Available online at www.sciencedirect.com

SciVerse ScienceDirect

ELSEVIER

Advances in Mathematics 231 (2012) 2593–2625

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

The Hall algebra approach to Drinfeld's presentation of quantum loop algebras

Rujing Dou [1], Yong Jiang *, Jie Xiao

Department of Mathematics, Tsinghua University, Beijing 100084, PR China

Received 8 February 2010; accepted 30 July 2012
Available online 7 September 2012

Communicated by Bertrand Toen

Abstract

The quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ was defined as a generalization of the Drinfeld's new realization of the quantum affine algebra to the loop algebra of any Kac–Moody algebra $\mathfrak{g}$. It has been shown by Schiffmann that the Hall algebra of the category of coherent sheaves on a weighted projective line is closely related to the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$, for some $\mathfrak{g}$ with a star-shaped Dynkin diagram. In this paper we study Drinfeld's presentation of $U_v(\mathcal{L}\mathfrak{g})$ in the double Hall algebra setting, based on Schiffmann's work. We explicitly find out a collection of generators of the double composition algebra $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$ and verify that they satisfy all the Drinfeld relations.

MSC: 14H60; 17B37; 18F20

Keywords: Quantum loop algebra; Drinfeld's presentation; Hall algebra; Weighted projective line; Coherent sheaf

1. Introduction

1.1

Let $\mathfrak{g}$ be a Kac–Moody algebra, $U(\mathfrak{g})$ be its universal enveloping algebra. The Drinfeld–Jimbo quantum group $U_v(\mathfrak{g})$ is defined by a collection of generators and relations (see 3.2), which is a

* Correspondence to: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.
E-mail addresses: drj05@mails.thu.edu.cn (R. Dou), yjiang@math.uni-bielefeld.de (Y. Jiang), jxiao@math.tsinghua.edu.cn (J. Xiao).

[1] Current address: Department of Mathematics, China University of Mining and Technology (Beijing), Beijing 100083, PR China.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.07.026

certain deformation of the Chevalley generators and Serre relations for $U(\mathfrak{g})$. When $\mathfrak{g}$ is affine, it is well-known that $\mathfrak{g}$ can be constructed as (a central extension of) the loop algebra $\mathcal{L}\mathfrak{g}_0$ of some simple Lie algebra $\mathfrak{g}_0$. In this case Drinfeld gave another set of generators and relations of $U_v(\mathfrak{g})$ known as *Drinfeld's new realization* of quantum affine algebras. This new presentation can be treated as a certain deformation of the loop algebra presentation of $\mathfrak{g}$. The isomorphism of the two presentations of $U_v(\mathfrak{g})$ was proved by Beck [2] (also see [14]). One can define the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ for any Kac–Moody algebra $\mathfrak{g}$ as a generalization of Drinfeld's presentation for quantum affine algebras (see 3.4).

1.2

The Ringel–Hall algebra approach to quantum groups has been developed since the 1990s, which shows a deep relationship between Lie theory and finite dimensional hereditary algebras. More precisely, let Q be the quiver whose underlying graph is the Dynkin diagram of the Kac–Moody algebra $\mathfrak{g}$. Consider the category of finite dimensional representations of Q over a finite field $k = \mathbb{F}_q$, denoted by mod(kQ). Due to Ringel and Green [21,11], the composition subalgebra of the Hall algebra $\mathbf{H}$ (mod kQ) is isomorphic to the positive part of the quantum group $U_v^+(\mathfrak{g})$ where v specializes to $\sqrt{q}$. This result was generalized to the whole quantum group by using the technique of Drinfeld double for Hopf algebras [25] (see 2.5).

Thus it is quite natural to consider the following problem.

Problem 1.1. How to understand Drinfeld's presentation of quantum affine algebras (and more generally, quantum loop algebras) in the Hall algebra setting?

One possible way to solve the problem for quantum affine algebras is to explain Beck's isomorphism in the language of Hall algebras. For type $\tilde{A}$ Hubery has given the answer for the positive part $U_v^+(\widehat{\mathfrak{sl}}_n)$ using nilpotent representations of cyclic quivers [13]. But it seems not easy to generalize his method to other types. We should also mention that McGerty [19] has given the Drinfeld generators for the positive part $U_v^+(\widehat{\mathfrak{sl}}_2)$ using representations of the Kronecker quiver.

1.3

On the other hand, in his remarkable paper [15] Kapranov observed that there are connections between the Hall algebra of the category of coherent sheaves on a smooth projective curve X and Drinfeld's new realization of the quantum affine algebra. And when X is the projective line, he constructed an isomorphism between a subalgebra of the Hall algebra and another positive part (compared with the standard one, see 3.2) of $U_q(\widehat{\mathfrak{sl}}_2)$ (also see [1]).

This result was generalized by Schiffmann [23] using Hall algebras of the categories of coherent sheaves on weighted projective lines (introduced in [10]) as follows.

When the Dynkin diagram Γ of $\mathfrak{g}$ is a star-shaped graph, he defined a certain positive part $U_v(\hat{\mathfrak{n}})$ of the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ (note that there is no standard positive part of $U_v(\mathcal{L}\mathfrak{g})$ since in general the loop algebra $\mathcal{L}\mathfrak{g}$ is not a Kac–Moody algebra). And he established an epimorphism from $U_v(\hat{\mathfrak{n}})$ to a subalgebra $\mathbf{C}(\mathrm{Coh}\,\mathbb{X})$ of the Hall algebra $\mathbf{H}(\mathrm{Coh}\,\mathbb{X})$, where $\mathbb{X}$ is the weighted projective line associated to Γ. Moreover, when $\mathbb{X}$ is of parabolic or elliptic type (the corresponding $\mathfrak{g}$ is of finite or affine type), the epimorphism is an isomorphism (see Theorem 5.4). This means that the Hall algebra for weighted projective lines is the right framework to consider Problem 1.1 for general quantum loop algebras.

However, the problem was not completely solved in Schiffmann's work. Namely not all Drinfeld's generators and relations were explicitly found out in the corresponding Hall algebra.

In fact the composition algebra $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$ is not generated by part of Drinfeld generators (some Chevalley generators are involved, see 5.6 for details). Moreover, only the positive half $U_v(\hat{\mathfrak{n}})$ was linked to the Hall algebra. Thus Drinfeld's presentation for the whole quantum loop algebra has not been fully understood yet.

1.4

In this paper we study the problem in the double Hall algebra $\mathbf{DH}(\mathrm{Coh}\,\mathbb{X})$, which is the reduced Drinfeld double of the Hall algebra $\mathbf{H}(\mathrm{Coh}\,\mathbb{X})$. We define a subalgebra $\mathbf{DC}(\mathrm{Coh}\,\mathbb{X})$ of $\mathbf{DH}(\mathrm{Coh}\,\mathbb{X})$, called double composition algebra, and show that a collection of generators of $\mathbf{DC}(\mathrm{Coh}\,\mathbb{X})$ satisfies all Drinfeld's relations (Theorem 5.5). Thus we have an epimorphism Ξ from the whole quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ to the double composition algebra $\mathbf{DC}(\mathrm{Coh}\,\mathbb{X})$.

Let us briefly explain our method. First we consider the generators and relations in $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$. Note that we assume that the Dynkin diagram of the Kac–Moody algebra $\mathfrak{g}$ is a star-shaped graph Γ (see 5.5). Each branch of Γ corresponds to a subalgebra isomorphic to $U_v(\widehat{\mathfrak{sl}}_n)$ for some n. Thus we can use the results in [13] to find out Drinfeld generators in such subalgebras. For the central vertex we keep the generators given in [23]. Then it remains to check the relations (some of them have been verified in [23]; see 7.1).

To check all Drinfeld's relations in the double Hall algebra $\mathbf{DH}(\mathrm{Coh}\,\mathbb{X})$, the key part is to verify the relations involving both positive and negative parts. For these relations we have to investigate the comultiplication in detail. This is much more complicated than the non-weighted case $\mathrm{Coh}(\mathbb{P}^1)$, which has been studied in [1,5]. However, we show that most terms appearing in the comultiplication of a Drinfeld generator corresponding to the central vertex do not affect our calculation (see the proof of Lemmas 8.2 and 8.7). Thus the problem can be solved.

Note that in a recent work [5] of Burban and Schiffmann, the case of $U_v(\widehat{\mathfrak{sl}}_2)$ has already been studied in the double Hall algebra setting. Our results coincide with theirs in this case. In particular, the epimorphism Ξ is an isomorphism.

1.5

The paper is organized as follows. In Section 2 we recall basic notions of Hall algebra and double Hall algebra, for details one can see [22,25]. In Section 3 we recall the definition of quantum loop algebra; see [23] or the Appendix of [24]. We give a brief review of the theory of coherent sheaves on weighted projective lines in Section 4, the main reference for this section is [10]. The main result of the paper (Theorem 5.5) is stated in Section 5. The proof of the main theorem consists of the next three sections. More precisely, in Section 6 we prove the relations satisfied by elements corresponding to torsion sheaves lying in a non-homogeneous tube, following [13]. Section 7 is devoted to the proof of relations in the positive Hall algebra. The remaining relations, especially those involving elements from both positive and negative parts, are proved in Section 8. In Section 9 we give some remarks for the quantum affine algebras. In particular, we explain that the homomorphism Ξ given by our main theorem is not induced from a derived equivalence functor for many cases.

2. Hall algebras and their Drinfeld doubles

2.1. Hereditary category

Let $k = \mathbb{F}_q$ be a finite field with q elements, and $\mathscr{A}$ be an essentially small abelian category. We assume that $\mathscr{A}$ is k-linear, Hom-finite and Ext-finite. That is, for all objects X, Y and Z in $\mathscr{A}$, the sets $\mathrm{Hom}(X, Y)$ and $\mathrm{Ext}^1(X, Y)$ are finite dimensional k-vector spaces and the composition of morphisms $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)$ is k-bilinear. We also assume that $\mathscr{A}$ is *hereditary*, i.e. $\mathrm{Ext}^i(-, -)$ vanishes for all $i \geqslant 2$.

Let $\mathcal{P}$ be the set of isomorphism classes of objects in $\mathscr{A}$ and $K_0(\mathscr{A})$ be the Grothendieck group of $\mathscr{A}$. For any $\alpha \in \mathcal{P}$ we choose a representative $V_\alpha \in \alpha$. And for each object M in $\mathscr{A}$, denote by $[M]$ its image in $K_0(\mathscr{A})$. Then the *Euler form*

$$\langle [V_\alpha], [V_\beta] \rangle := \dim_k \mathrm{Hom}_{\mathscr{A}}(V_\alpha, V_\beta) - \dim_k \mathrm{Ext}^1_{\mathscr{A}}(V_\alpha, V_\beta)$$

is a well-defined bilinear form on $K_0(\mathscr{A})$. The *symmetric Euler form* is defined by $(\mu, \nu) = \langle \mu, \nu \rangle + \langle \nu, \mu \rangle$, for $\mu, \nu \in K_0(\mathscr{A})$.

2.2. The Hall algebra

For any $\alpha \in \mathcal{P}$, denote by a_α the cardinality of the automorphism group of V_α. Set $v = \sqrt{q}$.

For any α, β and γ in $\mathcal{P}$, the *Hall number* $g^\gamma_{\alpha\beta}$ is defined to be the number of subobjects X of V_γ satisfying $X \in \beta$ and $V_\gamma / X \in \alpha$.

The *Hall algebra* associated to the category $\mathscr{A}$, denoted by $\mathbf{H}(\mathscr{A})$, is defined as follows. As a $\mathbb{C}$-vector space, it has a basis $\{u_\alpha | \alpha \in \mathcal{P}\}$. The multiplication is given by the following formula:

$$u_\alpha u_\beta = \sum_{\gamma \in \mathcal{P}} v^{\langle \alpha, \beta \rangle} g^\gamma_{\alpha\beta} u_\gamma, \quad \forall\, \alpha, \beta \in \mathcal{P}.$$

It is easy to see that $\mathbf{H}(\mathscr{A})$ is an associative algebra with $1 = u_0$.

2.3. The extended Hall algebra

The *extended Hall algebra* $\mathcal{H}(\mathscr{A})$ is defined by adding the Grothendieck group $K_0(\mathscr{A})$ as a torus to the Hall algebra $\mathbf{H}(\mathscr{A})$. More precisely, as a $\mathbb{C}$-vector space, $\mathcal{H}(\mathscr{A})$ has a basis $\{K_\mu u_\alpha | \mu \in K_0(\mathscr{A}),\ \alpha \in \mathcal{P}\}$. And the multiplication is given by the one inside $\mathbf{H}(\mathscr{A})$ together with the following additional rules:

$$K_\mu K_\nu = K_{\mu+\nu}, \quad \forall\, \mu, \nu \in K_0(\mathscr{A}),$$
$$K_\mu u_\alpha = v^{(\mu,\alpha)} u_\alpha K_\mu, \quad \forall\, \mu \in K_0(\mathscr{A}),\ \alpha \in \mathcal{P}.$$

If $\mathscr{A}$ is a *length category* (i.e. each object M has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_{i+1}/M_i is a simple object for any i), then $\mathcal{H}(\mathscr{A})$ has a Hopf algebra structure with the following comultiplication Δ, counit ϵ and antipode S:

$$\Delta(K_\mu) = K_\mu \otimes K_\mu,$$
$$\Delta(u_\gamma) = \sum_{\alpha,\beta \in \mathcal{P}} v^{\langle \alpha, \beta \rangle} \frac{a_\alpha a_\beta}{a_\gamma} g^\gamma_{\alpha\beta} u_\alpha K_\beta \otimes u_\beta,$$
$$\epsilon(u_\alpha) = \delta_{\alpha,0}, \quad \epsilon(K_\mu) = 1,$$
$$S(K_\mu) = K_{-\mu}, \quad S(u_0) = 0,$$
$$S(u_\gamma) = \sum_{m \geq 1} (-1)^m \sum_{\beta \in \mathcal{P}, \alpha_1 \cdots \alpha_m \in \mathcal{P}_1} v^{2 \sum_{i<j} \langle \alpha_i, \alpha_j \rangle} \frac{a_{\alpha_1} \cdots a_{\alpha_m}}{a_\gamma} g^\gamma_{\alpha_1 \cdots \alpha_m} g^\beta_{\alpha_1 \cdots \alpha_m} K_{-\gamma} u_\beta$$

where $\mathcal{P}_1 = \mathcal{P} \setminus \{0\}$ and $g^\gamma_{\alpha_1 \cdots \alpha_m}$ is the number of all filtrations

$$0 = M_m \subset M_{m-1} \subset \cdots \subset M_0 = V_\gamma$$

such that $M_{i-1}/M_i \simeq V_{\alpha_i}$ for any i.

Remark 2.1. (1) The Hall algebra can be defined for any finitary abelian category and it is both an algebra and a coalgebra. However, the heredity is needed to endow the Hall algebra with the structure of a bialgebra. For details one can see Schiffmann's lecture notes [24].

(2) The comultiplication (resp. antipode) was first defined by Green [11] (resp. the third-named author [25]) in the case that $\mathscr{A}$ is the category of finite dimensional modules over a finite dimensional hereditary algebra.

(3) If $\mathscr{A}$ is not a length category, then the comultiplication Δ takes value in the completion $\mathcal{H}(\mathscr{A})\widehat{\otimes}\mathcal{H}(\mathscr{A})$. And $\mathcal{H}(\mathscr{A})$ is a *topological bialgebra* (see [3]).

2.4. *The Drinfeld double of the Hall algebra*

Now we write $\mathcal{H}^+(\mathscr{A})$ for the extended Hall algebra $\mathcal{H}(\mathscr{A})$ defined above. And we also write the basis elements u_α^+ instead of u_α.

The "negative" extended Hall algebra, denoted by $\mathcal{H}^-(\mathscr{A})$, is defined to be the $\mathbb{C}$-algebra with basis $\{K_\mu u_\alpha^- : \mu \in K_0(\mathscr{A}), \alpha \in \mathcal{P}\}$ and the following multiplication rules:

$$u_\alpha^- u_\beta^- = v^{\langle\alpha,\beta\rangle} \sum_{\gamma\in\mathcal{P}} g_{\alpha\beta}^\gamma u_\gamma^-, \qquad K_\mu K_\nu = K_{\mu+\nu},$$

$$K_\mu u_\alpha^- = v^{-(\mu,\alpha)} u_\alpha^- K_\mu.$$

Similarly, if $\mathscr{A}$ is a length category, $\mathcal{H}^-(\mathscr{A})$ has a Hopf algebra structure:

$$\Delta(K_\mu) = K_\mu \otimes K_\mu,$$

$$\Delta(u_\gamma^-) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\beta,\alpha\rangle} \frac{a_\alpha a_\beta}{a_\gamma} g_{\beta\alpha}^\gamma u_\alpha^- \otimes u_\beta^- K_{-\alpha},$$

$$\epsilon(u_\alpha^-) = \delta_{\alpha,0}, \quad \epsilon(K_\mu) = 1,$$

$$S(K_\alpha) = K_{-\alpha}, \quad S(u_0^-) = 0,$$

$$S(u_\gamma^-) = \sum_{m\geq 1}(-1)^m \sum_{\beta\in\mathcal{P},\alpha_1,\ldots,\alpha_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\alpha_i,\alpha_j\rangle} \frac{a_{\alpha_1}\cdots a_{\alpha_m}}{a_\gamma} g_{\alpha_1\cdots\alpha_m}^\gamma g_{\alpha_m\cdots\alpha_1}^\beta u_\beta^- K_\gamma.$$

Actually in this case $\mathcal{H}^-(\mathscr{A})$ is the dual Hopf algebra of $\mathcal{H}^+(\mathscr{A})$ with opposite comultiplication.

Following Ringel [22], we define a bilinear form $\varphi : \mathcal{H}^+(\mathscr{A}) \times \mathcal{H}^-(\mathscr{A}) \longrightarrow \mathbb{C}$ by

$$\varphi(K_\mu u_\alpha^+, K_\nu u_\beta^-) = v^{-(\mu,\nu)-(\alpha,\nu)+(\mu,\beta)} \frac{1}{a_\alpha}\delta_{\alpha\beta},$$

for any μ, ν in $K_0(\mathscr{A})$ and α, β in $\mathcal{P}$.

The bilinear form defined above is a skew-Hopf pairing on $\mathcal{H}^+(\mathscr{A}) \times \mathcal{H}^-(\mathscr{A})$. It induces a Hopf algebra structure on $\mathcal{H}^+(\mathscr{A}) \otimes \mathcal{H}^-(\mathscr{A})$ as follows. Let $\widetilde{\mathbf{DH}}(\mathscr{A})$ be the free product of algebras $\mathcal{H}^+(\mathscr{A})$ and $\mathcal{H}^-(\mathscr{A})$ subject to the following relations $D(u_\alpha^-, u_\beta^+)$ for all $u_\alpha^- \in \mathcal{H}^-(\mathscr{A})$ and $u_\beta^+ \in \mathcal{H}^+(\mathscr{A})$:

$$\sum_{i,j} a_i^{(1)-} b_j^{(2)+} \varphi(b_j^{(1)+}, a_i^{(2)-}) = \sum_{i,j} b_j^{(1)+} a_i^{(2)-} \varphi(b_j^{(2)+}, a_j^{(1)-}),$$

where $\Delta(u_\alpha^-) = \sum_i a_i^{(1)-} \otimes a_i^{(2)-}$, $\Delta(u_\beta^+) = \sum_j b_j^{(1)+} \otimes b_j^{(2)+}$. Then $\widetilde{\mathbf{DH}}(\mathscr{A})$ is a (topological) Hopf algebra.

The *double Hall algebra* of the category $\mathscr{A}$ is defined to be the quotient of $\widetilde{\mathbf{DH}}(\mathscr{A})$ by the ideal generated by $\{K_\mu \otimes 1 - 1 \otimes K_{-\mu} : \mu \in K_0(\mathscr{A})\}$. This algebra is called the *reduced Drinfeld double* of the pairing $(\mathcal{H}^+(\mathscr{A}), \mathcal{H}^-(\mathscr{A}), \varphi)$, denoted by $\mathbf{DH}(\mathscr{A})$. For details see [25].

The double Hall algebra has the following triangular decomposition

$$\mathbf{DH}(\mathscr{A}) = \mathbf{H}^+(\mathscr{A}) \otimes \mathbf{T} \otimes \mathbf{H}^-(\mathscr{A}),$$

where $\mathbf{T}$ (resp. $\mathbf{H}^+(\mathscr{A})$, $\mathbf{H}^-(\mathscr{A})$) is the subalgebra of $\mathbf{DH}(\mathscr{A})$ generated by $\{K_\mu : \mu \in K_0(\mathscr{A})\}$ (resp. $\{u_\alpha^+ : \alpha \in \mathcal{P}\}$, $\{u_\alpha^- : \alpha \in \mathcal{P}\}$). It is easy to see that

$$\mathcal{H}^-(\mathscr{A}) \simeq \mathbf{T} \otimes \mathbf{H}^-(\mathscr{A}), \qquad \mathcal{H}^+(\mathscr{A}) \simeq \mathbf{H}^+(\mathscr{A}) \otimes \mathbf{T}.$$

2.5. The Hall algebra of mod kQ

Let Q be a finite quiver, I be the set of vertices of Q. Denote by mod kQ the category of finite dimensional nilpotent representations of Q over k. This is a hereditary length category. So we have the Hall algebra $\mathbf{H}$ (mod kQ) and the extended Hall algebra $\mathcal{H}$ (mod kQ). The *composition subalgebra* $\mathbf{C}$ (mod kQ) is defined to be the subalgebra of $\mathbf{H}$ (mod kQ) generated by u_{S_i} $(i \in I)$, where S_i is the simple module in mod kQ corresponding to the vertex i. The composition subalgebra provides a realization of the positive part of the quantum group $U_v(\mathfrak{g})$ (see the next section for definitions).

Theorem 2.2 (*Ringel [21], Green [11]*). *Let $\mathfrak{g}$ be the Kac–Moody algebra whose Dynkin diagram is the underlying graph of Q. Then we have an isomorphism*

$$\mathbf{C}\ (\text{mod } kQ) \simeq U_v^+(\mathfrak{g}),$$

where the image of the generator u_{S_i} is precisely the Chevalley generator E_i for each $i \in I$.

Furthermore, $\mathbf{C}$ (mod kQ) $\otimes\, \mathbf{T}$ is a Hopf subalgebra of $\mathcal{H}$ (mod kQ). So we can construct the reduced Drinfeld double $\mathbf{DC}$ (mod kQ), which is a subalgebra of $\mathbf{DH}$ (mod kQ). In this way the above Ringel–Green theorem is extended to the whole quantum group.

Theorem 2.3 (*[25]*). *The double composition algebra is isomorphic to the quantum group:*

$$\mathbf{DC}\ (\text{mod } kQ) \simeq U_v(\mathfrak{g}),$$

where $u_{S_i}^+ \mapsto E_i, u_{S_i}^- \mapsto -v^{-1}F_i$, $K_i \mapsto K_i$.

3. Drinfeld's presentation of quantum loop algebras

Throughout this section v is an indeterminate. This should not cause confusion with the previous notation $v = \sqrt{q}$ as later we will consider quantum groups (resp. quantum loop algebras) with v specialized to $\sqrt{q}$.

3.1. Kac–Moody algebras

Let $\mathcal{I}$ be a finite set, $C = (c_{ij})_{i,j \in \mathcal{I}}$ be a generalized Cartan matrix. In this paper we only consider the case that C is symmetric. Let $\mathfrak{g} = \mathfrak{g}(C)$ be the *Kac–Moody algebra* associated to C, which is the complex Lie algebra generated by $\{e_i, f_i, h_i : i \in \mathcal{I}\}$ with relations

$$[h_i, h_j] = 0, \quad \forall i, j \in \mathcal{I};$$
$$[h_i, e_j] = c_{ij} e_j, \qquad [h_i, f_j] = -c_{ij} f_j, \quad \forall i, j \in \mathcal{I};$$
$$[e_i, f_j] = \delta_{ij} h_i, \quad \forall i, j \in \mathcal{I};$$
$$(\mathrm{ad}\, e_i)^{1-c_{ij}} e_j = 0, \qquad (\mathrm{ad}\, f_i)^{1-c_{ij}} f_j = 0, \quad \forall i \in \mathcal{I} \text{ and } j \in \mathcal{I} \text{ with } i \neq j.$$

The root system of $\mathfrak{g}$ is denoted by Δ. The simple roots are denoted by $\alpha_i, i \in \mathcal{I}$. Let $Q = \oplus_{i \in \mathcal{I}} \mathbb{Z}\alpha_i$ be the root lattice, which is equipped with the Cartan bilinear form defined by $(\alpha_i, \alpha_j) = c_{ij}$.

3.2. Quantum groups

First we recall the following standard notations:

$$[m] = \frac{v^m - v^{-m}}{v - v^{-1}}, \quad \forall\, m \in \mathbb{Z},$$
$$[n]! = [n][n-1] \cdots [1], \quad \forall\, n \in \mathbb{N},$$
$$\begin{bmatrix} n \\ t \end{bmatrix} = \frac{[n]!}{[t]![n-t]!}, \quad \forall n, t \in \mathbb{N},\ t \leq n.$$

The Drinfeld–Jimbo *quantum group* (or the *quantized enveloping algebra*) $U_v(\mathfrak{g})$ of a Kac–Moody algebra $\mathfrak{g}$ is the $\mathbb{C}(v)$-algebra generated by $\{E_i, F_i : i \in \mathcal{I}\}$ and $\{K_\mu : \mu \in \mathbb{Z}\mathcal{I}\}$ with the following defining relations (see for example [17])

$$K_0 = 1, \qquad K_\mu K_\nu = K_{\mu+\nu}, \quad \forall \mu, \nu \in \mathbb{Z}\mathcal{I};$$
$$K_\mu E_i = v^{(\mu,i)} E_i K_\mu, \qquad K_\mu F_i = v^{-(\mu,i)} F_i K_\mu, \quad \forall i \in \mathcal{I}, \mu \in \mathbb{Z}\mathcal{I};$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v - v^{-1}}, \quad \forall i, j \in \mathcal{I};$$
$$\sum_{p=0}^{1-c_{ij}} (-1)^p \begin{bmatrix} 1 - c_{ij} \\ p \end{bmatrix} E_i^p E_j E_i^{1-c_{ij}-p} = 0, \quad \forall\, i \neq j \in \mathcal{I};$$
$$\sum_{p=0}^{1-c_{ij}} (-1)^p \begin{bmatrix} 1 - c_{ij} \\ p \end{bmatrix} F_i^p F_j F_i^{1-c_{ij}-p} = 0, \quad \forall\, i \neq j \in \mathcal{I}.$$

The quantized enveloping algebra has a natural triangular decomposition:

$$U_v(\mathfrak{g}) = U_v^-(\mathfrak{g}) \otimes U_v^0(\mathfrak{g}) \otimes U_v^+(\mathfrak{g}),$$

where $U_v^+(\mathfrak{g})$ (resp. $U_v^-(\mathfrak{g})$, $U_v^0(\mathfrak{g})$) is the subalgebra generated by E_i (resp. F_i, K_μ).

3.3. The Loop algebra of $\mathfrak{g}$

The *loop algebra* of a Kac–Moody algebra $\mathfrak{g}$, denoted by $\mathcal{L}\mathfrak{g}$, is defined to be the Lie algebra generated by $\{h_{i,k}, e_{i,k}, f_{i,k}, c : i \in \mathcal{I}, k \in \mathbb{Z}\}$ subject to the following relations:

c is central in $\mathcal{L}\mathfrak{g}$,

$$[h_{i,k}, h_{j,l}] = k\delta_{k,-l}c_{ij}c,$$

$$[e_{i,k}, f_{j,l}] = \delta_{i,j}h_{i,k+l} + k\delta_{k,-l}c,$$

$$[h_{i,k}, e_{j,l}] = c_{ij}e_{j,l+k}, \qquad [h_{i,k}, f_{j,l}] = -c_{ij}f_{j,l+k},$$

$$[e_{i,k+1}, e_{j,l}] = [e_{i,k}, e_{j,l+1}], \qquad [f_{i,k+1}, f_{j,l}] = [f_{i,k}, f_{j,l+1}],$$

$$[e_{i,k_1}, [e_{i,k_2}, [\ldots, [e_{i,k_n}, e_{j,l}]\ldots]]] = 0, \quad \text{for } n = 1 - c_{ij},$$

$$[f_{i,k_1}, [f_{i,k_2}, [\ldots, [f_{i,k_n}, f_{j,l}]\ldots]]] = 0, \quad \text{for } n = 1 - c_{ij}.$$

It is clear that there is an embedding of Lie algebras $\mathfrak{g} \hookrightarrow \mathcal{L}\mathfrak{g}$ assigning e_i, f_i, h_i to $e_{i,0}, f_{i,0}, h_{i,0}$ respectively.

Set $\widehat{Q} = Q \oplus \mathbb{Z}\delta$. We can extend the Cartan form to $\widehat{Q}$ by setting $(\delta, \alpha) = 0$ for all $\alpha \in \widehat{Q}$. $\mathcal{L}\mathfrak{g}$ is $\widehat{Q}$-graded by setting $\deg(e_{i,k}) = \alpha_i + k\delta$, $\deg(f_{i,k}) = -\alpha_i + k\delta$ and $\deg(h_{i,k}) = k\delta$. The root system of $\mathcal{L}\mathfrak{g}$ is $\widehat{\Delta} = \mathbb{Z}^*\delta \cup \{\Delta + \mathbb{Z}\delta\}$ (see [20]).

For each root $\alpha \in \widehat{Q}$, we call it *real* if $(\alpha, \alpha) = 2$, and *imaginary* if $(\alpha, \alpha) \le 0$.

Remark 3.1. (1) If $\mathfrak{g}$ is a complex simple Lie algebra, $\mathcal{L}\mathfrak{g}$ is isomorphic to $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$, which is an affine Kac–Moody algebra. The assignment $e_{i,k} \mapsto e_i t^k$, $f_{i,k} \mapsto f_i t^k$, $h_{i,k} \mapsto h_i t^k$ and $c \mapsto c$ gives the isomorphism (see [9]).

(2) In general, $\mathcal{L}\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ are not Kac–Moody algebras. And the above assignment only yields a surjective homomorphism $\mathcal{L}\mathfrak{g} \to \widehat{\mathfrak{g}}$, whose kernel was described in [20].

3.4. Quantum loop algebras

The *quantum loop algebra* (with zero central charge) $U_v(\mathcal{L}\mathfrak{g})$ is the $\mathbb{C}(v)$-algebra generated by $x^{\pm}_{i,k}$, $h_{i,l}$ and $K_i^{\pm 1}$ for $i \in \mathcal{I}$, $k \in \mathbb{Z}$, and $l \in \mathbb{Z}^*$ subject to the following relations:

$$[K_i, K_j] = [K_i, h_{j,l}] = 0, \tag{1}$$

$$[h_{i,l}, h_{j,k}] = 0, \tag{2}$$

$$K_i x^{\pm}_{j,k} K_i^{-1} = v^{\pm c_{ij}} x^{\pm}_{j,k}, \tag{3}$$

$$[h_{i,l}, x^{\pm}_{j,k}] = \pm\frac{1}{l}[lc_{ij}]x^{\pm}_{j,k+l}, \tag{4}$$

$$x^{\pm}_{i,k+1}x^{\pm}_{j,l} - v^{\pm c_{ij}}x^{\pm}_{j,l}x^{\pm}_{i,k+1} = v^{\pm c_{ij}}x^{\pm}_{i,k}x^{\pm}_{j,l+1} - x^{\pm}_{j,l+1}x^{\pm}_{i,k}, \tag{5}$$

$$\mathrm{Sym}_{k_1,\ldots,k_n} \sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix} x^{\pm}_{i,k_1} \cdots x^{\pm}_{i,k_t} x^{\pm}_{j,l} x^{\pm}_{i,k_{t+1}} \cdots x^{\pm}_{i,k_n} = 0, \tag{6}$$

where $i \neq j$, $n = 1 - c_{ij}$ and $\mathrm{Sym}_{k_1,\ldots,k_n}$ denotes symmetrization with respect to the indices $k_1, \ldots, k_n$. And

$$[x_{i,k}^{+}, x_{j,l}^{-}] = \delta_{ij} \frac{\psi_{i,k+l} - \varphi_{i,k+l}}{v - v^{-1}}, \tag{7}$$

where $\psi_{i,k}$ and $\varphi_{i,k}$ are defined by the following equations:

$$\sum_{k\geq 0} \psi_{i,k} u^k = K_i \exp\left((v - v^{-1}) \sum_{k=1}^{\infty} h_{i,k} u^k\right),$$

$$\sum_{k\geq 0} \varphi_{i,-k} u^{-k} = K_i^{-1} \exp\left(-(v - v^{-1}) \sum_{k=1}^{\infty} h_{i,-k} u^{-k}\right).$$

Remark 3.2. If $\mathfrak{g}$ is a simple Lie algebra, the above definition of $U_v(\mathcal{L}\mathfrak{g})$ is the so-called Drinfeld's new realization of quantum affine algebras. Namely in this case, $U_v(\mathcal{L}\mathfrak{g})$ is isomorphic to the Drinfeld–Jimbo quantized enveloping algebra $U_v(\widehat{\mathfrak{g}})$ (see [8,2] for details).

4. The category of coherent sheaves on weighted projective lines

Now we introduce the category of coherent sheaves on weighted projective lines as studied in [10]. In this section k denotes an arbitrary field.

4.1. Weighted projective lines

Let $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathbb{N}^n$. Consider the $\mathbb{Z}$-module

$$L(\mathbf{p}) = \mathbb{Z}\vec{x}_1 \oplus \mathbb{Z}\vec{x}_2 \oplus \cdots \oplus \mathbb{Z}\vec{x}_n / J$$

where J is the submodule generated by $\{p_1\vec{x}_1 - p_s\vec{x}_s \mid s = 2, \dots, n\}$. Set $\vec{c} = p_1\vec{x}_1 = \cdots = p_n\vec{x}_n \in L(\mathbf{p})$.

The polynomial ring $k[X_1, \dots, X_n]$ has a structure of $L(\mathbf{p})$-graded algebra by setting $\deg X_i = \vec{x}_i$. We will denote it by $S(\mathbf{p})$.

Let $\underline{\lambda} = \{\lambda_1, \dots, \lambda_n\}$ be a collection of distinct closed points (of degree 1) on the projective line $\mathbb{P}^1(k)$. Let $I(\mathbf{p}, \underline{\lambda})$ be the $L(\mathbf{p})$-graded ideal of $S(\mathbf{p})$ generated by $\{X_s^{p_s} - (X_2^{p_2} - \lambda_s X_1^{p_1}) \mid \dots\}$. Set $S(\mathbf{p}, \underline{\lambda}) = S(\mathbf{p})/I(\mathbf{p}, \underline{\lambda})$, which is also an $L(\mathbf{p})$-graded algebra. For any i, we denote by x_i the image of X_i in $S(\mathbf{p}, \underline{\lambda})$.

Let $\mathbb{X}_{\mathbf{p},\underline{\lambda}} = \operatorname{Specgr} S(\mathbf{p}, \underline{\lambda})$ be the set of all prime homogeneous ideals in $S(\mathbf{p}, \underline{\lambda})$. This is the so-called *weighted projective line*. The pair $(\mathbf{p}, \underline{\lambda})$ is called the *weight type* of $\mathbb{X}_{\mathbf{p},\underline{\lambda}}$. The number p_i is the *weight* of the point λ_i.

In the following we will fix a weight type $(\mathbf{p}, \underline{\lambda})$ and write $S = S(\mathbf{p}, \underline{\lambda})$, $\mathbb{X} = \mathbb{X}_{\mathbf{p},\underline{\lambda}}$ for short.

4.2. Coherent sheaves on $\mathbb{X}_{\mathbf{p},\underline{\lambda}}$

For any homogeneous element $f \in S$, let $V_f = \{\mathfrak{p} \in \mathbb{X} \mid f \in \mathfrak{p}\}$ and $D_f = \mathbb{X} \setminus V_f$.

The *structure sheaf* $\mathscr{O}_{\mathbb{X}}$ is defined to be the sheaf of $L(\mathbf{p})$-graded algebras on $\mathbb{X}$ associated to the presheaf $D_f \mapsto S_f$, where $S_f = \{g/f^l \mid g \in S, l \in \mathbb{N}\}$. We denote by $\mathscr{O}_{\mathbb{X}}$-Mod the category of sheaves of $L(\mathbf{p})$-graded $\mathscr{O}_{\mathbb{X}}$-modules on $\mathbb{X}$.

For any $\vec{x} \in L(\mathbf{p})$ and any $L(\mathbf{p})$-graded $\mathscr{O}_{\mathbb{X}}$-module $\mathscr{M}$, we denote by $\mathscr{M}(\vec{x})$ the *shift* of $\mathscr{M}$ by $\vec{x}$ (i.e. $\mathscr{M}(\vec{x})[\vec{y}] = \mathscr{M}[\vec{x} + \vec{y}]$). A sheaf $\mathscr{M}$ of $L(\mathbf{p})$-graded $\mathscr{O}_{\mathbb{X}}$-module is called *coherent* if there exists an open covering $\{U_i\}$ of $\mathbb{X}$ and for each i an exact sequence

$$\bigoplus_{s=1}^{N} \mathscr{O}_{\mathbb{X}}(\vec{l_s})|_{U_i} \to \bigoplus_{t=1}^{M} \mathscr{O}_{\mathbb{X}}(\vec{k_t})|_{U_i} \to \mathscr{M}|_{U_i} \to 0.$$

The category of coherent sheaves on $\mathbb{X}$, denoted by $\mathrm{Coh}(\mathbb{X})$, is a full subcategory of $\mathscr{O}_{\mathbb{X}}$-Mod. It has been proved in [10] that $\mathrm{Coh}(\mathbb{X})$ is a k-linear hereditary, Hom- and Ext-finite abelian category.

4.3. *The structure of the category* Coh($\mathbb{X}$)

Let $\mathscr{F}$ be the full subcategory of $\mathrm{Coh}(\mathbb{X})$ consisting of all locally free sheaves, and $\mathscr{T}$ be the full subcategory consisting of all torsion sheaves. Both $\mathscr{F}$ and $\mathscr{T}$ are extension-closed. Moreover, $\mathscr{T}$ itself is a hereditary length abelian category. The following lemma was proved in [10].

Lemma 4.1. (1) *For any sheaf $\mathscr{M} \in \mathrm{Coh}(\mathbb{X})$, it can be decomposed as $\mathscr{M}_t \oplus \mathscr{M}_f$ where $\mathscr{M}_t \in \mathscr{T}$ and $\mathscr{M}_f \in \mathscr{F}$.*
(2) $\mathrm{Hom}(\mathscr{M}_t, \mathscr{M}_f) = \mathrm{Ext}^1(\mathscr{M}_f, \mathscr{M}_t) = 0$*, for any $\mathscr{M}_t \in \mathscr{T}$ and $\mathscr{M}_f \in \mathscr{F}$.*

To describe $\mathscr{T}$ more precisely, we need a classification of the closed points in $\mathbb{X}$. Recall that $\mathbb{X} = \mathrm{Specgr}S(\mathbf{p}, \underline{\lambda})$. According to [10], each λ_i corresponds to the prime ideal generated by x_i, called an *exceptional point*. And any other homogeneous prime is given by a prime homogeneous polynomial $F(x_1^{p_1}, x_2^{p_2}) \in k[T_1, T_2]$, which is called an *ordinary points*.

Let $p = \mathrm{l.c.m}(p_1, \dots, p_n)$. The *degree* of a closed point is defined by setting $\deg(\lambda_i) = p/p_i$ for any i and $\deg(x) = pd$ for any ordinary point x corresponding to a prime homogeneous polynomial of degree d.

Let C_r be the cyclic quiver with r vertices. More precisely, for $r = 1$, C_1 is just the quiver with only one vertex and one loop arrow. For $r \geq 2$, the vertices of C_r are indexed by $\mathbb{Z}/r\mathbb{Z}$ and the arrows are from i to $i-1$ for each i. Denote by $\mathrm{rep}_0(C_r)_k$ the category of finite-dimensional nilpotent representations of C_r over the field k. The following lemma, due to [10], describes the structure of the subcategory $\mathscr{T}$.

Lemma 4.2. (1) *The category $\mathscr{T}$ decomposes as a coproduct $\mathscr{T} = \coprod_{x \in \mathbb{X}} \mathscr{T}_x$, where $\mathscr{T}_x$ is the subcategory of torsion sheaves with support at x.*
(2) *For any ordinary point x of degree d, let k_x denote the residue field at x. Then $\mathscr{T}_x$ is equivalent to the category* $\mathrm{rep}_0(C_1)_{k_x}$.
(3) *For any exceptional point λ_i $(1 \leq i \leq n)$, the category $\mathscr{T}_{\lambda_i}$ is equivalent to* $\mathrm{rep}_0(C_{p_i})_k$.

4.4. *Indecomposable objects in* $\mathscr{T}$

We first give a description of the simple objects in $\mathscr{T}$.

For any ordinary point x of degree d, let π_x denote the prime homogeneous polynomial corresponding to x. Then multiplication by π_x gives the exact sequence

$$0 \to \mathscr{O}_{\mathbb{X}} \to \mathscr{O}_{\mathbb{X}}(d\vec{c}) \to S_x \to 0.$$

S_x is the unique (up to isomorphism) simple sheaf in the category $\mathscr{T}_x$. Moreover, for any $\vec{k} \in L(\mathbf{p})$ we have $S_x(\vec{k}) = S_x$.

For any exceptional point λ_i, multiplication by x_i yields the exact sequence

$$0 \to \mathscr{O}_{\mathbb{X}}((j-1)\vec{x}_i) \to \mathscr{O}_{\mathbb{X}}(j\vec{x}_i) \to S^i_j \to 0, \quad \text{for each } j,\ 1 \le j \le p_i.$$

And $\{S^i_j | 1 \le j \le p_i\}$ is a complete set of pairwise non-isomorphic simple sheaves in the category $\mathscr{T}_{\lambda_i}$, for any i $(1 \le i \le n)$. Moreover, for any $\vec{k} = \sum k_i \vec{x}_i$ we have $S^i_j(\vec{k}) = S^i_{j+k_i \pmod{p_i}}$.

Now we describe the indecomposable objects. Recall the following well-known results on representation theory of cyclic quivers.

(1) In the category $\mathrm{rep}_0(C_1)$, the set of isomorphism classes of indecomposables is $\{S(a) | a \in \mathbb{N}\}$, where $S = S(1)$ is the only simple representation, and $S(a)$ is the unique indecomposable representation of length a.

For any partition $\mu = (\mu_1 \ge \cdots \ge \mu_t)$, let $S(\mu) = \bigoplus_i S(\mu_i)$. Then any object in $\mathrm{rep}_0(C_1)$ is isomorphic to $S(\mu)$ for some μ.

(2) In the category $\mathrm{rep}_0(C_r)$, we have r simple representations $S_j = S_j(1)$ $(1 \le j \le r)$. Denote by $S_j(a)$ the unique indecomposable representation with top S_j and length a. Then $\{S_j(a) | 1 \le j \le r, a \in \mathbb{N}\}$ is the set of all isomorphism classes of indecomposables in $\mathrm{rep}_0(C_r)$.

The above results, combined with Lemma 4.2, give a classification of indecomposable objects in $\mathscr{T}$. We denote by $S_x(a)$ the unique indecomposable object of length a in $\mathscr{T}_x$, for any ordinary point x. And for any exceptional point λ_i, the indecomposable objects in $\mathscr{T}_{\lambda_i}$ are denoted by $S^i_j(a)$ $(1 \le j \le p_i, a \in \mathbb{N})$.

4.5. The Grothendieck group and the Euler form

The following proposition (see [10]) gives an explicit description of the Grothendieck group of $\mathrm{Coh}(\mathbb{X})$.

Proposition 4.3.

$$K_0(\mathrm{Coh}(\mathbb{X})) \cong (\mathbb{Z}[\mathscr{O}_{\mathbb{X}}] \oplus \mathbb{Z}[\mathscr{O}_{\mathbb{X}}(\vec{c})] \oplus \bigoplus_{i,j} \mathbb{Z}[S^i_j])/I$$

where I is the subgroup generated by $\{\sum_{j=1}^{p_i} [S^i_j] + [\mathscr{O}_{\mathbb{X}}] - [\mathscr{O}_{\mathbb{X}}(\vec{c})] | i = 1, \ldots, n\}$.

In the following we simply write $\mathscr{O}$ for $\mathscr{O}_{\mathbb{X}}$. Set

$$\delta = [\mathscr{O}(\vec{c})] - [\mathscr{O}] = \sum_{j=1}^{p_i} [S^i_j], \quad \text{for } i = 1, \ldots, n.$$

To calculate the Euler form on $K_0(\mathrm{Coh}(\mathbb{X}))$, we have the following lemma (see [23]).

Lemma 4.4. *The Euler form* $\langle -, -\rangle$ *on* $K_0(\mathrm{Coh}(\mathbb{X}))$ *is given by*

$$\langle [\mathscr{O}], [\mathscr{O}]\rangle = 1, \qquad \langle \mathscr{O}, \delta\rangle = 1, \qquad \langle \delta, [\mathscr{O}]\rangle = -1,$$

$$\langle \delta, \delta\rangle = 0, \qquad \langle \delta, [S_j^i]\rangle = 0, \qquad \langle [S_j^i], \delta\rangle = 0,$$

$$\langle [\mathscr{O}], [S_j^i]\rangle = \begin{cases} 1 & \text{if } j = p_i \\ 0 & \text{if } j \neq p_i \end{cases}$$

$$\langle [S_j^i], [\mathscr{O}]\rangle = \begin{cases} -1 & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases}$$

$$\langle [S_j^i], [S_{j'}^{i'}]\rangle = \begin{cases} 1 & \text{if } i = i',\ j = j' \\ -1 & \text{if } i = i',\ j \equiv j' + 1 \ (\mathrm{mod}\ p_i) \\ 0 & \text{otherwise.} \end{cases}$$

5. Main results

5.1

From now on we fix a finite field $k = \mathbb{F}_q$ and set $v = \sqrt{q}$. And we also fix a weight type (see 4.1):

$$\mathbf{p} = (p_1, \ldots, p_n), \qquad \underline{\lambda} = \{\lambda_1, \ldots, \lambda_n\}.$$

Consider the weighted projective line $\mathbb{X} = \mathbb{X}_{\mathbf{p},\underline{\lambda}}$ and the category of coherent sheaves $\mathrm{Coh}(\mathbb{X})$. We keep the notions in the last section.

Since $\mathrm{Coh}(\mathbb{X})$ is a k-linear hereditary, Hom- and Ext-finite abelian category, we have the Hall algebra $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$ and the double Hall algebra $\mathbf{DH}(\mathrm{Coh}(\mathbb{X}))$, as in Sections 2.2 and 2.4.

The subcategory $\mathscr{T}$ (resp. $\mathscr{T}_x$, for any closed point x in $\mathbb{X}$) is a hereditary length abelian category. Thus we can define the Hall algebra $\mathbf{H}(\mathscr{T})$ (resp. $\mathbf{H}(\mathscr{T}_x)$). It is clear that $\mathbf{H}(\mathscr{T})$ and $\mathbf{H}(\mathscr{T}_x)$ are sub-Hopf algebras of $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$, since the categories $\mathscr{T}$ and $\mathscr{T}_x$ are stable under taking extensions, subobjects and quotients.

5.2

In this and the next subsection we define some element $T_r \in \mathbf{H}(\mathscr{T})$, for any $r \in \mathbb{N}$, following [23].

For any ordinary point x of degree d, we know that $\mathscr{T}_x$ is equivalent to the category of finite-dimensional nilpotent representations of the quiver C_1 over k_x (see Lemma 4.2). Since $k = \mathbb{F}_q$ we know that $k_x = \mathbb{F}_{q^d}$. Thus we have the following isomorphism:

$$\Theta_x : \mathbf{H}(C_1)_{\mathbb{F}_{q^d}} \to \mathbf{H}(\mathscr{T}_x),$$

where $\mathbf{H}(C_1)_{\mathbb{F}_{q^d}}$ is the Hall algebra associated to the category $\mathrm{rep}_0(C_1)_{\mathbb{F}_{q^d}}$.

Let $\Lambda = \mathbb{C}[z_1, z_2, \ldots]^{\mathfrak{S}_\infty}$ be Macdonald's ring of symmetric functions (see [18]). The following result is well-known, due to Hall.

Theorem 5.1. *There exists an isomorphism of algebras $\Upsilon : \Lambda \simeq \mathbf{H}(C_1)_{\mathbb{F}_q}$ given by $e_r \mapsto v^{r(r-1)} S_x(1^r)$, for any $r \geq 1$, where e_r is the r-th elementary symmetric function.*

Let p_r denote the r-th power-sum symmetric function and set $\mathbf{h}_r = \frac{[r]}{r}\Upsilon(p_r)$. Then

$$\mathbf{h}_r = \frac{[r]}{r} \sum_{|\mu|=r} n(l(\mu)-1) S_x(\mu),$$

where $n(l) = \Pi_{i=1}^{l}(1-v^{2i})$ (see [18]).

For each $r \in \mathbb{N}$, set

$$\mathbf{h}_{r,x} = \begin{cases} 0 & \text{if } r \nmid d \\ \Theta_x(\mathbf{h}_{r/d}) & \text{if } r \mid d. \end{cases}$$

5.3

For any exceptional point λ_i, $\mathscr{T}_{\lambda_i}$ is equivalent to the category $\mathrm{rep}_0(C_{p_i})_k$. Thus we have an isomorphism

$$\Theta_{\lambda_i} : \mathbf{H}(C_{p_i})_k \to \mathbf{H}(\mathscr{T}_{\lambda_i}),$$

where $\mathbf{H}(C_{p_i})_k$ is the Hall algebra associated to the category $\mathrm{rep}_0(C_{p_i})_k$.

For any positive integer m, there is a natural fully faithful functor $\iota_m : \mathrm{rep}_0(C_m) \to \mathrm{rep}_0(C_{m+1})$ whose image is the full subcategory consisting of all objects X such that $\mathrm{Hom}(X, S_m) = \mathrm{Hom}(S_{m+1}, X) = 0$. The functor ι_m induces an embedding of Hall algebras $\mathbf{H}(C_m) \to \mathbf{H}(C_{m+1})$:

$$u_{S_i} \mapsto \begin{cases} u_{S_i} & \text{for } 1 \leq i < m \\ u_{S_{m+1}(2)} = v u_{S_{m+1}} u_{S_m} - u_{S_m} u_{S_{m+1}} & \text{for } i = m. \end{cases}$$

Hence the composition $\iota_{m-1} \circ \cdots \circ \iota_2 \circ \iota_1$ of functors gives an embedding of categories $\mathrm{rep}_0(C_1) \hookrightarrow \mathrm{rep}_0(C_m)$, which induces an embedding of Hall algebras:

$$\Psi : \mathbf{H}(C_1) \longrightarrow \mathbf{H}(C_m).$$

Set

$$\mathbf{h}_{r,\lambda_i} = \Theta_{\lambda_i} \circ \Psi(\mathbf{h}_r).$$

Finally we define

$$T_r = \sum_{x \in \mathbb{X}} \mathbf{h}_{r,x} \in \mathbf{H}(\mathrm{Coh}(\mathbb{X})),$$

where the sum is taken over all closed points x on $\mathbb{X}$.

5.4

We also need some notations in [13] for the Hall algebra of $\mathbf{H}(C_m)$. For any $1 \leq l \leq m$, let $\mathcal{M}_{l,\alpha}$ be the set of all isomorphism classes of modules M in $\mathrm{rep}_0(C_m)$ such that $\underline{\dim}\, M = \alpha$ and $\mathrm{soc}(M) \subseteq S_1 \oplus \cdots \oplus S_l$.

Let δ_m be the sum of dimension vectors of all simple modules, i.e. the minimal imaginary root for C_m. For any $r \in \mathbb{N}$, set

$$c_{l,r} = (-1)^r v^{-2lr} \sum_{M \in \mathcal{M}_{l,r\delta_m}} (-1)^{\dim \mathrm{End(M)}} a_M u_M \in \mathbf{H}(C_m).$$

Then define $p_{l,r} \in \mathbf{H}(C_m)$ via the following generating function

$$\sum_{r \geq 1} (1 - v^{-2lr}) p_{l,r} T^{r-1} = \frac{d}{dT} \log C_l(T),$$

where $C_l(T) = 1 + \sum_{r \geq 1} c_{l,r} T^r$.

And define

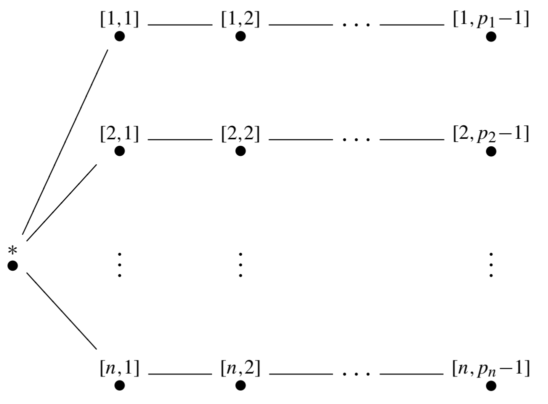
$$\pi_{l,r} := \frac{[lr]}{r} p_{l,r}.$$

For any exceptional point λ_i on $\mathbb{X}$, let $\pi^i_{j,k} = \Theta_{\lambda_i}(\pi_{j,k})$. It is not difficult to see that $\pi^i_{1,r} = \mathbf{h}_{r,\lambda_i}$.

Remark 5.2. It was proved in [12] that there is an algebra isomorphism $\mathbf{H}(C_m) \simeq U_v^+(\widehat{sl}_m) \otimes \mathcal{Z}$, where $\mathcal{Z} = \mathbb{C}[p_{m,1}, p_{m,2}, \ldots]$ is a central subalgebra of $\mathbf{H}(C_m)$ and the element $p_{m,r}$ is homogeneous of degree $r\delta_m$.

5.5

We can associate a star-shaped graph Γ to the weight type $(\mathbf{p}, \underline{\lambda})$:



As marked in the graph, the central vertex is denoted by $*$. There are n branches and in each branch there are $p_i - 1$ $(1 \leq i \leq n)$ vertices respectively. We denote by $[i, j]$ the jth vertex in the ith branch. Let $\Gamma_0 = \{*, [i, j] | 1 \leq i \leq n, 1 \leq j \leq p_i - 1\}$ denote the set of vertices of Γ.

Consider the Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(\Gamma)$ associated to the graph Γ. As in 3.3 and 3.4, we have the loop algebra $\mathcal{L}\mathfrak{g}$ and its quantized enveloping algebra $U_v(\mathcal{L}\mathfrak{g})$. The root systems of $\mathfrak{g}$ and $\mathcal{L}\mathfrak{g}$ are denoted by Δ and $\hat{\Delta}$ respectively. In view of the graph Γ, the simple roots in Δ are denoted by α_* and α_{ij}, for $1 \leq i \leq n$ and $1 \leq j \leq p_i - 1$. We also know that $\hat{\Delta} = \mathbb{Z}^* \delta \cup \{\Delta + \mathbb{Z}\delta\}$.

From Proposition 4.3, there is a natural isomorphism of $\mathbb{Z}$-modules $K_0(\mathrm{Coh}(\mathbb{X})) \cong \hat{Q}$ given by

$$[S^i_j] \mapsto \alpha_{ij}, \quad \text{for } 1 \le j \le p_i - 1,\ 1 \le i \le n,$$
$$[S^i_{p_i}] \mapsto \delta - \sum_{j=1}^{p_i-1} \alpha_{ij}, \quad \text{for } 1 \le i \le n,$$
$$[\mathscr{O}(k\vec{c})] \mapsto \alpha_* + k\delta, \quad \text{for } k \in \mathbb{Z}.$$

Now by Lemma 4.4 we have the following result.

Lemma 5.3. *The symmetric Euler form on* $K_0(\mathrm{Coh}(\mathbb{X}))$ *coincides with the Cartan form on* $\hat{Q}$.

5.6

In [23] (also see [24]) Schiffmann has proved that the Hall algebra $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$ provides a realization of the quantized enveloping algebra of a certain nilpotent subalgebra of $\mathcal{L}\mathfrak{g}$, denoted by $U_v(\widehat{\mathfrak{n}})$.

Let us recall the definition of $U_v(\widehat{\mathfrak{n}})$. First, for each i, we denote by U_i the subalgebra of $U_v(\mathcal{L}\mathfrak{g})$ generated by $x^{\pm}_{[i,j],k}$, $h_{[i,j],l}$ and $K^{\pm 1}_{[i,j]}$ $(1 \le j \le p_i - 1,\ k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\})$. It is clear that this subalgebra is isomorphic to $U_v(\widehat{\mathfrak{sl}}_{p_i})$. Denote the Chevalley generators of $U_v(\widehat{\mathfrak{sl}}_{p_i})$ by E^i_j and F^i_j $(1 \le j \le p_i)$. Then the image of E^i_j under Beck's isomorphism is $x^+_{[i,j],0}$, for $1 \le j \le p_i - 1$. But the image of $E^i_{p_i}$, which we denote by ε_i, is not a Drinfeld generator. Now let U^+_i be the subalgebra generated by $x^+_{[i,j],0}$ and ε_i. Thus U^+_i is isomorphic to the standard positive part of $U_v(\widehat{\mathfrak{sl}}_{p_i})$. Finally, $U_v(\widehat{\mathfrak{n}})$ is defined to be the subalgebra of $U_v(\mathcal{L}\mathfrak{g})$ generated by $x^+_{*,k}$, $h_{*,r}$ and U^+_i $(k \in \mathbb{Z}, r \ge 1, 1 \le i \le n)$.

Now let $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$ be the subalgebra of $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$ generated by $u_{\mathscr{O}(k\vec{c})}, u_{S^i_j}$ and T_r for $k \in \mathbb{Z}, 1 \le i \le n, 1 \le j \le p_i$ and $r \in \mathbb{N}$. It is called the *composition algebra* of $\mathrm{Coh}(\mathbb{X})$. The following theorem was proved by Schiffmann.

Theorem 5.4 (*[23]*). *The assignment* $x^+_{[i,j],0} \mapsto u_{S^i_j}$ *for* $1 \le j \le p_i - 1, \varepsilon_i \mapsto u_{S^i_{p_i}}, x_{*,k} \mapsto u_{\mathscr{O}(k\vec{c})}, h_{*,r} \mapsto T_r$ *gives an epimorphism of algebras*

$$\Phi : U_v(\widehat{\mathfrak{n}}) \twoheadrightarrow \mathbf{C}(\mathrm{Coh}(\mathbb{X})).$$

Moreover, if $\mathfrak{g}$ *is of finite or affine type,* Φ *is an isomorphism.*

The subalgebra $U_v(\widehat{\mathfrak{n}})$ can be viewed as a certain “positive part” of $U_v(\mathcal{L}\mathfrak{g})$. But by definition it is not generated by part of Drinfeld generators. Thus the correspondence between generators of $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$ and Drinfeld generators of $U_v(\mathcal{L}\mathfrak{g})$ is not completely explicit.

As in 2.4, we have the double Hall algebra $\mathbf{DH}(\mathrm{Coh}(\mathbb{X}))$. We define the *double composition algebra* $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$ to be the subalgebra of $\mathbf{DH}(\mathrm{Coh}(\mathbb{X}))$ generated by $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$, $\mathbf{C}^-(\mathrm{Coh}(\mathbb{X}))$ (the subalgebra of $\mathbf{H}^-(\mathrm{Coh}(\mathbb{X}))$ defined similar to $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$) and the torus $\mathbf{T}$. Recall that $\mathbf{T} = \{K_\alpha | \alpha \in K_0(\mathrm{Coh}(\mathbb{X}))\}$. The following notations will be used: $K_* = K_{[\mathscr{O}]}$, $K_{[i,j]} = K_{[S^i_j]}$ for $1 \le i \le n, 1 \le j \le p_i - 1$.

We will show that the Drinfeld generators and relations for the whole quantum loop algebra can be fully understood in the double composition algebra.

5.7

We keep the notations in the previous subsections.

Note that in 5.3 and 5.4 we have defined the elements $T_k, \pi^i_{j,k}$ in the Hall algebra $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$ for any $1 \le i \le n$, $1 \le j \le p_i - 1$ and $k \ge 1$. Similarly we can define the elements $T^-_k, \pi^{-i}_{j,k}$ in the negative Hall algebra $\mathbf{H}^-(\mathrm{Coh}(\mathbb{X}))$.

Moreover, we define

$$\eta^+_{i,j} = v^{1-j}\,\Theta_{\lambda_i}\left(\sum_{M\in\mathcal{M}_{j+1,\delta-e_j}} (1-v^2)^{\dim \mathrm{End}(M)-1} u^+_M\right) K_{[i,j]},$$

$$\eta^-_{i,j} = -v^{-j}\,\Theta^-_{\lambda_i}\left(\sum_{M\in\mathcal{M}_{j+1,\delta-e_j}} (1-v^2)^{\dim \mathrm{End}(M)-1} u^-_M\right) K^-_{[i,j]}.$$

The following theorem is the main result of this paper.

Theorem 5.5. *For any star-shaped graph Γ, let $\mathfrak{g}$ be the Kac–Moody algebra and $\mathbb{X}$ be the weighted projective line associated to Γ respectively. Then the following elements together with $\{K_s \in \mathbf{T} | s \in \Gamma_0\}$ in the double Hall algebra $\mathbf{DH}(\mathrm{Coh}(\mathbb{X}))$ satisfy the defining relations of the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ (see 3.4).*

$$h_{s,r} = \begin{cases} T_r & s=*,\ r>0 \\ -T^-_{-r} & s=*,\ r<0 \\ \pi^i_{j+1,r} - (v^r+v^{-r})\pi^i_{j,r} + \pi^i_{j-1,r} & s=[i,j],\ r>0 \\ -\pi^{-i}_{j+1,-r} + (v^r+v^{-r})\pi^{-i}_{j,-r} - \pi^{-i}_{j-1,-r} & s=[i,j],\ r<0 \end{cases}$$

$$x^+_{s,t} = \begin{cases} u^+_{\mathscr{O}(t\vec{c})} & s=*,\ t\in\mathbb{Z} \\ u^+_{S^i_j} & s=[i,j],\ t=0 \\ \dfrac{t}{[2t]}[h_{[i,j],t}, x^+_{[i,j],0}] & s=[i,j],\ t\ge 1 \\ \eta^-_{i,j} & s=[i,j],\ t=-1 \\ \dfrac{-k}{[-2k]}[h_{[i,j],-k}, x^+_{[i,j],-1}] & s=[i,j],\ t=-k-1,\ k>0 \end{cases}$$

$$x^-_{s,t} = \begin{cases} -vu^-_{\mathscr{O}(-t\vec{c})} & s=*,\ t\in\mathbb{Z} \\ \eta^+_{i,j} & s=[i,j],\ t=1 \\ \dfrac{-k}{[2k]}[h_{[i,j],k}, x^-_{[i,j],1}] & s=[i,j],\ t=k+1,\ k>0 \\ -vu^-_{S^i_j} & s=[i,j],\ t=0 \\ \dfrac{k}{[-2k]}[h_{[i,j],-k}, x^-_{[i,j],0}] & s=[i,j],\ t=-k,\ k>0. \end{cases}$$

The proof of this theorem consists of the next three sections.

Remark 5.6. (1) By the above theorem we have an algebra homomorphism

$$\Xi : U_v(\mathcal{L}\mathfrak{g}) \to \mathbf{DH}(\mathrm{Coh}(\mathbb{X})).$$

The image of Ξ is in fact the double composition algebra $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$. This can be easily seen from the following results proved in [13]:

$$u_{S^i_{p_i}} = (-1)^n[-vu^-_{S^i_{p_i-1}}, \ldots, [-vu^-_{S^i_2}, \eta^+_{i,1}]_{v^{-1}}]_{v^{-1}} K_{[S^i_{p_i}]},$$

where the v-commutator is defined as in Lemma 6.4. And for any i, the elements $\pi^i_{j+1,r} - (v^r + v^{-r})\pi^i_{j,r} + \pi^i_{j-1,r}$ and $\eta^+_{i,j}$ are in $\Theta_i(\mathbf{C}(C_{p_i})) \subset \mathbf{C}(\mathrm{Coh}(\mathbb{X}))$.

(2) Shortly after the first version of this paper appeared in arXiv, Burban and Schiffmann proved in [4] that the extended composition algebra $\mathbf{C}(\mathrm{Coh}(\mathbb{X})) \otimes \mathbf{T}$ is a topological sub-bialgebra of $\mathbf{H}(\mathrm{Coh}(\mathbb{X})) \otimes \mathbf{T}$. Thus we can construct the reduced Drinfeld double of the algebra $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$. Now it is clear that our double composition algebra $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$ coincides with the reduced Drinfeld double of $\mathbf{C}(\mathrm{Coh}(\mathbb{X}))$. However, we do not need this result throughout the paper.

(3) We expect that our homomorphism Ξ is injective, namely $\Xi : U_v(\mathcal{L}\mathfrak{g}) \simeq \mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$, at least in the case that $\mathfrak{g}$ is of finite or affine type. Note that it was shown in [4] that the two algebras $U_v(\mathcal{L}\mathfrak{g})$ and $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$ are isomorphic in the finite type case, where the isomorphism is induced by a derived equivalence functor of the category $\mathrm{Coh}(\mathbb{X})$ and representations of a certain tame quiver. But it is difficult to understand Drinfeld's presentation by such isomorphisms. In particular, even if our Ξ is an isomorphism, it cannot be the isomorphism induced by a derived equivalence functor for most cases. We will give a more detailed explanation in Section 9.

6. Relations in the subalgebras isomorphic to $U_v(\widehat{\mathfrak{sl}}_{p_i})$

6.1. Relations in each tube $\mathscr{T}_{\lambda_i}$

Recall the star-shaped graph Γ in 5.5. We can see that for any fixed $i \in \{1, 2, \ldots, n\}$, the full subgraph consisting of vertices $\{[i, j] | 1 \le j \le p_i - 1\}$ is a Dynkin diagram of type A_{p_i-1}. Thus the relations to be satisfied by the elements $x^{\pm}_{[i,j],k}, h_{[i,j],r}$ for all $1 \le j \le p_i - 1$, $k \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\}$ are actually the defining relations of $U_v(\widehat{\mathfrak{sl}}_{p_i})$. We will prove them in this section.

Note that by definition the elements $x^+_{[i,j],k}, x^-_{[i,j],l}, h_{[i,j],r}$ for $1 \le j \le p_i - 1$, $k \in \mathbb{N}$, $l, r \in \mathbb{N}^*$ are all in the subalgebra $\mathbf{H}(\mathscr{T}_{\lambda_i})$, which is isomorphic to $\mathbf{H}(C_{p_i})$. Thus we can use the method developed by Hubery in [13], where he explicitly wrote down the elements in $\mathbf{H}(C_m)$ satisfying Drinfeld relations of $U_v^+(\widehat{\mathfrak{sl}}_m)$. Then by the isomorphism Θ_{λ_i}, we can transfer the result to $\mathbf{H}(\mathscr{T}_{\lambda_i}) \subset \mathbf{H}(\mathrm{Coh}(\mathbb{X}))$. Namely we have the following.

Proposition 6.1 (*[13]*). *For any fixed $i \in \{1, 2, \ldots, n\}$, the elements $x^+_{[i,j],k}, x^-_{[i,j],l}, h_{[i,j],r}$ for $1 \le j \le p_i - 1, k \in \mathbb{N}, l, r \in \mathbb{N}^*$ satisfy the Drinfeld relations of $U_v^+(\widehat{\mathfrak{sl}}_{p_i})$.*

This result can be easily extended to $U_v(\widehat{\mathfrak{sl}}_{p_i})$.

Corollary 6.2. *For any fixed $i \in \{1, 2, \ldots, n\}$, the elements $x^{\pm}_{[i,j],k}, h_{[i,j],r}$ for all $1 \le j \le p_i - 1, k \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\}$ satisfy the Drinfeld relations for $U_v(\widehat{\mathfrak{sl}}_{p_i})$.*

The proof will be given in 6.3.

6.2. Beck's isomorphism for $U_v(\widehat{\mathfrak{sl}}_m)$

In this subsection we briefly recall Beck's isomorphism in [2].

Let W be the Weyl group of $\mathfrak{sl}_m$. The simple reflections $s_1, \ldots, s_{m-1}$ generate W. Let P be the weight lattice and Q be the root lattice. The fundamental weights are denoted by $\omega_1, \ldots, \omega_{m-1}$.

The *extended affine Weyl group* is defined to be the semi-direct product $\widetilde{W} = P \rtimes W$, where $(x, \omega)(x', \omega') = (x + \omega(x'), \omega\omega')$. And the *affine Weyl group* associated to $\widehat{\mathfrak{sl}}_m$ is the subgroup $\widehat{W} = Q \rtimes W$. We have the decomposition $\widetilde{W} = \widehat{W} \rtimes (\mathbb{Z}/m\mathbb{Z})$, where the cyclic group $\mathbb{Z}/m\mathbb{Z}$ has a generator $\tau = (\omega_1, s_1 s_2 \cdots s_{m-1})$.

Set $s_m := (\omega_1 + \omega_{m-1}, s_1 s_2 \cdots s_{m-1} \cdots s_2 s_1)$. Then $\{s_1, s_2, \ldots, s_m\}$ is a set of generators of the affine Weyl group $\widehat{W}$. We can extend the length function on $\widehat{W}$ to $\widetilde{W}$ by setting $l(\tau) = 0$.

Note that the fundamental weights, considered as elements in $\widetilde{W}$, have the following reduced expressions in terms of the generators s_i and τ:

$$\omega_i = \tau^i (s_{m-i} \cdots s_{m-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i).$$

The braid group associated to $\widetilde{W}$ is the group with generators T_ω ($\omega \in \widetilde{W}$) the relations $T_\omega T'_\omega = T_{\omega\omega'}$ if $l(\omega) + l(\omega') = l(\omega\omega')$. Following Lusztig, it acts on $U_v(\widehat{sl}_m)$ (see 3.2) via the following rules:

$$T_i(E_i) = -F_i K_i, \qquad T_i(F_i) = -K_i^{-1} E_i, \qquad T_i(K_\alpha) = K_{s_i(\alpha)},$$
$$T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)}, \quad \text{for } i \neq j,$$
$$T_i(F_j) = \sum_{r+s=-c_{ij}} (-1)^r v^r F_i^{(r)} F_j F_i^{(s)}, \quad \text{for } i \neq j$$
$$T_\tau(K_i) = K_{i+1}, \qquad T_\tau(E_i) = E_{i+1}, \qquad T_\tau(F_i) = F_{i+1}$$

where $E_i^{(r)} = E_i^r/[r]!$, $F_i^{(r)} = F_i^r/[r]!$ and $(c_{ij})_{1 \le i,j \le m}$ is the Cartan matrix associated to $\widehat{sl}_m$.

For $1 \le i \le m-1$ and $j \in \mathbb{Z}$, let

$$x_{i,j}^- = (-1)^{ij} v^{mj} T_{\omega_i}^j (F_i), \qquad x_{i,j}^+ = (-1)^{ij} v^{mj} T_{\omega_i}^{-j} (E_i).$$

For $1 \le i \le m-1,\ k > 0$, define $h_{i,k}$ via the following generating functions

$$K_i \exp((v - v^{-1}) \Sigma_{k>0} h_{i,k} u^k) = \Sigma_{l \ge 0} \psi_{i,l} u^l,$$

where $\psi_{i,l} = (v - v^{-1})[E_i, T_{\omega_i}^l (F_i)]$ for $l > 0$ and $\psi_{i,0} = K_i$.

Similarly, define $h_{i,-k}$ via

$$K_i^{-1} \exp((v^{-1} - v) \Sigma_{k>0} h_{i,-k} u^k) = \Sigma_{l \ge 0} \psi_{i,-l} u^l$$

where $\varphi_{i,-l} = (v - v^{-1})[F_i, T_{\omega_i}^l (E_i)]$ for $l > 0$ and $\varphi_{i,0} = K_i^{-1}$.

The following is now well-known; see [2] for the proof.

Theorem 6.3. *$U_v(\widehat{sl}_m)$ is generated by the elements $x_{i,j}^\pm, h_{i,k}, K_i^{\pm 1}$, where $1 \le i \le m-1$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z} \setminus \{0\}$. The defining relations are Drinfeld relations for $U_v(\mathcal{L}\mathfrak{sl}_m)$. Thus the Drinfeld–Jimbo presentation of $U_v(\widehat{sl}_m)$ is isomorphic to the Drinfeld presentation of $U_v(\mathcal{L}\mathfrak{sl}_m)$.*

6.3. *The proof of Proposition 6.1 and Corollary 6.2*

Proposition 6.1 is a result of Hubery [13]. Corollary 6.2 follows easily by a similar method.

Now we briefly recall the arguments in [13]. Let C_m be the cyclic quiver with m vertices and consider the Hall algebra $\mathbf{H}(C_m)$. The composition algebra $\mathbf{C}(C_m)$ is the subalgebra of $\mathbf{H}(C_m)$ generated by u_{S_i} for $1 \le i \le m$. We know that the composition subalgebra $\mathbf{C}(C_m)$ is isomorphic to $U_v^+(\widehat{\mathfrak{sl}}_m)$ (Theorem 2.2) and the reduced Drinfeld double $\mathbf{DC}(C_m)$ is isomorphic to $U_v(\widehat{\mathfrak{sl}}_m)$ (Theorem 2.3). And the isomorphism is given by

$$u_{S_i} \mapsto E_i, \qquad -vu_{S_i}^- \mapsto F_i.$$

We also know that $U_v(\widehat{\mathfrak{sl}}_m)$ is isomorphic to $U_v(\mathcal{L}\mathfrak{sl}_m)$. Let

$$\Upsilon : \mathbf{DC}(C_m) \simeq U_v(\mathcal{L}\mathfrak{sl}_m)$$

be the composition of two isomorphisms mentioned above.

Now if we can find the inverse images of the elements $x_{i,j}^{\pm}$, $h_{i,k}$ under Υ, they should certainly satisfy the Drinfeld relations.

By Theorem 6.3 we have

$$x_{i,1}^- = (-1)^i v^m T_{\omega_i}(F_i), \qquad x_{i,-1}^+ = (-1)^{-i} v^{-m} T_{\omega_i}(E_i).$$

Recall that $\omega_i = \tau^i (s_{n-i} \cdots s_{n-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i)$. By induction, we have the following result.

Lemma 6.4. *For $1 \le i \le m-1$, we have*

$$T_{\omega_i}(F_i) = -K_i[E_m, E_{m-1}, \ldots, E_{i+1}, E_1, E_2, \ldots, E_{i-1}]_{v^{-1}},$$
$$T_{\omega_i}(E_i) = -[F_{i-1}, \ldots, F_2, F_1, F_{i+1}, \ldots, F_{m-1}, F_m]_v K_i^{-1},$$

where $[a,b]_v = ab - vba$, and $[a,b,c]_v = [[a,b]_v, c]_v$.

We identify E_i (resp. F_i) with $u_{s_i}^+$ (resp. $-vu_{s_i}^-$). Further calculations yield

$$T_{\omega_i}(F_i) = -v^{-m+i+1} K_i [u_{s_m(m-i)}^+, u_{s_1}^+, u_{s_2}^+ \ldots, u_{s_{i-1}}^+]_{v^{-1}},$$
$$T_{\omega_i}(F_i)K_i^{-1} = (-1)^i v^{1-m-i} \sum_{M \in \mathcal{M}_{i+1,\delta-e_j}} (1-v^2)^{\dim \operatorname{End}(M)-1} u_M^+.$$

Therefore, we have

$$x_{i,0}^+ = u_{S_i}^+,$$
$$x_{i,1}^- = v^{1-i} \sum_{M \in \mathcal{M}_{i+1,\delta-e_i}} (1-v^2)^{\dim \operatorname{End}(M)-1} u_M^+ K_i.$$

In a similar way, we have

$$x_{i,0}^- = u_{S_i}^-,$$
$$x_{i,-1}^+ = -v^{-1} v^{1-i} \sum_{M \in \mathcal{M}_{i+1,\delta-e_i}} (1-v^2)^{\dim \operatorname{End}(M)-1} u_M^- K_i^-.$$

The inverse image of the elements $h_{i,k}$ in the Hall algebra can be found by induction using the fact that $\pi_{n,r}$ (see 5.4) is central and primitive in the Hall algebra. And the result is

$$h_{i,k} = \pi_{i+1,k} - (v^k + v^{-k})\pi_{i,k} + \pi_{i-1,k}, \quad \text{for } k > 0.$$

Again by a similar method we have

$$h_{i,-k} = -(\pi^-_{i+1,k} - (v^k + v^{-k})\pi^-_{i,k} + \pi^-_{i-1,k}), \quad \text{for } k > 0.$$

Finally, the elements $x^+_{i,j}$ $(j \neq 0, -1)$ and $x^-_{i,j}$ $(j \neq 0, 1)$ are determined by the relation 3.4 (4).

7. Relations in $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$

In this section we focus on the elements $x^+_{[i,j],k}, x^-_{[i,j],k+1}, h_{[i,j],l}, x^+_{*,r}$, and $h_{*,t}$ where $1 \le i \le n$, $1 \le j \le p_i - 1$, $k \ge 0$, $l, t \in \mathbb{N}$ and $r \in \mathbb{Z}$. These are the elements belonging to the positive Hall algebra $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$.

7.1. *The known relations*

First, for reader's convenience, we list the relations which have already been proved in [23].

(a) For $1 \le i \le n, r,\ r_1 \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$[T_{*,r}, u_{\mathcal{O}(k\vec{c})}] = \frac{[2r]}{r} u_{\mathcal{O}((k+r)\vec{c})},$$
$$[T_{*,r}, x^+_{[i,1],r_1}] = -\frac{[r]}{r} x^+_{[i,1],r_1+r}.$$

(b) For $t_1, t_2 \in \mathbb{Z}$,

$$u_{\mathcal{O}((t_1+1)\vec{c})}u_{\mathcal{O}(t_2\vec{c})} - v^2 u_{\mathcal{O}(t_2\vec{c})}u_{\mathcal{O}((t_1+1)\vec{c})} = v^2 u_{\mathcal{O}(t_1\vec{c})}u_{\mathcal{O}((t_2+1)\vec{c})} - u_{\mathcal{O}((t_2+1)\vec{c})}u_{\mathcal{O}(t_1\vec{c})}.$$

(c) For $1 \le i \le n, r, r_1, r_2 \in \mathbb{N}$ and $t, t_1, t_2 \in \mathbb{Z}$,

$$\mathrm{Sym}_{r_1,r_2}\{x^+_{[i,1],r_1}x^+_{[i,1],r_2}u_{\mathcal{O}(t\vec{c})} - [2]x^+_{[i,1],r_1}u_{\mathcal{O}(t\vec{c})}x^+_{[i,1],r_2} + u_{\mathcal{O}(t\vec{c})}x^+_{[i,1],r_1}x^+_{[i,1],r_2}\} = 0,$$
$$\mathrm{Sym}_{t_1,t_2}\{u_{\mathcal{O}(t_1\vec{c})}u_{\mathcal{O}(t_2\vec{c})}x^+_{[i,1],r} - [2]u_{\mathcal{O}(t_1\vec{c})}x^+_{[i,1],r}u_{\mathcal{O}(t_2\vec{c})} + x^+_{[i,1],r}u_{\mathcal{O}(t_1\vec{c})}u_{\mathcal{O}(t_2\vec{c})}\} = 0.$$

(d) For $1 \le i \le n, r, r_1, r_2 \in \mathbb{N}$ and $t \in \mathbb{Z}$,

$$u_{\mathcal{O}((t+1)\vec{c})}x^+_{[i,1],r} - v^{-1}x^+_{[i,1],r}u_{\mathcal{O}((t+1)\vec{c})} = v^{-1}u_{\mathcal{O}(t\vec{c})}x^+_{[i,1],r+1} - x^+_{[i,1],r+1}u_{\mathcal{O}(t\vec{c})},$$
$$x^+_{[i,1],r_1+1}x^+_{[i,1],r_2} - v^2 x^+_{[i,1],r_2}x^+_{[i,1],r_1+1} = v^2 x^+_{[i,1],r_2+1}x^+_{[i,1],r_1} - x^+_{[i,1],r_1}x^+_{[i,1],r_2+1}.$$

For any two elements arising from different tubes, we know that they commute with each other as there are no non-trivial extensions between torsion sheaves belonging to different tubes.

Moreover, from the last section we know that the elements in each $\mathbf{H}(\mathscr{T}_{\lambda_i})$ $(1 \le i \le n)$ satisfy the required relations.

7.2. *The remaining relations*

Lemma 7.1. (1) $[h_{[i,j],k}, h_{[i,l],m}] = 0$, *for any* $1 \le i \le n$, $1 \le j, l \le p_i - 1$, $k, m \in \mathbb{N}$.
(2) $[h_{[i,j],l}, h_{*,k}] = 0$, *for any* $1 \le i \le n$, $1 \le j \le p_i - 1$, $l, k \in \mathbb{N}$.

Proof. We deduce $[\pi^i_{j,k}, \pi^i_{l,m}] = 0$ by the embedding of algebras

$$\mathbf{H}_v(C_1) \hookrightarrow \mathbf{H}_v(C_2) \cdots \hookrightarrow \mathbf{H}_v(C_{p_i})$$

and the fact that $\pi^i_{j,k}$ is in the center of $\mathbf{H}_v(C_j)$ (see the remark in 5.4). The first equation follows.

For the second one, we have $[h_{[i,j],l}, h_{*,k}] = [h_{[i,j],l}, \pi^i_{1,k}] = 0$. □

Lemma 7.2. $[h_{*,l}, x^+_{[i,j],k}] = 0$, *for any* $1 \le i \le n, 2 \le j \le p_i - 1,\ l, k \in \mathbb{N}$.

Proof. First we prove the case $k = 0$, we know that $x^+_{[i,j],0} = u_{S^i_j}$. Assume that $u_{\mathscr{G}}$ is a term with non-zero coefficient in the expression of $h_{*,l} = T_l$, where $\mathscr{G}$ is a torsion sheaf. We only need to consider the direct summand $\mathscr{F}$ of $\mathscr{G}$ belonging to $\mathscr{T}_{\lambda_i}$. We have $[\mathscr{F}] = l\delta$ and $\mathrm{top}(\mathscr{F}) = S^i_0 \oplus S^i_0 \cdots \oplus S^i_0$, $\mathrm{soc}(\mathscr{F}) = S^i_1 \oplus S^i_1 \cdots \oplus S^i_1$. It is clear that $[u_{\mathscr{F}}, u_{S^i_j}] = 0$. Thus we get $[h_{*,l}, x^+_{[i,j],0}] = 0$.

For the general case, one just need to apply $\mathrm{ad}(h_{[i,j],k})$ to the above formula. □

Lemma 7.3. $[h_{*,l}, x^-_{[i,1],k}] = \frac{[l]}{l} x^-_{[i,1],1+k}$, *for any* $l \in \mathbb{N}$, $1 \le i \le n$ *and* $k \ge 1$.

Proof. Again we first consider the simplest case, namely the case $k = 1$.

We know that $x^-_{[i,1],1} = u_{S^i_0(p_i-1)} K_{[i,1]}$. It is not difficult to see that $[\pi^i_{2,k}, x^-_{[i,1],1}] = 0$. Hence we have

$$[h_{*,l}, x^-_{[i,1],1}] = [\mathbf{h}_{l,\lambda_i}, x^-_{[i,1],1}] = \frac{1}{-(v^l + v^{-l})}[h_{[i,1],l}, x^-_{[i,1],1}] = \frac{[l]}{l} x^-_{[i,1],l+1}.$$

For $k > 1$, we have

$$\begin{aligned}[h_{*,l}, x^-_{[i,1],k}] &= \frac{-(k-1)}{[2(k-1)]}[h_{*,l}, [h_{[i,1],k-1}, x^-_{[i,1],1}]] \\ &= \frac{-(k-1)}{[2(k-1)]}[h_{[i,1],k-1}, [h_{*,l}, x^-_{[i,1],1}]] \\ &= \frac{-(k-1)}{[2(k-1)]}\frac{[l]}{l}[h_{[i,1],k-1}, x^-_{[i,1],l+1}] = \frac{[l]}{l} x^-_{[i,1],k+l}.\end{aligned}$$ □

Lemma 7.4. $[h_{*,l}, x^-_{[i,j],k}] = 0$, *for any* $1 \le i \le n, 2 \le j \le p_i - 1,\ l \in \mathbb{N}$ *and* $k \ge 1$.

Proof. The case $k = 1$ can be proved in the same way as Lemma 7.2. For $k > 1$ we just apply $\mathrm{ad}(h_{[i,j],k})$. □

Lemma 7.5. $[x^+_{*,k}, x^-_{[i,j],1}] = 0$, *for any* $k \in \mathbb{Z}$, $1 \le i \le n,\ 1 \le j \le p_i - 1$.

Proof. We first consider the case $j = 1$. Note that $x^-_{[i,1],1} = u_{S^i_0(p_i-1)} K_{[i,1]}$, so we have

$$\begin{aligned}[x^+_{*,k}, u_{S^i_0(p_i-1)} K_{[i,1]}] &= x^+_{*,k} u_{S^i_0(p_i-1)} K_{[i,1]} - u_{S^i_0(p_i-1)} K_{[i,1]} x^+_{*,r} \\ &= (x^+_{*,k} u_{S^i_0(p_i-1)} - v^{-1} u_{S^i_0(p_i-1)} x^+_{*,r}) K_{[i,1]}.\end{aligned}$$

We know that

$$\begin{aligned}&\mathrm{Hom}(\mathscr{O}(k\vec{c}), S^i_0(p_i - 1)) = k, \\ &\mathrm{Ext}^1(\mathscr{O}(k\vec{c}), S^i_0(p_i - 1)) = \mathrm{Hom}(S^i_0(p_i - 1), \mathscr{O}(k\vec{c})) = 0.\end{aligned}$$

Thus we have

$$x^+_{*,k} u_{S_0^i(p_i-1)} = v u_{\mathscr{O}(k\vec{c}) \oplus S_0^i(p_i-1)}, \qquad u_{S_0^i(p_i-1)} x^+_{*,k} = v^2 u_{\mathscr{O}(k\vec{c}) \oplus S_0^i(p_i-1)}.$$

Then it follows that $[x^+_{*,k}, u_{S_0^i(p_i-1)} K_{[i,1]}] = 0$.

Now assume $j \geq 2$, in this case we know that

$$x^-_{[i,j],1} = (-v)^{j+1} K_{[i,j]} [u_{S_0^i(p_i-j)}, u_{S_1^i}, \ldots, u_{S_{j-1}^i}]_{v^{-1}}.$$

It suffices to prove $[x^+_{*,k}, [u_{S_0^i(p_i-j)}, u_{S_1^i}]_{v^{-1}}] = 0$, which can be deduced using the following identities:

$$u_{S_0^i(p_i-j)} u_{\mathscr{O}(k\vec{c})} = v u_{\mathscr{O}(k\vec{c})} u_{S_0^i(p_i-j)},$$
$$u_{S_1^i} u_{\mathscr{O}(k\vec{c})} = v^{-1}(u_{\mathscr{O}(k\vec{c})} u_{S_1^i} + u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}),$$
$$u_{S_0^i(p_i-j)} u_{\mathscr{O}(k\vec{c}+\vec{x}_i)} = u_{\mathscr{O}(k\vec{c}+\vec{x}_i)} u_{S_0^i(p_i-j)}. \qquad \square$$

Lemma 7.6. $[h_{[i,1],l}, x^+_{*,k}] = \frac{-[l]}{l} x^+_{*,k+l}$, *for any* $1 \leq i \leq n$, $l \in \mathbb{N}$ *and* $k \in \mathbb{Z}$.

Proof. For fixed i, we know that the following relation holds (see Corollary 6.2):

$$[x^+_{[i,j],k}, x^-_{[i,j],l}] = \frac{\psi_{[i,j],k+l} - \varphi_{[i,j],k+l}}{v - v^{-1}}.$$

Now we set

$$\xi_r^i = [x^+_{[i,1],r-1}, x^-_{[i,1],1}] K^{-1}_{[i,1]} = \frac{1}{v - v^{-1}} \psi_{[i,1],r} K^{-1}_{[i,1]}.$$

By the definition of $\psi_{[i,1],r}$ we have

$$r h_{[i,1],r} = r\xi_r^i - \sum_{s=1}^{r-1} (v - v^{-1}) s h_{[i,1],s} \xi^i_{r-s}. \tag{$*1$}$$

The definition of ξ_r^i yields

$$\xi_r^i = x^+_{[i,1],r-1} u_{S_0^i(p_i-1)} - v^2 u_{S_0^i(p_i-1)} x^+_{[i,1],r-1}.$$

Thus we have

$$\begin{aligned} \xi_r^i u_{\mathscr{O}(k\vec{c})} &= x^+_{[i,1],r-1} u_{S_0^i(p_i-1)} u_{\mathscr{O}(k\vec{c})} - v^2 u_{S_0^i(p_i-1)} x^+_{[i,1],r-1} u_{\mathscr{O}(k\vec{c})} \\ &= v x^+_{[i,1],r-1} u_{\mathscr{O}(k\vec{c})} u_{S_0^i(p_i-1)} - v^2 u_{S_0^i(p_i-1)} x^+_{[i,1],r-1} u_{\mathscr{O}(k\vec{c})}. \end{aligned}$$

We claim that the following identity holds:

$$\begin{aligned} \xi_r^i u_{\mathscr{O}(k\vec{c})} &= u_{\mathscr{O}(k\vec{c})} \xi_r^i + (v^{-1} - v) u_{\mathscr{O}((k+1)\vec{c})} \xi^i_{r-1} + v^{-1}(v^{-1} - v) u_{\mathscr{O}((k+2)\vec{c})} \xi^i_{r-2} \\ &\quad + \cdots + v^{-(r-2)}(v^{-1} - v) u_{\mathscr{O}((k+r-1)\vec{c})} \xi^i_1 - v^{-(r-1)} u_{\mathscr{O}((k+r)\vec{c})}. \end{aligned} \tag{$*2$}$$

Now we prove the claim by induction on r.

When $r = 1$, we have $\xi_1^i = u_{S_1^i} u_{S_0^i(p_i-1)} - v^2 u_{S_0^i(p_i-1)} u_{S_1^i}$.

Also we have

$$\begin{aligned}
&u_{S_1^i}u_{\mathscr{O}(k\vec{c})} = v^{-1}(u_{\mathscr{O}(k\vec{c})\oplus S_1^i} + u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}),\\
&u_{\mathscr{O}(k\vec{c})}u_{S_1^i} = u_{\mathscr{O}(k\vec{c})\oplus S_1^i},\\
&u_{S_0^i(p_i-1)}u_{\mathscr{O}(k\vec{c}+\vec{x}_i)} = v^{-1}(u_{\mathscr{O}(k\vec{c}+\vec{x}_i)\oplus S_0^i(p_i-1)} + u_{\mathscr{O}((k+1)\vec{c})}),\\
&u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}u_{S_0^i(p_i-1)} = u_{\mathscr{O}(k\vec{c}+\vec{x}_i)\oplus S_0^i(p_i-1)}.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
\xi_1^i u_{\mathscr{O}(k\vec{c})} &= vu_{S_1^i}u_{\mathscr{O}(k\vec{c})}u_{S_0^i(p_i-1)} - v^2 u_{S_0^i(p_i-1)}u_{S_1^i}u_{\mathscr{O}(k\vec{c})}\\
&= u_{\mathscr{O}(k\vec{c})}u_{S_1^i}u_{S_0^i(p_i-1)} + u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}u_{S_0^i(p_i-1)}\\
&\quad - vu_{S_0^i(p_i-1)}u_{\mathscr{O}(k\vec{c})}u_{S_1^i} - vu_{S_0^i(p_i-1)}u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}\\
&= u_{\mathscr{O}(k\vec{c})}u_{S_1^i}u_{S_0^i(p_i-1)} + u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}u_{S_0^i(p_i-1)}\\
&\quad - v^2 u_{\mathscr{O}(k\vec{c})}u_{S_0^i(p_i-1)}u_{S_1^i} - u_{\mathscr{O}(k\vec{c}+\vec{x}_i)}u_{S_0^i(p_i-1)} - u_{\mathscr{O}((k+1)\vec{c})}\\
&= u_{\mathscr{O}(k\vec{c})}\xi_1^i - u_{\mathscr{O}((k+1)\vec{c})}.
\end{aligned}$$

Then we assume that (∗2) holds for $r-1$. By [23] 4.13, we have

$$\begin{aligned}
\xi_r^i u_{\mathscr{O}(k\vec{c})} &= (u_{\mathscr{O}(k\vec{c})}\xi_{r-1}^i + \xi_{r-2}^i u_{\mathscr{O}((k+1)\vec{c})} - vu_{\mathscr{O}((k+1)\vec{c})}\xi_{r-2}^i)u_{S_0^i}\\
&\quad - vu_{S_0^i}(u_{\mathscr{O}(k\vec{c})}\xi_{r-1}^i + \xi_{r-2}^i u_{\mathscr{O}((k+1)\vec{c})} - vu_{\mathscr{O}((k+1)\vec{c})}\xi_{r-2}^i)\\
&= u_{\mathscr{O}(k\vec{c})}\xi_r^i - vu_{\mathscr{O}((k+1)\vec{c})}\xi_{r-1}^i + v^{-1}\xi_{r-1}^i u_{\mathscr{O}((k+1)\vec{c})}.
\end{aligned}$$

This completes the proof of (∗2). And the lemma is a consequence of (∗2) and (∗1). □

Lemma 7.7. (1) $[x^+_{[i,j],l}, x^+_{*,k}] = 0$, *for* $1 \le i \le n$, $2 \le j \le p_i - 1$, $l \ge 0$ *and* $k \in \mathbb{Z}$.
(2) $[h_{[i,j],l}, x^+_{*,k}] = 0$, *for* $1 \le i \le n$, $2 \le j \le p_i - 1$, $l \in \mathbb{N}$ *and* $k \in \mathbb{Z}$.

Proof. First we observe that (2) is a consequence of (1), since $[h_{[i,j],l}, x^+_{*,k}] = 0$ if and only if $[[x^+_{[i,j],l-1}, x^-_{[i,j],1}], x^+_{*,k}] = 0$ and we know that $[x^-_{[i,j],1}, x^+_{*,k}] = 0$ by Lemma 7.5.

Note that $[x^+_{[i,j],0}, x^+_{*,k}] = 0$, because there is no non-trivial extension between $\mathscr{O}(k\vec{c})$ and S^i_j for $j \ge 2$.

Now we argue by induction on j. When $j = 2$, using Lemma 7.6 we have

$$\begin{aligned}
[x^+_{[i,2],l}, x^+_{*,k}] &= \frac{-l}{[l]}[[h_{[i,1],l}, u_{S_2^i}], x^+_{*,k}]\\
&= \frac{-l}{[l]}[[h_{[i,1],l}, x^+_{*,k}], u_{S_2^i}] = 0.
\end{aligned}$$

Assume that for $1 < j < m$ the relation holds, we also have $[h_{[i,j],l}, x^+_{*,k}] = 0$ for $1 < j < m$. Hence

$$\begin{aligned}
[x^+_{[i,m],l}, x^+_{*,k}] &= \frac{-l}{[l]}[[h_{[i,m-1],l}, x^+_{[i,m],0}], x^+_{*,k}]\\
&= \frac{-l}{[l]}[[h_{[i,m-1],l}, x^+_{*,k}], x^+_{[i,m],0}] = 0. \quad \square
\end{aligned}$$

Lemma 7.8. $[x^-_{[i,j],l}, x^+_{*,k}] = 0$, *for any* $1 \le i \le n$, $1 \le j \le p_i - 1$, $l \ge 1$ *and* $k \in \mathbb{Z}$.

Proof. For the case $l = 1$ this is just Lemma 7.5. For $l > 1$ we apply $\mathrm{ad}(h_{[i,j],r})$ and use Lemmas 7.6 and 7.7(2). □

8. Relations in DH(Coh($\mathbb{X}$))

We prove all the Drinfeld relations in this section. In the last section we have proved the relations in $\mathbf{H}(\mathrm{Coh}(\mathbb{X}))$. In the same way we can prove the relations in $\mathbf{H}^-(\mathrm{Coh}(\mathbb{X}))$. So we focus on the relations involving both positive and negative elements.

8.1

We first investigate the comultiplication of $h_{*,r}$ and $u_{\mathscr{O}(k\vec{c})}$ ($r \ge 0, k \in \mathbb{Z}$) in detail, which is crucial for our calculations. By definition we have

$$\Delta(h_{*,r}) = h_{*,r} \otimes 1 + K_{r\delta} \otimes h_{*,r} + \sum_{0<[A],[B]<r\delta} f(A,B) u_A K_{[B]} \otimes u_B,$$

$$\Delta(u_{\mathscr{O}(k\vec{c})}) = u_{\mathscr{O}(k\vec{c})} \otimes 1 + \sum_{r=0}^{\infty} \theta_{*,r} K_{\alpha_*+(k-r)\delta} \otimes u_{\mathscr{O}((k-r)\vec{c})} + \sum_{\vec{x}\in L(\mathbf{p})_+, \vec{x}\ne t\delta,\ \forall t\in\mathbb{N}} \theta_{\vec{x}} K_{[\mathscr{O}((k-r)\vec{c}-\vec{x})]} \otimes u_{\mathscr{O}((k-r)\vec{c}-\vec{x})},$$

where $\{\theta_{*,r}\}_{r\ge1}$ is defined by the following generating series

$$\sum_{k\ge0} \theta_{*,r} u^k = \exp\left((v - v^{-1}) \sum_{k=1}^{\infty} h_{*,k} u^k\right).$$

In the following, for simplicity we will call $\sum_{0<[A],[B]<r\delta} f(A,B) u_A K_{[B]} \otimes u_B$ the remaining terms in $\Delta(h_{*,r})$ and $\sum_{\vec{x}\in L(\mathbf{p})_+, \vec{x}\ne t\delta,\ \forall t\in\mathbb{N}} \theta_{\vec{x}} K_{[\mathscr{O}((k-r)\vec{c}-\vec{x})]} \otimes u_{\mathscr{O}((k-r)\vec{c}-\vec{x})}$ the remaining terms in $\Delta(u_{\mathscr{O}(k\vec{c})})$.

8.2

In this subsection we prove the relation $[h_{s,r}, h_{t,m}] = 0$ for $1 \le s, t \le n$ and $rm < 0$. We assume that $m > 0$ and $r < 0$.

Lemma 8.1. *For fixed* i, *we have*

$$[\pi^{+i}_{l_1,k_1}, \pi^{-i}_{l_2,k_2}] = 0,$$

where $1 \le l_1, l_2 \le p_i$ *and* $k_1, k_2 \in \mathbb{N}$.

Proof. For simplicity, we will omit i and write p for p_i. Also we write $h_{j,k}$ for $h_{[i,j],k}$, for any $1 \le j \le p_i - 1$.

For $1 \le k \le p - 1$, we have the relation

$$[\pi^+_{p,k_1}, h_{p-k,-k_2}] = 0.$$

This implies

$$[\pi^+_{p,k_1}, \pi^-_{p-k+1,k_2} - (v^{k_2} + v^{-k_2})\pi^-_{p-k,k_2} + \pi^-_{p-k-1,k_2}] = 0.$$

Note that $[\pi^+_{p,k_1}, \pi^-_{p,k_2}] = 0$. Hence we get a system of homogeneous linear equations $AX = 0$, where

$$A = \begin{pmatrix} a & 1 & 0 & 0 & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & 0 \\ 0 & 1 & a & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a & 1 \\ 0 & 0 & \cdots & 0 & 1 & a \end{pmatrix}$$

$$X = (b_{p,p-1}, b_{p,p-2}, \dots, b_{p,2}, b_{p,1})^t,$$

and $a = -(v^{k_2} + v^{-k_2})$, $b_{p,l} = [\pi^+_{p,k_1}, \pi^-_{l,k_2}]$.

It is clear that A is a nonsingular matrix; thus $[\pi^+_{p,k_1}, \pi^-_{l,k_2}] = 0$ holds for any $1 \le l \le p-1$.

Similarly, the relations $[h_{l_1,k_1}, h_{l_2,-k_2}] = 0$ $(1 \le l_1 \le p-1, 1 \le l_2 \le p-1)$ induce a system of homogeneous linear equations with $(p-1) \times (p-1)$ variables $[\pi^+_{l_1,k_1}, \pi^-_{l_2,k_2}]$ and the coefficient matrix is also nonsingular. Therefore, for any $1 \le l_1 \le p-1$, $1 \le l_2 \le p-1$ and $k_1, k_2 \in \mathbb{N}$, we have $[\pi^+_{l_1,k_1}, \pi^-_{l_2,k_2}] = 0$. □

Now we can deduce that

$$[h_{*,r}, h_{*,m}] = -\sum_{i=1}^{n} [\pi^{-i}_{1,-r}, \pi^{+i}_{1,m}] = 0,$$

$$[h_{*,r}, h_{[i,j],m}] = -[\pi^{-i}_{1,-r}, \pi^{+i}_{j+1,m} - (v^m + v^{-m})\pi^{+i}_{j,m} + \pi^{+i}_{j-1,m}] = 0.$$

8.3

In this subsection we deal with the following relation for the remaining cases

$$[h_{s,r}, x^{\pm}_{t,m}] = \pm\frac{1}{r}[la_{st}]x^{\pm}_{t,r+m}.$$

Lemma 8.2. $[h_{*,-l}, x^+_{*,k}] = \frac{1}{l}[2l]x^+_{*,-l+k}$, *for any* $l > 0$ *and* $k \in \mathbb{Z}$.

Proof. Recall that

$$\Delta(u_{\mathcal{O}(k\vec{c})}) = u_{\mathcal{O}(k\vec{c})} \otimes 1 + \sum_{r=0}^{\infty} \theta_{*,r} K_{\alpha_* + (k-r)\delta} \otimes u_{\mathcal{O}((k-r)\vec{c})} + \text{remaining terms}.$$

We prove that if A is a quotient of $\mathcal{O}(k\vec{c})$ as well as a subsheaf of a torsion sheaf occurring with nonzero coefficient in $h_{*,l}$, then $[A] = s\delta$ for some $s \in \mathbb{N}$.

Otherwise, assume that $[A]$ is not a multiple of δ, then there exists an exact sequence

$$0 \to \mathcal{O}(\vec{x}) \to \mathcal{O}(k\vec{c}) \to A \to 0,$$

where $\vec{x}$ is not a multiple of $\vec{c}$.

This implies

$$\text{Ext}^1(S_1^m, \mathcal{O}(\vec{x})) = \text{Hom}(\mathcal{O}(\vec{x}), S_0^m) = 0, \quad \text{for some } 1 \le m \le n.$$

There is always an injective map from S_1^m to A. Thus we have the following commutative diagram where the right square is a pull-back diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}(\vec{x}) & \longrightarrow & \mathscr{O}(\vec{x}) \oplus S_1^m & \xrightarrow{p_2} & S_1^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f & & \downarrow i & & \\
0 & \longrightarrow & \mathscr{O}(\vec{x}) & \longrightarrow & \mathscr{O}(k\vec{c}) & \xrightarrow{g} & A & \longrightarrow & 0
\end{array}$$

Since there is no non-zero morphism from S_1^m to $\mathscr{O}(k\vec{c})$, we have $gf = 0$. But ip_2 is clear not zero, which is a contradiction.

This means we do not need to consider the remaining terms in $\Delta(u_{\mathscr{O}(k\vec{c})})$.

Now using the definition of the Drinfeld double and note that $(\theta_{*,r}, h_{*,r}) = \frac{[2r]}{r}$, the lemma follows. $\square$

Lemma 8.3. (1) $[h_{*,l}, x^+_{[i,1],k}] = \frac{1}{l}[-l]x^+_{[i,1],k+l}$, *for* $l < 0$, $1 \le i \le n$ *and* $k \ge 0$.
(2) $[h_{*,l}, x^-_{[i,1],k}] = -\frac{1}{l}[-l]x^-_{[i,1],k+l}$, *for* $l < 0$, $1 \le i \le n$ *and* $k \ge 1$.

Proof. (1) We have shown that $[\pi^{+i}_{2,-l}, x^-_{[i,1],1}] = 0$. Similarly we have

$$[\pi^{-i}_{2,-l}, x^+_{[i,1],-1}] = 0.$$

Applying $\mathrm{ad}(h_{[i,1],k+1})$ to the above formula, we have $[\pi^{-i}_{2,-l}, x^+_{[i,1],k}] = 0$.
So we can deduce that

$$[h_{*,l}, x^+_{[i,1],k}] = [\pi^{-i}_{1,-l}, x^+_{[i,1],k}] = \frac{-1}{v^l + v^{-l}}[h_{[i,1],l}, x^+_{[i,1],k}] = \frac{1}{l}[-l]x^+_{[i,1],k+l}.$$

(2) We have shown that $[\pi^{+i}_{2,-l}, x^+_{[i,1],0}] = 0$. Similarly we have

$$[\pi^{-i}_{2,-l}, x^-_{[i,1],0}] = 0.$$

Applying $\mathrm{ad}(h_{[i,1],k})$ to the above formula, we have $[\pi^{-i}_{2,-l}, x^-_{[i,1],k}] = 0$.
Hence we have

$$[h_{*,l}, x^-_{[i,1],k}] = [\pi^{-i}_{1,-l}, x^-_{[i,1],k}] = \frac{-1}{v^l + v^{-l}}[h_{[i,1],l}, x^-_{[i,1],k}] = -\frac{1}{l}[-l]x^-_{[i,1],k+l}. \quad \square$$

Lemma 8.4. (1) $[h_{*,l}, x^+_{[i,m],k}] = 0$, *for* $1 \le i \le n$, $2 \le m \le p_i - 1$, $l < 0$ *and* $k \ge 0$.
(2) $[h_{*,l}, x^-_{[i,m],k}] = 0$, *for* $1 \le i \le n$, $2 \le m \le p_i - 1$, $l < 0$ *and* $k \ge 1$.

Proof. For the first equation, apply $\mathrm{ad}(h_{[i,m],k+1})$ to $[h_{*,l}, x^+_{[i,m],-1}] = 0$. And for the second one, just apply $\mathrm{ad}(h_{[i,m],k})$ to $[h_{*,l}, x^-_{[i,m],0}] = 0$. $\square$

Lemma 8.5. $[h_{[i,1],l}, x^+_{*,k}] = \frac{1}{l}[-l]x^+_{*,k+l}$, *for* $1 \le i \le n$, $l < 0$ *and* $k \in \mathbb{Z}$.

Proof. Set $\xi^i_l = [x^+_{[i,1],-l}, x^-_{[i,1],2l}]K_{[i,1]}$. We have shown that

$$[x^+_{[i,1],-l}, x^-_{[i,1],2l}] = \frac{\varphi_{[i,1],l}}{v - v^{-1}}.$$

By the definition of φ (see 3.4 (7)), we have for any $r < 0$,

$$-rh_{[i,1],r} = -r\xi_l^i + \sum_{s=1}^{-r-1}(v - v^{-1})sh_{[i,1],-s}\xi_{r+s}^i. \qquad (*3)$$

We claim that the following identity holds:

$$\begin{aligned}\xi_l^i x_{*,k}^+ &= x_{*,k}^+\xi_l^i + (v - v^{-1})x_{*,k-1}^+\xi_{l+1}^i + v(v - v^{-1})x_{*,k-2}^+\xi_{l+2}^i \\ &\quad + \cdots + v^{(-r-2)}(v - v^{-1})x_{*,k+l+1}^+\xi_{-1}^i - v^{(-r-1)}x_{*,k+l}^+. \qquad (*4)\end{aligned}$$

We prove the claim by induction. When $l = -1$, we have

$$\begin{aligned}\xi_{-1}^i x_{*,k}^+ &= [x_{[i,1],0}^+, x_{[i,1],-1}^-]K_{[i,1]}x_{*,k}^+ \\ &= v^{-1}[x_{[i,1],0}^+, x_{[i,1],-1}^-]x_{*,k}^+K_{[i,1]} \\ &= x_{*,k}^+\xi_{-1}^i - v^{-1}x_{*,k-1}^+\xi_0^i + v\xi_0^i x_{*,k-1}^+ \\ &= x_{*,k}^+\xi_{-1}^i + x_{*,k-1}^+.\end{aligned}$$

Now assume (∗4) holds for $l = -r + 1$. We will prove the case $l = -r$. By [23, 4.13] and $[x_{[i,1],2l}^-, x_{*,k}^+] = 0$, which will be proved in Lemma 8.8 independent of this lemma, we deduce that

$$\begin{aligned}\xi_l^i x_{*,k}^+ &= [x_{[i,1],-l}^+, x_{[i,1],2l}^-]K_{[i,1]}x_{*,k}^+ \\ &= v^{-1}[x_{[i,1],-l}^+, x_{[i,1],2l}^-]x_{*,k}^+K_{[i,1]} \\ &= v^{-1}(x_{[i,1],-l}^+x_{[i,1],2l}^-x_{*,k}^+ - x_{[i,1],2l}^-x_{[i,1],-l}^+x_{*,k}^+)K_{[i,1]} \\ &= v^{-1}(x_{[i,1],-l}^+x_{*,k}^+x_{[i,1],2l}^- - x_{[i,1],2l}^-x_{[i,1],-l}^+x_{*,k}^+)K_{[i,1]} \\ &= v^{-1}((vx_{[i,1],-l+1}^+x_{*,k-1}^+ + vx_{*,k}^+x_{[i,1],-l}^+ - x_{*,k-1}^+x_{[i,1],-l+1}^+)x_{[i,1],2l}^- \\ &\quad + x_{[i,1],2l}^-(vx_{[i,1],-l+1}^+x_{*,k-1}^+ + vx_{*,k}^+x_{[i,1],-l}^+ - x_{*,k-1}^+x_{[i,1],-l+1}^+))K_{[i,1]} \\ &= x_{*,k}^+\xi_l^i - v^{-1}x_{*,k-1}^+\xi_{l+1}^i + v\xi_{l+1}^i x_{*,k-1}^+.\end{aligned}$$

The lemma is now a consequence of (∗3) and (∗4). □

Lemma 8.6. $[h_{[i,m],l}, x_{*,k}^+] = 0$, *for* $1 \le i \le n$, $2 \le m \le p_i - 1$, $l < 0$ *and* $k \in \mathbb{Z}$.

Proof. When $m = 2$, we have proved that

$$[x_{[i,2],1}^-, x_{*,k}^+] = 0, \qquad [x_{[i,2],0}^+, x_{*,k}^+] = 0.$$

Applying $\mathrm{ad}(h_{[i,1],l-1})$ to the second formula, we get

$$[x_{[i,2],l-1}^+, x_{*,k}^+] = 0.$$

Then we have $[\varphi_{[i,2],l}, x_{*,k}^+] = 0$, which is the same as $[h_{[i,2],l}, x_{*,k}^+] = 0$.

Assume for any $m < p_i$ the relation $[x_{[i,m],l}^+, x_{*,k}^+] = 0$ holds. We shall prove the case $m + 1$. Note that the following still holds:

$$[x_{[i,m+1],1}^-, x_{*,k}^+] = 0, \qquad [x_{[i,m+1],0}^+, x_{*,k}^+] = 0.$$

Now applying $\mathrm{ad}(h_{[i,m],l-1})$ to the second formula, we get

$$[x^+_{[i,m+1],l-1}, x^+_{*,k}] = 0.$$

Then we deduce that $[\varphi_{[i,m+1],l}, x^+_{*,k}] = 0$, which is the same as $[h_{[i,m+1],l}, x^+_{*,k}] = 0$. □

8.4

Next we consider the relation

$$[x^+_{s,k}, x^-_{t,l}] = \delta_{st}\frac{\psi_{s,k+l} - \varphi_{s,k+l}}{v - v^{-1}}.$$

Lemma 8.7. *For any k, l such that $k + l \geq 0$,*

$$[x^+_{*,k}, x^-_{*,l}] = \frac{\psi_{*,k+l} - \varphi_{*,k+l}}{v - v^{-1}}.$$

Proof. Recall the comultiplication:

$$\Delta(u_{\mathscr{O}(k\vec{c})}) = u_{\mathscr{O}(k\vec{c})} \otimes 1 + \sum_{r=0}^{\infty}\theta_{*,r}K_{\alpha_*+(k-r)\delta} \otimes u_{\mathscr{O}((k-r)\vec{c})} + \text{remaining terms.}$$

$$\Delta(u_{\mathscr{O}(-l\vec{c})}) = u_{\mathscr{O}(-l\vec{c})} \otimes 1 + \sum_{r=0}^{\infty}\theta_{*,r}K_{\alpha_*+(-l-r)\delta} \otimes u_{\mathscr{O}((-l-r)\vec{c})} + \text{remaining terms.}$$

For any $A_1 = u_{B_1}K_{[C_1]} \otimes u_{C_1}$ (resp. $A_2 = u_{B_2}K_{[C_2]} \otimes u_{C_2}$) appearing in the remaining terms of $\Delta(u_{\mathscr{O}(k\vec{c})})$ (resp. $\Delta(u_{\mathscr{O}(-l\vec{c})})$), B_1 is a nonzero sheaf of finite length and C_2 is a nonzero line bundle. So they are not isomorphic to each other. And similarly, C_1 is a nonzero line bundle and B_2 is a nonzero sheaf of finite length. They are not isomorphic to each other. Thus we do not need to consider the remaining terms.

Then the lemma can be deduced by the definition of the Drinfeld double. □

Lemma 8.8. $[x^+_{*,k}, x^-_{[i,j],l}] = 0$, *for any $k \in \mathbb{Z}$, $1 \leq i \leq n$, $1 \leq j \leq p_i - 1$, $l \leq 0$.*

Proof. By Lemma 7.5, we have $[x^+_{*,k}, x^-_{[i,j],1}] = 0$.

For $j = 1$, we have

$$[x^+_{*,k}, x^-_{[i,1],l}] = 0,$$

by applying $\mathrm{ad}(h_{*,l-1})$ to $[x^+_{*,k}, x^-_{[i,1],1}] = 0$.

For $2 \leq j \leq p_i - 1$, we deduce $[x^+_{*,k}, x^-_{[i,j],l}] = 0$ by applying $\mathrm{ad}(h_{[i,j-1],l-1})$ to $[x^+_{*,k}, x^-_{[i,j],1}] = 0$. □

8.5

Now we consider the following relation:

$$x^{\pm}_{s,k+1}x^{\pm}_{t,l} - v^{\pm a_{st}}x^{\pm}_{t,l}x^{\pm}_{s,k+1} = v^{\pm a_{st}}x^{\pm}_{s,k}x^{\pm}_{t,l+1} - x^{\pm}_{t,l+1}x^{\pm}_{s,k}.$$

Lemma 8.9. *For any $1 \leq i \leq n$, $l < 0$ and $k \in \mathbb{Z}$,*

$$x^+_{*,k+1}x^+_{[i,1],l} - v^{-1}x^+_{[i,1],l}x^+_{*,k+1} = v^{-1}x^+_{*,k}x^+_{[i,1],l+1} - x^+_{[i,1],l+1}x^+_{*,k}.$$

Proof. We have already known that

$$x^+_{*,k+1}x^+_{[i,1],0} - v^{-1}x^+_{[i,1],0}x^+_{*,k+1} = v^{-1}x^+_{*,k}x^+_{[i,1],1} - x^+_{[i,1],1}x^+_{*,k}.$$

The required result can be obtained by applying $\mathrm{ad}(h_{[i,1],l-1})$ to the above formula. □

Lemma 8.10. $[x^+_{*,k}, x^+_{[i,j],l}] = 0$ *for* $1 \le i \le n, 2 \le j \le p_i - 1,\ l < 0$ *and* $k \in \mathbb{Z}$.

Proof. Just apply $\mathrm{ad}(h_{[i,1],l-1})$ to $[x^+_{*,k}, x^+_{[i,j],0}] = 0$. □

8.6

Finally we deal with the following relation:

$$\mathrm{Sym}_{k_1,\dots,k_n} \sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix} x^\pm_{i,k_1} \cdots x^\pm_{i,k_t} x^\pm_{j,l} x^\pm_{i,k_{t+1}} \cdots x^\pm_{i,k_n} = 0,$$

where $i \neq j$ and $n = 1 - a_{ij}$.

Lemma 8.11. *For any* $k_1 \le 0, 1 \le i \le n, k_2,\ t \in \mathbb{Z}$

$$\mathrm{Sym}_{k_1,k_2}\{x^+_{[i,1],k_1}x^+_{[i,1],k_2}x^+_{*,t} - [2]x^+_{[i,1],k_1}x^+_{*,t}x^+_{[i,1],k_2} + x^+_{*,t}x^+_{[i,1],k_1}x^+_{[i,1],k_2}\} = 0.$$

Proof. For $k_1 = 0, k_2 = 0$ and $t \in \mathbb{Z}$, we know that

$$\mathrm{Sym}_{0,0}\{x^+_{[i,1],0}x^+_{[i,1],0}x^+_{*,t} - [2]x^+_{[i,1],0}x^+_{*,t}x^+_{[i,1],0} + x^+_{*,t}x^+_{[i,1],0}x^+_{[i,1],0}\} = 0.$$

Applying $\mathrm{ad}(h_{*,k_1})$, we get

$$\begin{aligned} 0 = {} & -\frac{[k_1]}{k_1}\mathrm{Sym}_{k_1,0}\{x^+_{[i,1],k_1}x^+_{[i,1],0}x^+_{*,t} - [2]x^+_{[i,1],k_1}x^+_{*,t}x^+_{[i,1],0} + x^+_{*,t}x^+_{[i,1],k_1}x^+_{[i,1],0}\} \\ & + \frac{[2k_1]}{k_1}\mathrm{Sym}_{0,0}\{x^+_{[i,1],0}x^+_{[i,1],0}x^+_{*,t+k_1} \\ & - [2]x^+_{[i,1],0}x^+_{*,t+k_1}x^+_{[i,1],0} + x^+_{*,t+k_1}x^+_{[i,1],0}x^+_{[i,1],0}\}. \end{aligned}$$

Hence we have

$$\mathrm{Sym}_{k_1,0}\{x^+_{[i,1],k_1}x^+_{[i,1],0}x^+_{*,t} - [2]x^+_{[i,1],k_1}x^+_{*,t}x^+_{[i,1],0} + x^+_{*,t}x^+_{[i,1],k_1}x^+_{[i,1],0}\} = 0.$$

Then apply $\mathrm{ad}(h_{*,k_2})$ to the above equation, which yields

$$\begin{aligned} 0 = {} & -\frac{[k_2]}{k_2}\mathrm{Sym}_{k_1+k_2,0}\{x^+_{[i,1],k_1+k_2}x^+_{[i,1],0}x^+_{*,t} \\ & - [2]x^+_{[i,1],k_1+k_2}x^+_{*,t}x^+_{[i,1],0} + x^+_{*,t}x^+_{[i,1],k_1+k_2}x^+_{[i,1],0}\} \\ & + \frac{[2k_2]}{k_2}\mathrm{Sym}_{k_1,0}\{x^+_{[i,1],k_1}x^+_{[i,1],0}x^+_{*,t+k_2} \\ & - [2]x^+_{[i,1],k_1}x^+_{*,t+k_2}x^+_{[i,1],0} + x^+_{*,t+k_2}x^+_{[i,1],k_1}x^+_{[i,1],0}\} \\ & - \frac{[k_2]}{k_2}\mathrm{Sym}_{k_1,k_2}\{x^+_{[i,1],k_1}x^+_{[i,1],k_2}x^+_{*,t} - [2]x^+_{[i,1],k_1}x^+_{*,t}x^+_{[i,1],k_2} + x^+_{*,t}x^+_{[i,1],k_1}x^+_{[i,1],k_2}\}. \end{aligned}$$

The proof is completed. □

Lemma 8.12. *For any $r < 0$, $1 \le i \le n$, $t_1, t_2 \in \mathbb{Z}$*

$$\mathrm{Sym}_{t_1,t_2}\{x^+_{*,t_1}x^+_{*,t_2}x^+_{[i,1],r} - [2]x^+_{*,t_1}x^+_{[i,1],r}x^+_{*,t_2} + x^+_{[i,1],r}x^+_{*,t_1}x^+_{*,t_2}\} = 0.$$

Proof. For any t_1, t_2, the following holds:

$$\mathrm{Sym}_{t_1,t_2}\{x^+_{*,t_1}x^+_{*,t_2}x^+_{[i,1],0} - [2]x^+_{*,t_1}x^+_{[i,1],0}x^+_{*,t_2} + x^+_{[i,1],0}x^+_{*,t_1}x^+_{*,t_2}\} = 0.$$

Now applying $\mathrm{ad}(h_{*,l})$, we get

$$\begin{aligned}0 = &-\frac{[l]}{l}\mathrm{Sym}_{t_1+l,t_2}\{x^+_{*,t_1+l}x^+_{*,t_2}x^+_{[i,1],0} - [2]x^+_{*,t_1+l}x^+_{[i,1],0}x^+_{*,t_2} + x^+_{[i,1],0}x^+_{*,t_1+l}x^+_{*,t_2}\}\\ &-\frac{[l]}{l}\mathrm{Sym}_{t_1,t_2+l}\{x^+_{*,t_1}x^+_{*,t_2+l}x^+_{[i,1],0} - [2]x^+_{*,t_1}x^+_{[i,1],0}x^+_{*,t_2+l} + x^+_{[i,1],0}x^+_{*,t_1}x^+_{*,t_2+l}\}\\ &+\frac{[2l]}{l}\mathrm{Sym}_{t_1,t_2}\{x^+_{*,t_1}x^+_{*,t_2}x^+_{[i,1],l} - [2]x^+_{*,t_1}x^+_{[i,1],l}x^+_{*,t_2} + x^+_{[i,1],l}x^+_{*,t_1}x^+_{*,t_2}\}.\end{aligned}$$

Thus the relation holds for all $l < 0$. □

9. Remarks on derived equivalence and PBW-basis

In this section we restrict to the case that $\mathfrak{g}$ is of finite type, i.e. $\mathcal{L}\mathfrak{g}$ is an affine Kac–Moody algebra.

9.1. Derived equivalences and double Hall algebras

In this case, the associated star-shaped Dynkin diagram Γ is of type A-D-E. Denote by $\widehat{\Gamma}$ the corresponding extended Dynkin diagram. We know that the category $\mathrm{Coh}(\mathbb{X})$ is derived equivalent to mod Λ where Λ is the path algebra of $\widehat{\Gamma}$ (hence Λ is a tame hereditary algebra). More precisely, let μ be the *slope* function for coherent sheaves and χ be the *Euler characteristic* of the weighted projective line $\mathbb{X}$ (see [10] for missing definitions). We have the following theorem.

Theorem 9.1 (*[10]*). *The direct sum T of a representative system of indecomposable bundles E with slope $0 \le \mu(E) < \chi$ is a tilting object of* $\mathrm{Coh}(\mathbb{X})$ *whose endomorphism ring is isomorphic to Λ. Thus we have*

$$\mathscr{D}^b(\mathrm{Coh}(\mathbb{X})) \simeq \mathscr{D}^b\,(\mathrm{mod}\ \Lambda).$$

Recently Cramer has proved the following result, which asserts that the double Hall algebra is invariant under derived equivalences.

Theorem 9.2 (*[6]*). *Let $\mathscr{A}$ and $\mathscr{B}$ be two k-linear finitary hereditary categories. Assume that there is an equivalence of triangulated categories $D^b(\mathscr{A}) \xrightarrow{\mathbb{F}} D^b(\mathscr{B})$. Let $R(\mathscr{A}) \xrightarrow{\widehat{\mathbb{F}}} R(\mathscr{B})$ be the induced equivalence of the root categories. Then there is an algebra isomorphism $\mathbb{F}: \mathbf{DH}(\mathscr{A}) \longrightarrow \mathbf{DH}(\mathscr{B})$ uniquely determined by the following property. For any object X in $\mathscr{A}$ such that $\mathbb{F}(X) \simeq \widehat{X}[-n_{\mathbb{F}}(X)]$ with $\widehat{X}$ in $\mathscr{B}$ and $n_{\mathbb{F}}(X) \in \mathbb{Z}$ we have:*

$$\mathbb{F}(u^{\pm}_X) = v^{-n_{\mathbb{F}}(X)\langle\widehat{X},\widehat{X}\rangle}u^{\pm\overline{n_{\mathbb{F}}(X)}}_{\widehat{X}}K^{\pm n_{\mathbb{F}}(X)}_{\widehat{X}}$$

where $\overline{n_{\mathbb{F}}(X)} = +$ (resp. -) if $n_{\mathbb{F}}(X)$ is even (resp. odd). For $\alpha \in K_0(\mathscr{A})$, we have $\mathbb{F}(K_\alpha) = K_{\mathbb{F}(\alpha)}$.

9.2. Incompatibility of some homomorphisms

Now let us consider the following diagram

$$\begin{array}{ccccc} U_v(\hat{\mathfrak{g}}) & \xrightarrow{\Omega} & \mathbf{DC}(\mathrm{mod}\Lambda) & \xrightarrow{\iota_1} & \mathbf{DH}(\mathrm{mod}\Lambda) \\ \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle\mathbb{F}} \\ U_v(\mathcal{L}\mathfrak{g}) & \xrightarrow{\Xi} & \mathbf{DC}(\mathrm{Coh}(\mathbb{X})) & \xrightarrow{\iota_2} & \mathbf{DH}(\mathrm{Coh}(\mathbb{X})) \end{array}$$

where ψ is the Drinfeld–Beck isomorphism, Ω is the isomorphism in Theorem 2.3, Ξ is the epimorphism given by Theorem 5.5, ι_1, ι_2 are natural embeddings and $\mathbb{F}$ is the isomorphism in Theorem 9.2.

When $\mathfrak{g} = \mathfrak{sl}_2$, we know that Λ is the path algebra of the Kronecker quiver and $\mathbb{X}$ is the (non-weighted) projective line $\mathbb{P}^1$. In this case it has been proved in [5] that Ξ is an isomorphism and the above diagram is commutative. This is equivalent to say that the restriction of the isomorphism $\mathbb{F}$ to $\mathbf{DC}$ (mod Λ) gives the isomorphism

$$\Xi \circ \psi \circ \Omega^{-1} : \mathbf{DC}\ (\mathrm{mod}\ \Lambda) \simeq \mathbf{DC}(\mathrm{Coh}(\mathbb{X})).$$

However, for the other cases, the diagram may not be commutative even if Ξ is an isomorphism. The reason is as follows.

Denote by E_m the Chevalley generators of the standard positive part $U_v^+(\hat{\mathfrak{g}})$. Here $m \in \Gamma_0 \cup \{e\}$, where e denotes the extending vertex of $\widehat{\Gamma}$. By definition of Ω, the image of each E_m in $\mathbf{DH}$ (mod Λ) is a simple Λ-module.

On the other hand, the Drinfeld–Beck isomorphism ψ sends E_m to $x_{m,0}^+$ for all m except the extending vertex. Now if m is not the central vertex $*$, by Theorem 5.5, the image of $x_{m,0}^+$ in $\mathbf{DH}(\mathrm{Coh}(\mathbb{X}))$ under the homomorphism Ξ is a simple sheaf lying on the bottom of some non-homogeneous tube.

Thus if the diagram is commutative, we should have a derived equivalence functor $\mathbb{G} : \mathscr{D}^b$ (mod Λ) $\simeq \mathscr{D}^b(\mathrm{Coh}(\mathbb{X}))$ assigning all simple Λ-modules, except two (corresponding to the vertex $*$ and e), to sheaves on the bottom of non-homogeneous tubes. However, this is impossible for types D and E.

Note that in [4] it has been proved that the isomorphism $\mathbb{F}$ restricts to an isomorphism $\mathbf{DC}$ (mod Λ) $\simeq \mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$, which we still denote by $\mathbb{F}$. Then the composition $\mathbb{F} \circ \Omega \circ \psi^{-1}$ gives an isomorphism $U_v(\mathcal{L}\mathfrak{g}) \simeq \mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$. But it is difficult to explicitly find out Drinfeld's generators and relations in $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$ through this isomorphism.

9.3. Two PBW-type bases

Now we have two realizations of the quantum affine algebra $U_v(\hat{\mathfrak{g}})$ arising from Hall algebras of two different hereditary categories which are derived equivalent. Note that the derived equivalence $\mathscr{D}^b(\mathrm{Coh}(\mathbb{X})) \simeq \mathscr{D}^b$ (mod Λ) is "visible" if one looks at the Auslander–Reiten-quivers. Recall that the AR-quiver of $\mathrm{Coh}(\mathbb{X})$ consists of two components, the locally free part $\mathscr{F}$ and the torsion part $\mathscr{T}$, while there are three components in the AR-quiver of mod Λ, namely the preprojective component $\mathscr{P}$, the preinjective one $\mathscr{I}$ and the regular one $\mathscr{R}$. Roughly speaking, the derived equivalence is given by splitting the component $\mathscr{F}$ into two pieces corresponding to $\mathscr{P}$ and $\mathscr{I}[-1]$ respectively, and identifying $\mathscr{T}$ with $\mathscr{R}$.

The Hall algebra approach has many advantages. For example, the Hall algebra has a natural basis indexed by the isomorphism classes of objects in the category. Moreover, the structure of the category (e.g. the AR-quiver) gives us more information. In particular, one can construct a PBW-type basis encoding the structure of the category. This has been done in [16] for the Hall algebra of mod Λ.

Proposition 9.3 (*[16]*). *The following set of elements*

$$\{u_P E_{\pi_1 \mathbf{c}} E_{\pi_2 \mathbf{c}} \cdots E_{\pi_n \mathbf{c}} T_\omega u_I\}$$

is a basis of the composition algebra $\mathbf{C}$ (mod Λ)*, i.e. a basis of the (standard) positive part* $U_v^+(\hat{\mathfrak{g}})$.

Let us briefly explain the notations in the above proposition: P runs over all preprojective modules and I runs over all preinjective modules. Hence u_P, u_I are basis elements arising from preprojective and preinjective components respectively. For each i, the $E_{\pi_i \mathbf{c}}$ is a certain element in $\mathbf{H}(\mathscr{T}_{\lambda_i})$, where $\mathscr{T}_{\lambda_i}$ is a non-homogeneous tube. These elements were first constructed in [7]; we omit the explicit definition here. $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t)$ runs over all partitions of positive integers. And $T_\omega = T_{\omega_1} T_{\omega_2} \cdots T_{\omega_t}$, where T_r is the element defined in 5.3. Note that the regular component $\mathscr{R}$ is equivalent to the torsion part $\mathscr{T}$.

Similarly we can construct a PBW-type basis for another positive part of $U_v(\hat{\mathfrak{g}})$ using the Hall algebra of Coh($\mathbb{X}$).

Proposition 9.4. *The following set of elements*

$$\{u_V E_{\pi_1 \mathbf{c}} E_{\pi_2 \mathbf{c}} \cdots E_{\pi_n \mathbf{c}} T_\omega\}$$

is a basis of the composition algebra $\mathbf{C}$(Coh($\mathbb{X}$))*, i.e. a basis of* $U_v(\hat{\mathfrak{n}})$.

In this proposition, V runs over all vector bundles in $\mathscr{F}$. Other notations are the same as Proposition 9.3.

Comparing the above two propositions, we can see that the PBW-type bases of two different halves of the quantum affine algebra are related by the derived equivalence of the two categories mod Λ and Coh($\mathbb{X}$).

Acknowledgments

We would like to thank Professor Olivier Schiffmann for his reading and remarks. We are also very grateful to the anonymous referee for his/her valuable comments and suggestions.

The research was supported in part by NSF of China (No. 10631010) and by NKBRPC (No. 2006CB805905). Y. Jiang was also partly supported by SFB/TR 45 in Universität Bonn and SFB 701 in Universität Bielefeld, Germany.

References

[1] P. Baumann, C. Kassel, The Hall algebra of the category of coherent sheaves on the projective line, J. Reine Angew. Math. 533 (2001) 207–233.

[2] J. Beck, Braid group action and quantum affine algebras, Comm. Math. Phys. 165 (1994) 555–568.

[3] I. Burban, O. Schiffmann, On the Hall algebra of an elliptic curve I, Duke Math. J. 161 (2012) 1171–1231.

[4] I. Burban, O. Schiffmann, The composition Hall algebra of a weighted projective line, J. Reine Angew. Math. (2012) http://dx.doi.org/10.1515/crelle.2012.023.

[5] I. Burban, O. Schiffmann, Two descriptions of the quantum affine algebra $U_v(\hat{\mathfrak{sl}}_2)$, Glasg. Math. J. 54 (2012) 283–307.

[6] T. Cramer, Double Hall algebras and derived equivalences, Adv. Math. 224 (2010) 1097–1120.

[7] B. Deng, J. Du, J. Xiao, Generic extensions and canonical bases for cyclic quivers, Canad. J. Math. 59 (2007) 1260–1283.

[8] V. Drinfeld, A new realization of Yiangians and quantum affine algebras, Dokl. Akad. Nauk SSSR 296 (1987) 14–17.

[9] H. Garland, The arithmetic theory of loop groups, Publ. Math. Inst. Hautes Éudes Sci. 52 (1980) 5–136.

[10] W. Geigle, H. Lenzing, A class of weighed projective curves arising in the representation theory of finite-dimensional algebras, in: Singularities, Representations of Algebras and Vector Bundles (Lambrecht, Germany, 1985), in: Lecture Note in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.

[11] J.A. Green, Hall algebras, hereditary algebras and quantum groups, Invent. Math. 120 (1995) 361–377.

[12] A. Hubery, Symmetric functions and the centre of the Ringel–Hall algebra of a cyclic quiver, Math. Z. 251 (2005) 705–719.

[13] A. Hubery, Three presentations of the Hopf algebra $U_v(\widehat{\mathfrak{gl}}_n)$, Preprint. Available at: http://www.maths.leeds.ac.uk/~ahubery/DrinPres.pdf.

[14] N. Jing, On Drinfeld realization of quantum affine algebras, in: The Monster Lie Algebras, vol. 7, Ohio State Univ. Math. Res. Inst. Publ., 1998, pp. 195–206.

[15] M. Kapranov, Eisenstein series and quantum affine algebras, J. Math. Sci. 84 (1997) 1311–1360.

[16] Z. Lin, J. Xiao, G. Zhang, Representations of tame quivers and affine canonical bases, Publ. Res. Inst. Math. Sci. 47 (2011) 825–885.

[17] G. Lusztig, Introduction to Quantum Groups, in: Progress in Mathematics, vol. 110, Birkhäuser, Boston, 1993.

[18] I.G. Macdonald, Symmetric Functions and Hall Polynomials, in: Oxford Math. Monogr., Oxford Univ. Press, New York, 1979.

[19] K. McGerty, The Kronecker quiver and bases of quantum affine $\mathfrak{sl}_2$, Adv. Math. 197 (2005) 411–429.

[20] R.V. Moody, S.E. Rao, T. Yokonuma, Toroidal Lie algebras and vertex representations, Geom. Dedicata 35 (1990) 283–307.

[21] C.M. Ringel, Hall algebras and quantum groups, Invent. Math. 101 (1990) 583–592.

[22] C.M. Ringel, Green's theorem on Hall algebras, in: Representation Theory of Algebras and Related Topics (Mexico City, 1994), in: CMS Conference Proceedings, vol. 19, American Mathematical Socity, Providence, RI, 1996, pp. 185–245.

[23] O. Schiffmann, Noncommutative projective curves and quantum loop algebras, Duke Math. J. 121 (2004) 113–167.

[24] O. Schiffmann, Lectures on Hall algebras, arXiv:math/0611617.

[25] J. Xiao, Drinfeld double and Ringel-Green theory of Hall algebra, J. Algebra 190 (1997) 100–144.