MATHEMATICS

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Manifold schemes and foliations on the 2-torus and the Klein bottle. III

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11. IRREDUCIBLE G-DOMAINS

We assume that B is an irreducible G-domain, i.e. B contains no proper G-subdomains except the empty one. G is supposed to be a finitely generated abelian group.

THEOREM 11.1. *Under the above hypotheses B is tame and hence the description of theorem* 9.1 *is applicable. The case $B \simeq \mathbb{R}$ falls apart into the following subcases:*

(a) *There is a unique G-stable point o, and G contains an orientation reversing element. Each component of $B-\{o\}$ is an irreducible H-domain, where H denotes the subgroup of orientation preserving elements.*

(b) *G acts as a group of orientation preserving transformations; there is a $g \in G$ such that for any $a \in B$ the sequence $(g^n a)$ is strictly monotone and divergent, i.e. the cyclic group generated by g acts freely on B.*

REMARK. In view of this result it would be desirable to have a relatively simple a priori argument to establish the tameness of B. Although for the special case of abelian finitely generated G some reduction in the preceding sections could be attained, we do not know at the moment a line of argument which is essentially simpler.

PROOF. By proposition 10.2.1 and theorem 10.2.1 there is an irreducible convex G-set C of one of the following types: (i) C is a point, (ii) C is a cluster, (iii) $C \simeq \mathbb{R}$, (iv) C is an irreducible G-set of non-trivial type. In addition the remark after lemma 10.2.2 states that in the cases (iii) and (iv) C is also Γ-irreducible for some cyclic subgroup $\Gamma \subset G$. We now discuss the various cases separately.

(ii) If C is a cluster, there is a unique component U of $B-C$ such that $C = bdry\ U$. Hence if C is G-invariant, U is also G-invariant. Therefore this case does not occur if B is an irreducible G-domain.

(i) Putting $C = \{c\}$, an element $g \in G$ will either interchange or leave invariant the components U_0, U_1 of $B - \{c\}$. The last possibility does not occur for every $g \in G$ since this would contradict the irreducibility of B as a domain. Suppose $g \in G$ interchanges U_0 and U_1. Observe that g and H generate G, and that $g^2 \in H$. This implies that if $V_0 \subset U_0$ is an H-domain, then $V_0 \cup gV_0$ is a G-invariant open set. If c is a boundary point of V_0, c is also a boundary point of gV_0, and $V_0 \cup gV_0 \cup \{c\}$ is then a G-invariant domain, and hence, by the irreducibility of B, it follows that $V_0 = U_0$. If c is not a boundary point of V_0, it is neither a boundary point of gV_0, and $B - \overline{V_0 \cup gV_0}$ is then a G-invariant open set containing c. The component of c in this open set is then a G-invariant domain, and therefore $= B$, i.e. $\overline{V_0 \cup gV_0}$ is empty and hence V_0 is empty. This shows that U_0 (and similarly U_1) are H-irreducible domains. Consequently there is no H-fixpoint in U_0, since otherwise the domain bounded by c and the fixpoint would be an H-invariant proper subdomain of U_0.

Furthermore by the discussion of (ii) an H-irreducible cluster can not occur in U_0. The existence of an irreducible convex H-set E of non trivial type is ruled out, since such a set would be a unique irreducible convex H-set in B, whereas $gE \subset U_1$ would be another irreducible convex H-set. Therefore by theorem 10.2.1 U_0, and consequently also U_1, is a copy of $\mathbb{R}$, and hence $B = U_0 \cup U_1 \cup \{c\} \simeq \mathbb{R}$.

(iv) By proposition 10.2.1, G acts as a translation group on the set $\tilde{C}$ of the components C_i of C, where we assume that the elements of $\tilde{C}$ are labeled by the integers in a convexity preserving fashion. Let U_i be the domain bounded by the endpoint of C_i and the initial point of C_{i+1}. Then $U = \bigcup_{i \in \mathbf{Z}} U_i$ is G-invariant, and since $C \cup U$ is open and connected (proposition 10.1.4) and G-invariant, it follows that $C \cup U = B$. Let $H \subset G$ be the subgroup which acts trivially on $\tilde{C}$. Then each U_i is automatically H-invariant, and the boundary points of U_i are H-fixpoints.

Suppose that V_0 is an H-subdomain of U_0 with $bdry\ U_0 \subset bdry\ V_0$. Then $C \cup (GV_0) \cup (\bigcup_{i \not\equiv 0 \bmod k} U_i)$, where k is the minimal shift in index effected by the action of G, is a G-invariant domain. By the irreducibility of B as a domain this implies that $V_0 = U_0$. If $bdry\ U_0 \not\subset bdry\ V_0$, then $bdry\ V_0$ would be disjoint from $bdry\ U_0$. Then the component of $U_0 - (\overline{V}_0 \cap U_0)$, the boundary of which contains the pair of associate nodes bounding U_0, is H-invariant, and hence by the preceding argument would coincide with U_0. Hence U_0 (and similarly each U_i) is an irreducible H-domain. Applying theorem 10.2.1 to the H-domain U_0, the case of an

H-fixpoint is ruled out since such a point together with *bdry* U_0 would determine a proper non-empty H-subdomain of U_0. The case of an H-cluster is taken care of by the discussion of (ii) of the present theorem.

The case of a non-trivial irreducible convex H-set in U_0 would lead to a copy of it in U_k (k the minimal shift effected by G as a translation group) by applying a suitable $g \in G$, which contradicts the uniqueness of such an irreducible set. Hence U_0 (and similarly each U_i) is an irreducible H-domain $\cong \mathbb{R}$.

12. FOLIATIONS ON THE 2-TORUS AND KLEIN BOTTLE

By § 6 the quotient scheme of the 2-torus M with respect to a foliation F is a tree-manifold B divided out by the action of a free abelian group G of rank 2. In order to make the results of § 11 applicable we observe

PROPOSITION. *B is an irreducible G-domain.*

PROOF. Let $p: \tilde{M} = \mathbb{R}^2 \to \tilde{M}/\tilde{F} = B$ be the quotient map of the universal covering $\tilde{M} = \mathbb{R}^2$ of the torus with respect to the lifted foliation $\tilde{F}$. For any $a \in B$, $p^{-1}(a)$ is a closed set homeomorphic to $\mathbb{R}$ [8]. It is known that the foliation $\tilde{F}$ is a locally trivial fibration over B [20]. Therefore for any subdomain $U \subset B$, $p^{-1}(U)$ is locally trivially fibered by simply connected fibres over U. Since a tree-manifold is simply connected (e.g. in the sense of admitting only trivial coverings), it follows that $p^{-1}(U)$ is connected and simply connected for any subdomain $U \subset B$. If U would be in addition a G-domain, then $p^{-1}(U)$ would be a connected and simply connected G-subdomain of $\mathbb{R}^2$. Therefore $p^{-1}(U)/G$ would have the same homology in dimension 2 as the torus. Hence $p^{-1}(U)/G$ would be the whole torus, i.e. $p^{-1}(U) = \mathbb{R}^2$, and $U = B$.

This shows that theorems 9.1 and 11.1 apply, and we obtain

THEOREM 12.1. *The quotient scheme of a foliation on the 2-torus T^2 may be represented as B/G, where B is a tree-manifold and $G = \pi_1(T^2)$ is a free abelian group of rank 2 acting on B. For the pair B, G one of the following statements holds:*

(i) *$B \cong \mathbb{R}$, and there is a primitive element $g \in G$ acting freely on $B \cong \mathbb{R}$.*

(iii) *B is a tame tree-manifold; its branching tree Σ_N, N = set of nodes, is an infinite simplicial 1-manifold. G acts as a group of translations on Σ_N; G/H is infinite cyclic where H is the subgroup leaving Σ_N elementwise fixed; H is infinite cyclic and acts freely on each component $U_j \cong \mathbb{R}$ of $B - N$ which is bounded by a pair of associate nodes.*

PROOF. The statements are consequences of the theorem 9.1 and 11.1 provided one can exclude the case of a G-fixpoint in B. Such a point would correspond to a member of the family $\tilde{F}$ which is G-invariant.

Since G operates freely on $\mathbf{R}^2$, G would also operate freely on any G-invariant leaf, which is impossible for dimensional reasons.

REMARK. A g-fixpoint in B for some $g \in G$, $g \neq e$, corresponds to a closed curve of the foliation F on the torus. The nodes of B are fixpoints relative H; they correspond to closed curves which are usually called *separatrices*. The above result shows that a foliation on the torus contains at most finitely many separatrices.

As a consequence of the preceding theorem one obtains

THEOREM 12.2. (Kneser [10]) *A foliation on the Klein bottle K^2 contains at least one closed leaf.*

PROOF. Let $\hat{G}$ be the fundamental group of K^2 and $G \subset \hat{G}$ the normal subgroup of the orientable double covering T^2 of K^2, and B the quotient of the universal covering of K^2 (and T^2) with respect to the lifted foliation. Theorem 12.1 holds with respect to B and G. Hence in case (iii) the existence of a closed leaf on T^2, and therefore also on K^2, is guaranteed by the above remark.

Suppose now that $B \simeq \mathbf{R}$, and let $(g_1, g_2 : g_1 g_2 g_1^{-1} g_2)$ be a presentation of G. If g_2 operates without a fixpoint on $B \simeq \mathbf{R}$, then g_2 acts orientation preserving and freely on $\mathbf{R}$. Hence the direction of the vector $(a, g_2 a)$, $a \in \mathbf{R}$, is independent of a. However $g_1(a, g_2 a) = (g_1 a, g_1 g_2 a) = (g_1 a, g_2^{-1} g_1 a) = (g_2 b, b)$ with $b = g_2^{-1} g_1 a$, i.e. g_1 reverses the orientation on $\mathbf{R}$ and will therefore have a fixpoint. Hence either g_2 or g_1 has a fixpoint on B.

The above proof shows a little more. Let as before g_1, g_2 be a set of generators of $\hat{G}$ with $g_1 g_2 g_1^{-1} g_2 = 1$. Any element of $\hat{G}$ can be uniquely written as $g_2^{n_2} g_1^{n_1}$ or $g_1^{n_1} g_2^{m_2}$ with $m_2 = (-)^{n_1} n_2$, i.e. G is the semi-direct product $G_2 \times G_1$ where G_i is the cyclic group generated by g_i. An element $g_2^n g_1$ is conjugate to g_1 iff $n \equiv 0 \bmod 2$ and it is conugate to $g_1' = g_2 g_1$ otherwise.

Assume that G_2 acts freely on B. Then the orbit $\hat{G}a$ of a g_1-fixpoint a does not contain any g_1'-fixpoint and vice versa. Indeed since a is a g_1-fixpoint, $\hat{G}a = G_2 a$, and $g_2^n a$ is a fixpoint of $g_2^n g_1 g_2^{-n} = g_2^{2n} g_1$. Hence if $a' = g_2^n a$ would be a g_1'-fixpoint too, it would be a simultaneous fixpoint of g_1' and g_2^{2n-1}. Since $2n-1 \neq 0$ this would contradict the free action of G_2. Without the assumption of the free action of G_2, the argument still yields the assertion if we take into account that B is the base of a $\hat{G}$-invariant foliation of the plane. A simultaneous g_2', g_2^{2n-1}-fixpoint would correspond to a $\hat{G}$-invariant 1-dimensional leaf with a free $\hat{G}$-action which is impossible for homological reasons.

Returning to the above proof, assume that $B \simeq \mathbf{R}$ and that g_2 acts freely. Then, because of $g_1 g_2 g_1^{-1} g_2 = 1 = g_1' g_2 g_1'^{-1} g_2$, both g_1 and g_1' act with fixpoints a and a' say. Since the orbits of a and a' are disjoint it follows that in this situation there are at least two closed leafs on K^2. (The situation considered here is a special case of a more general situation considered in the fixpoint theory of J. Nielsen [13].) Since furthermore g_1 and g_1' are orientation reversing, a and a' are the unique respective fixpoints. These correspond to curves in $\mathbb{R}^2$ which are g_1-invariant and g_1'-invariant respectively. The closed curve on K^2 which corresponds to a g_1-invariant (g_1'-invariant) curve will be called a *short Möbius circle*, and a curve on K^2 which corresponds to a g_1^2-invariant curve in $\mathbb{R}^2$ will be called a *Möbius circle*. Observe that $g_1^2 = g_1'^2$ is a generator of the infinite cyclic centre of $\hat{G}$; therefore the notion of Möbius circle is an intrinsic one. Observe furthermore that the stability subgroup $\subset \hat{G}$ of a curve in $\mathbb{R}^2$ that covers a curve in K^2 is determined up to conjugacy, and that an automorphism of $\hat{G}$ either leaves the conjugacy class of the subgroups generated by g_1 and g_1' invariant or else interchanges the two classes. Therefore the notion of short Möbius circle is also intrinsic.

A special case of a foliation on K^2 for which the above situation applies, is the Möbius foliation which may be described as follows.

Take $\hat{G}$ to be the group in the plane generated by $g_1\colon (x, y) \mapsto (x+1, -y)$ and $g_2\colon (x, y) \mapsto (x, y+1)$. K^2 is to be the quotient space $\mathbb{R}^2/\hat{G}$. The fibration in $\mathbb{R}^2$ by the horizontal lines defines on K^2 the Möbius foliation, the leaves of which are Möbius circles, with two short Möbius circles corresponding to $y=0$ and $y=\frac{1}{2}$ respectively.

A sufficiently small perturbation of this foliation, e.g. by perturbing the associated field of tangent line elements (preserving the continuous differentiability of the field), leads to a perturbation of the fibration in $\mathbb{R}^2$ which will still result in a fibration over $\mathbb{R}$ with a free action of g_2. Therefore we obtain

ADDENDUM 12.2.1. *A (sufficiently small) perturbation of the Möbius foliation yields a foliation with exactly two short Möbius circles among the leaves.*

More generally one may state

ADDENDUM 12.2.2. *If F is the family of integral curves of a nowhere vanishing C^1 vector field, and if F contains a Möbius circle, then F contains exactly two short Möbius circles.*

REMARK. Since the Möbius foliation is generated by the horizontal vector field, it is clear that the statement 12.2.1 is a consequence of this one.

We sketch the proof. By theorem 12.1 the free abelian group G generated by g_1^2, g_2 – the fundamental group of the orientable double covering of K^2 – acts irreducibly on B, and contains an element $g = g_2^n(g_1^2)^m$ which acts freely on B. Since we assume that g_1^2 has a fixpoint, n is non-zero and therefore also g_2^n acts freely, but then g_2 does so. Therefore if $B \simeq \mathbf{R}$, the existence of exactly two short Möbius circles has been established. In case B has nodes let Σ be the branching tree. Σ is an infinite simplicial 1-manifold. Since g_2 acts freely on Σ, the same argument as before shows that g_1 and g_1' act orientation reversing and therefore each will have a unique fixsimplex δ, δ'. The G-orbits of δ and δ' are disjoint. If δ is a vertex, the two cobounding 1-simplices of δ are interchanged, i.e. g_1 leaves invariant an open domain $U \simeq \mathbf{R}$ in B, and U is bounded by a pair of associate nodes which are interchanged. The geometric argument below will establish that the corresponding situation in the plane is incompatible with the assumption that the foliation in the plane is the family of integral curves of a $\hat{G}$-equivariant C^1 vector field. The same applies to g_1', hence δ and δ' are 1-simplices, which establishes the existence of exactly two short Möbius circles in F.

Let X be a $\hat{G}$-equivariant nowhere vanishing C^1 vector field in the plane and F the corresponding family of integral curves. We assume the action of $\hat{G}$ in the plane to be given by the formulae written down before. Suppose that $\gamma = g_1^2\gamma$ and $g_1\gamma$ is a pair of associate separatrices of $\tilde{F}$. Since γ is g_1^2-invariant, there is a $p \in \gamma$ with a horizontal tangent, i.e. $X(p)$ is horizontal. By the g_1-invariance of horizontal vectors, $X(g_1p) = X(p)$ holds. On the other hand, γ and $g_1\gamma$ being associate separatrices, the flow along γ and $g_1\gamma$ is in opposite directions, and hence also $X(g_1p) = -X(p)$, i.e. $X(p) = 0$ which contradicts the hypothesis.

REMARK. Two examples of a regular family F with a single Möbius circle are easily constructed as follows.

Take F to be a $\hat{G}$-equivariant foliation in the plane with the set of horizontal lines $y = k/3$, $k \in \mathbb{Z}$, $k \not\equiv 0 \bmod 3$, as set of separatrices. The resulting family F on K^2 contains just one Möbius circle, which divides the family F into two basins.

Similarly if one takes the lines $y = (2k+1)/2$, $k \in \mathbb{Z}$ as separatrices, one gets a family with a single Möbius circle which is in addition short; F is a single basin.

13. DIFFERENTIABILITY STRUCTURE

The preceding sections describe the quotients of the torus and the Klein bottle as C^0 schemes. As has been observed in § 3 (examples 2 and 3) it may happen that two C^k schemes ($k \geqslant 1$) are equivalent as C^0 schemes without being C^k equivalent. The problem of "moduli" that arises thus, will only be partially solvable in special cases. We mention a few such partial solutions pertinent to the case of the torus.

Suppose that $B \simeq \mathbf{R}$ and that the free abelian group G acts without fixpoints. The classical results of Bohl, Denjoy, Siegel [3, 5, 15], translated into scheme language, just state that in the C^2 case the scheme $A=(\mathbf{R};G)$ is C^0 equivalent to a Poincaré torus Poinc $(\mathbf{R};G)$ of dimension 1 and rank 2. As we observed in § 3 the "module" τ of such a Poincaré torus may be taken to be a point of the real projective line up to an integral unimodular transformation. A module of Poinc $(\mathbf{R};G)$ can be obtained by taking a basis g_1, g_2 for G, factoring out $\mathbf{R}$ by g_1 so as to get an S^1, and taking the rotation number τ (normalized as a number mod 1) of g_2 acting on S^1. In case $(\mathbf{R};G)$ is a C^ω or a C^∞ scheme the results of Arnol'd and Herman [1, 4, 12, 14] give conditions for C^ω and C^∞ equivalence respectively in terms of the degree of approximability of τ by rationals. To be more precise:

THEOREM (Herman). *The C^∞ scheme $(\mathbf{R};G)$ is C^∞ equivalent to Poinc $(\mathbf{R};G)$ if the continued fraction expansion of a module of the latter has bounded coefficients.*

THEOREM (Arnol'd). *Let $(\mathbf{R};T)$ be a Poincaré torus, where T is the group generated by translations g_1, g_τ, with shift 1 and τ (irrational) respectively. Suppose that G_f is the abelian group generated by g_1 and $g_{\tau,f}$: $x \mapsto x+\tau+f(x,\varepsilon)$, where f is analytic in (x,ε), has period 1 in x and satisfies a Lipschitz condition of order 1 with respect to ε and valid in a strip along the real $(=x\text{-})$ axis in $\mathbf{R}$, and $f(x,0)=0$. Then, if $|\tau-m/n|\geqslant K/n^3$, for suitable $K>0$ irrespective of m, n, the C^ω scheme $(\mathbf{R};G_f)$ is C^ω equivalent to Poinc $(\mathbf{R};G_f)$; however Poinc $(\mathbf{R};G_f)$ is possibly degenerate, i.e. the module may be rational. Furthermore Poinc $(\mathbf{R};G_f)$ is an analytic family (parametrized by ε), and the equivalence $(\mathbf{R};G_f)\to$ Poinc $(\mathbf{R};G_f)$ depends analytically on ε.*

A case which is much simpler to handle is the following.

Assume that B is a tree manifold, and that the irreducible convex G-set is just the set N of nodes of B. In particular this implies that $B-N$ decomposes into copies $\mathbf{R}_i$ of $\mathbf{R}$, and that every $\mathbf{R}_i$ is bounded by a pair of associate nodes which are labeled by $(i-1)$ and i respectively; of course we assume as usual that the indices $i\in\mathbb{Z}$ are assigned in a convexity preserving fashion. We select a basis g_1, g_2 for G such that g_1 acts trivially on N and that g_2 induces a translation on N over k. Finally assume that $(B;G)$ is a C^ω scheme, and that in a neighbourhood of every node the action of g_1 is described in terms of a local coordinate x by $g_1(x)= =\alpha_{1i}x+\alpha_{2i}x^2+\dots$ with $\alpha_{1i}\neq 0$ everywhere. This assumption implies in this particular situation that $\alpha_{1i}>0$ for every i. Suppose furthermore that for some i, $0<\alpha_{1i}<1$, then it follows that for every j, $0<\alpha_{1j}<1$ holds too. From now on we make this assumption. Siegel showed [16, 17 chap. III] that one can choose a C^ω equivalent local coordinate such that $g_1(x)=a_{1i}x$. From this one obtains that $(B;G)$ is C^ω equivalent to a scheme of which the pages are copies $\mathbf{R}_i$ of $\mathbf{R}$ and for which the generating set of transitions

$\{g_{1i}, g_{2i}, \tau_{i,i+1}\}$ is the following:

$$g_{1i}: x_i \mapsto x_i' = \alpha_i x_i;\ x_i, x_i' \in \mathbf{R}_i;\ 0 < \alpha_i < 1;\ \alpha_i = \alpha_{i+k},$$

k being the shift induced by g_2;

$$g_{2i}: x_i \mapsto x_{i+k} = \beta_i x_i;\ x_i \in \mathbf{R}_i,\ x_{i+k} \in \mathbf{R}_{i+k};\ \beta_i \neq 0,$$
$$\operatorname{sign} \beta_i = (-)^k;\ \beta_i = \beta_{i+k};\ \mu_{i+1}/\mu_i = \lambda_{i+1}/\lambda_i =:\ \varepsilon_i,$$

where $\mu_i = \log \beta_i$, $\lambda_i = \log \alpha_i$;

$$\tau_{i+1,i}: x_i \mapsto x_{i+1} = \exp(\psi_i(\log |x_i|)),\ x_i > 0,\ i \equiv 0 \bmod 2$$
$$= -\exp(\psi_i(\log |x_i|)),\ x_i < 0,\ i \equiv 1 \bmod 2;$$

$\psi_i(\xi) = \varepsilon_i \xi + \phi_{i+1}(\xi)$, $\xi \in \mathbf{R}$; ϕ_{i+1} is real analytic and satisfies the conditions $\phi_{i+1}(0) = 0$, $\phi_{i+1}' > -1$, $\phi_{i+1}(\xi + \lambda_{i+1}) = \phi_{i+1}(\xi) = \phi_{i+1}(\xi + \mu_{i+1})$; therefore if $\varkappa = \mu_{i+1}/\lambda_{i+1}$ ($\varkappa$ is independent of i) is irrational, $\phi_{i+1} = 0$; otherwise ϕ_{i+1} may be a non-trivial periodic function.

In case the manifold scheme arises from an analytic foliation on the torus, the shift number k is the winding number of the field of tangent line elements to the foliation (for the definition and properties of the winding number cf. e.g. [18] *).

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* The results of this paper were rediscovered independently in 1968 by Mr. C. van Egmond.

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Added in proof: The notion of quotient manifold scheme of a foliation was also developed under the name of "feuillage" in P. Molino: Sur la géométrie transverse des feuilletages, Ann. Inst. Fourier **25**, 279–284, (1975).

A notion close to that of manifold scheme is that of "receuil", cf. J. M. Souriau: Géométrie et Relativité (Hermann, Paris, 1964).

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