar system having a critical point with purely complex eigenvalues \(\pm a_0i\), \(0 \neq a_0 \in \mathbb{R}\). Then there exists an analytic change of variables and of the time such that in polar coordinates it writes as

\[
\begin{align*}
\dot{r} &= rb(r^2), \\
\dot{\theta} &= a(r^2),
\end{align*}
\]  

(1.4)

being \(a\) and \(b\) analytic functions at 0, with \(a(0) = a_0\) and \(b(0) = 0\). Furthermore:

1. If \(b(r^2) = r^{2\ell}(b_{\ell} + O(r^2))\), with \(b_{\ell} \neq 0\), then its first non-zero Lyapunov constant is \(V_{2\ell+1} = 2\pi b_{\ell}\).
2. The origin is a center if and only if \(b(r^2) \equiv 0\).

Our result is:

\footnotesize
\(^2\) The first part of the theorem collects all the results explained above. It is included in the statement for the sake of completeness.
**Theorem A** (Computation of the generalized Lyapunov constants). Consider an analytic planar system having a monodromic nilpotent critical point. Then there exists an analytic change of variables such that it writes as

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^{2n-1} + yb(x),
\end{align*}
\]  

(1.5)

being \(b(x) \equiv 0\) or \(b(x) = \sum_{j \geq \beta} b_j x^j\), with \(b_\beta \neq 0\), and satisfying one of the following conditions:

(i) \(\beta > n - 1\),
(ii) \(\beta = n - 1\), and \(b_\beta^2 - 4n < 0\).

Furthermore:

(1) If \(b(x) = b^\circ(x) + x^{2\ell}(b_{2\ell} + O(x))\), with \(b_{2\ell} \neq 0\), being \(b^\circ(x) := (b(x) - b(-x))/2\), then its first significative generalized Lyapunov constant is

(a) \(V_{2-n+2\ell} = Kb_{2\ell}\) when either \(\beta > n - 1\), or \(\beta = n - 1\) and \(\beta\) is odd. Here \(K = K(n, \ell, b_{n-1})\) is a positive constant given in the proof.

(b) \(V_1 = \exp\left(-\frac{2b_\beta\pi}{n\sqrt{4n-b_\beta^2}}\right)\) when \(\beta = 2\ell = n - 1\).

(2) The origin is a center if and only if \(b^e(x) := b(x) - b^\circ(x) \equiv 0\).

**Remark 1.2.** As we will see in the last section of the paper, for a practical use of Theorem A, it is not necessary to get the complete normal form. It suffices to write the system as

\[
\begin{align*}
\dot{x} &= -y + O(||(x, y)||^r), \\
\dot{y} &= x^{2n-1} + yb_r(x) + O(||(x, y)||^r),
\end{align*}
\]  

(1.6)

for a suitable \(r\), and the polynomial \(b_r(x)\) has also as even coefficients the first degenerate Lyapunov constants.

Although useful, the above approach based on the obtention of the normal form is computationally expensive even to determine the stability of the nilpotent critical point. So, whenever it is possible, it is very interesting to get conditions on the general form (1.1) to ensure that the origin is monodromic (see Theorem 2.1) and in this case to obtain its stability. In several cases, this problem is solved in [1, Theorem B]. That paper gives the first stability conditions from the study of the Poincaré return map. There is one case there that has resisted that approach, concretely the case (ii) of Theorem A. In next result we can solve it by using Theorem A.

**Theorem B.** Given system (1.1),

\[
\begin{align*}
\dot{x} &= -y + X_2(x, y), \\
\dot{y} &= Y_2(x, y),
\end{align*}
\]

define the following functions:
\[ f(x) := Y_2(x, F(x)), \]
\[ \phi(x) := \frac{\partial X_2(x, F(x))}{\partial x} + \frac{\partial Y_2(x, F(x))}{\partial y}, \]
\[ Z_1(x, y) := X_2(x, F(x) + y) - X_2(x, F(x)), \]
\[ Z_2(x, y) := \frac{W(x, F(x) + y) - W(x, F(x)) - \frac{\partial W(x, F(y))}{\partial y} y}{y^2}. \]

where \( y = F(x) \) is the solution of \(-y + X_2(x, y) = 0\) passing through \((0, 0)\) and \(W(x, y) = Y_2(x, y) - F'(x)X_2(x, y)\). Assume that

\[ f(x) = x^{2n-1} + ax^{2n} + O(x^{2n+1}), \]
\[ \phi(x) = bx^{n-1} + b_0x^n + O(x^{n+1}), \]
\[ Z_2(x, y) = c + O(|x, y|), \]
\[ Z_1(x, y) = dx + d_0y + O(|x, y|^2), \]

with \( b^2 - 4n < 0, n \geq 2 \). Then the origin of the system is monodromic and its stability is given by:

1. The sign of \( b \) when \( n \) is odd.
2. The sign of \( b_0 + b\frac{ad-c-a(n+1)}{2n+1} \) when \( n \) is even.

We end the paper by applying Theorem A to generate limit cycles and to study the center problem for two families of planar vector fields. The first one is included in the so called Kukles system, see for instance [9]. As far as we know this problem has been widely studied when the origin is a non-degenerate monodromic point, but our results for the nilpotent case given in Theorem 4.1 are new. The second example studies a simple, but not easy, family of cubic systems, see Theorem 4.2.

2. Preliminary results

We begin this section stating the Monodromy Theorem for nilpotent critical points (for the original proof by Andreev see [2] and, for a shorter one, [1]).

**Theorem 2.1.** Consider system (1.1) and assume that the origin is an isolated singularity. Define the functions

\[ f(x) := Y_2(x, F(x)) = ax^\alpha + O(x^{\alpha+1}), \quad a \neq 0, \alpha \geq 2, \quad \text{and} \]
\[ \phi(x) := \frac{\partial X_2(x, F(x))}{\partial x} + \frac{\partial Y_2(x, F(x))}{\partial y}, \]

where \( y = F(x) \) is the solution of \(-y + X_2(x, y) = 0\) passing through \((0, 0)\). Write \( \phi(x) = bx^\beta + O(x^{\beta+1}), \ b \neq 0 \) and \( \beta \geq 1, \) or \( \phi(x) \equiv 0. \)
Then, the origin of (1.1) is monodromic if and only if \( a > 0 \), \( \alpha \) is an odd number \((\alpha = 2n - 1)\), and one of the following three conditions holds:

1. \( \beta > n - 1 \),
2. \( \beta = n - 1 \) and \( b^2 - 4an < 0 \),
3. \( \phi \equiv 0 \).

Let us introduce the generalized Lyapunov constants. Firstly, we define them for an analytic periodic system defined on a cylinder \((r, \theta) \in \mathbb{R} \times \mathbb{R}/[0, T]\). Consider the differential equation

\[
\frac{dr}{d\theta} = S(r, \theta) = \sum_{i=1}^{\infty} S_i(\theta)r^i, \tag{2.1}
\]

where the \( S_i(\theta), i \geq 1 \), are \( T \)-periodic functions. Denote by \( r(\theta; (0, \rho)) \) the solution of the above equation such that \( r = \rho \) for \( \theta = 0 \). It can be written

\[
r(\theta; (0, \rho)) = \sum_{i=1}^{\infty} u_i(\theta)\rho^i, \tag{2.2}
\]

with \( u_1(0) = 1, u_k(0) = 0 \), for all \( k \geq 2 \). Hence, the return map is given by the series

\[
P(\rho) = \sum_{i=1}^{\infty} u_i(T)\rho^i.
\]

Fixed a system, the only significative term is the first that makes the return map differ from the identity map, and it will determine the stability of the solution \( r = 0 \). On the other hand, if we consider a family of systems depending on parameters, each of the \( u_i(T) \) depends on these parameters. We will call \( k \)th generalized Lyapunov constant \( V_k = u_k(T) \) the value of this expression assuming \( u_1(T) = 1, u_2(T) = \cdots = u_{k-1}(T) = 0 \).

Notice that a neighbourhood of the solution \( r = 0 \) of system (1.1) belongs to a continuous of periodic solutions if and only if \( P(\rho) \equiv \rho \), i.e., when \( V_1 = 1 \) and \( V_k = 0 \) for all \( k \geq 2 \). In this case we will say that \( r = 0 \) is a center.

One of the main tools to prove Theorem 2.1 and also for the proof of Theorem A are the generalized polar coordinates. These coordinates also will be utilized to write a system having a nilpotent singularity in the form (2.1). Before introducing them we need to define the generalized trigonometrical functions, \( x(\theta) = C_1(\theta), y(\theta) = S_n(\theta) \), firstly considered by Lyapunov, see [10]. These functions are defined as the unique solution of the Cauchy problem

\[
\begin{cases}
\dot{x} = dx/d\theta = -y, \\
\dot{y} = dy/d\theta = x^{2n-1},
\end{cases} \tag{2.3}
\]

with initial conditions \( x(0) = 1, y(0) = 0 \).

We list some of their properties in the next proposition. See [8] or [10] for a proof.
Proposition 2.2.

1. \( \text{Cs}^{2n}(\theta) + n \text{Sn}^2(\theta) = 1 \).
2. \( \text{Cs}(\theta) \) and \( \text{Sn}(\theta) \) are \( T \)-periodic functions where
   \[ T = 2\sqrt{\frac{\pi}{n}} \frac{\Gamma(\frac{1}{2n})}{\Gamma(\frac{p+1}{2n})}. \]
3. \( \int_0^T \text{Sn}^p(\theta) \text{Cs}^q(\theta) d\theta = 0 \) if \( p \) or \( q \) are odd.
4. \( \int_0^T \text{Sn}^p(\theta) \text{Cs}^q(\theta) d\theta = \frac{2}{\sqrt{np+1}} \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+1}{2n} + \frac{q+1}{2n})} \) if both \( p \) and \( q \) are even.

Given any natural number \( n \in \mathbb{N} \), consider the generalized polar coordinates: \( x = r \text{Cs}(\theta), \ y = r^n \text{Sn}(\theta) \). In this system of coordinates a planar system \( \dot{x} = X(x, y), \ \dot{y} = Y(x, y) \) writes as

\[ \dot{r} = \frac{x^{2n-1}X(x, y) + yY(x, y)}{r^{2n-1}} \quad \text{and} \quad \dot{\theta} = \frac{xY(x, y) - nyX(x, y)}{r^{n+1}}. \] (2.4)

As we will see, given any monodromic system of the form (1.1), always exists a suitable \( n \) such that in the corresponding generalized polar coordinates it writes as a system on the cylinder of the form (2.1). The degenerate Lyapunov constants associated to this new system, will be by definition, the generalized Lyapunov constants for a nilpotent singularity. Notice that when the singularity has purely imaginary eigenvalues, the above definition, taking \( n = 1 \), gives the usual Lyapunov constants. In Proposition 2.6 we recall how these generalized constants can be used to generate limit cycles from a critical point.

We also need the following two known results, see [8,14]. Their proofs are included for the sake of completeness.

Lemma 2.3. Near the origin and after a reparametrization of the time, system (1.2),

\[ \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= a(x) + y\tilde{b}(x), \end{aligned} \]

with \( a(x) = x^{2n-1}(a_{2n-1} + O(x)) \), \( a_{2n-1} > 0 \), can be transformed into:

\[ \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x^{2n-1} + yb(x), \end{aligned} \] (2.5)

where \( b(x) \) is an analytic function given in the proof.

Proof. We make the following change in the variables \( x \) and time:

\[ u = 2^n \sqrt{\frac{2n}{\int_0^x a(s) \, ds}} := \Phi(x) = x^{2n}\sqrt{a_{2n-1} + O(x)}, \]

\[ \frac{dt}{dt_1} = \frac{u^{2n-1}}{a(x)} = \frac{(a_{2n-1}x^{2n} + O(x^{2n+1}))(2n-1)/(2n)}{a_{2n-1}x^{2n-1} + O(x^{2n})} = a_{2n-1}^{-(1/(2n))} + O(x). \]
Some easy computations lead us to
\[
\begin{align*}
  u' &= -y, \\
  y' &= u^{2n-1} + y\left(\frac{u^{2n-1}b(\Phi^{-1}(u))}{a(\Phi^{-1}(u))}\right).
\end{align*}
\]
Notice that the above vector field is also analytic, as we wanted to prove. \(\square\)

**Remark 2.4.** For instance, in the above lemma, if \(a(x) = x^{2n-1}(1 + \alpha x + O(x^2))\) and \(\tilde{b}(x) = x^{n-1}(b + \beta x + O(x^2))\), then \(b(x) = x^{n-1}(b + [\beta - \alpha^{n+1}]x + O(x^2))\).

**Proposition 2.5** (Characterization of the centers). Consider the analytic system
\[
\begin{align*}
  \dot{x} &= -y, \\
  \dot{y} &= x^{2n-1} + yb(x),
\end{align*}
\]
being \(b(x) = \sum_{j \geq \beta} b_j x^j\) and having the origin as a monodromic critical point, i.e., satisfying one of the following conditions:

(i) \(\beta > n - 1\),
(ii) \(\beta = n - 1\) and \(b_{\beta}^2 - 4n < 0\),
(iii) \(b(x) \equiv 0\).

Then the origin is a center if and only if \(b^e(x) := \frac{b(x) + b(-x)}{2} \equiv 0\).

**Proof.** Because of Theorem 2.1 we have the monodromy conditions for system (2.6).

If \(b^e(x) \equiv 0\) by applying the change of variables and time \(x = -\tilde{x}, t = -\tilde{t}\), we obtain the same system. Hence using the reversibility criterium of Poincaré, we have proved that the origin of system (2.6) is a center. Let us prove the converse. Assume that \(b^e(x) = x^{2\ell}(b_{2\ell} + O(x))\), for some \(b_{2\ell} \neq 0\). Set \(b^o(x) = b(x) - b^e(x)\). We already know that a system with \(b(x)\) odd has a center around the origin. Given any system we can take the level curves of the system with the odd part of \(b(x)\) as a kind of Lyapunov function for the system with the whole \(b(x)\). Concretely, note that
\[
\begin{align*}
  -y \left( x^{2n-1} + yb(x) \right) &\equiv -y^2\left( b^o(x) - b(x) \right) = y^2 x^{2\ell}(b_{2\ell} + O(x)),
\end{align*}
\]
does not change sign in a neighbourhood of the origin. Thus the origin is not a center and the proposition follows. \(\square\)

To end this section we recall how the expressions of the generalized Lyapunov constants can be used to generate limit cycles from a nilpotent critical point. Next result is common knowledge and extends to nilpotent critical points the usual degenerate Andronov–Hopf bifurcation.

**Proposition 2.6.** Let \(X(\lambda)\) be an analytic family of planar vector fields depending on some parameters \(\lambda \in \mathbb{R}^m\), and let \(\Lambda \subset \mathbb{R}^m\) be such that for all \(\lambda \in \Lambda\) the origin is a nilpotent monodromic critical point of \(X(\lambda)\). Assume that there is a function \(c: \mathbb{R}^\ell \rightarrow \Lambda\) and an
integer number \( m_{\ell+1} \), such that, until \( V_{m_{\ell+1}} \), all the generalized Lyapunov constants associated to the vector field \( X(c(\alpha_1, \alpha_2, \ldots, \alpha_\ell)) \) are zero, except \( V_{m_i} \), \( i = 1, 2, \ldots, \ell+1 \). Moreover, assume also that:

(i) these constants are

\[
V_{m_1} = h_1(\alpha_1, \alpha_2, \ldots, \alpha_\ell), \quad V_{m_2} = h_2(\alpha_2, \alpha_3, \ldots, \alpha_\ell), \ldots, \\
V_{m_{\ell-1}} = h_{\ell-1}(\alpha_{\ell-1}, \alpha_\ell), \quad V_{m_\ell} = h_\ell(\alpha_\ell), \quad V_{m_{\ell+1}} = H_\ell+1,
\]

being \( 2 \leq m_1 < m_2 < \cdots < m_\ell < m_{\ell+1} \) integer numbers and \( H_{\ell+1} \neq 0 \),

(ii) \( h_1(0, \alpha_2, \ldots, \alpha_\ell) = h_2(0, \alpha_3, \ldots, \alpha_\ell) = h_{\ell-1}(0, \alpha_\ell) = h_\ell(0) = 0 \),

(iii) \( H_1, H_2, \ldots, H_\ell, H_{\ell+1} \) alternate sign, where \( H_i = \frac{\partial h_i}{\partial \alpha_i}(0, 0, \ldots, 0) \).

Then, if we take the values \( \alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell \) satisfying

\[
|\alpha_1| \ll |\alpha_2| \ll \cdots \ll |\alpha_{\ell-1}| \ll |\alpha_\ell| \ll |H_{\ell+1}|,
\]

then the ordinary differential equation associated to \( X(c(\alpha_1, \alpha_2, \ldots, \alpha_\ell)) \) has at least \( \ell \) limit cycles in a sufficiently small neighbourhood of the origin.

The parametrization \( c(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \) given in each of the families studied in Section 4 is found by studying the concrete expressions of the generalized Lyapunov constants.

3. Proof of Theorems A and B

To prove Theorem A, we give the following key result, inspired in [6,17]. It provides a way of computing the first significative generalized constant for a system defined on a cylinder taking advantage of a center “near” the given system.

**Proposition 3.1.** Consider an analytic ordinary differential equation of the form

\[
\frac{dr}{d\theta} = S(r, \theta) + r^m(v(\theta) + O(r)), \quad m \geq 1,
\]

defined on a cylinder \((r, \theta) \in \mathbb{R} \times \mathbb{R}/[0, T]\) and assume that the equation

\[
\frac{dr}{d\theta} = S(r, \theta) = r(S_1(\theta) + O(r)),
\]

is such that \( r = 0 \) is a periodic solution, and furthermore that all the solutions near this one are also periodic \((r = 0 \text{ is a center})\). Thus, if \( m = 1 \), \( V_1 = \exp\left(\int_0^Tv(\theta)d\theta\right) \). If \( m > 1 \) then \( V_1 = 1 \), all the generalized constants for (3.1) from \( V_2 \) until \( V_{m-1} \) are zero, and

\[
V_m = \int_0^Tv(\theta)e^{(m-1)\int_0^\theta S_1(\psi)d\psi}d\theta.
\]

**Proof.** Following [6,17] we will express (3.1) in a new coordinate system \((R, \theta)\), being \( R = F(r, \theta) \) a change of variables such that \( dR/d\theta \equiv 0 \). Let us construct this change. The
geometrical idea is to assign to all the points living in a periodic orbit of Eq. (3.2) the value of the intersection of the orbit with the line $\theta = 0$. More concretely, let $\varphi(\theta; (\theta_0, r_0))$ the solution of (3.2) such that $r = r_0$ when $\theta = \theta_0$. Then in the variable

$$ R = F(\theta, r) := \varphi(-\theta; (\theta, r)). $$

(3.4)

Eq. (3.2) writes as $dR/d\theta = 0$. The inverse of the above change is given by

$$ r = G(\theta; R) := \varphi(\theta; (0, R)). $$

(3.5)

Writing $\varphi(\theta; (0, R)) = R(u_1(\theta) + O(R))$, we get that $u_1(\theta) = \exp(\int_0^\theta S_1(\psi)\,d\psi)$. This equality proves that (3.4) and (3.5) define an actual change of variables and also that

$$ r = G(\theta, R) = R(u_1(\theta) + O(R)) \quad \text{and} \quad R = F(\theta, r) = r\left(\frac{1}{u_1(\theta)} + O(r)\right). $$

Let us write Eq. (3.1) in this new system of coordinates:

$$ \frac{dR}{d\theta} = \frac{\partial F(\theta, r)}{\partial \theta} + \frac{\partial F(\theta, r)}{\partial r} \frac{dr}{d\theta} \\
= \frac{\partial F(\theta, r)}{\partial \theta} + \frac{\partial F(\theta, r)}{\partial r} \left\{ S(r, \theta) + r^m \left( v(\theta) + O(r) \right) \right\} \\
= \frac{\partial F(\theta, r)}{\partial r} \left\{ r^m \left( v(\theta) + O(r) \right) \right\} = r^m \left( \frac{v(\theta)}{u_1(\theta)} + O(r) \right) \\
= R^m \left( u_1^{m-1}(\theta)v(\theta) + O(R) \right). $$

(3.6)

Since at $\theta = 0$ the two coordinates $r$ and $R$ coincide, we calculate the Poincaré map from $\theta = 0$ to $\theta = T$ in the last variable. Write the solution starting at $R = r = \rho$ when $\theta = 0$ as

$$ R(\theta, \rho) = \sum_{i \geq 1} w_1(\theta)\rho^i, $$

with $w_1(0) = 1$ and $w_i(0) = 0$ for $i \geq 1$. If $m = 1$ by replacing it in the above differential equation we obtain that $w_1(\theta) = v(\theta)w_1(\theta)$, and so $w_1(\theta)$ can be easily obtained. If $m > 1$ then $w_1(\theta) \equiv 1, w_2(\theta) \equiv \cdots \equiv w_{m-1}(\theta) \equiv 0$ and

$$ w_m(\theta) = \int_0^\theta u_1^{m-1}(\psi)v(\psi)\,d\psi. $$

Since $V_m = w_m(T)$, the result follows. □

**Proof of Theorem A.** As we have already explained, the first part of the statement is proved in [14]. To give the expression of the generalized Lyapunov constants of the normal form (1.5) let us write the system in the generalized polar coordinates, $(x, y) = (r\,Cs(\theta), r^n\,Sn(\theta))$. We get

$$ \frac{dr}{d\theta} = T(r, \theta) := \frac{\sum_{j \geq 0} b_j \,Sn^2(\theta)Cs^j(\theta)r^{2-n+j}}{1 + \sum_{j \geq 0} b_j \,Sn(\theta)Cs^{j+1}(\theta)r^{1-n+j}}. $$

(3.6)
Consider the new differential equation

$$\frac{dr}{d\theta} = S(r, \theta) := \sum_{j \geq \beta, j \text{ odd}} b_j \frac{S_n^2(\theta)C_{s_j}(\theta)r^{2-n+j}}{1 + \sum_{j \geq \beta, j \text{ odd}} b_j \frac{S_n(\theta)C_{s_j+1}(\theta)r^{1-n+j}}{1 + b_{j-1}S_n(\theta)C_s(\theta)}}.$$  \hfill (3.7)

By using Proposition 2.5, we know that the origin is a center for Eq. (3.7). In order to apply Proposition 3.1, let us compute $T(r, \theta) - S(r, \theta)$. Recall that we assume that $b_{2\ell} \neq 0$ and that $2\ell \geq n - 1$.

$$T(r, \theta) - S(r, \theta) = b_{2\ell}r^{2-n+2\ell}(v(\theta) + O(r)),$$

where

$$v(\theta) = \begin{cases} S_n^2(\theta)C_{s_{2\ell}}(\theta), & \text{when } \beta > n - 1, \\ \frac{S_n^2(\theta)C_{s_{2\ell}}(\theta)}{(1 + b_{n-1}S_n(\theta)C_s(\theta))^2}, & \text{when } \beta = n - 1, \text{ and } \beta \text{ is odd,} \\ \frac{S_n^2(\theta)C_{s_{2\ell}}(\theta)}{1 + b_{n-1}S_n(\theta)C_s(\theta)}, & \text{when } \beta = 2\ell = n - 1. \end{cases}$$

We also need to compute, in each of the cases, $S(r, \theta) = r(S_1(\theta) + O(r))$. For the first and the third situation $S_1(\theta) \equiv 0$. In the second one, namely $\beta = n - 1$ and $\beta$ odd, we get that

$$S_1(\theta) = \frac{b_{n-1}S_n^2(\theta)C_{s_{n-1}}(\theta)}{1 + b_{n-1}S_n(\theta)C_s(\theta)}.$$  \hfill (3.8)

Thus, if $\beta = 2\ell = n - 1$ then

$$V_1 = \exp \left[ \int_0^\theta \frac{b_{n-1}S_n^2(\theta)C_{s_{2\ell}}(\theta)}{1 + b_{n-1}S_n(\theta)C_s(\theta)} d\theta \right].$$  \hfill (3.9)

Otherwise, we have $V_1 = 1$, $V_2 = \cdots = V_{1-n+2\ell} = 0$ and $V_{2-n+2\ell} = b_{2\ell} \int_0^\theta w(\theta) d\theta$, where

$$w(\theta) = \begin{cases} S_n(\theta)C_{s_{2\ell}}(\theta), & \text{when } \beta > n - 1, \\ \frac{S_n(\theta)C_{s_{2\ell}}(\theta)\exp((1-n+2\ell)\int_0^\theta S_1(\psi)\psi)\psi)}{(1 + b_{n-1}S_n(\theta)C_s(\theta))^2}, & \text{when } \beta = n - 1, \text{ and } \beta \text{ odd,} \end{cases}$$

where $S_1(\theta)$ is given in (3.8). The above expressions give a proof of statement (a) because notice that $w(\theta)$ is a non-negative function. To end the proof we explicitly compute the above integrals in two of the three cases. We have not been able to obtain the exact expression when $\beta = n - 1$ and $\beta$ is odd.

For computing the expression (3.9) we introduce the variable $x = \frac{S_n(\theta)}{C_s(\theta)}$. We obtain

$$V_1 = \exp \left( \int_{-\infty}^{+\infty} \frac{2b_{n-1}x^2 dx}{(1 + b_{n-1}x + nx^2)(1 + nx^2)} \right) = \exp \left( \frac{2b_{n-1}\pi}{n\sqrt{4n - b_{n-1}^2}} \right).$$
as we wanted to prove. When $\beta > n - 1$ we have that

$$V_{2-n+2\ell} = b_{2\ell} \int_0^T \text{Sn}^2(\theta) \text{Cs}^{2\ell}(\theta) \, d\theta = \frac{\sqrt{\pi} \Gamma\left(\frac{2\ell+1}{2n}\right)}{\sqrt{n^3} \Gamma\left(\frac{3n+2\ell+1}{2n}\right)} b_{2\ell},$$

where we have used Proposition 2.2.

**Proof of Theorem B.** Consider system (1.1). By introducing the change of variables $x = x$ and $u = y - F(x)$, and renaming again $u$ as $y$ we obtain system

$$\begin{cases}
\dot{x} = y(-1 + Z_1(x, y)), \\
\dot{y} = f(x) + y\phi(x) + y^2 Z_2(x, y),
\end{cases} \quad (3.10)$$

where the above functions are defined in the statement of the theorem. Note that $Z_1(x, y) = O((|x, y|^2))$. From Theorem 2.1 we already know that the origin is a monodromic critical point. Let us try to compute its first significant generalized Lyapunov constant straightly from the definition given in Section 2.

System (3.10), in the generalized polar coordinates $(x, y) = (r \text{Cs}(\theta), r^n \text{Sn}(\theta))$ can be written as

$$\frac{dr}{d\theta} = \frac{A(r, \theta)}{B(r, \theta)} = S_1(\theta)r + S_2(\theta)r^2 + O(r^3),$$

where

$$A(r, \theta) = b \text{Cs}^{n-1}(\theta) \text{Sn}^2(\theta)r$$

$$+ \left[ (a + d) \text{Cs}^{2n}(\theta) \text{Sn}(\theta) + b_0 \text{Cs}^n(\theta) \text{Sn}^2(\theta) + c \text{Sn}^3(\theta) \right] r^2 + O(r^3),$$

$$B(r, \theta) = 1 + b \text{Cs}^n(\theta) \text{Sn}(\theta)$$

$$+ \left[ a \text{Cs}^{2n+1}(\theta) + b_0 \text{Cs}^{n+1}(\theta) \text{Sn}(\theta) + (c - nd) \text{Cs}(\theta) \text{Sn}^2(\theta) \right] r + O(r^2).$$

Notice that

$$S_1(\theta) = \frac{b \text{Cs}^{n-1}(\theta) \text{Sn}^2(\theta)}{1 + b \text{Cs}^n(\theta) \text{Sn}(\theta)},$$

and $S_2$ can also be easily obtained. Write a solution of the above equation as

$$r(\theta; (0, \rho)) = \sum_{i=1}^{\infty} u_i(\theta) \rho^i,$$

with $u_1(0) = 1$, $u_k(0) = 0$, for all $k \geq 2$. Then

$$V_1 = u_1(T) = \exp\left( \int_0^T S_1(\theta) \, d\theta \right) = \begin{cases}
\exp\left( \frac{2b\pi}{\sqrt{4n-b^2}} \right), & \text{if } n \text{ odd}, \\
0, & \text{if } n \text{ even}.
\end{cases}$$

Thus the case $n$ odd follows. If $n$ is even we have to compute $V_2$. By Proposition 3.1,

$$V_2 = \int_0^T S_2(\theta) \left( \exp\int_0^\theta S_1(\phi) \, d\phi \right) \, d\theta.$$
We have not been able to compute the above expression, but from it we realize that \( V_2 \) can also be computed by studying the truncated system

\[
\begin{align*}
\dot{x} &= y(-1 + dx), \\
\dot{y} &= x^{2n-1} + ax^{2n} + y(bx^{n-1} + b_0x^n) + cy^2.
\end{align*}
\] (3.11)

It is well known that the above system can be transformed into a Liénard system, see [16, Theorem 15.15]. Consider the change of variables

\[x = x, \quad z = y(1 - dx)\Psi(x) \text{ and } \frac{dt}{d\tau} = \Psi(x),\]

where

\[\Psi(x) = \frac{1}{1-dx} \exp \int_0^x \frac{c}{1-du} \, du.\]

In these coordinates the system writes as \((z \text{ is again } y)\)

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^{2n-1} + (a + d + 2c)x^{2n} + O(x^{2n+1}) \\
&\quad + y(bx^{n-1} + (b_0 + b(d + c))x^n + O(x^{n+1})).
\end{align*}
\]

By using Remark 2.4 the above system is transformed into

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^{2n-1} + y(bx^{n-1} + [b_0 + b\frac{nd-c-a(n+1)}{2n+1}]x^n + O(x^{n+1})).
\end{align*}
\]

From Theorem A the result follows. \(\square\)

4. Applications

This section is devoted to apply Theorem A to generate limit cycles for several families of planar vector fields having a nilpotent singularity.

We start with a simple example to understand how our method works. Consider the system

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^5 + ax^6 + y(bx^3 + cx^4).
\end{align*}
\]

By using Lemma 2.3, we get:

\[
\begin{align*}
x_1' &= -y, \\
y' &= x_1^5 + y(bx_1^3 + (c - \frac{5}{7}ab)x^4 + (\frac{36}{39}a^2b - 6ac)x^5 \\
&\quad + \frac{13}{294}a^2(21c - 19ab)x^6 + \frac{80}{1029}a^3(13ab - 14c)x^7 + O(x^8)).
\end{align*}
\]

Applying Theorem A, we know that its first generalized Lyapunov constants are \( V_3 = K_3(c - 5/7ab) \) and \( V_5 = K_5a^2(21c - 19ab) \), with \( K_i > 0, \ i = 3, 5 \). In order to have a center both generalized constants must be zero, then \( ab = c = 0 \). It is easy to check that under these conditions the system has actually a center. Let us prove that there are systems inside this family having at least one limit cycle surrounding the origin. Consider the next 1-parameter family

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^5 + x^6 + y(x^3 + (\frac{5}{7} + \alpha)x^4).
\end{align*}
\]

For this family, \( V_3 = K_3\alpha \) and \( V_5 = -4K_5 \). Then by using Proposition 2.6 we know that if we choose \( \alpha > 0 \) and \( \alpha \ll 1 \) then the system has, at least, one limit cycle around the origin.
We study now a family of the so-called Kukles system, with 6 parameters. We characterize the centers of the family and get a lower bound for its cyclicity. Kukles systems having the origin as a non-degenerate singularity have been widely studied but, as far as we know, our results on the case of a degenerate singularity are new.

**Theorem 4.1.** Consider next system of Kukles type

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,
\end{align*}
\]

(4.1)

with \(a_{30} > 0\) and \(a_{11}^2 - 8a_{30} < 0\) (these two conditions are the monodromy conditions).

The only families inside (4.1) with a center at the origin are \(a_{21} = a_{03} = a_{11}a_{02} = 0\).

Moreover, there exist systems of the form (4.1) with 3 limit cycles around the origin.

**Proof.** By applying Theorem 2.1, we know that, if \(a_{30} > 0\) and \(a_{11}^2 - 8a_{30} < 0\), system (4.1) is monodromic. We transform it into its Takens normal form and then we apply Lemma 2.3 for having it into its simplest form, i.e.,

\[
\begin{align*}
\dot{x} &= -y + O(10), \\
\dot{y} &= x^3 + y(\sum_{i=1}^{8} b_ix^i) + O(10),
\end{align*}
\]

(4.2)

where we have explicitly calculated all the \(b_i\) for \(i = 1, \ldots, 8\).

Applying Theorem A we know the first generalized Lyapunov constant of system (4.2), and then, of system (4.1), is

\[V_2 = K_2b_2 = K_2 \left( \frac{a_{02}a_{11}}{5} + a_{21} \right),\]

with \(K_2 > 0\).

If it is zero, then the second generalized Lyapunov constant is \(V_4 = K_4b_4\); using the fact that \(V_2 = 0\), we obtain

\[V_4 = K_4 \left( a_{03} \left( 3 - \frac{2}{7}a_{11}^2 \right) - \frac{1}{175}a_{02}a_{11} \left( a_{02}^2 - 30a_{12} \right) \right).\]

If this expression has to be zero, we have two possibilities (verifying that the system is monodromic): the first one \(a_{03} = a_{02}a_{11} = 0\) which is a reversible center or \(a_{11}a_{02} \neq 0\) and \(a_{12} = (-525a_{03} + a_{02}^3a_{11} + 50a_{03}a_{11}^2)/(30a_{02}a_{11})\). As \(a_{02} \neq 0\) we can make a linear change of coordinates and consider \(a_{02} = 1\). We substitute these conditions in \(b_6\) and we obtain

\[V_6 = K_6 \frac{628a_{11}^2 + 1875a_{03}a_{11}(33 + 10a_{11}^2) - 15625a_{03}^2(1197 - 156a_{11}^2 + 4a_{11}^4)}{506250a_{11}}.\]

We again have two possibilities for this constant to be zero:

\[a_{03} = \frac{495a_{11} + 150a_{11}^2 \pm \sqrt{3251889a_{11}^2 - 243372a_{11}^4 + 32548a_{11}^6}}{250(1197 - 156a_{11}^2 + 4a_{11}^4)}\]

(we remark that the denominator above does not vanish under the monodromy conditions). In each of them, if we substitute in \(b_8\), the conditions for having a center are the solutions.
of a polynomial of degree 10. We study them and there are only 4 real solutions and none of them verifies the monodromy condition \( a_{11}^2 < 8 \); it means there are no families of centers verifying the previous conditions and then the only ones are \( a_{03} = a_{21} = a_{11}a_{02} = 0 \).

In order to prove the second part of the theorem, consider the system

\[
\begin{align*}
    \dot{x} &= -y, \\
    \dot{y} &= xy + y^2 + x^3 + \left( \frac{1}{5} + \alpha_2 \right)x^2y + \left( \frac{-95 + \sqrt{3041065}}{16500} + \alpha_6 \right)xy^2 \\
    &\quad + \left( \frac{645 - \sqrt{3041065} - 16500\alpha_6}{261250} + \alpha_4 \right)y^3. 
\end{align*}
\] (4.3)

This system is monodromic because it verifies the hypothesis of Theorem 2.1.

Its generalized Lyapunov constants are

\[
\begin{align*}
    V_2 &= K_2\alpha_2, \\
    V_4 &= K_4\alpha_4, \\
    V_6 &= K_6 \left( -\frac{\alpha_6(\sqrt{3041065} + 8250\alpha_6)}{64125} \right), \\
    V_8 &= -K_8, \\
\end{align*}
\]

with \( K_i > 0, i \in \{2, 4, 6, 8\} \).

Applying Proposition 2.6, if we choose \( \alpha_6, \alpha_4 < 0 \), and \( \alpha_2 > 0 \) in such a way that \( |\alpha_2| \ll |\alpha_4| \ll |\alpha_6| \ll 1 \), we obtain a system with, at least, 3 limit cycles around the origin. \( \square \)

In the next theorem we study a simple 3-parameter cubic family, without centers, for which we are also able to generate 3 limit cycles from the origin.

**Theorem 4.2.** Consider the next system

\[
\begin{align*}
    \dot{x} &= -y + Ax^2 + Bxy + Cy^2, \\
    \dot{y} &= x^3 + xy^2 + y^3, 
\end{align*}
\] (4.4)

with \( A^2 < 2 \) (the monodromy condition). The origin of the system is always a focus. Moreover, there are systems inside this family with 3 limit cycles around the origin.

**Proof.** By Theorem 2.1 the monodromy condition is \( A^2 < 2 \).

We compute the normal form for the system up to order 9 and according to Theorem A we get the first Lyapunov constant:

\[
V_2 = K_2 \left( \frac{9}{5} AB \right).
\]

There are two possibilities for being zero. We investigate both separately.

**Case** \( B = 0 \). In this case, we compute the second Lyapunov constant and we get

\[
V_4 = K_4 \left( 3 + A^2 - \frac{10}{7} A^4 \right).
\]
We have again two possibilities in order to have a center, \( A = \pm \sqrt{\frac{7+\sqrt{889}}{20}} \). For each of them we compute the third generalized Lyapunov constant, obtaining:

\[
V_6 = K_6 \left(\frac{150121 + 4363\sqrt{889}}{27000} \mp 45\sqrt{5(7 + \sqrt{889})(313 + 19\sqrt{889})C}\right).
\]

In each case, we impose that it has to be zero and compute the next Lyapunov constant. In both cases we get

\[
V_8 = K_8 \left(-\frac{2(21192358735517 + 715954002551\sqrt{889})}{928125(7 + \sqrt{889})(313 + 19\sqrt{889})^2}\right) < 0.
\]

Thus when \( B = 0 \) system (4.4) can not have a center at the origin.

**Case \( A = 0 \) (and \( B \neq 0 \)).** We compute the second generalized Lyapunov constant,

\[
V_4 = K_4(3 + BC).
\]

The only possibility to be zero is \( C = -\frac{3}{B} \). In this case,

\[
V_6 = K_6 \left(-\frac{7}{5}(2 + B^2)\right) < 0,
\]

and again, it can not have a center at the origin.

In order to prove the second part of the theorem, consider system (4.4) with the following coefficients:

\[
A = -\sqrt{\frac{7 + \sqrt{889}}{20}} + \alpha_4, \quad B = \alpha_2,
\]

\[
C = -\frac{150121 + 4363\sqrt{889}}{(45\sqrt{5(7 + \sqrt{889})})(313 + 19\sqrt{889})} + \alpha_6,
\]

where \( \alpha_2, \alpha_4 \) and \( \alpha_6 \) are small parameters.

The first Lyapunov constants are:

\[
V_2 = K_2 \left(-\frac{9\alpha_2}{50} \left(\sqrt{5(7 + \sqrt{889})} - 10\alpha_4\right)\right),
\]

\[
V_4 = K_4 \left(\frac{\alpha_4}{35} \left(\sqrt{4445(7 + \sqrt{889})} - 5(14 + 3\sqrt{889})\alpha_4
\right.
\]

\[
\left.\quad + 20\sqrt{5(7 + \sqrt{889})\alpha_4^2} - 50\alpha_4^3\right)\right),
\]

\[
V_6 = K_6 \left(\frac{\alpha_6}{60(313 + 19\sqrt{889})} \sqrt{\frac{7 + \sqrt{889}}{5}} (209449 + 5947\sqrt{889}) \right),
\]

\[
V_8 = K_8 \left(-\frac{2(21192358735517 + 715954002551\sqrt{889})}{928125(7 + \sqrt{889})(313 + 19\sqrt{889})^2}\right),
\]

where \( K_i > 0 \) for \( i \in \{2, 4, 6, 8\} \).
If we choose $\alpha_2, \alpha_4 < 0$ and $\alpha_6 > 0$ such that $|\alpha_2| \ll |\alpha_4| \ll |\alpha_6| \ll 1$ then, applying Proposition 2.6, we get that system (4.4) has, at least 3 limit cycles around the origin, as we wanted to prove. □

References