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A Discrete Fourier Method for Numerical Solution of Strongly Coupled Mixed Parabolic Systems

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Abstract—This paper provides a discrete Fourier method for numerical solution of strongly coupled mixed parabolic systems which avoids solving large algebraic systems. After discretization using Crank-Nicholson difference scheme, the exact solution of the discretized problem is constructed and stability is analyzed.

Keywords—Coupled parabolic problem, Crank-Nicholson, Numerical solution.

1. INTRODUCTION

Coupled parabolic partial differential systems are frequent in diffusion processes [1,2], microwave engineering [1], and geomechanics [3] between other type of problems. Difference methods are widely used in the literature for the scalar case [4] but for the coupled case the stability conditions proposed in [5,6] can be improved using other difference schemes. Dealing with difference methods there are two main approaches, the Fourier method and the algebraic which is based on solving the discretized algebraic problem, see [4] for details. Here, we develop a Fourier method to construct numerical solutions of strongly coupled parabolic systems of the form,

$$u_t(x,t) - Au_{xx}(x,t) = 0, \qquad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 0, \qquad t > 0, \tag{2}$$

$$Bu(1,t) + Cu_x(1,t) = 0, \qquad t > 0, \tag{3}$$

$$u(x,0) = F(x), \qquad 0 \le x \le 1, \tag{4}$$

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where $F(x)$ and the unknown $u(x,t)$ are vectors in $\mathbb{C}^s$ and A, B, C are matrices in $\mathbb{C}^{s\times s}$ with C invertible and

$$C^{-1}B \text{ has real eigenvalues} \tag{5}$$

and

$$\operatorname{Re}(z) > 0, \qquad \text{for all eigenvalue of } A. \tag{6}$$

Exact solution series of problem (1)–(4) have been proposed in [7] but its computation is difficult because of the infiniteness of the series and the computation of matrix exponentials [8]. The algebraic difference method is not appropriated to be used in problem (1)–(4) because of the existence of several matrices in the coupled boundary conditions as well as in equation (1) makes unmanageable the stability conditions in terms of the data.

This paper is organized as follows. Section 2 deals with the discretization of problem (1)–(4) using Crank-Nicholson scheme and the study of the discretized boundary value problem. Section 3 deals with the solution of the mixed discretized problem in the case where the spectrum of the matrix $-C^{-1}B$ has one real eigenvalue. Section 4 deals with the study of the stability of the constructed solutions in Section 3, assuming the existence of one real eigenvalue in the matrix $-C^{-1}B$. In Section 5, the method is extended to the case where the matrix $-C^{-1}B$ has several real eigenvalues.

Throughout this paper, the set of all the eigenvalues of a matrix A in $\mathbb{C}^{s\times s}$ is denoted by $\sigma(A)$. The spectral radius of A, denotated by $\rho(A)$, is the maximum of the set, $\{|w|; w \in \sigma(A)\}$. Let $\alpha(A)$ and $\beta(A)$ be the following numbers,

$$\alpha(A) = \max\{\operatorname{Re}(z); z \in \sigma(A)\}, \qquad \beta(A) = \min\{\operatorname{Re}(z); z \in \sigma(A)\}. \tag{7}$$

If P is a matrix in $\mathbb{C}^{s\times p}$ we denoted by $P^\dagger$ the Moore-Penrose pseudoinverse of P. An account of examples, properties and applications of this concept may be found in [9,10] and $P^\dagger$ can be efficiently computed with MATLAB package. The kernel of a matrix D of $\mathbb{C}^{s\times s}$, denotated by $\ker D$, coincides with the image of the matrix $I - D^\dagger D$ denotated by $\operatorname{Im}(I - D^\dagger D)$ where I is the identity matrix in $\mathbb{C}^{s\times s}$, see [9]. We say that a subspace E of $\mathbb{C}^s$ is invariant by matrix A of $\mathbb{C}^{s\times s}$ if $A(E) \subset E$. Hence, property $A(\ker G) \subset \ker G$ is equivalent to the condition $GA(I - G^\dagger G) = 0$, where G is a matrix of $\mathbb{C}^{s\times s}$. If $Q \in \mathbb{C}^{s\times s}$, $f(z)$ is an holomorphic function defined on the open set Ω of the complex plane and $\sigma(Q)$ lies in Ω, then, the holomorphic matrix functional calculus defines $f(Q)$ as a matrix that may be computed as a polynomial in Q, in fact as a polynomial in Q of degree smaller than the minimal polynomial of Q, see [11, p. 567; 12].

2. THE DISCRETIZED BOUNDARY VALUE PROBLEM

Let M be a positive integer, $h = \Delta x$, $\Delta t = k$, $r = k/h^2$, and divide the domain $[0,1] \times [0,\infty[$ into equal rectangles of sides Δx, Δt. If we introduce coordinates of a typical mesh point (mh, nk) with $U(m,n) = u(mh, nk)$ and use Crank-Nicholson scheme, [13] to approximate (1)–(3) one gets the discrete boundary value problem,

$$\begin{aligned} &\frac{rA}{2}\left[U(m+1,n) + U(m-1,n)\right] + (I - rA)\,U(m,n) \\ &= -\frac{rA}{2}\left[U(m+1,n+1) + U(m-1,n+1)\right] + (I + rA)\,U(m,n+1), \end{aligned} \tag{8}$$

$$U(0,n) = 0, \qquad n > 0, \tag{9}$$

$$BU(M,n) + MC\left[U(M,n) - U(M-1,n)\right] = 0, \qquad n > 0. \tag{10}$$

Let $\{T(n)\}$, $\{X(m)\}$ be solutions of the difference equations,

$$\left(I + \frac{\rho A}{2}\right)T(n) - \left(I - \frac{\rho A}{2}\right)T(n+1) = 0, \qquad T(n) \in \mathbb{C}^{s\times s}, \quad n > 0, \tag{11}$$

$$X(m+1) - \left(\frac{2r+\rho}{r}\right) X(m) + X(m-1) = 0, \qquad X(m) \in \mathbb{C}^s, \quad 1 \le m \le M-1, \tag{12}$$

where $\rho \in \mathbb{R}$. Then, it is easy to check that $\{U(m,n)\}$ defined by

$$U(m,n) = T(n)\, X(m), \tag{13}$$

satisfies (8). Let us take ρ with

$$-4r < \rho < 0. \tag{14}$$

Then, the solution set of (12) can be written in the form, see [5],

$$X(m) = \cos(m\theta)\, c + \sin(m\theta)\, d, \tag{15}$$

where c, d lie in $\mathbb{C}^s$ and

$$0 < \theta < \pi, \qquad \cos\theta = \frac{2r+\rho}{2r}, \quad \rho = -4r\sin^2\left(\frac{\theta}{2}\right). \tag{16}$$

If $X(0) = 0$, then, $\{U(m,n)\}$ defined by (13) satisfies (9). The solution set of (12), satisfying $X(0) = 0$, takes the form,

$$X(m) = \sin(m\theta)\, d, \qquad d \in \mathbb{C}^s. \tag{17}$$

Under hypothesis (6), the matrix $I + 2r\sin^2(\theta/2)A$ is invertible by the spectral mapping theorem [11, p. 564]. The solution of (11), satisfying $T(0) = I$, is given by

$$T(n) = \left[\left(I + 2r\sin^2\left(\frac{\theta}{2}\right)A\right)^{-1}\left(I - 2r\sin^2\left(\frac{\theta}{2}\right)A\right)\right]^n, \qquad n \ge 0. \tag{18}$$

By (17),(18), the functions of the form (13) satisfying (9) are given by

$$U(m,n) = \left[\left(I + 2r\sin^2\left(\frac{\theta}{2}\right)A\right)^{-1}\left(I - 2r\sin^2\left(\frac{\theta}{2}\right)A\right)\right]^n \sin(m\theta)\, d, \qquad d \in \mathbb{C}^s. \tag{19}$$

By imposing the condition (10), one gets that nonzero vectors d must verify

$$\begin{aligned}&\{B\sin(M\theta) + MC\left[\sin(M\theta) - \sin((M-1)\theta)\right]\}\\ &\cdot\left[\left(I + 2r\sin^2\left(\frac{\theta}{2}\right)A\right)^{-1}\left(I - 2r\sin^2\left(\frac{\theta}{2}\right)A\right)\right]^n d = 0.\end{aligned} \tag{20}$$

Fixed $r > 0$, the function of the complex variable z, defined as

$$v(z) = \frac{1 - 2r\sin^2(\theta/2)\, z}{1 + 2r\sin^2(\theta/2)\, z},$$

is holomorphic in the disk $|z| < 1/2r$. Given A satisfying (6), if $r < (2\alpha(A))^{-1}$, then, it follows that $\sigma(A)$ is contained in $|z| < 1/2r$ and by the properties of the holomorphic matrix functional calculus [11, Ch. VII], the matrix $(I+2r\sin^2(\theta/2)A)^{-1}(I-2r\sin^2(\theta/2)A)$ as well as all its positive powers, can be computed as a polynomial of degree $p-1$, being p, the degree of the minimal polynomial of A. Hence, condition (20) is equivalent to

$$\{B\sin(M\theta) + MC\left[\sin(M\theta) - \sin((M-1)\theta)\right]\} A^j d = 0, \qquad 0 \le j \le p-1. \tag{21}$$

Condition (21) means that

$$L(\theta) = B\sin(M\theta) + MC\left[\sin(M\theta) - \sin((M-1)\theta)\right] \text{ is singular}, \qquad 0 < \theta < \pi. \tag{22}$$

If (22) holds, then, $\sin(M\theta) \neq 0$ and (22) is equivalent to the condition,

$$C^{-1}B + \frac{M\left[\sin(M\theta) - \sin((M-1)\theta)\right]}{\sin(M\theta)} I \text{ is singular.} \tag{23}$$

Under hypothesis (5), condition (23) holds if there exists $\mu \in \mathbb{R}$, such that

$$\frac{M\left[\sin(M\theta) - \sin((M-1)\theta)\right]}{\sin(M\theta)} = \mu, \qquad 0 < \theta < \pi. \tag{24}$$

Let us introduce the matrices $G(\mu)$ and $\tilde{G}(\mu)$ defined by

$$G(\mu) = C^{-1}B + \mu I, \qquad \tilde{G}(\mu) = \begin{bmatrix} G(\mu) \\ G(\mu)A \\ \vdots \\ G(\mu)A^{p-1} \end{bmatrix} \in \mathbb{C}^{sp\times s}. \tag{25}$$

Hence, condition (21) can be written

$$\tilde{G}(\mu)\, d = 0, \qquad d \in \mathbb{C}^s, \tag{26}$$

and this system admits nonzero solutions if

$$\operatorname{rank}\left(\tilde{G}(\mu)\right) < s. \tag{27}$$

Under condition (27), the solution set of (26) can be described in the form

$$d = \left(I - \tilde{G}(\mu)^{\dagger}\tilde{G}(\mu)\right)\tilde{d}, \qquad \tilde{d} \in \mathbb{C}^s \sim \{0\}, \tag{28}$$

see theorem of [14, p. 24]. With respect to equation (24), by [5] for the case $\mu < 1$, for each integer ℓ with $1 \le \ell \le M-1$, there exists a solution $\theta_\ell \in J_\ell =](\ell-1)\pi/M, \ell\pi/M[$. For the case, $\mu \ge 1$, by [5], we know the existence of a root $\theta_\ell \in J_\ell$, for $2 \le \ell \le M-1$.

If $\rho \in \mathbb{R}$ is an eigenvalue of the scalar discrete Sturm-Liouville problem,

$$\begin{gathered} x(m+1) - \left(\frac{2r+\rho}{r}\right) x(m) + x(m-1) = 0, \qquad 1 \le m \le M-1, \\ x(0) = 0, \qquad x(M) = \frac{M}{M-\mu} x(M-1). \end{gathered} \tag{29}$$

then, by (24), one gets that ρ is also an eigenvalue of the vector problem,

$$\begin{gathered} X(m+1) - \left(\frac{2r+\rho}{r}\right) X(m) + X(m-1) = 0, \qquad 1 \le m \le M-1, \\ X(0) = 0, \qquad BX(M) + MC\left[X(M) - X(M-1)\right] = 0. \end{gathered} \tag{30}$$

Since by [15, Ch. 11], problem (29) has $M-1$ eigenvalues, we need to find another eigenvalue of (30), for the case $\mu \ge 1$. To this end, we consider values of ρ outside of the interval (14).

If $\mu = 1$, consider $\rho = 0$ in (12) obtaining the solution set $X(m) = c + md$ with $c, d \in \mathbb{C}^s$. By imposing the condition $X(0) = 0$, one gets

$$X(m) = md, \qquad d \in \mathbb{C}^s, \quad 1 \le m \le M-1. \tag{31}$$

Taking $\rho = 0$ into (11), one gets $T(n) = I$ and thus,

$$U(m,n) = T(n)\, X(m) = md, \tag{32}$$

is a solution of (8),(9). The boundary condition (10) implies that vector d of (32) must verify $(B + C)\, d = 0$, or

$$G(1)d = \left(C^{-1}B + I\right)\, d = 0. \tag{33}$$

Equation (33) admits nonzero solutions if

$$\mu = 1 \in \sigma\left(-C^{-1}B\right), \tag{34}$$

and by the theorem of [14, p. 24], the solution set of (33) takes the form,

$$d = \left(I - G(1)^{\dagger}\, G(1)\right) \tilde{d}, \qquad \tilde{d} \in \mathbb{C}^s \sim \{0\}. \tag{35}$$

The corresponding solution $\{U(m,n)\}$ of problem (8)–(10) takes the form,

$$U_1(m,n,1) = m\left(I - G(1)^{\dagger}\, G(1)\right) \tilde{d}, \qquad \tilde{d} \in \mathbb{C}^s \sim \{0\}, \quad 1 \le m \le M-1, \quad n > 0. \tag{36}$$

For the case $\mu > 1$, consider $\rho > 0$ and let

$$R = R(\rho) = \frac{2r + \rho}{2r} > 1. \tag{37}$$

If we denote by

$$w_1 = R + \sqrt{R^2 - 1}, \qquad w_2 = R - \sqrt{R^2 - 1}, \tag{38}$$

the characteristic roots of equation $z^2 - ((2r + \rho)/r)z + 1 = 0$ associated to (12), then, the solution set of (12) in this case, satisfying $X(0) = 0$, is given by

$$X(m) = \left(w_1^m - w_2^m\right) d, \qquad d \in \mathbb{C}^s. \tag{39}$$

The solution set of (11), for $\rho > 0$, satisfying $T(0) = I$, for $\rho < 2/\alpha(A)$ is given by

$$T(n) = \left[\left(I - \frac{\rho A}{2}\right)^{-1} \left(I + \frac{\rho A}{2}\right)\right]^n, \qquad n \ge 0. \tag{40}$$

By (39),(40) a set of solutions of the problem (8),(9) is given by

$$U(m,n) = \left[\left(I - \frac{\rho A}{2}\right)^{-1} \left(I + \frac{\rho A}{2}\right)\right]^n \left(w_1^m - w_2^m\right) d, \qquad n \ge 0, \quad 1 \le m \le M-1. \tag{41}$$

By imposing condition (10) to $\{U(m,n)\}$ and taking into account that, for

$$\rho < \frac{2}{\alpha(A)}, \tag{42}$$

the matrix $[(I - \rho A/2)^{-1}(I + \rho A/2)]^n$ can be written as a polynomial in A of degree $p - 1$, condition (10) holds if

$$C^{-1}B\left(w_1^M - w_2^M\right) + M\left[\left(w_1^M - w_2^M\right) - \left(w_1^{M-1} - w_2^{M-1}\right)\right] A^j\, d = 0, \qquad 0 \le j \le p-1, \tag{43}$$

or there exists $R > 1$, such that

$$G(M\gamma(R)) = C^{-1}B + M\gamma(R), \qquad \text{is singular}, \tag{44}$$

where

$$\gamma(R) = 1 - \frac{w_1^{M-1} - w_2^{M-1}}{w_1^M - w_2^M}. \tag{45}$$

Since $\gamma :]1, \infty[\longrightarrow \mathbb{R}$ is continuous and

$$\lim_{R \to 1^+} \gamma(R) = \frac{1}{M}; \qquad \lim_{R \to +\infty} \gamma(R) = 1, \qquad 1 < \mu < M,$$

there exists $R_1 \in]1, \infty[$, such that

$$\gamma(R_1) = \frac{\mu}{M}. \tag{46}$$

Taking this value of R_1 into (44), one gets that $G(M\gamma(R_1)) = G(\mu)$ and hence, the corresponding $\rho_1 = \rho(R_1) = 2r(R_1 - 1)$ is an eigenvalue of (29) and also of (30) if (27) holds, for $\mu > 1$. In this case, the obtained eigenfunctions set takes the form

$$\left.\begin{aligned} U_1(m,n,\mu) &= \left[\left(I - \frac{\rho_1(r)A}{2}\right)^{-1}\left(I + \frac{\rho_1(r)A}{2}\right)\right]^n (w_1^n - w_2^n)\, d, \\ d &= \left[I - \tilde{G}(\mu)^\dagger \tilde{G}(\mu)\right]\tilde{d}, \qquad \tilde{d} \in \mathbb{C}^s \sim \{0\}, \\ &\quad 1 \le m \le M-1, \qquad n > 0, \end{aligned}\right\} \tag{47}$$

where $\rho_1 = \rho(r) = 2r(R_1 - 1)$ is taken with $r > 0$ small enough so that

$$r(R_1 - 1)\alpha(A) = r(R_1 - 1)\max\{\operatorname{Re}(z),\ z \in \sigma(A)\} < 1, \tag{48}$$

which guarantees the existence of $(I - \rho_1(r)A/2)^{-1}$. Summarizing the following result has been established.

THEOREM 2.1. *Assume hypotheses (5),(6) and let $\mu \in \mathbb{R} \cap \sigma(-C^{-1}B)$. Let p be the degree of the minimal polynomial of A, let M be a positive integer $M > \mu$, $r = k/h^2$ and let $G(\mu)$, $\tilde{G}(\mu)$ be defined by (25) satisfying (27) and $r < (2\alpha(A))^{-1}$.*

(i) *If $\mu < 1$, then, for each ℓ integer with $1 \le \ell \le M-1$, there is a solution θ_ℓ of equation (24) in $J_\ell =](\ell-1)\pi/M, \ell\pi/M[$. The eigenfunction solutions $U_\ell(m,n,\mu)$ are given by*

$$\left.\begin{aligned} U_\ell(m,n,\mu) &= \left[\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1}\left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right]^n \sin(m\theta_\ell)\, d_\ell, \\ d_l &= \left[I - \tilde{G}(\mu)^\dagger \tilde{G}(\mu)\right]\tilde{d}, \qquad \tilde{d} \in \mathbb{C}^s \sim \{0\} \\ &\quad 1 \le m \le M-1, \qquad n > 0. \end{aligned}\right\} \tag{49}$$

If $\mu = 0$, $\theta_\ell = ((2\ell-1)/(2M-1))\pi$, $1 \le \ell \le M-1$ and (49) is obtained substituting θ_ℓ by this concrete value for each ℓ.

(ii) *If $\mu = 1$, then, there are two types of eigenfunctions, for each ℓ with $2 \le \ell \le M-1$, $U_\ell(m,n,\mu)$ defined by (49) with $\mu = 1$ and, for $\ell = 1$, $U_1(m,n,1)$ defined by (36).*

(iii) *If $\mu > 1$, there are also two types of eigenfunctions, for each ℓ with $2 \le \ell \le M-1$, $U_\ell(m,n,\mu)$ defined by (49) and for $\ell = 1$, taking r small enough so that (48) holds, $U_1(m,n,\mu)$ defined by (47) where w_1 and w_2 are given by (38).*

3. THE MIXED DISCRETE PROBLEM

In this section, we superpose the solutions of the discretized boundary value problem (8)–(10) in order to construct a solution satisfying the initial condition,

$$U(m,0) = F(mh) = f(m), \qquad 0 \le m \le M. \tag{50}$$

Taking into account that the scalar Sturm-Liouville problem (29) has the same eigenvalues as (30) and that eigenfunctions of (30) have in each entry an eigenfunction of (29), we can apply the Fourier method to each component and then guarantee that resulting vector Fourier coefficients lie in $\ker \tilde{G}(\mu)$.

We consider three different cases. First, let $\mu < 1$. By imposing the initial condition (50) to the finite series candidate solution,

$$U(m,n) = \sum_{\ell=1}^{M-1} \left[\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1} \left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right]^n \sin(m\theta_\ell)\, d_\ell, \tag{51}$$

where d_ℓ are vectors to be determined, one gets

$$f(m) = U(m,0) = \sum_{\ell=1}^{M-1} \sin(m\theta_\ell)\, d_\ell. \tag{52}$$

By the theory of discrete Fourier series, see [15, Ch. 11], and since the set $\{\sin(m\theta_\ell)\}_{\ell=1}^{M-1}$ is an orthogonal set of eigenfunctions of the scalar Sturm-Liouville problem, working component by component one gets that (52) holds if

$$d_\ell = \frac{\sum_{\eta=1}^{M-1} \sin(\eta\theta_\ell)\, f(\eta)}{\sum_{\eta=1}^{M-1} \sin^2(\eta\theta_\ell)}, \tag{53}$$

but in order to be each term of the sum of (51) an eigenfunction of (30), vectors d_ℓ must belong to $\ker \tilde{G}(\mu)$. This condition is satisfied if

$$\{f(m), 1 \le m \le M-1\} \subset \ker \tilde{G}(\mu), \tag{54}$$

or alternatively,

$$\{f(m), 1 \le m \le M-1\} \subset \ker G(\mu) \qquad \text{and} \qquad G(\mu)A\left(I - G(\mu)^\dagger G(\mu)\right) = 0, \tag{55}$$

where the second condition of (55) means that $\ker G(\mu)$ is an invariant subspace by matrix A, see [7] for details.

Consider now the case $\mu = 1$. In this case, we superpose two types of eigenfunctions according with the Theorem 2.1. If we seek a candidate solution of the mixed problem (8)–(10),(50) of the form,

$$U(m,n) = md_1 + \sum_{\ell=2}^{M-1} \left[\left(I + 2r\,A\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1} \left(I - 2r\,A\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right]^n \sin(m\theta_\ell)\, d_\ell, \tag{56}$$

and by imposing (50) one gets

$$f(m) = m\,d_1 + \sum_{\ell=2}^{M-1} \sin(m\theta_\ell)\, d_\ell. \tag{57}$$

Taking into account that $\{h_\ell(m)\}_{\ell=1}^{M-1}$ defined by

$$h_\ell(m) = \begin{cases} m, & \ell = 1, \\ \sin(m\theta_\ell), & 2 \le \ell \le M-1, \end{cases}$$

are orthogonal eigenfunctions of (29) with respect to $w(m) = 1$, $1 \le m \le M-1$, by the theory of discrete Fourier series, one gets

$$\begin{aligned} d_1 &= \frac{\sum_{\eta=1}^{M-1} \eta f(\eta)}{(M^3/3 - M^2/2 + M/6)}, \\ d_\ell &= \frac{\sum_{\eta=1}^{M-1} f(\eta)\sin(\eta\theta_\ell)}{\sum_{\eta=1}^{M-1} \sin^2(\eta\theta_\ell)}, \qquad 2 \le \ell \le M-1. \end{aligned} \tag{58}$$

Note that in order to guarantee that vectors d_ℓ given by (58) lie in $\ker \tilde{G}(1)$, it is sufficient that condition (54) or (55) holds, for $\mu = 1$.

Finally, for the case $\mu > 1$, if we define $\{h_\ell(m)\}_{\ell=1}^{M-1}$ by

$$h_\ell(m) = \begin{cases} (w_1^m - w_2^m), & \ell = 1, \\ \sin(m\theta_\ell), & 2 \le \ell \le M-1, \end{cases}$$

where w_1, w_2 are defined by (38), and proceed as in the case $\mu = 1$, one gets that

$$\begin{aligned} U(m,n) &= (w_1^m - w_2^m)\left[\left(I - \frac{\rho_1(r)A}{2}\right)^{-1}\left(I + \frac{\rho_1(r)A}{2}\right)\right]^n d_1 \\ &+ \sum_{\ell=2}^{M-1}\left[\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1}\left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right]^n \sin(m\theta_\ell)\, d_\ell, \end{aligned} \tag{59}$$

is a solution of (8)–(10) and (50) if (54) or (55) holds, for $\mu > 1$, and

$$\begin{aligned} d_1 &= \frac{\sum_{\eta=1}^{M-1} (w_1^m - w_2^m) f(\eta)}{\sum_{\eta=1}^{M-1} (w_1^m - w_2^m)^2}, \\ d_\ell &= \frac{\sum_{\eta=1}^{M-1} \sin(\eta\theta_\ell) f(\eta)}{\sum_{\eta=1}^{M-1} \sin^2(\eta\theta_\ell)}, \qquad 2 \le \ell \le M-1, \end{aligned} \tag{60}$$

because under condition (54) or (55), vectors d_ℓ defined by (60) lie in $\ker \tilde{G}(\mu)$.

Summarizing the following result has been established.

THEOREM 3.1. *Under hypotheses and notation of Theorem 2.1 assume that condition (55) holds, for any $\mu \in \sigma(-C^{-1}B) \cap \mathbb{R}$. Then, we have the following.*

(i) *If $\mu < 1$, the eigenfunction $\{U(m,n)\}$ defined by (51), where vectors d_ℓ, for $1 \le \ell \le M-1$ are defined by (53), is a solution of the mixed problem (8)–(10) and (50).*

(ii) *If $\mu = 1$, the eigenfunction $\{U(m,n)\}$ defined by (56), with vectors d_1 and d_ℓ, for $2 \le \ell \le M-1$ defined by (58), is a solution of the mixed problem (8)–(10) and (50).*

(iii) *If $\mu > 1$, let R_1 be solution of (45) and consider $\rho_1(r) = 2r(R_1 - 1)$. Then, the eigenfunction $\{U(m,n)\}$ defined by (59), with vectors d_1 and d_ℓ, for $2 \le \ell \le M-1$ defined by (60), is a solution of the mixed problem (8)–(10) and (50).*

4. STABILITY

We begin this section by introducing the concept of stability to be used.

DEFINITION 4.1. *We say that a solution $\{U(m,n)\}$ of (8) is stable in the fixed station sense with respect to the time, if given $h > 0$, $T > 0$, and M integer, both fixed with $Mh = 1$, then, the sequence $\{U(m,n)\}$ remains bounded, for all (m,n) with $1 \le m \le M-1$, $1 \le n \le N$ and all $k > 0$, N integer satisfying $Nk = T$.*

If $\{U(m,n)\}$ remains bounded, for all $n > 0$ and $1 \le m \le M-1$, $Mh = 1$, with M, h fixed, then, $\{U(m,n)\}$ is said to be uniformly stable with respect to the time.

We will study the stability of sequences $\{U_\ell(m,n,\mu)\}$ introduced in Theorem 2.1. By the spectral mapping theorem [11, p. 564], it follows that

$$\sigma\left(\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1}\left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right) = \left\{\frac{1-2ra\sin^2(\theta_\ell/2)}{1+2ra\sin^2(\theta_\ell/2)},\ a \in \sigma(A)\right\}. \tag{61}$$

Let $a = a_1 + ia_2 \in \sigma(A)$ with a_1, a_2 real numbers and $a_1 > 0$, and note that

$$\begin{aligned}\left|\frac{1-2ra\sin^2(\theta_\ell/2)}{1+2ra\sin^2(\theta_\ell/2)}\right|^2 &= 1 - \frac{8ra_1\sin^2(\theta_\ell/2)}{1+4ra_1\sin^2(\theta_\ell/2)+4r^2|a|^2\sin^4(\theta_\ell/2)} \\ &\le 1 - \frac{8r\beta(A)\sin^2(\theta_\ell/2)}{1+4r\alpha(A)+4r^2(\rho(A))^2},\end{aligned} \tag{62}$$

where $\alpha(A)$ and $\beta(A)$ are defined in (7). By (61),(62), one gets

$$\begin{aligned}&\rho\left(\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1}\left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right) \\ &\quad \le \left(1 - \frac{8r\beta(A)\sin^2(\theta_\ell/2)}{1+4r\alpha(A)+4r^2(\rho(A))^2}\right)^{1/2} < 1.\end{aligned} \tag{63}$$

By [10, p. 23, Theorem 1.3.6], there exists a matrix norm $\|.\|_r$, such that

$$\left\|\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1}\left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right\|_r < 1. \tag{64}$$

Hence, for all $n \ge 0$, $r > 0$,

$$\left\|\left[\left(I + 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)^{-1}\left(I - 2rA\sin^2\left(\frac{\theta_\ell}{2}\right)\right)\right]^n\right\|_r < 1. \tag{65}$$

This proves that $\{U_\ell(m,n,\mu)\}$ introduced in Theorem 2.1, is uniformly stable with respect to the time, for all μ if $2 \le \ell \le M-1$ and also, for $\{U_1(m,n,\mu)\}$ if $\mu < 1$. Let us consider now the cases $\{U_1(m,n,\mu)\}$, for $\mu \ge 1$. If $\mu = 1$, then, we have $U_1(m,n,1) = m\,d_1$ and thus, is uniformly stable with respect to the time, because it does not depend on n.

Now, let $\mu > 1$ and take r small enough so that

$$r(R_1 - 1)\|A\| < 1. \tag{66}$$

By the perturbation lemma [11], under condition (66), one gets

$$\left\|\left(I - \rho_1(r)\frac{A}{2}\right)^{-1}\right\| \le (1 - r(R_1-1)\|A\|)^{-1} = \mathrm{O}(1), \qquad \text{as } k \to 0. \tag{67}$$

Since $\|I + \rho_1(r)(A/2)\| = 1 + \mathrm{O}(k)$, as $k \to 0$, by (67), it follows that

$$\left\|\left[\left(I + \rho_1(r)\frac{A}{2}\right)\left(I - \rho_1(r)\frac{A}{2}\right)^{-1}\right]^n\right\| \le (1+\mathrm{O}(k))^n \le (1+\mathrm{O}(k))^N \le e^{NkL} = e^{TL}, \tag{68}$$

for some $L > 0$ and all n with $1 \le n \le N$. Since $h_1(m) = w_1^m - w_2^m$, is bounded, for $1 \le m \le M-1$, by (68) and (47) one gets that $\{U_1(m,n,\mu)\}$ is stable in the fixed station sense with respect to the time, for $\mu > 1$. Summarizing the following result has been established.

THEOREM 4.1. *With the hypotheses of Theorem 3.1, the solution $\{U(m,n)\}$ of problem (8)–(10),(50) given by Theorem 3.1 is uniformly stable with respect to the time, for the case $\mu \le 1$. If $\mu > 1$, then, the solution given by Theorem 3.1 is stable with respect to the time, in the fixed station sense, for small enough values of $k > 0$ so that (66) holds.*

In the following example, we show that hypotheses of Theorem 4.1 are easy to check.

EXAMPLE 4.1. Consider problem (8)–(10),(50) with data

$$f(x) = \left(\frac{17}{10}x, -\frac{3}{5}x, x\right)^{\mathsf{T}}, \qquad A = \begin{bmatrix} \frac{3}{10} & \frac{17}{10} & 0 \\ -\frac{3}{5} & \frac{13}{5} & 0 \\ -1 & -2 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 6 & -2 & 9 \\ 4 & 22 & 20 \\ -14 & 23 & -10 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 3 \\ -7 & 6 & -8 \end{bmatrix}.$$

Then, C is invertible with

$$C^{-1}B = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \sigma\left(-C^{-1}B\right) = \{-5, -2, 0\}, \qquad \sigma(A) = \left\{2, \frac{9}{10}\right\}.$$

The minimal polynomial of A is $m(\lambda) = (1/10)(\lambda - 2)^2(9 - 10\lambda)$, and thus, $p = 3$. Let $\mu = 0$, $G(0)$ and $\tilde{G}(0)$ be the matrices,

$$G(0) = C^{-1}B + 0I = C^{-1}B, \qquad \tilde{G}(0) = \left[\begin{array}{ccc} 2 & 1 & 4 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \\ \hline 4 & 2 & 8 \\ 0 & 10 & 6 \\ 0 & 0 & 0 \\ \hline 8 & 4 & 16 \\ 0 & 20 & 12 \\ 0 & 0 & 0 \end{array}\right],$$

with $\operatorname{rank}(\tilde{G}(0)) = 2 < 3$. Hence, for $\mu = 0$, hypothesis (27) of Theorem 3.1 are satisfied and also condition (55) because the initial vector $f(m) = ((17/10)mh, -(3/5)mh, mh)^{\mathsf{T}}$ lies in $\ker G(0)$ and $\ker G(0)$ is an invariant subspace of A because

$$G(0)^{\dagger} = \begin{bmatrix} \frac{4}{25} & -\frac{2}{25} & 0 \\ -\frac{3}{25} & \frac{88}{425} & 0 \\ \frac{1}{5} & -\frac{1}{85} & 0 \end{bmatrix} \qquad \text{and} \qquad G(0)\,A\left[I - G(0)^{\dagger} G(0)\right] = 0.$$

The solution of the problem is given by (51) and (53) with $\theta_\ell = ((2\ell - 1)/(2M - 1))\pi$, $1 \le \ell \le M - 1$, and by Theorem 4.1, it is uniformly stable with respect to the time. It is easy to check that, for $\mu = -5$ and $\mu = 2$, the $\operatorname{rank}(\tilde{G}(\mu)) = 3$ and thus, hypothesis (27) of Theorem 3.1 does not hold.

5. THE PROJECTION METHOD

In Section 3, a solution of the mixed discretized problem (8)–(10),(50) is constructed assuming that there exists $\mu \in \sigma(-C^{-1}B) \cap \mathbb{R}$ and, in this case, the range $\{f(m);\ 0 \leq m \leq M\}$ lies in $\ker G(\mu)$, see condition (55). In this section, we assume that

$$\mu_j \in \sigma\left(-C^{-1}B\right) \cap \mathbb{R}, \qquad \mu_i \neq \mu_j, \quad i \neq j, \quad 1 \leq j \leq q. \tag{69}$$

Let us consider the matrices in $\mathbb{C}^{s\times s}$ defined by

$$\left.\begin{aligned} G(\mu_j) &= C^{-1}B + \mu_j I, \\ T(\mu_j) &= G(\mu_1)G(\mu_2)\cdots G(\mu_{j-1})G(\mu_{j+1})\cdots G(\mu_q), \qquad 1 \leq j \leq q, \end{aligned}\right\} \tag{70}$$

and the polynomial,

$$L(x) = (x-\mu_1)(x-\mu_2)\cdots(x-\mu_q). \tag{71}$$

Since the polynomials $x-\mu_j$ are mutually coprime, by the descomposition theorem [12, p. 536], it follows that

$$S = \ker L\left(-C^{-1}B\right) = \ker G(\mu_1) \oplus \ker G(\mu_2) \oplus \cdots \oplus \ker G(\mu_q). \tag{72}$$

Let us consider the scalars α_j defined by

$$\alpha_j = \left\{ \prod_{\gamma=1,\gamma\neq j}^{q} (\mu_j - \mu_\gamma) \right\}^{-1}, \tag{73}$$

and the coprime polynomials of degree $q-1$ defined by

$$Q_j(x) = \prod_{\gamma=1,\gamma\neq j}^{q} (x-\mu_\gamma). \tag{74}$$

By (73),(74) and Bezout's theorem [12, p. 538], it follows that

$$Q(x) = \sum_{j=1}^{q} \alpha_j Q_j(x). \tag{75}$$

Note that $Q(x)$ is the Lagrange interpolating polynomial satisfying $Q(\mu_j) = 1$, for $1 \leq j \leq q$. By (75) and the properties of the matrix functional calculus acting on the matrix $-C^{-1}B$, it follows that

$$\begin{aligned} I = Q\left(-C^{-1}B\right) &= \sum_{j=1}^{q} \alpha_j\, Q_j\left(-C^{-1}B\right) \\ &= (-1)^{q-1} \sum_{j=1}^{q} \alpha_j\, G(\mu_1)\, G(\mu_2)\cdots G(\mu_{j-1}) G(\mu_{j+1})\cdots G(\mu_q), \end{aligned}$$

$$I = (-1)^{q-1} \sum_{j=1}^{q} \alpha_j T(\mu_j), \tag{76}$$

Given the problem (8)–(10),(50), let us consider the projection of the sequence $\{f(m)\}_{m=1}^{M-1}$ on the subspace $\ker G(\mu_j)$ defined by

$$\hat{f}_j(m) = (-1)^{q-1} \alpha_j T(\mu_j) f(m), \qquad 0 \le m \le M, \quad 1 \le j \le q. \tag{77}$$

Note that by (76),(77), one gets

$$\begin{aligned} G(\mu_j) \hat{f}_j(m) &= (-1)^{q-1} \alpha_j G(\mu_j) T(\mu_j) f(m) \\ &= -\alpha_j L\left(-C^{-1}B\right) f(m) = 0, \qquad 1 \le j \le q, \quad 0 \le m \le M, \end{aligned}$$

or,

$$\hat{f}_j(m) \in \ker G(\mu_j), \qquad 1 \le j \le q, \quad 0 \le m \le M. \tag{78}$$

Let us denote (P_j), the problem defined by (8)–(10), together with the initial condition,

$$U(m, 0) = \hat{f}_j(m), \qquad 0 \le m \le M, \quad 1 \le j \le q. \tag{79}$$

Under the hypothesis,

$$G(\mu_j) A \left(I - G(\mu_j)^{\dagger} G(\mu_j)\right) = 0, \qquad 1 \le j \le q, \tag{80}$$

the solution of problem (P_j) is given by Theorem 3.1, where the Fourier coefficients d_ℓ should be replaced by $d_\ell^{(j)} = (-1)^{q-1}\alpha_j T(\mu_j)\, d_\ell$.

Summarizing the following result has been established.

THEOREM 5.1. *Let $\mu_1, \mu_2, \dots, \mu_q$ distinct real eigenvalues of the matrix $-C^{-1}B$. Let $L(x)$ be the polynomial defined by (71) and let $G(\mu_j)$ and $T(\mu_j)$ be the matrices in $\mathbb{C}^{s\times s}$ defined by (70). Let p be the degree of the minimal polynomial of A and assume the hypotheses and notation of Theorem 3.1 with respect to matrices C and A. If $\operatorname{rank}(\tilde{G}(\mu_j)) < s$, for $1 \le j \le q$, condition (80) holds and $\{U_j(m,n)\}$ is the solution of problem (P_j) where $\hat{f}_j(m)$ is given by (77), then,*

$$U(m, n) = \sum_{j=1}^{q} U_j(m, n), \qquad 1 \le m \le M-1, \quad n > 0, \tag{81}$$

is a solution of the mixed problem (8)–(10),(50). This solution is uniformly stable if $\mu_j \le 1$, for $1 \le j \le q$ and stable in the fixed station sense with respect to the time if $\mu_{j_0} > 1$, for some j_0 with $1 \le j_0 \le q$.

EXAMPLE 5.1. Consider problem (8)–(10),(50) with data

$$A = \begin{bmatrix} \frac{3}{4} & 0 & 1 \\ 3 & 1 & -12 \\ -\frac{1}{4} & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -\frac{2}{3} \\ \frac{7}{6} & -\frac{1}{2} & -5 \\ -\frac{4}{3} & \frac{3}{2} & 14 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 0 & 2 \\ 1 & -1 & 0 \\ 1 & 3 & 3 \end{bmatrix},$$

$f(m) = (f_1(m), f_2(m), f_3(m))^{\top} = (mh, m^2h^2 - mh, mh)^{\top}$, $h = 1/M$, with $1 \le m \le M-1$.

In this case, we have that C is invertible, $\sigma(A) = \{1, 7/4\}$ and $p = 2$,

$$-C^{-1}B = \begin{bmatrix} -\frac{1}{6} & 0 & 0 \\ 1 & -\frac{1}{2} & -5 \\ -\frac{1}{2} & 0 & \frac{1}{3} \end{bmatrix}, \qquad \sigma\left(-C^{-1}B\right) = \left\{-\frac{1}{2}, -\frac{1}{6}, \frac{1}{3}\right\},$$

$$\mu_1 = -\frac{1}{2}, \qquad \mu_2 = -\frac{1}{6}, \qquad \mu_3 = \frac{1}{3}.$$

$$G(\mu_1) = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ -1 & 0 & 5 \\ \frac{1}{2} & 0 & -\frac{5}{6} \end{bmatrix}, \qquad G(\mu_2) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & \frac{1}{3} & 5 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix},$$

$$\tilde{G}(\mu_1) = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ -1 & 0 & 5 \\ \frac{1}{2} & 0 & -\frac{5}{6} \\ \hline -\frac{1}{4} & 0 & -\frac{1}{3} \\ -2 & 0 & 9 \\ \frac{7}{12} & 0 & -\frac{7}{6} \end{bmatrix}, \qquad \tilde{G}(\mu_2) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & \frac{1}{3} & 5 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \hline 0 & 0 & 0 \\ -1 & \frac{1}{3} & 5 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix},$$

$\operatorname{rank}(\tilde{G}(\mu_j)) = 2 < 3$, for $j = 1, 2$,

$$G(\mu_1)^\dagger = \begin{bmatrix} -\frac{111}{73} & \frac{18}{73} & \frac{108}{73} \\ 0 & 0 & 0 \\ -\frac{117}{365} & \frac{18}{73} & \frac{102}{365} \end{bmatrix}, \qquad G(\mu_2)^\dagger = \begin{bmatrix} 0 & \frac{18}{73} & \frac{181}{73} \\ 0 & \frac{3}{73} & \frac{18}{73} \\ 0 & \frac{18}{73} & \frac{35}{73} \end{bmatrix}.$$

$\ker G(\mu_j)$ in an invariant subspace of A, for $j = 1, 2$, because

$$G(\mu_j)\, A \left(I - G(\mu_j)^\dagger\, G(\mu_j)\right) = 0, \qquad j = 1, 2.$$

The eigenvalue $\mu_3 = 1/3$ is disregarded because $\operatorname{rank}(\tilde{G}(\mu_3)) = 3$. Consider Theorem 5.1 with $q = 2$. Taking into account $G(\mu_1)$ and $G(\mu_2)$, we have

$$\hat{f}_1(m) = \begin{bmatrix} 0 \\ m^2h^2 \\ 0 \end{bmatrix}, \qquad \hat{f}_2(m) = \begin{bmatrix} mh \\ -mh \\ mh \end{bmatrix},$$

$$\hat{f}_j(m) \in \ker G(\mu_j), \qquad j = 1, 2,$$

$$f(m) \in \ker L\left(-C^{-1}B\right) = \ker G(\mu_1) \oplus \ker G(\mu_2).$$

If $U_j(m,n)$ is the solution of problem (8)–(10),(50) with initial condition,

$$U_j(m, 0) = \hat{f}_j(m), \qquad j = 1, 2, \quad 0 \le m \le M,$$

then,

$$U(m,n) = U_1(m,n) + U_2(m,n),$$

is the solution of (8)–(10),(50). Since $\mu_1 < 1$, $\mu_2 < 1$, this solution is uniformly stable with respect to the time.

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