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# Almost principal minors of inverse $M$-matrices

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**Abstract**

It is well known that if an inverse $M$-matrix has a 0 entry, then it must be reducible and thus have many more 0 entries. This property is actually a special case of a deeper phenomenon that might be loosely described as relations among vanishing almost principal minors in an inverse $M$-matrix. This phenomenon encompasses both minors of nested dimension (a certain loose monotonicity) and minors of the same size in loosely related positions. This phenomenon is limited to almost principal minors and, where possible, converses and examples are given to show the limit of the extent of this phenomenon. It is also shown that if one almost principle minor is contained in another, then the magnitude of the former is larger than that of the latter.


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## 1. Introduction

By an *$M$-matrix*, we mean an $n$-by-$n$ matrix $A$ with nonpositive off-diagonal entries that is invertible and has an entry-wise nonnegative inverse; this is equivalent to $A$ being of the form $\alpha I - B$, in which $B$ is entry-wise nonnegative and $\alpha > \rho(B)$,

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PII: S 0 0 2 4 - 3 7 9 5 ( 0 1 ) 0 0 3 5 2 - 4

the spectral radius of $B$. A nonnegative matrix that occurs as the inverse of an $M$-matrix is called an *inverse M-matrix*; we denote the $M$-matrices by $\mathscr{M}$ and the inverse $M$-matrices by $\mathscr{I}M$. Much is known about both important classes [2,4,7,8]. For any $m$-by-$n$ matrix $A$, we denote the submatrix lying in rows $\alpha$ and columns $\beta$ by $A[\alpha, \beta]$, where $\alpha \subseteq M = \{1, 2, \ldots, m\}$ and $\beta \subseteq N = \{1, 2, \ldots, n\}$. If $m = n$ and $\alpha = \beta$, the principal submatrix $A[\alpha, \alpha]$ is abbreviated $A[\alpha]$. If $A$ is square and $A[\alpha]$ is invertible, the *Schur complement* $A/A[\alpha]$ is defined as $A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c]$ in which $\alpha^c$ denotes the complement of $\alpha$ in $M = N$. It is known that both $\mathscr{M}$ and $\mathscr{I}M$ are closed under extractions of principal submatrices or of Schur complements [8]. We let $|\alpha|$ denote the cardinality of $\alpha$.

We shall make use of the Schur complement form of the inverse [6] given in the following form. Let the square matrix $A$ be partitioned as

$$\mathbf{A} = \begin{bmatrix} A[\alpha] & A[\alpha, \alpha^c] \\ A[\alpha^c, \alpha] & A[\alpha^c] \end{bmatrix}, \tag{1.1}$$

in which $A$, $A[\alpha]$, and $A[\alpha^c]$ are all invertible. Then

$$\begin{aligned} \mathbf{A}^{-1} &= \begin{bmatrix} (A/A[\alpha^c])^{-1} & -A[\alpha]^{-1}A[\alpha, \alpha^c](A/A[\alpha])^{-1} \\ -(A/A[\alpha])^{-1}A[\alpha^c, \alpha]A[\alpha]^{-1} & (A/A[\alpha])^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (A/A[\alpha^c])^{-1} & -(A/A[\alpha^c])^{-1}A[\alpha, \alpha^c]A[\alpha^c]^{-1} \\ -A[\alpha^c]^{-1}A[\alpha^c, \alpha](A/A[\alpha^c])^{-1} & (A/A[\alpha])^{-1} \end{bmatrix}. \end{aligned} \tag{1.2}$$

From (1.2) we see that if $A \in IM$, then

$$\begin{aligned} &A[\alpha^c]^{-1}A[\alpha^c, \alpha],\ A[\alpha^c, \alpha]A[\alpha]^{-1}, \\ &\quad (A/A[\alpha^c])^{-1}A[\alpha, \alpha^c],\ A[\alpha, \alpha^c](A/A[\alpha])^{-1} \geqslant 0, \end{aligned}$$

a fact we will use later.

We will also need a special case of Sylvester's identity for determinants (see [6]). Let $A$ be an $n$-by-$n$ matrix, $\alpha \subseteq N$, and suppose $|\alpha| = k$. Define the $(n-k)$-by-$(n-k)$ matrix $B = (b_{ij})$, with $i, j \in \alpha^c$, by setting $b_{ij} = \det A[\alpha + i, \alpha + j]$, for every $i, j \in \alpha^c$. Then Sylvester's identity states that for each $\delta, \gamma \subseteq \alpha^c$, with $|\delta| = |\gamma| = m$,

$$\det B[\delta, \gamma] = (\det A[\alpha])^{m-1} \det A[\alpha \cup \delta, \alpha \cup \gamma]. \tag{1.3}$$

A special case that we use is the following. Let $A$ be an $n$-by-$n$ matrix partitioned as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12}^{\mathrm{T}} & a_{13} \\ a_{21} & A_{22} & a_{23} \\ a_{31} & a_{32}^{\mathrm{T}} & a_{33} \end{bmatrix}, \tag{1.4}$$

in which $A_{22}$ is $(n-2)$-by-$(n-2)$ and $a_{11}$, $a_{33}$ are scalars. Define the matrices

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12}^{\mathrm{T}} \\ a_{21} & A_{22} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a_{12}^{\mathrm{T}} & a_{13} \\ A_{22} & a_{23} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} a_{21} & A_{22} \\ a_{31} & a_{32}^{\mathrm{T}} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} A_{22} & a_{23} \\ a_{32}^{\mathrm{T}} & a_{33} \end{bmatrix}.$$

If we let $b = \det B$, $c = \det C$, $d = \det D$, and $e = \det E$, then by (1.3) it follows that

$$\det \begin{bmatrix} b & c \\ d & e \end{bmatrix} = \det A_{22} \ \det A.$$

Hence, provided $\det A_{22} \neq 0$, we have

$$\det A = \frac{\det B \ \det E - \det C \ \det D}{\det A_{22}}. \tag{1.5}$$

Square submatrices that are defined by index sets differing in only one index, or the minors that are their determinants, are called *almost principal*. For simplicity we abbreviate "almost principal minor" ("principal minor") to APM (PM). APMs are special for a variety of reasons including that, in the co-factor form of the inverse, they are exactly the numerators of off-diagonal entries of inverses of principal submatrices. So, if $A \in \mathscr{M}$ or $A \in \mathscr{I}M$, an APM is 0 if and only if an off-diagonal entry of the inverse of a principal submatrix equals 0. Using the informal notation $\alpha + i$ $(\alpha - i)$ to denote the augmentation of the set $\alpha$ by $i \notin \alpha$ (deletion of $i \in \alpha$ from $\alpha$), almost principal submatrices are of the form $A[\alpha + i, \alpha + j]$, $i, j \notin \alpha$ $(A[\alpha - i, \alpha - j]$, $i, j \in \alpha)$, $i \neq j$. All PMs in $A \in \mathscr{M}$ or $A \in \mathscr{I}M$ are positive. Because of inheritance (of the property $\mathscr{I}M$ under extraction of principal submatrices [8]) the sign of every nonzero APM in $A \in \mathscr{M}$ or $A \in \mathscr{I}M$ is determined entirely by its position. Specifically, if $\alpha \subseteq N$ and $i, j \in \alpha$, $\operatorname{sgn}(\det A[\alpha - i, \alpha - j])$ equals $(-1)^{r+s}$ if $A \in \mathscr{M}$ and $(-1)^{r+s+1}$ if $A \in \mathscr{I}M$ in which $r$ (respectively, $s$) is the number of indices in $\alpha$ less than or equal to $i$ (respectively, $j$). An analogous statement can be made concerning $\det A[\alpha + i, \alpha + j]$.

Our purpose here is to present more subtle information about APMs of an $\mathscr{I}M$ matrix: certain inequalities, and relations among those that may be 0. To clarify our interest, consider the following example.

**Example 1.1.** Consider the $\mathscr{I}M$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0.4 & 0.2 \\ 0.8 & 1 & 0.8 & 0.4 \\ 0.6 & 0.5 & 1 & 0.4 \\ 0.2 & 0.2 & 0.25 & 1 \end{bmatrix}.$$

Notice that the only vanishing APMs are the determinants of $A[\{1,2\},\{2,3\}]$, $A[\{1,2\},\{2,4\}]$, $A[\{1,2,3\},\{2,3,4\}]$, and $A[\{1,2,4\},\{2,3,4\}]$. Thus, the $(1,3)$ entry of each of $A[\{1,2,3\}]^{-1}$ and $A[\{1,2,4\}]^{-1}$ is 0 while both the $(1,3)$ and $(1,4)$

entries of $A^{-1}$ are 0 and these are the only entries that vanish in the inverse of any principal submatrix.

The above example leads to three questions.

(1) If the inverse of a proper principal submatrix of an $\mathscr{I}M$ matrix contains a block of 0's, does this imply that the inverse of the matrix itself has a larger block of 0's (and, somehow, in related positions)?
(2) If the inverse of an $\mathscr{I}M$ matrix contains a block of 0's, does this imply that there is a block of 0's in the inverse of some other principal submatrix?
(3) If the inverse of a proper principal submatrix of an $\mathscr{I}M$ matrix contains a 0, does this imply that the inverse of some other proper principal submatrix of the same size must also contain a 0?

(1) and (3) are certainly not true for invertible matrices in general.

**Example 1.2.** Consider the invertible matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 4 & 1 \\ 4 & 3 & 2 & 5 \end{bmatrix}.$$

The (3, 1) minor of $A[\{1, 2, 3\}]$ is 0, but no other minor of $A$ is 0. In fact, no other minor of any principal submatrix of $A$ is 0.

Of course, there may be "isolated" 0's in the inverse of an $\mathscr{I}M$ matrix, and examples are easily found in which there is a single 0 entry in its inverse. All three questions will be answered affirmatively in Section 4, with precise descriptions of these phenomena. Moreover, (2) will be answered for invertible matrices in general.

Because of Jacobi's determinantal identity, there are often analogous statements about matrices in $\mathscr{M}$. It should be noted that besides PMs and APMs no other minors have deterministic signs throughout $\mathscr{M}$ or throughout $\mathscr{I}M$; analogously, there seem to be no results like those we present beyond PMs and APMs.

## 2. Inverses and principal submatrices

The two operations of inverting and extracting a principal submatrix do not, of course, in general commute when applied to a given matrix (for which both are defined). There are, however, several interesting, elementary, entry-wise inequalities when these operations are applied to $A \in \mathscr{M}$ or $A \in \mathscr{I}M$ (for which they are always defined) in various orders. We record these here for reference and reflection. (Compare to inequalities in the positive semidefinite ordering for positive definite matrices [6, p. 474].) Throughout $\leqslant$ or $<$ should be interpreted entry-wise.

**Theorem 2.1.** *If* $A \in \mathscr{M}$ *or* $A \in \mathscr{I}M$ *and* $\emptyset \neq \alpha \subseteq N$, *then*
(i) $(A^{-1}[\alpha])^{-1} \leqslant A[\alpha]$; *and*
(ii) $A[\alpha]^{-1} \leqslant A^{-1}[\alpha]$.

**Proof.** Notice that (i) holds for $A \in \mathscr{M}$ since

$$(A^{-1}[\alpha])^{-1} = A/A[\alpha^c] = A[\alpha] - A[\alpha, \alpha^c]A[\alpha^c]^{-1}A[\alpha^c, \alpha] \leqslant A[\alpha].$$

(The latter inequality holds because $A[\alpha^c]^{-1} \geqslant 0$.) (i) holds for $A \in \mathscr{I}M$ by the same argument since $A[\alpha^c]^{-1}A[\alpha^c, \alpha] \geqslant 0$. (The former fact has been noted previously [1], while the latter fact does not seem to have appeared in the literature.) Since (ii) for $A \in \mathscr{M}$ ($A \in \mathscr{I}M$) is just a restatement of (i) for $A \in \mathscr{I}M$ ($A \in \mathscr{M}$), the theorem holds. □

It follows from (Theorem 2.1(ii)) that if there are some 0 off-diagonal entries in $A[\alpha]^{-1}$, $A \in \mathscr{I}M$, then there are 0 entries in $A^{-1}$ in the corresponding positions. In particular, if a certain APM in $A[\alpha]$ vanishes, then a larger (i.e., more rows and columns) APM in a corresponding position vanishes in $A$. Actually, more can be said, as we shall see later. A hint of this is the following. An entry of a square matrix is, itself, an APM; if an entry of $A \in \mathscr{I}M$ is 0, it is known that $A$ must be reducible [8] and thus, if $n > 2$, other entries (i.e., other APM's of the same size) must be 0.

## 3. Principal and almost principal minor inequalities

It follows specifically from the observations of the last section that if $A \in \mathscr{I}M$, $\alpha \subseteq \beta \subseteq N$, and the APM

$$\det A[\alpha + i, \alpha + j] = 0,$$

then

$$\det A[\beta + i, \beta + j] = 0,$$

$i, j \notin \beta$. Thus, a "smaller" vanishing APM implies that any "larger" one containing it also vanishes. This suggests that there may, in general, be inequalities between such minors. We note, in advance, that Theorem 2.1(ii) gives some inequalities, but we give additional ones here.

We call $A = (a_{ij})$ *normalized* if $a_{ii} = 1$, $i = 1, 2, \ldots, n$. We first give inequalities for the normalized case that may be paraphrased as saying that "larger" minors are smaller. The first inequality is a special case of Fischer's inequality while the second is new.

**Theorem 3.1.** *If* $A \in \mathscr{I}M$ *is normalized and* $\emptyset \neq \alpha \subseteq \beta \subseteq N - i - j$, *then*
(i) $\det A[\beta] \leqslant \det A[\alpha]$; *and*
(ii) $|\det A[\beta + i, \beta + j]| \leqslant |\det A[\alpha + i, \alpha + j]|$.

**Proof.** Noting that the determinantal inequalities of Fischer and Hadamard hold for inverse $M$-matrices [7, p. 127], we have $\det A[\beta] \leqslant \det A[\beta - \alpha] \; \det A[\alpha] \leqslant \det A[\alpha]$ which establishes (i).

Now assume that $i \neq j$, $B = A[\beta + i + j]$, and $\gamma = \alpha + i + j$. Then $B \in \mathscr{IM}$ also, and, from Theorem 2.1(ii), we have $|(B^{-1})_{ij}| \leqslant |(B[\gamma]^{-1})_{ij}|$ or equivalently,

$$\left|\frac{\det B[\beta + i, \beta + j]}{\det B}\right| \leqslant \left|\frac{\det B[\alpha + i, \alpha + j]}{\det B[\gamma]}\right|.$$

Thus,

$$\begin{aligned} |\det B[\beta + i, \beta + j]| &\leqslant \frac{\det B}{\det B[\gamma]}\, |\det B[\alpha + i, \alpha + j]| \\ &\leqslant (\det B[\beta + i + j - \gamma])\, |\det B[\alpha + i, \alpha + j]| \\ &\qquad\qquad\qquad\qquad\qquad \text{(by Fischer)} \\ &\leqslant |\det B[\alpha + i, \alpha + j]| \quad \text{(by Hadamard)}. \end{aligned}$$

Since $B$ is a principal submatrix of $A$, (ii) follows. □

Since $A = (a_{ij}) \in \mathscr{IM}$ may be normalized via multiplication by $D = \operatorname{diag}(a_{11}, \ldots, a_{nn})^{-1}$, we may easily obtain nonnormalized inequalities from Theorem 3.1.

**Corollary 3.2.** *If* $A = (a_{ij}) \in \mathscr{IM}$ *and* $\emptyset \neq \alpha \subseteq \beta \subseteq N - i - j$, *then*
(i) $\det A[\beta] \leqslant \det A[\alpha] \prod_{i \in \beta - \alpha} a_{ii}$; *and*
(ii) $|\det A[\beta + i, \beta + j]| \leqslant |\det A[\alpha + i, \alpha + j]| \prod_{i \in \beta - \alpha} a_{ii}$.

## 4. Vanishing almost principal minors

From prior discussion we know that, for $A \in \mathscr{IM}$, if an entry of $A[\alpha]^{-1}$ is 0, then a corresponding entry of $A^{-1}$ is 0. However, if $\alpha \subseteq N$ properly, one quickly finds that, unlike for general matrices, it is problematic to construct an example in which an entry of $A[\alpha]^{-1}$ is 0 and just one entry of $A^{-1}$ (the corresponding one) is 0. We present here two results that show there is a good reason for this. In one, it is shown that any block of 0's in $A[\alpha]^{-1}$ implies a "larger" block of 0's in $A^{-1}$, and in the other it is shown that any 0 entry in $A[\alpha]^{-1}$ implies 0's in certain other matrices $A[\beta]^{-1}$ when $|\beta| = |\alpha| < n$. Both results lead to interpretations in terms of rank of submatrices of inverse *M*-matrices rather like the row/column inclusion results for positive semidefinite and other matrices, noted in [3].

Since a 0 entry in $A \in \mathscr{IM}$ implies that $A$ is reducible and thus has other 0 entries (if $n > 2$), it follows that if $A \in \mathscr{IM}$ and $A[\alpha]^{-1}$ has a 0 entry, $|\alpha| = 2$, then $A^{-1}$ is reducible and the 0 entry is actually part of a 0 block in $A^{-1}$. This generalizes substantially even when $A \in \mathscr{IM}$ is positive and answers question (1).

**Theorem 4.1.** *Suppose that* $A \in \mathscr{IM}$ *and that* $\gamma = N - i$ *for some* $i \in N$. *Then, if* $A[\gamma]^{-1}$ *has a p-by-q* 0 *submatrix,* $A^{-1}$ *has either a p-by-*$(q + 1)$ *or a* $(p + 1)$*-by-q*

0 *submatrix. Specifically, if* $A[\gamma]^{-1}[\alpha, \beta] = 0$ *in which* $|\alpha| = p$ *and* $|\beta| = q$, *then either* $A^{-1}[\alpha, \beta + i] = 0$ *or* $A^{-1}[\alpha + i, \beta] = 0$.

**Proof.** Let $A \in \mathscr{I}M$ and $\gamma = N - i$ for some $i \in N$. Without loss of generality, assume that $A$ has the partitioned form

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{4.1}$$

in which $A_{11} = a_{ii}$ for some $i$, $1 \leqslant i \leqslant n$, and $A_{22} = A[\gamma]$. If an invertible matrix $A$ is partitioned as in (4.1) with $A_{22}$ invertible, then

$$\mathbf{A}^{-1} = \begin{bmatrix} s^{-1} & -s^{-1}u^{\mathrm{T}} \\ -s^{-1}v & A_{22}^{-1} + s^{-1}vu^{\mathrm{T}} \end{bmatrix} \tag{4.2}$$

in which $s = A/A_{22}$, $u^{\mathrm{T}} = A_{12}A_{22}^{-1}$, and $v = A_{22}^{-1}A_{21}$. Thus, if $A_{22} \in \mathscr{I}M$, it is easily seen (and was first noticed in [8, Theorem 8]) that $A \in \mathscr{I}M$ if and only if (i) $s > 0$, (ii) $u^{\mathrm{T}} \geqslant 0$, (iii) $v \geqslant 0$, and (iv) $A_{22}^{-1} \leqslant -s^{-1}vu^{\mathrm{T}}$, except for diagonal entries. Now $A_{22}^{-1} \in \mathscr{M}$ and hence is in $Z$. Assume that $A_{22}^{-1}$ has a $p$-by-$q$ submatrix of 0's, say $A_{22}^{-1}[\alpha, \beta] = 0$ in which $|\alpha| = p$ and $|\beta| = q$. If $v_r = 0$ for all $r \in \alpha$, then $A^{-1}[\alpha, \beta + i]$ is a $p$-by-$(q + 1)$ 0 submatrix of $A^{-1}$. On the other hand, if $v_r \neq 0$ for some $r \in \alpha$, then it follows from (iv) that $u_s = 0$ for all $s \in \beta$ and hence $A^{-1}[\alpha + i, \beta]$ is a $p$-by-$(q + 1)$ 0 submatrix of $A^{-1}$. This completes the proof. □

In regard to Example 1.1 we see that for $\gamma = \{1, 2, 3\}$ or $\{1, 2, 4\}$, a 1-by-1 block of 0's in $A[\gamma]^{-1}$ leads to a 1-by-2 block of 0's in $A^{-1}$. It is easy to construct examples such that both possibilities for the 0 block of $A^{-1}$ occur and also such that exactly one of the possibilities occurs.

Let the *measure of irreducibility* $m(A)$ of an invertible $n$-by-$n$ matrix $A$ be defined as

$$m(A) = \max_{(p,q)\in S}(p + q)$$

in which $S = \{(p, q) \mid A \text{ contains a } p\text{-by-}q \text{ off-diagonal zero submatrix}\}$. Thus, an $n$-by-$n$ matrix $A$ is reducible if and only if $m(A) = n$. Moreover, in Theorem 4.1, we have shown that if $B$ is an $(n - 1)$-by-$(n - 1)$ principal submatrix of $A \in \mathscr{I}M$, then $m(A^{-1}) \geqslant m(B^{-1}) + 1$. In general, if $B$ is a $k$-by-$k$ principal submatrix of $A \in \mathscr{I}M$, then $m(A^{-1}) \geqslant m(B^{-1}) + n - k$. (Note that the latter statement implies that if $B$ is a reducible principal submatrix of $A \in \mathscr{I}M$, then $A$ is reducible also. And, in particular, if $A \in \mathscr{I}M$ has a 0 entry, then $A$ is reducible, a fact noted previously.) The question is: when are these inequalities in fact equalities?

In order to establish a converse to Theorem 4.1 we will need the following result on "complementary nullities". This fact has origins in [5] and has been refined, for example in [9]. Let Null($A$) denote the (right) null space of a matrix $A$, nullity($A$) denote the dimension of Null($A$), and r($A$) denote the rank of $A$.

**Theorem 4.2** (Complementary nullities). *Let $A$ be an $n$-by-$n$ invertible matrix and $\emptyset \neq \alpha, \beta \subseteq N$ with $\alpha \cap \beta = \emptyset$. Then*, $\mathrm{null}(A^{-1}[\alpha, \beta]) = \mathrm{null}(A[\beta^c, \alpha^c])$.

We use Theorem 4.4 to prove a general result on 0 patterns of inverses. We let $M_n(S)$ denote the $n$-by-$n$ matrices with entries from a set S.

**Theorem 4.3.** *Let $\emptyset \neq \alpha, \beta \subseteq N$ with $\alpha \cap \beta = \emptyset$, and let $A \in M_n(F)$, where $F$ is an arbitrary field. If $A^{-1}[\alpha, \beta] = 0$, then, for any $\gamma \subseteq N$ such that $\alpha \cap \gamma \neq \emptyset$, $\beta \cap \gamma \neq \emptyset$, $(\alpha \cup \beta)^c \subseteq \gamma$, and $A[\gamma]$ is invertible, we have $A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] = 0$.*

**Proof.** Suppose that $A \in M_n(F)$ and $A^{-1}[\alpha, \beta] = 0$ in which $\emptyset \neq \alpha, \beta \subseteq N$ with $\alpha \cap \beta = \emptyset$. Then nullity$(A^{-1}[\alpha, \beta]) = |\beta|$; hence, by complementary nullity, nullity $(A[\beta^c, \alpha^c]) = |\beta|$. Therefore, $r(A[\beta^c, \alpha^c]) = |\alpha^c| - |\beta| = n - |\alpha| - |\beta| = |(\alpha \cup \beta)^c|$. Further, suppose that $\gamma \subseteq N$ satisfies $\alpha \cap \gamma \neq \emptyset$, $\beta \cap \gamma \neq \emptyset$, $(\alpha \cup \beta)^c \subseteq \gamma$, and $A[\gamma]$ is invertible. Since $\gamma - \alpha \cap \gamma = \gamma \cap \alpha^c$ and $\gamma - \beta \cap \gamma = \gamma \cap \beta^c$, $r(A[\gamma][\gamma - \beta \cap \gamma, \gamma - \alpha \cap \gamma]) \leqslant r(A[\beta^c, \alpha^c]) = |(\alpha \cup \beta)^c|$. Thus, nullity $(A[\gamma][\gamma - \beta \cap \gamma, \gamma - \alpha \cap \gamma]) \geqslant |\gamma - \alpha \cap \gamma| - |(\alpha \cup \beta)^c| = |\beta \cap \gamma| + |(\alpha \cup \beta)^c| - |(\alpha \cup \beta)^c| = |\beta \cap \gamma|$. Hence, by complementary nullity, nullity$(A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma]) \geqslant |\beta \cap \gamma|$. This last inequality must be an equality, i.e., $A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] = 0$ which completes the proof. □

We can make similar statements concerning the minors of $A$ and $A[\gamma]$ even if $A$ is singular.

It is easy to show that the condition $(\alpha \cup \beta)^c \subseteq \gamma$ is necessary. For instance, in Example 1.1, $A^{-1}[\alpha, \beta] = 0$ in which $\alpha = \{1\}$ and $\beta = \{3, 4\}$. But if $\gamma = \alpha \cup \beta$ (and thus $\alpha \cap \gamma \neq \emptyset$ and $\beta \cap \gamma \neq \emptyset$ while it is not the case that $(\alpha \cup \beta)^c \subseteq \gamma$), then $A[\gamma]^{-1}$ has no 0 entries.

We are now able to answer question (2).

**Corollary 4.4.** *Let $A \in \mathscr{I}M$ with $m(A^{-1}) = p + q$, say $A^{-1}[\alpha, \beta] = 0$ in which $\emptyset \neq \alpha, \beta \subseteq N$ with $|\alpha| = p$ and $|\beta| = q$. Further, let $\gamma = N - i$ for some $i \in N$, and assume that $\alpha \cap \gamma \neq \emptyset$ and $\beta \cap \gamma \neq \emptyset$. Then, $A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] = 0$ if and only if $(\alpha \cup \beta)^c \subseteq \gamma$.*

**Proof.** Assume the hypothesis holds and observe that $\alpha \cap \beta = \emptyset$ since $A \in \mathscr{I}M$.

Firstly, suppose that $A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] = 0$. Then $m(A^{-1}) \geqslant |\alpha \cap \gamma| + |\beta \cap \gamma|$. By the remarks after Theorem 4.1,

$$m(A^{-1}) \geqslant m(A[\gamma]^{-1}) + n - |\gamma|$$

and so

$$|\alpha| + |\beta| \geqslant |\alpha \cap \gamma| + |\beta \cap \gamma| + n - |\gamma|.$$

Rearranging, we have

$$(|\alpha| - |\alpha \cap \gamma|) + (|\beta| - |\beta \cap \gamma|) + |\gamma| \geqslant n$$

or equivalently,

$$|\alpha - \gamma| + |\beta - \gamma| + |\gamma| \geqslant n.$$

The latter holds (and with equality) if and only if $(\alpha \cup \beta)^{\mathrm{c}} \subseteq \gamma$.

The converse follows from Theorem 4.3 since all principal submatrices of $A$ are invertible. □

In regard to Example 1.1, $A^{-1}[\alpha, \beta] = 0$ in which $\alpha = \{1\}$ and $\beta = \{3, 4\}$. For $\gamma = \{1, 2, 3\}$ or $\{1, 2, 4\}$, we have $\alpha \cap \gamma \neq \emptyset$, $\beta \cap \gamma \neq \emptyset$, and $(\alpha \cup \beta)^{\mathrm{c}} \subseteq \gamma$. In each case $A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] = 0$. On the other hand, for $\gamma = \{1, 3, 4\}$, we have $\alpha \cap \gamma \neq \emptyset$, $\beta \cap \gamma \neq \emptyset$, but $(\alpha \cup \beta)^{\mathrm{c}} \subseteq \gamma$ does not hold and $A[\gamma]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] \neq 0$.

So we see that the inequality $m(A^{-1}) \geqslant m(B^{-1}) + 1$, where $B$ is an $(n - 1$-by-$(n - 1)$ principal submatrix of $A$ (noted in the discussion after Theorem 4.1) is an equality as long as $B = A[\gamma]$ in which $\alpha \cap \gamma \neq \emptyset$, $\beta \cap \gamma \neq \emptyset$, and $(\alpha \cup \beta)^{\mathrm{c}} \subseteq \gamma$. And Example 1.1 (with $\gamma = \{2, 3, 4\}$) shows that, otherwise, the inequality may be strict.

We next discuss vanishing APMs and in response to question (3) establish the fact that they imply that other APMs, of the same size, vanish. We will use the following lemma.

**Lemma 4.5.** *Let $A = (a_{ij})$ be an n-by-n $\mathscr{I}M$ matrix ($n \geqslant 3$) and $j \in N$. If $a_{i_1 j}, a_{i_2 j}, \ldots, a_{i_t j} = 0$, then, for all $k \notin \{j, i_1, \ldots, i_t\}$, either $a_{kj} = 0$ or $a_{i_1 k}, a_{i_2 k}, \ldots, a_{i_t k} = 0$.*

**Proof.** This follows from the fact [8,12] that if $A$ is an $n$-by-$n$ $\mathscr{I}M$ matrix ($n \geqslant 3$), then for all $i, j, k \in N$, $a_{ik}a_{kj} \leqslant a_{ij}a_{kk}$. □

**Theorem 4.6.** *Let $A = (a_{ij})$ be an n-by-n $\mathscr{I}M$ matrix ($n \geqslant 3$); $i, j, k$ be distinct indices in N, and let $\delta$ be a subset of $N - i - j - k$. Then, if* $\det A[\delta + i, \delta + j] = 0$,

(i) $\det A[\delta + i, \delta + k] = 0$ *or*

(ii) $\det A[\delta + k, \delta + j]| = 0$.

**Proof.** Let $A = (a_{ij})$ be an $n$-by-$n$ $\mathscr{I}M$ matrix ($n \geqslant 3$); $i, j, k$ be distinct indices in $N$, let $\delta$ be a subset of $N - i - j - k$, and assume that $\det A[\delta + i, \delta + j] = 0$. If $\delta = \emptyset$, the result follows since $A[\{i, j, k\}]$ must be reducible. So assume $\delta \neq \emptyset$. By permutation similarity we may assume that $i = i_1$, $\delta = \{i_2, \ldots, i_{p-1}\}$, and $j = i_p$. Let $A_1 = A[\{i_1, \ldots, i_p\}]$. Since $0 = \det A[\delta + i, \delta + j] = \det B$, where $B = A[\{i_1, \ldots, i_{p-1}\}, \{i_2, \ldots, i_p\}]$, we see that the $(i_p, i_1)$ minor of $A_1$ is 0. Further, since $A[\delta]$ is a $(p - 2)$-by-$(p - 2)$ principal submatrix of $A$ lying in the lower left corner of $B = [b_1, \ldots, b_{p-1}]$, $\{b_1, \ldots, b_{p-2}\}$ is linearly independent and thus $b_{p-1} = \sum_{i=1}^{p-2} \beta_i b_i$. If $b_{p-1} = 0$, then, by Lemma 4.5, either $a_{ki_p} = a_{kj} = 0$ or $a_{i_1 k}, a_{i_2 k}, \ldots, a_{i_{p-1} k} = 0$.

The former case implies $\det A[\delta + k, \delta + j] = 0$ while the latter implies $\det A[\delta + i, \delta + k] = 0$. So assume $b_{p-1} > 0$. Then $\beta_i \neq 0$ for some $i$, $1 \leqslant i \leqslant p-2$. By simultaneous permutation of rows and columns indexed by $\delta$, we may assume $\beta_1 \neq 0$.

Let $k = i_{p+1}$ and consider the submatrix $A_2 = A[\{i_1, \ldots, i_{p+1}\}]$ of $A$. Since $A_1$ is a principal submatrix of $A_2$, the $(i_p, i_1)$ minor of $A_2$ is 0 also, i.e.,

$$0 = \det A[\{i_1, \ldots, i_{p-1}, i_{p+1}\}, \{i_2, \ldots, i_{p+1}\}] = \det A_3.$$

Now the $(p-1)$-by-$(p-1)$ submatrix lying in the upper left corner of $A_3$ is $B$ and $\det B = 0$. Therefore, applying Sylvester's identity for determinants to $\det A_3$, we see that either

(1) $\det A[\{i_1, \ldots, i_{p-1}\}, \{i_3, \ldots, i_{p+1}\}] = 0$ or
(2) $\det A[\{i_2, \ldots, i_{p-1}, i_{p+1}\}, \{i_2, \ldots, i_p\}] = 0$.

**Case I.** Suppose (1) holds. Then

$$\begin{aligned} 0 &= \det A[\{i_1, \ldots, i_{p-1}\}, \{i_3, \ldots, i_{p+1}\}] \\ &= \det [b_2, \ldots, b_{p-2}, b_{p-1}, b_p] \\ &= \det \left[ b_2, \ldots, b_{p-2}, \sum_{i=1}^{p-2} \beta_i b_i, b_p \right] \\ &= \det [b_2, \ldots, b_{p-2}, \beta_1 b_1, b_p] \\ &= (-1)^{p-3} \beta_1 \det[b_1, b_2, \ldots, b_{p-2}, b_p]. \end{aligned}$$

Since $\beta_1 \neq 0$,

$$\begin{aligned} 0 &= \det [b_1, b_2, \ldots, b_{p-2}, b_p] \\ &= \det A[\{i_1, \ldots, i_{p-1}\}, \{i_2, \ldots, i_{p-1}, i_{p+1}\}] \\ &= \det A[\delta + i, \delta + k]. \end{aligned}$$

This establishes (i).

**Case II.** Suppose (2) holds. Then

$$\begin{aligned} 0 &= \det A[\{i_2, \ldots, i_{p-1}, i_{p+1}\}, \{i_3, \ldots, i_{p+1}\}] \\ &= \det A[\delta + k, \delta + j]. \end{aligned}$$

This establishes (ii) and completes the proof. □

We can also prove Theorem 4.6 using Theorems 4.1 and 4.3 as follows. Let $\rho = \delta + i + j$ and $\tau = \rho + k$. Since $\det A[\delta + i, \delta + j] = 0$, $A[\rho]^{-1}[i, j] = 0$. So, by Theorem 4.1, either $A[\tau]^{-1}[i, j + k] = 0$ or $A[\tau]^{-1}[i + k, j] = 0$. Suppose that $A[\tau]^{-1}[i, j + k] = A[\tau]^{-1}[\alpha, \beta] = 0$. Let $\gamma = \delta + i + k = \tau - j$. Then, in $A[\delta + i + j + k]$, we have $\alpha \cap \gamma \neq \emptyset$, $\beta \cap \gamma \neq \emptyset$, and $(\alpha \cup \beta)^{\mathrm{c}} = \delta \subseteq \gamma$. Thus, by Theorem 4.3, $A[\tau]^{-1}[\alpha \cap \gamma, \beta \cap \gamma] = A[\gamma]^{-1}[i, k] = 0$ which implies that $\det A[\delta + i, \delta + k] = 0$. Similarly, it can be shown that if $A[\gamma]^{-1}[i + k, j] = 0$, then $\det A[\delta + k, \delta + j] = 0$.

In regard to Example 1.1 the {1, 2},{2, 3} minor of $A$ vanishes. Hence, by Theorem 4.6, either the {1, 2},{2, 4} minor or the {2, 4},{2, 3} minor must vanish. The former was true.

*M*-matrices and inverse *M*-matrices are known for their similarities with the positive definite matrices and these results are no exception. In the positive definite matrices Theorem 4.6 has an obvious analog (namely that the $\delta + i, \delta + j$ minor is 0 if and only if the $\delta + j, \delta + i$ minor is 0) while the implication of Corollary 4.4 that follows from Theorem 4.3 has an identical analog. Less obvious is an analog to Theorem 4.1, yet there is one.

Recall that positive semidefinite matrices have an interesting property that may be called "row and column inclusion" [3]. If $A$ is an $n$-by-$n$ positive semidefinite matrix, then, for any index set $\alpha \subseteq N$ and any index $i \notin N - \alpha$, the row $A[i, \alpha]$ (and thus the column $A[\alpha, i]$) lies in the row space of $A[\alpha]$ (column space of $A[\alpha]$). Of course, this is interesting only in the event that $A[\alpha]$ is singular, and, so, there is no complete analog for $\mathscr{I}M$ matrices, in which every principal submatrix is necessarily invertible. However, just as there are slightly weakened analogs in the case of totally nonnegative matrices ("row *or* column inclusion" [3]), Theorem 4.1 implies certain analogs for $\mathscr{I}M$ matrices; now the role of principal submatrices is replaced by almost principal submatrices. We will need the following lemma. Here Row($A$) (Col($A$)) denotes the row (column) space of a matrix $A$.

**Lemma 4.7.** *Let the n-by-n matrix B have the partitioned form*

$$\mathbf{B} = \begin{bmatrix} C & d \\ e^{\mathrm{T}} & f \end{bmatrix}$$

*in which f is a scalar and* $r(C) = n - 2$. *Then, if* $d \notin \mathrm{Col}(C)$ *and* $e^{\mathrm{T}} \notin \mathrm{Row}(C)$, *B is invertible.*

**Proof.** Since $d \notin \mathrm{Col}(C)$, $r\left(\begin{bmatrix} C & d \end{bmatrix}\right) = n - 1$ and since $e^{\mathrm{T}} \notin \mathrm{Row}(C)$, $\begin{bmatrix} e^{\mathrm{T}} & f \end{bmatrix} \notin \mathrm{Row}\left(\begin{bmatrix} C & d \end{bmatrix}\right)$. This implies $r(B) = n$, i.e., $B$ is invertible. □

**Corollary 4.8.** *Let A be an n-by-n* $\mathscr{I}M$ *matrix,* $\alpha \subseteq N$, *and* $i, j \in N - \alpha$. *Then, for each* $k \notin \alpha + i + j$, *either* $A[k, \alpha + j]$ *lies in the row space of* $A[\alpha + i, \alpha + j]$ *or* $A[\alpha + i, k]$ *lies in the column space of* $A[\alpha + i, \alpha + j]$.

**Proof.** The corollary certainly holds if $A[\alpha + i, \alpha + j]$ is invertible. So assume not. Then $\det A[\alpha + i, \alpha + j] = 0$ and $r(A[\alpha + i, \alpha + j]) = |\alpha|$. Thus, $A[\alpha + i + j]^{-1}$ has a 0 in the $(i, j)$ position and if $k \notin \alpha + i + j$, it follows from Theorem 4.1 that $A[\alpha + i + j + k]^{-1}$ also has a 0 in the $(i, j)$ position. Hence, $\det A[\alpha + i + k, \alpha +$

$j + k] = 0$. With $B = A[\alpha + i + k, \alpha + j + k]$ and $C = A[\alpha + i, \alpha + j]$, we see that the result follows upon applying Lemma 4.7. □

We note that in the statement of Corollary 4.8 the almost principal submatrix $A[\alpha + i, \alpha + j]$ could be replaced by $A[\alpha, \alpha + j]$ in the row case and by $A[\alpha + i, \alpha]$ in the column case to yield a stronger, but less symmetric, statement.

In the row or column inclusion results for totally nonnegative matrices, a bit more is true, the either/or statement is validated either always by rows or always by columns (or both), and this is (trivially) also so in the positive semidefinite case. Thus, it is worth noting that such phenomenon does *not* carry over to the almost principal $\mathscr{I}M$ case.

**Example 4.9.** Consider the $\mathscr{I}M$ matrix

$$\mathbf{A} = \begin{bmatrix} 14 & 4 & 1 & 1 & 4 \\ 6 & 16 & 4 & 4 & 6 \\ 6 & 6 & 14 & 4 & 6 \\ 6 & 6 & 4 & 14 & 6 \\ 4 & 4 & 1 & 1 & 14 \end{bmatrix}.$$

The hypothesis of the corollary is satisfied with $\alpha = \{2\}$, $i = 1$, and $j = 3$ and the conclusion is satisfied for $k = 4$ by columns and not rows and for $k = 5$ by rows and not columns.

Examples in which satisfaction is always via columns or always via rows are easily constructed.

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