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A perfect GO-space which cannot densely embed in any perfect orderable space

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Abstract

In this paper, we prove that there exists a perfect GO-space which cannot densely embed in any perfect orderable space. This result answers an open question: "Does every perfect GO-space have an orderable perfect space in which it densely embeds?"

Keywords: GO-space; LOTS; Orderable d-extension; Perfect GO-space

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1. Introduction

A *linearly ordered topological space* (abbreviated *LOTS*) is a triple $\langle X, \lambda, \leqslant\rangle$, where $\langle X, \leqslant\rangle$ is a linearly ordered set and λ is the usual interval topology defined by $\leqslant$ (i.e., λ is the topology generated by $\{(a,\rightarrow)(\leqslant)\colon a \in X\} \cup \{(\leftarrow, a)(\leqslant)\colon a \in X\}$ as a subbase), where $(a,\rightarrow)(\leqslant) = \{x \in X\colon a < x\}$ and $(\leftarrow, a)(\leqslant) = \{x \in X\colon x < a\}$. Similarly $(a,b)(\leqslant) = \{x \in X\colon a < x < b\}$, $[a,b)(\leqslant) = \{x \in X\colon a \leqslant x < b\}$, etc. When it is needed to emphasize on which set the ordering or the topology is defined, we will write $\leqslant_X$, λ_X instead of $\leqslant$, λ. A *generalized ordered space* (abbreviated *GO-space*) is a triple $\langle X, \mathcal{T}, \leqslant\rangle$, where $\langle X, \leqslant\rangle$ is a linearly ordered set and $\mathcal{T}$ is a topology on X such that $\lambda \subset \mathcal{T}$ and $\mathcal{T}$ has a base consisting of order convex sets, where a subset A of X is called *order convex* or simply *convex* if $x \in A$ for every x lying between two points of A. A LOTS $\langle Y, \lambda_Y, \leqslant_Y\rangle$ is called a *linearly ordered extension* of a GO-space $\langle X, \mathcal{T}, \leqslant_X\rangle$ if

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$X \subset Y$, $\mathcal{T} = \lambda_Y|X$ $(= \{U \cap X: U \in \lambda_Y\})$ and $\leqslant_X = \leqslant_Y |X$ (i.e., $\leqslant_Y$ is an extending ordering of $\leqslant_X$), and we also say that the GO-space $\langle X, \mathcal{T}, \leqslant_X\rangle$ can order-preservingly embed in the LOTS $\langle Y, \lambda_Y, \leqslant_Y\rangle$. Furthermore if X is also dense (respectively, closed) in the space $\langle Y, \lambda_Y, \leqslant_Y\rangle$, then $\langle Y, \lambda_Y, \leqslant_Y\rangle$ is said to be a *linearly ordered d-extension* (respectively, *c-extension*) of $\langle X, \mathcal{T}, \leqslant_X\rangle$. A topological space $\langle X, \mathcal{T}\rangle$ is called *orderable* if there exists a linear ordering $\leqslant$ on X such that the open interval topology defined by $\leqslant$ coincides with $\mathcal{T}$. An orderable space $\langle Y, \lambda_Y\rangle$ is called an *orderable* (respectively, *d*-, *c*-) *extension* of a GO-space $\langle X, \mathcal{T}, \leqslant\rangle$ if X is a (respectively, dense, closed) subset of Y and $\mathcal{T} = \lambda_Y|X$ (where we do not require that the ordering on Y extends the ordering $\leqslant$ on X).

It is well known that a topological space $\langle X, \mathcal{T}\rangle$ is a GO-space together with some ordering $\leqslant_X$ on X if and only if $\langle X, \mathcal{T}\rangle$ is a topological subspace of some LOTS $\langle Y, \lambda, \leqslant_Y\rangle$ with $\leqslant_X = \leqslant_Y|X$, so any GO-space has an orderable extension. Naturally we are interested in that what topological properties which a GO-space has can be inherited by some orderable extension of the GO-space. It is known that metrizability and (hereditary) paracompactness of a GO-space can be inherited by some of its linearly ordered c-extensions [6]. But for the perfectness (a space is *perfect* if every open subset of the space is a F_σ-set) the following problem posed in [3] remains open.

Problem 1.1. Does every perfect GO-space have a perfect orderable extension?

Related to this problem, it is known that the Sorgenfrey line S is a perfect GO-space which has no perfect orderable c-extension [6, Theorem 5.9], but S has a perfect linearly orderable d-extension [3]. The following problem which is a special case of Problem 1.1 was posed in [2, "Posed problems" 8] or [7, Question (V)].

Problem 1.2. Does every perfect GO-space have a perfect orderable d-extension?

Some partial answers have been obtained. Miwa and Kemoto proved that there exists a perfect GO-space which does not have any perfect linearly ordered d-extension [9]. Also Bennett, Hosobuchi and Miwa [1] gave some conditions under which Problem 1.1 and Problem 1.2 have affirmative answers.

In this paper, we shall give a negative answer to Problem 1.2 by proving the following theorem:

Theorem 1.3. *There exists a perfect GO-space which has no perfect orderable d-extension.*

Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{Q}$ denotes the set of all rational numbers. For undefined terminologies we refer the reader to [4].

2. Preliminaries for the proof of Theorem 1.3

First we state the construction of the space which satisfies Theorem 1.3. Throughout this paper in the below, we use the notations of this section.

Example 2.1. Let X be the set of all real numbers, K the Cantor set and $\leqslant$ the usual ordering on X. Put $T = \bigcup\{K+q\colon\ q \in \mathbb{Q}\}$, where $K+q = \{x+q\colon\ x \in K\}$. Let the topology $\mathcal{T}$ on X have a base as follows:

$$\{[x, x+\varepsilon)(\leqslant)\colon\ \varepsilon > 0,\ x \in X - T\} \cup \{\{x\}\colon\ x \in T\}.$$

Then $\langle X, \mathcal{T}, \leqslant\rangle$ is a GO-space (this space was constructed in [8]). Since T is clearly a σ-discrete dense set of $\langle X, \mathcal{T}, \leqslant\rangle$, $\langle X, \mathcal{T} \leqslant\rangle$ is perfect (see [5, Theorem 2.4.5 and the proof on p. 51]).

Let $\langle Y, \lambda_Y\rangle$ be an orderable d-extension of $\langle X, \mathcal{T}, \leqslant\rangle$. Then there exists an ordering $\leqslant_Y^{\sim}$ on Y such that $\langle Y, \lambda_Y, \leqslant_Y^{\sim}\rangle$ is a LOTS and satisfies $X \subset Y$, $\mathcal{T} = \lambda_Y|X$ and $\mathrm{cl}_Y X = Y$. Then the LOTS $\langle Y, \lambda_Y, \leqslant_Y^{\sim}\rangle$ is a linearly ordered d-extension of the GO-space $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$, where $\leqslant^{\sim} = \leqslant_Y^{\sim}|X$. It is known that for the GO-space $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ there is a minimal linearly ordered d-extension $\langle \widetilde{X}, \lambda_{\widetilde{X}}, \leqslant_{\widetilde{X}}^{\sim}\rangle$ constructed as follows [9]:

Let λ_X be the order topology on $\langle X, \leqslant^{\sim}\rangle$, and let

$$\begin{aligned}\widetilde{X} = X \times \{0\} &\cup \{\langle x, -1\rangle\colon\ x \in X \text{ and } [x, \rightarrow)(\leqslant_X^{\sim}) \in \mathcal{T} - \lambda\}\\ &\cup \{\langle x, 1\rangle\colon\ x \in X \text{ and } (\leftarrow, x](\leqslant_X^{\sim}) \in \mathcal{T} - \lambda\}\end{aligned}$$

be a subset of the lexicographic product $X \times \{-1, 0, 1\}$. Let $\leqslant_{\widetilde{X}}^{\sim}$ be the lexicographic ordering on $\widetilde{X}$ and let $\tilde{\lambda}_{\widetilde{X}}$ be the associated order topology on $\langle \widetilde{X}, \leqslant_{\widetilde{X}}^{\sim}\rangle$.

It is proved in [9] that $\langle \widetilde{X}, \tilde{\lambda}_{\widetilde{X}}, \leqslant_{\widetilde{X}}^{\sim}\rangle$ can be densely embedded by an order preserving homeomorphism in any linearly ordered d-extension of $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$. Because perfectness is a hereditary property, to prove that $\langle Y, \lambda_Y, \leqslant_Y^{\sim}\rangle$ is not perfect it will be enough to prove $\langle \widetilde{X}, \tilde{\lambda}_{\widetilde{X}}, \leqslant_{\widetilde{X}}^{\sim}\rangle$ is not perfect. We will denote $\langle \widetilde{X}, \tilde{\lambda}_{\widetilde{X}}, \leqslant_{\widetilde{X}}^{\sim}\rangle$ by $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ (cf. [1]).

Because $\langle X, \mathcal{T}, \leqslant\rangle$ and $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ have the same topology $\mathcal{T}$, we will simply say that a subset of X is open or closed in $\langle X, \mathcal{T}\rangle$ without mentioning the ordering $\leqslant$ or $\leqslant^{\sim}$. In the following the concepts "dense" and "nowhere dense" will be always in the sense of the Euclidean topology on the real line.

Lemma 2.2. *For $x \in X$, if $\{x\}$ is open in $\langle X, \mathcal{T}\rangle$, then $\{x\}$ is open in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$.*

Proof. By the construction of $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$, if $\{x\}$ is open in $\langle X, \mathcal{T}\rangle$, then it is easy to see that x has a predecessor and a successor in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$, so $\{x\}$ is open in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$. □

In the following discussion we assume that $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ is perfect. By Lemma 2.2, T is an open set in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$, so T is an F_σ-set. So

$$T = \bigcup_{n \in \mathbb{N}} F_n,$$

where each F_n is closed in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$. Let $T_i = K + q_i$, $i \in \mathbb{N}$, where $\{q_i\colon i \in \mathbb{N}\}$ is an enumeration of $\mathbb{Q}$. Put $T_{in} = T_i \cap F_n$, then $T = \bigcup\{T_{in}\colon i, n \in \mathbb{N}\}$.

Lemma 2.3. *T_{in} is closed in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ for $i, n \in \mathbb{N}$.*

Proof. Notice that $T_{in} \subset F_n \subset X$ and T_{in} is closed in $\langle X, \mathcal{T}\rangle$. $\square$

Recall that a *gap* in a linearly ordered set $\langle Z, \leqslant\rangle$ is an ordered pair (A, B) of subsets of Z satisfying

(1) $Z = A \cup B$;

(2) $a < b$ for all $a \in A$ and $b \in B$;

(3) A has no right endpoint and B has no left endpoint (see [10]).

Let $Z^+ = Z \cup \{c\colon c \text{ is a gap in } \langle Z, \leqslant\rangle\}$. Now define a linear ordering $\leqslant_{Z^+}$ on Z^+ as follows:

If $a, b \in Z$, then $a \leqslant_{Z^+} b$ if and only if $a \leqslant b$.

If $c = (A, B)$, $a \in A$ and $b \in B$, then $a <_{Z^+} c <_{Z^+} b$.

For a pair of gaps $c_1 = (A_1, B_1)$ and $c_2 = (A_2, B_2)$, $c_1 \leqslant c_2$ if and only if $A_1 \subset A_2$.

Then $\langle Z^+, \leqslant_{Z^+}\rangle$ is also a linearly ordered set. It is called the Dedekind completion of $\langle Z, \leqslant\rangle$. In the following, for the linearly ordered set $\langle X, \leqslant^{\sim}\rangle$, we write $\leqslant^{\sim}$ instead of $\leqslant^{\sim}_{X^+}$ for convenience.

Lemma 2.4. *If $x \in T_{in}$ and x is not the right endpoint of T_{in} in $\langle X, \leqslant^{\sim}\rangle$, then either x has a successor in the ordering $\leqslant^{\sim}$ in T_{in} or there exists a gap $\alpha(x) \in X^+$ satisfying*

$$x <^{\sim} \alpha(x), \qquad (x, \alpha(x))(\leqslant^{\sim}) \cap T_{in} = \emptyset$$

and for any $y \in X$ with $\alpha(x) <^{\sim} y$,

$$(\alpha(x), y)(\leqslant^{\sim}) \cap T_{in} \neq \emptyset.$$

Proof. Let $x \in T_{in}$ and x be not the right endpoint of T_{in} in $\langle X, \leqslant^{\sim}\rangle$. If x has no successor in the ordering $\leqslant^{\sim}$ in T_{in}, then there must be a point $x_0 \in X$ with $x <^{\sim} x_0$ such that $(x, x_0)(\leqslant^{\sim}) \cap T_{in} = \emptyset$; otherwise, for any $y \in X$ with $x <^{\sim} y$,

$$(x, y)(\leqslant^{\sim}) \cap T_{in} \neq \emptyset,$$

which implies that $\{x\}$ is not open in the sense of the interval topology of the ordering $\leqslant^{\sim}$ on X. Since $\{x\}$ is open in $\langle X, \mathcal{T}\rangle$, we have $\langle x, 1\rangle \in L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ and

$$\langle x, 1\rangle \in \mathrm{cl}_{L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle} T_{in},$$

but $\langle x, 1\rangle \notin T_{in}$, which contradicts the closedness of T_{in} in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ (see Lemma 2.3). Hence x_0 exists.

Let

$$A = \{y \in X\colon\ y <^{\sim} z \text{ for all } z \in T_{in} \text{ with } x <^{\sim} z\}, \text{ and}$$

$$B = \{y \in X\colon\ z \leqslant^{\sim} y \text{ for some } z \in T_{in} \text{ with } x <^{\sim} z\}.$$

Then clearly (1) $X = A \cup B$; (2) if $a \in A$ and $b \in B$, then $a <^{\sim} b$; and (3) B has no left endpoint in $\langle X, \leqslant^{\sim}\rangle$ from the construction of B and the fact that x has no successor in T_{in} with respect to the ordering $\leqslant^{\sim}$. We claim that A has no right endpoint. Suppose that u_0 is the right endpoint of A. By the fact that T_{in} is closed in $\langle X, \mathcal{T}\rangle$, $(\leftarrow, u_0](\leqslant^{\sim})$ is open. Since u_0 has no successor in $\langle X, \leqslant^{\sim}\rangle$ from the construction of A and the fact that x has no successor in T_{in} with respect to $\leqslant^{\sim}$, $(\leftarrow, u_0](\leqslant^{\sim})$ is not open in the interval topology of the ordering $\leqslant^{\sim}$. It follows that $\langle u_0, 1\rangle \in L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$. Thus $\langle u_0, 1\rangle \in \mathrm{cl}_{L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle} T_{in}$ which contradicts the closedness of T_{in} in $L\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$. Thus (A, B) is a gap in $\langle X, \leqslant^{\sim}\rangle$. Let $\alpha(x) = (A, B)$. Then it is easy to check that $\alpha(x)$ is the required point. □

Now we define an equivalence relation R on the set T as follows: For $x, y \in T$, $x R y$ if and only if there is no $z \in X - T$ such that $x <^{\sim} z <^{\sim} y$ or $y <^{\sim} z <^{\sim} x$. Let $\mathcal{S} = \{S\colon\ S$ is an equivalence class of R and S contains at least two points of $T\}$.

Lemma 2.5. *$\mathcal{S} \neq \emptyset$ and for any $a, b \in X$ with $a < b$, there exists an $S \in \mathcal{S}$ such that $S \subset (a, b)(\leqslant)$.*

Proof. Since $X - T$ is dense, there exists an $r \in (X - T) \cap (a, b)(\leqslant)$. It follows that there exists an open and convex set C in $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ such that $r \in C \subset (a, b)(\leqslant)$ since $(a, b)(\leqslant)$ is an open neighborhood of r in $\langle X, \mathcal{T}\rangle$. Moreover there exists an $\varepsilon > 0$ such that $[r, r + \varepsilon)(\leqslant) \subset C \subset (a, b)(\leqslant)$. Take $r_1, r_2 \in (r, r + \varepsilon)(\leqslant) \cap (X - T)$ with $r_1 \neq r_2$. Then $\{r, r_1, r_2\} \subset C$. We may assume that $r <^{\sim} r_1 <^{\sim} r_2$. It follows that $(r, r_2)(\leqslant^{\sim})$ is an open neighborhood of r_1 in $\langle X, \mathcal{T}\rangle$. So there exists $\varepsilon_1 > 0$ such that $[r_1, r_1 + \varepsilon_1)(\leqslant) \subset (r, r_2)(\leqslant^{\sim}) \subset C$. Since $(r_1, r_1 + \varepsilon_1)(\leqslant) \cap T$ is uncountable and

$$T = \bigcup \{T_{in}\colon\ i, n \in \mathbb{N}\},$$

there exist i_0, $n_0 \in \mathbb{N}$ such that $(r_1, r_1 + \varepsilon_1)(\leqslant) \cap T_{i_0 n_0}$ is uncountable. So

$$(r, r_2)(\leqslant^{\sim}) \cap T_{i_0 n_0}$$

is uncountable. For each $x \in (r, r_2)(\leqslant^{\sim}) \cap T_{i_0 n_0}$ satisfying that x is not the right endpoint of $T_{i_0 n_0}$ in $\langle X, \leqslant^{\sim}\rangle$, we define

$$W(x) = \begin{cases} (x, x^+)(\leqslant^{\sim}), & \text{if } x \text{ has a successor } x^+ \text{ of the ordering } \leqslant^{\sim} \text{ in } T_{i_0 n_0}; \\ (x, \alpha(x))(\leqslant^{\sim}), & \text{if } x \text{ has no successor of the ordering } \leqslant^{\sim} \text{ in } T_{i_0 n_0}; \end{cases}$$

where $\alpha(x)$ is as defined in Lemma 2.4.

Clearly if $x \neq y$, $W(x) \cap W(y) = \emptyset$ and each $W(x)$ is open in $\langle X, \mathcal{T}\rangle$. Notice that, if $W(x) \cap (X - T) \neq \emptyset$, there must be some open interval $(c, d)(\leqslant) \subset W(x)$. Hence there exist at most countably many $W(x)$'s such that $W(x) \cap (X - T) \neq \emptyset$ since the subspace $(X - T)$ of the real line has countable cellularity in its Euclidean topology. It follows that there exists an $x \in (r, r_2)(\leqslant^\sim) \cap T_{i_0 n_0}$ such that $W(x) \cap (X - T) = \emptyset$. If $W(x) = (x, x^+)(\leqslant^\sim)$, then $x, x^+ \in T_{i_0 n_0}$ and $(x, x^+)(\leqslant^\sim) \cap (X - T) = \emptyset$, so $x R x^+$ which implies that there exists an $S(x) \in \mathcal{S}$ such that $\{x, x^+\} \subset S(x)$. If $W(x) = (x, \alpha(x))(\leqslant^\sim)$, from Lemma 2.4 we know that $\alpha(x)$ is a gap (A, B). By the definition of the gap, A has no right endpoint in $\langle X, \leqslant^\sim\rangle$. So $(x, \alpha(x))(\leqslant^\sim) \neq \emptyset$ since $x \in A$. Thus $(x, \alpha(x))(\leqslant^\sim) \subset T$ since

$$\big(x, \alpha(x)\big)(\leqslant^\sim) \cap (X - T) = W(x) \cap (X - T) = \emptyset.$$

Take a $y \in (x, \alpha(x))(\leqslant^\sim)$, then $x R y$. Hence there exists an $S(x) \in \mathcal{S}$ such that $\{x, y\} \subset S(x)$. Noticing that $x \in (r, r_2)(\leqslant^\sim) \cap T_{i_0 n_0}$, we have

$$S(x) \cap (r, r_2)(\leqslant^\sim) \neq \emptyset.$$

By the definition of the equivalence relation R,

$$S(x) \subset (r, r_2)(\leqslant^\sim)$$

since $\{r, r_2\} \subset X - T$. Therefore we have $S(x) \subset (r, r_2)(\leqslant^\sim) \subset C \subset (a, b)(\leqslant)$. □

Lemma 2.6. *Let C be a convex set in $\langle X, \leqslant^\sim\rangle$. If there exist three distinct elements S_1, S_2, S_3 of $\mathcal{S}$ such that $C \cap S_i \neq \emptyset$ for $i = 1, 2, 3$, then C must contain one of the S_i.*

Proof. Take $x_i \in C \cap S_i$ for $i = 1, 2, 3$, then x_1, x_2 and x_3 are distinct since S_1, S_2 and S_3 are distinct. We may assume $x_1 <^\sim x_2 <^\sim x_3$. By the definition of R there exist $r_1, r_2 \in X - T$ such that $x_1 <^\sim r_1 <^\sim x_2 <^\sim r_2 <^\sim x_3$. Thus $S_2 \subset (r_1, r_2)(\leqslant^\sim)$. So for any $x \in S_2$, $x_1 <^\sim x <^\sim x_3$. Since $x_1, x_3 \in C$ and C is convex in $\langle X, \leqslant^\sim\rangle$, we have $x \in C$. Thus $S_2 \subset C$. □

3. The proof of Theorem 1.3

In order to prove Theorem 1.3 it is sufficient to show that the perfect GO-space $\langle X, \mathcal{T}, \leqslant\rangle$ constructed in Example 2.1 has no perfect orderable d-extension. By the argument in Section 2 it is sufficient to show that $L\langle X, \mathcal{T}, \leqslant^\sim\rangle$ is not perfect.

Proof of Theorem 1.3. Suppose that $L\langle X, \mathcal{T}, \leqslant^\sim\rangle$ is perfect and take the collection $\mathcal{S}$ defined as in Section 2. For each $S \in \mathcal{S}$, take two distinct points $\beta(S), \gamma(S) \in S$ such that $\beta(S) < \gamma(S)$. Let $\mathcal{A} = \{(\beta(S), \gamma(S))(\leqslant)\colon S \in \mathcal{S}\}$. Then $\mathcal{A}$ is a collection of open intervals of the ordering $\leqslant$. We will show that there exists an uncountable subset of $X - T$ which is dense in X and each point of which belongs to infinitely many elements of $\mathcal{A}$.

For each $n \in \mathbb{N}$, let

$$B_n = \{x \in X - T\colon x \text{ is contained exactly in } n \text{ elements of } \mathcal{A}\}.$$

Then each B_n is nowhere dense in X with its Euclidean topology. In fact, for any open interval $(a,b)(\leqslant)$, if $(a,b)(\leqslant) \cap B_n \neq \emptyset$ take an $x \in (a,b)(\leqslant) \cap B_n$, then

$$U = (a,b)(\leqslant) \cap \left(\bigcap\{A \in \mathcal{A}\colon\ x \in A\}\right)$$

is an open interval on the ordering $\leqslant$ which contains x. By Lemma 2.5, the set $\{\beta(S)\colon\ S \in \mathcal{S}\}$ is dense in X. So there exists an $S \in \mathcal{S}$ such that $\beta(S) \in U$. Then $(\beta(S), \gamma(S))(\leqslant) \in \mathcal{A}$ and for each $y \in V = (\beta(S), \gamma(S))(\leqslant) \cap U$, we have $|\{A \in \mathcal{A}\colon\ y \in A\}| > n$. Thus $y \notin B_n$. Hence $V \subset (a,b)(\leqslant)$ and $V \cap B_n = \emptyset$. Since V is a nonempty open interval of the ordering $\leqslant$, B_n is nowhere dense in X with its Euclidean topology.

Let $\mathrm{cl}_{X(E)} B_n$ be the closure of B_n in the sense of the Euclidean topology on X. Then

$$F = T \cup \left(\bigcup_{n \in \mathbb{N}} \mathrm{cl}_{X(E)} B_n\right)$$

is a σ-nowhere dense set of X in the sense of the Euclidean topology, so $X - F$ is uncountable and dense in X such that $X - F \subset X - T$ and for any $x \in X - F$, $\{A \in \mathcal{A}\colon\ x \in A\}$ is infinite.

Next we consider two cases:

Case 1: *There exists an* $x_0 \in X - F$ *such that* $\{x_0\} = \bigcap\{A \in \mathcal{A}\colon\ x_0 \in A\}$.

Notice that

$$\{A \in \mathcal{A}\colon\ x_0 \in A\} = \big\{(\beta(S), \gamma(S))(\leqslant)\colon\ x_0 \in (\beta(S), \gamma(S))(\leqslant),\ S \in \mathcal{S}\big\}.$$

For any open neighborhood of x_0 in $\langle X, \mathcal{T}\rangle$ of the form $[x_0, x_0 + \varepsilon)(\leqslant)$, where $\varepsilon > 0$, we have

$$[x_0, x_0 + \varepsilon)(\leqslant) \cap \big\{\gamma(S)\colon\ x_0 \in (\beta(S), \gamma(S))(\leqslant),\ S \in \mathcal{S}\big\}$$

is infinite and

$$[x_0, x_0 + \varepsilon)(\leqslant) \cap \big\{\beta(S)\colon\ x_0 \in (\beta(S), \gamma(S))(\leqslant),\ S \in \mathcal{S}\big\} = \emptyset.$$

Since $[x_0, x_0 + \varepsilon)(\leqslant)$ is open in $\langle X, \mathcal{T}\rangle$ there exists an open convex set C in $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ such that $x_0 \in C \subset [x_0, x_0 + \varepsilon)(\leqslant)$. Again because C is open in $\langle X, \mathcal{T}\rangle$ there exists an $\varepsilon' > 0$ such that $[x_0, x_0 + \varepsilon')(\leqslant) \subset C \subset [x_0, x_0 + \varepsilon)(\leqslant)$. It follows that there exists an infinite subset $\mathcal{S}'$ of $\mathcal{S}$ such that $\gamma(S) \in C$, but $\beta(S) \notin C$ for $S \in \mathcal{S}'$. By Lemma 2.6, C is not convex in $\langle X, \leqslant^{\sim}\rangle$, which is a contradiction.

Case 2: *For any* $x \in X - F$, $\bigcap\{A \in \mathcal{A}\colon\ x \in A\} \neq \{x\}$.

Let $D = \{\beta(S)\colon\ S \in \mathcal{S}\}$. Rewrite $F = \bigcup F_n$ instead of

$$F = T \cup \left(\bigcup_{n \in \mathbb{N}} \mathrm{cl}_{X(E)} B_n\right),$$

where each F_n is nowhere dense and closed in the sense of Euclidean topology on the real line. For $x_0 \in X - F$, take a_0 and b_0 satisfying $a_0 < b_0$, $[a_0, b_0](\leqslant) \subset \bigcap\{A \in \mathcal{A}\colon\ x_0 \in A\}$

and $[a_0, b_0](\leqslant) \cap F_1 = \emptyset$. In the following we will inductively choose $[a_n, b_n](\leqslant)$ and $S_n \in \mathcal{S}$ for every $n \in \mathbb{N}$ satisfying:

$$\begin{aligned} &[a_0, b_0](\leqslant) \supset \left[\beta(S_1), \gamma(S_1)\right](\leqslant) \supset [a_1, b_1](\leqslant) \supset \cdots \\ &\quad \supset \left[\beta(S_n), \gamma(S_n)\right](\leqslant) \supset [a_n, b_n](\leqslant) \supset \cdots, \\ &0 < b_n - a_n < \frac{b_0 - a_0}{2^n} \quad \text{and} \quad [a_n, b_n](\leqslant) \cap F_{n+1} = \emptyset. \end{aligned} \tag{$*$}$$

Observe that, in the following process of defining $[a_n, b_n](\leqslant)$ and S_n, the choice of S_n depends only on $[a_{n-1}, b_{n-1}](\leqslant)$ and the choice of $[a_n, b_n](\leqslant)$ depends only on S_n. So to initialize the induction we only need to define $[a_0, b_0]$ which we have done. Now assume that for each $i < n$ we have chosen $[a_i, b_i](\leqslant)$ and $S_i \in \mathcal{S}$ satisfying:

$$\begin{aligned} &[a_0, b_0](\leqslant) \supset \left[\beta(S_1), \gamma(S_1)\right](\leqslant) \supset [a_1, b_1](\leqslant) \supset \cdots \\ &\quad \supset \left[\beta(S_{n-1}), \beta(S_{n-1})\right](\leqslant) \supset [a_{n-1}, b_{n-1}](\leqslant), \\ &0 < b_i - a_i < \frac{b_0 - a_0}{2^i} \quad \text{and} \quad [a_i, b_i](\leqslant) \cap F_{i+1} = \emptyset, \end{aligned}$$

for $i = 1, 2, \ldots, n-1$.

By Lemma 2.5, $D = \{\beta(S)\colon\ S \in \mathcal{S}\}$ is dense. Let

$$M_{n-1} = \left\{\beta(S) \in (a_{n-1}, b_{n-1})(\leqslant) \cap D\colon\ \gamma(S) \geqslant b_{n-1}\right\}.$$

If M_{n-1} is dense in some

$$(c, d)(\leqslant) \subset (a_{n-1}, b_{n-1})(\leqslant),$$

then for $y_0 \in (c, d)(\leqslant) \cap (X - F)$ and arbitrary $\varepsilon > 0$ with $[y_0, y_0 + \varepsilon)(\leqslant) \subset (c, d)(\leqslant)$,

$$[y_0, y_0 + \varepsilon)(\leqslant) \cap M_{n-1}$$

is infinite. Since $[y_0, y_0 + \varepsilon)(\leqslant)$ is an open neighborhood of y_0 in $\langle X, \mathcal{T}\rangle$ there exists an open convex set C in $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ such that $y_0 \in C \subset [y_0, y_0 + \varepsilon)(\leqslant)$. Furthermore, there exists $\varepsilon' > 0$ such that

$$[y_0, y_0 + \varepsilon')(\leqslant) \subset C \subset [y_0, y_0 + \varepsilon)(\leqslant).$$

Because that $[y_0, y_0 + \varepsilon')(\leqslant) \cap M_{n-1}$ is infinite and if $\beta(S) \in [y_0, y_0 + \varepsilon')(\leqslant) \cap M_{n-1}$, then $\beta(S) \in C$ and

$$\gamma(S) \notin (a_{n-1}, b_{n-1})(\leqslant) \supset [y_0, y_0 + \varepsilon)(\leqslant) \supset C.$$

By Lemma 2.6, C is not convex in $\langle X, \mathcal{T}, \leqslant^{\sim}\rangle$ which is a contradiction. Therefore M_{n-1} is nowhere dense in $(a_{n-1}, b_{n-1})(\leqslant)$. Let

$$M_{n-1}^{\sim} = \left((a_{n-1}, b_{n-1})(\leqslant) \cap D\right) - M_{n-1}.$$

Then $M_{n-1}^{\sim}$ is dense in $(a_{n-1}, b_{n-1})(\leqslant)$ since D is dense. Take a $\beta(S_n) \in M_{n-1}^{\sim}$, then

$$\left[\beta(S_n), \gamma(S_n)\right](\leqslant) \subset (a_{n-1}, b_{n-1})(\leqslant).$$

Pick an $x_n \in (\beta(S_n), \gamma(S_n))(\leqslant) \cap (X - F)$, then

$$\{x_n\} \neq \bigcap\{A \in \mathcal{A}\colon\ x_n \in A\} \subset \big(\beta(S_n), \gamma(S_n)\big)(\leqslant) \subset [a_{n-1}, b_{n-1}](\leqslant).$$

Hence we may choose a_n and b_n in $\bigcap\{A \in \mathcal{A}\colon\ x_n \in A\}$ satisfying

$$0 < b_n - a_n < \frac{b_0 - a_0}{2^n}, \qquad [a_n, b_n](\leqslant) \cap F_{n+1} = \emptyset$$

and

$$[a_n, b_n](\leqslant) \subset \bigcap\{A \in \mathcal{A}\colon\ x_n \in A\}.$$

So by the induction we may define $[a_n, b_n]$ and $S_n \in \mathcal{S}$ satisfying $(*)$ for each n. Hence there exists a $z_0 \in X - F$ such that

$$z_0 \in [a_n, b_n](\leqslant) \subset \big[\beta(S_n), \gamma(S_n)\big](\leqslant) \quad \text{for every } n \in \mathbb{N}.$$

It follows that $\{z_0\} = \bigcap\{A \in \mathcal{A}\colon\ z_0 \in A\}$. This contradicts the assumption of this case. □

Remark. Unfortunately, by Example 2.1 we cannot answer Problem 1.1. In fact there exists a perfect linearly ordered extension of $\langle X, \mathcal{T}, \leqslant\rangle$ according to the result in [1].

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