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Research on Necessary and Sufficient Condition for Hamilton Graph

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Abstract

An important concept, "closed domain" is proposed in this paper. In the same time, necessary and sufficient lemma for closed domain, R, is proved on which necessary and sufficient theorem for judging whether a general graph G is a Hamilton graph is proposed and proved. All instances in this paper are judged by comparatively using the theorem proposed herein and the original necessary condition theorem and sufficient condition theorem to prove the correctness of the method proposed in this paper.

*Keywords:*graph, spanning sub-graph, loop, closed domain, inside and outside (yin yang)boundary, Hamilton graph

1. Introduction

In 1859, Irish mathematician William Hamilton firstly proposed a mathematic game on dodecahedron. The question is whether a loop containing all peaks can be found? If each peak is assimilated to a city and the side between two peaks is regarded as a traffic line, then, this question is also called "circumnavigation" question. The question is called Hamilton graph by mathematicians and is defined as follow:

For given graph, G, if a route passing all nodes in the graph for only once, this route is called Hamilton route. If a loop passing all nodes in the graph for only once, this is called Hamilton loop (circle). Graph containing Hamilton loop is called Hamilton graph.

Research results show that finding out the necessary and sufficient condition for Hamilton graph is a puzzle in graph theory without satisfying solution until now. Necessary condition theorem (theorem 1) and sufficient theorem (theorem 2) which were found recently for judging Hamilton graph are unsuitable for solving this question due to unrestricted or excessive restrict condition. It is just as the point proposed by Professor Wang Shuhe, mathematician of our country. "It is not easy to judge whether a graph is a

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Hamilton graph", he said: "Although there are some sufficient or necessary conditions for judging Hamilton graph, these conditions also can not solve the question. The question is actually one of the significant difficult questions in mathematics and computer science."

Theorem 1:

Necessary condition on which G is a Hamilton graph is

$\forall S \subset V(G), S \neq \phi$ Then, W (G-S) ≤ |S|.

Where, W (G-S) is the number of connecting zones.

Theorem 2:

Suppose |V(G)| ≥ 3, if d(u)+d(v)≥|V(G)|-1 is true for any pare of peak u and v, then, there is Hamilton track in G; if d(u)+d(v) ≥|V(G)|, then G is a Hamilton graph.

This theorem was established by ORE in 1960 as necessary and sufficient condition for Hamilton graph.

By analyzing the applicable cases for theorem 1 and 2, especially the instances that use these theorems to judge whether the graphs are Hamilton graphs, it can be found out that some graphs meet the condition in theorem 1 although originally they are not Hamilton graphs which indicates that the necessary condition is unrestricted; for some graphs, for example "knight graph" in chess which is originally Hamilton graph, but it can not meet the ORE sufficient condition for which it can not be judge by using this theorem. It indicates that the ORE condition is too restricted.

To overcome the shortcomings of above two conditions and to find out necessary and sufficient condition for scientifically and reasonably judging a Hamilton graph, the author finds out and extracts an important concept named in this paper as "closed domain" and suggestively proposes a necessary and sufficient condition for judging Hamilton graphs for discussion from the results of research by means of observation and analysis on instances of Hamilton graphs.

2. Basis of discussion

2.1 Definition of closed domain

For given connected graph G = <V, E>, where

$V = \{v_1, \ldots, v_n\} \quad E = \{e_1, \ldots, e_m\}$

Suppose number of nodes |V| = n, and number of sides |E| = m, if graph R = <v, e> is a spanning spanning sub-graph of connected graph G = <V, E> and |v| = |e| = |V| = n, then, R = <v, e> is called a closed domain of G = <V, E> and named $R = [G]$.

2.2 Lemma

For connected graphs G = <V, E> and R = <v, e>, suppose R is a closed domain of G, then, necessary and sufficient condition for R=[G] is that R and G shall meet the following requirements in the same time:

- $|v| = |e| = |V| = n$ (condition 1)
- $\deg(R) = |V| = n$ (condition 2)
- $\deg(V_i) = 2 \qquad (i = 1, 2, \ldots, n)$ (condition 3)
- R divides G into an internal/yin plane and an external/yang plane, and all n nodes locate at border line of internal plane and external plane.(condition 4)
- R is a spanning sub-graph of G and W(R) = 1. (condition 5)

2.3 Theorem

The connected graph is a Hamilton graph if and only if all spanning sub-graphs are closed domain R.

2.4 Proving

1) *Lemma proving*

a) Necessity

2.5 Suppose that the connected graph $G=<V,E>$ *has a closed domain* $R=<v,e>$*, i.e.* $R=[G]$*, then* $|v|=|e|=|V|=n$*. So condition 1 of lemma in this paper is met according to definition of closed domain in this paper. On the basis of this result, it can be know that* $R=<v,e>$ *is a graph with n nodes and n sides. The number planes bounded by these nodes and sides is n, i.e.* $\deg(R)=|V|=n$*. On the basis of this conclusion, condition 2 of lemma in this paper is met.*

Because each node of $R=<v,e>$ is only related to two sides, so, $\deg(V_i)=2$ which indicates that condition 3 of lemma in this paper is met.

It is known from Euler's theorem that if G is a connected planar graph with v nodes, e sides and r planes, then the Euler's formula becomes $v-e+r=2$.

It is known from definition of closed domain in this paper that $R=<v,e>$ is a spanning sub-graph of $G=<V,E>$, i.e. $|v|=|V|=n$， $|e|\leq|E|$. In fact, spanning sub-graph R divides plane at which G locates into two parts: interior of graph R is called internal/yin plane and exterior of R is called external/yang plane. So, it can be known from $r=2$ and $v-e+r=2$ obtained from Euler's formula that $v=e$. Then, according to definition of Hamilton loop, it is known that $|v|=|e|=|V|=n$. This indicates the reason why $R=<v,e>$ is a spanning sub-graph of $G=<V,E>$ and why $|V|=|e|=n$ is defined.

Deriving $|v|=|e|$ from $r=2$ and then obtaining $|v|=|V|$ from definition of Hamilton loop to derive $|v|=|e|=|V|=n$ is a theoretic critical point and breach. Spanning sub-graph $R=<v,e>$ divides G into internal/yin plane and external/yang plane. In graph G, n nodes relate to n sides, so, these n nodes unaffectedly locate along the border line of internal and external planes. It indicates that condition 4 of lemma in this paper is met.

According to the definition of loop, Hamilton loop consequentially pass all nodes of G, so, if a graph is a Hamilton graph, it must have loop containing all nodes. It indicates that condition 5 of lemma in necessary.

Above all, necessity of lemma in this paper can be derived from R = [G].

b) Sufficiency

For given graph $G=<V,E>$ and $R=<v,e>$, if all conditions through condition 1 to 5 are met, then, R completely meets the definition of closed domain is a spanning sub-graph of G and meets the provision $|v|=|e|=|V|=n$, i.e. sufficiency of R = [G] is proved, so, the lemma is proved.

2) Theorem proving

a) Necessity

Suppose that connected graph $G=<V,E>$ is a Hamilton graph. According to definition of Hamilton graph, a Hamilton loop exists in G. Suppose that the area bounded by this loop is $R=<v,e>$, it can be known that $|v|=|V|=n$ and $e=|v|=n$ according to definition of Hamilton graph. This indicates that R is a spanning sub-graph of G. R is a closed domain bounded by loop of G and a spanning sub-graph of G as well, i.e. $R=[G]$, on which the necessity is proved.

b) Sufficiency

Suppose that $R=<v,e>$ is the connected graph $G=<V,E>$, where $|v|=n$, then， $|v|=|V|=n$， $|e|\leq|E|$， i.e. R has n nodes and e sides. Suppose that $R=<v,e>$ is a closed domain of graph $G = < V, E >$, it must meet all conditions through condition 1 to 5 of closed domain lemma in this paper.

Graph $R = < v, e >$ meeting all conditions must be a Hamilton loop and R is also a spanning sub-graph of G. So, G has Hamilton loop, namely, G is a Hamilton graph.

On the basis of this conclusion, the necessity is proved; in further, the necessity and sufficiency are all proved. So the theorem is proved.

3. Instances

3.1 Example 1

Judge whether regular dodecahedron planar embedding (figure 1) and regular icosahedron planar embedding (figure 2) are Hamilton graph?

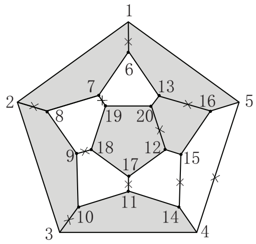

Figure 1. Regular dodecahedron planar embedding

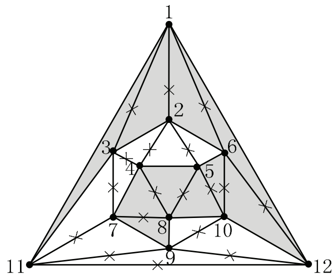

Figure 2. Regular icosahedron planar embedding

Solution: Graph in figure 1 is a regular dodecahedron planar embedding graph and graph in figure 2 is a Regular icosahedron planar embedding graph.

These two graphs will be judged using necessary and sufficient condition (closed domain) theorem in this paper and theorem 1(necessary condition) and theorem 2(sufficient condition) for comparing.

3) Judge using necessary and sufficient condition (closed domain) theorem in this paper

For graph in figure 1, |e|=30，|V|=20. In closed domain R, E|=|V|=20, deg(V_i)=3 (i=1,2,⋯20). Each node needs to subtract 1 degree and two nodes cutting off one side need to subtract 1 degree. So the sides need to be cut off are 4-5, 14-15, 11-17, 9-18, 7-19, 3-10, 2-8, 1-6, 13-16, 20-12, totally 10 sides (sides marked with "×" are sides to be cut off). On the basis of this, a closed domain is obtained. Then fill internal/yin plane with gray and external/yang plane with white. 20 nodes are all locate along the border line. For each node of R, deg(V)=2，deg(R)=20. Comparing R with graph in figure 1, the nodes are same, but number of sides in R (20) is less than that of in graph in figure 1 (30), so R is a spanning sub-graph of graph in figure 1 and meets the necessary and sufficient condition proposed herein.

In a similar way, cut off |E|-|V|=18 sides (marked with symbol "×" in the graph) from the graph in figure2 Closed domain of the graph is filled with gray. This graph meets the necessary and sufficient condition, so it is a Hamilton graph.

4) Judge by means of theorem 1 (necessary condition)

For graph in figure 1, cut off 15 nodes from 6 to 20, and |S|=15， W(G-S)=1, so, W(G-S)<|S| and meets the condition of theorem 1. Because it only meets the necessary condition, it is probably not a Hamilton graph.

For graph in figure 2, cut off 9 nodes from 2 to 10, and |S|=15， W(G-S)=1, so, W(G-S)<|S| and meets the condition of theorem 1. But it probably not a Hamilton graph.

5) Judge by means of theorem 2 (sufficient condition)

For graph in figure 1, f (u) + d (v) = 6 < n = 20, it can not meet the theorem 2 and is maybe not a Hamilton graph.

For graph in figure 2, f (u) + d (v) = 6 < n = 12, it can not meet the theorem 2 and is maybe not a Hamilton graph.

Conclusion:

- Judged by means of necessary and sufficient condition proposed herein, graph in figure 1 is a Hamilton graph. Its loop is 1-2-3-4-14-11-10-9-8-7-6-13-20-19-18-17-12-15-16-5-1. And graph in figure 1 is also a Hamilton graph. Its loop is 1-11-3-2-6-5-4-7-9-8-10-12-1.
- Judged by means of theorem 1 and theorem 2, figur1 and graph in figure 2 are all uncertainly Hamilton graph, i.e. it can not be judged.

3.2 Example 2

Judge whether graph in figure 3 is a Hamilton graph?

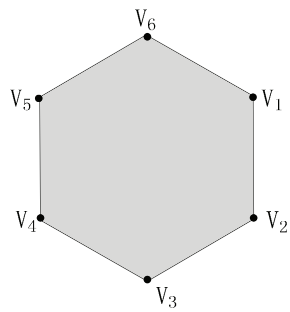

Figure 3. Hexagon

3.2.1 Judge by means of necessary and sufficient condition proposed herein

- |v|=|e|=|V|=n=6
- deg(R)=|V|=6
- $deg(V_i)=2$ (i=1,2,⋯6)
- R=<6,6> ,G=<6,6>, so R is a spanning sub-graph of G and is a connected graph.

Six nodes V_1 to V_6 all locate along border line of the closed domain, i.e. they all locate along internal/yin and external/yang border line. So graph in figure 2 meets the necessary and sufficient condition proposed herein, it is a Hamilton graph.

3.2.2 Judge by means of theorem 1

Let S = (V_1, V_2), then, |S|=2 W (G-S) = 1, so W (G-S) <|S|. It meets the condition of theorem 1 and is maybe not a Hamilton graph.

3.2.3 Judge by means of theorem 2

Randomly take two points u and v, d(n)+d(v)=4<6-1, It doesn't meet the condition of theorem 2 and is maybe not a Hamilton graph.

This indicates that the graph can not be judged by means of theorem 1 and theorem 2. Whereas the graph is judged a Hamilton graph by means of necessary and sufficient condition proposed herein, and V_1-V_2-V_3-V_4-V_5-V_6-V_1 is a loop of it.

3.3 Example 3

Judge whether the Knight graph in figure 4 is a Hamilton graph?

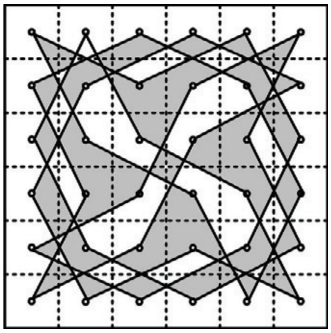

Figure 4. Knight graph

3.3.1 Judge by means of necessary and sufficient condition proposed herein

- $|v|=|e|=|V|=n=36$
- $\deg(R)=|V|=n=36$
- $\deg(V_i)=2 \qquad (i=1,2,\ldots,36)$
- R divides G into internal (yin) plane and external (yang) plane shown in figure 4. All nodes through 1 to 36 all locate along the border line of internal plane and external plane.
- Compare R and G. Then, |U|=|V|=36, |E|=|e|=36. So, R is a spanning sub-graph and a connected graph.

Based on the above evidence satisfy and meet the lemma herein. On the basis of this, R is a closed domain of G, consequently, the graph in figure 4 meets the theorem herein and is a Hamilton graph. Its loop is 1-2-3-4-5-6-7-8-9-10-11-12-13-14-14-16-17-18-19-20-21-22-23-24-25-26-27-28-29-30-31-32-33-34-35-36-1.

3.3.2 Judge by means of theorem 1

Cut off node 1 through node 34, W(G-S)=1，|S|=34, so, W(G-S)<|S|. It meets theorem 1 but it is not necessarily a Hamilton graph.

3.3.3 Judge by means of theorem 2

Suppose u_i and v_j are any two nodes (i (j)=1,2,⋯⋯,36) of the graph in figure 4. Then for any u_i and v_j, d(u_i)+d(v_j)=4<36. It can not meet the condition of theorem 2, so it is maybe not a Hamilton graph.

Above all, the graph can not be correctly judged by means of theorem 1 and theorem 2. But it is definitely judged as a Hamilton graph by means of necessary and sufficient condition proposed herein. The loop of the graph is shown in figure 4.

3.4 Example 4

Judge whether the graph in figure 5 is a Hamilton graph?

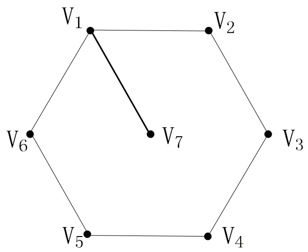

Figure 5. common graph

Judge by means of necessary and sufficient condition proposed herein:

Because $\deg(V_1)=3, \deg(V_7)=1, \deg(V_2)=\deg(V_3)=\deg(V_4)=\deg(V_5)=2$.

Cut off side V_1 to V_7, then deg (V_1) =2, but deg (V_7) =0, it can not meet the lemma herein, i.e. it cannot meet the necessary and sufficient condition proposed herein. So, the graph in figure 5 is not a Hamilton graph.

3.5 Example 5

Judge whether the graph in figure 6 is a Hamilton graph?

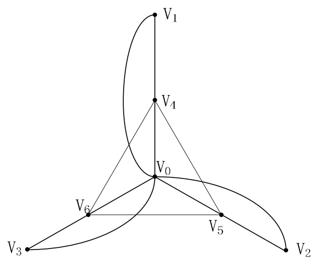

Figure 6. common graph

3.5.1 Judge by means of necessary and sufficient condition proposed herein.

In first, Judge whether it is a closed domain by means of the lemma herein.

- $|v|=7, |e|=12$
- $\deg(R)=|V|=7$
- $\deg(V_i)=2$ (i=1,2,3)

$\deg(V_i)=4$(i=4,5,6) $\deg(V_0)=6$. If the degree of V_0 is 2, it is needed to cut off 4 sides relating to v. There must be one point with degree 1 among V_1,V_2,V_3. So, degrees of 7 nodes (V_0 to V_6) must not all be 2 and there is no closed domain R in the graph in figure 6, i.e. it can not meet the necessary and sufficient condition proposed herein and not a Hamilton graph.

Literature [1] uses other method to prove that the graph in figure 6 is not a Hamilton graph.

3.5.2 Judge by means of theorem 1

Let $S=(V_1,V_2,V_3,V_0)$, then $|S|=4$ $W(G-S)=1$, so $W(G-S)=1<|S|$. It meets the theorem 1, so the graph in figure 6 is not necessarily a Hamilton graph, i.e. it can not be judged.

3.5.3 Judge by means of theorem 2

Take any two points V_1 and V_2, then $d(n)+d(v)=4<7$, it can not meet the theorem 2. The graph in figure 6 is maybe not a Hamilton graph, i.e. it can not be judged.

4. Conclusion

- Lemma and necessary and sufficient theorem proposed in this paper are closed to the definition of Hamilton graph and have strong basis background. The author has found that the Hamilton loop (circle) divides the plane where connected graph G locates into internal part and external part (which is called yin plane and yang plane). On this basis, it can be known that r = 2. According to Euler's formula v-e+ r =2, then, v=e is obtained.The author considers that the Hamilton loop (circle) resulting from r = 2 locates along the border line which divides the yin plane and yang plane. On view of yin in yang or yang in yin in *Book* of *Changes*, Hamilton graph can be considered a graph with balance between yin and yang. If this consideration is correct, judge whether a connected graph G is a Hamilton loop is equivalent to judge whether states of balance between yin and yang exist in graph G. The graph with balance between yin and yang is considered that Hamilton graph is contained, whereas the graph with imbalance between yin and yang is considered that a Hamilton graph is not contained. Namely, the graph G is a Hamilton graph if it is balance between yin and yang; otherwise, it is not a Hamilton graph.
- The instances validations show that the lemma and necessary and sufficient theorem proposed in this paper are correct. Where as necessary condition (theorem 1) and sufficient condition (theorem 2, ORE theorem) often can not give a definite judging conclusion. So, it can be shown that the lemma and necessary and sufficient theorem proposed in this paper are correct, and easy and feasible for use.
- Finding out the necessary and sufficient condition for Hamilton graph is a very difficult problem. The lemma and necessary and sufficient theorem proposed in this paper are only suggestive and preliminary discussion. Advices and directives of experts are welcome for mistakes in this paper.

References

[1] Wang Shuhe. Graph theory. Science Pres, 2008, pp. 130-137

[2] Tian Wencheng, Zhou Luxin. 2001,pp. 87-225.

[3] Xiang Jie, Yu Zhongwen. Discrete Mathematics. Tianjin University Press, 2001, pp.248-292.

[4] Ainouche A, Kouider M. Hamiltonism and partially square graphs[J].Graphs and Combinatorics,1999,15(3):257 265.

[5] Liu,Yiping,Tian,Feng,Wu,Zhengsheng. "Sequence concerning Hamiltonicity of graphs" Journal of Nanjing Normal University(Natural Science),1995,18(1):19~28.

[6] Min Xu "Several New Results on Hamiltonicity of Graphs" Outstanding Dissertation of Chinese text databases 2002,(02)

[7] li-li Zhang "Several Sufficient Conditions on Hamiltonian Properties in Graphs" Outstanding Dissertation of Chinese text databases, 2002,(02)

[8] Jin Yang "Several Sufficient Conditions for Hamiltonian Graphs" Outstanding Dissertation of Chinese text databases, 2003,(02)